Green Technology Adoption and the Business Cycle

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Abstract

We analyze the adoption of green technology in a dynamic economy affected by random shocks where demand spillovers are the main driver of technological improvements. Firms' beliefs and consumers' anticipations drive the path of the economy. We derive the optimal policy of investment subsidy and the expected time and likelihood of reaching a targeted level of environmental quality under economic uncertainty. This allows us to estimate the value that should be given to the environment in order to avoid an environmental catastrophe as a function of the strength of spillover effects.

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"I believe that ultimately the electric motor will be universally used for trucking in all large cities, and that the electric automobile will be the family carriage of the future. All trucking must come to electricity. I am convinced that it will not be long before all the trucking in New York City will be electric."

Thomas Edison, Automobile Topics, May 1914.

1 Introduction

The increasing number of environmental issues that the world is facing has triggered a wide debate on how to switch toward sustainable development paths. Adoption of green technologies (AGT) is one amongst the main channels through which countries will be able to avoid environmental disasters without harming too much their well-being. However, in addition to the high level of uncertainty attending most environmental issues, adopting new, cleaner technologies is risky for firms. At first, they incur a switching cost: green technologies are often more expensive, less productive, the workforce may not have the skills to operate the new technology, etc. Moreover, in the long run, investment choices may reveal to be inefficient, harming the firms' profitability. Hence, even if a technology is available (at the turn of the 20th century, 38% of american automobiles were powered by electricity) and is endorsed by a preeminent inventor and businessman like Thomas Edison, that does not guarantee that it is the best choice to be made. Both Network externalities and technological spillovers play an important role in determining what is the optimal technology that firms must adopt.[1] Public policies may be designed to help firms to overcome this risk and to invest in green technologies, but the extent of this investment effort also depends on the prevailing economic environment. If the economy is facing a recession, available funds may be scarce, leading firms to postpone investment projects. We may thus expect that the path toward environmental sustainability to be stochastically affected by the same economic shocks that generate the business cycle.

[1]Consider for example that a firm decides to replace its fleet of fuel vehicles by electric ones (EV). If many firms expect EV to be used nationwide in the near future and decide to do the same, it is likely that the network of EV charging stations spreads broadly and using EV may become very convenient and cheap. If instead the rest of the economy turns to hydrogen vehicles, then the few firms that have chosen EV may end-up being penalized.

In this paper, we analyze this problem using a simple AGT model, focusing on an environmental policy that takes the form of subsidizing investment in green technologies. We determine the relationship between the volatility in the adoption path of technology and the value that should be given to the environmental quality (EQ) in order to avoid an environmental disaster. We consider a setting where industrial production continuously harms the environment, which has an intrinsic ability to partly regenerate and recover from past damages as long as EQ has not dropped below a tipping point. By investing in less-polluting technologies, firms contribute to lowering their impact on the environment. Firms' profit depends on their past and present investment choices that are subsumed in a "technology mix" index, a parameter that measures the pollution intensity of their production process. A key feature of our framework is that a firm equipped at a given date with the optimal production process does not means that its machines are the most recent one or the most efficient or innovative. Instead, the profitability of the various production processes depends also on the skills of the workforce that use the technology and the availability of the inputs and maintenance services required by the technology.[2] The more a technology is used, the better the diffusion of knowledge and the networks of related services, implying that the "optimal technology mix" depends on the investment choices made by all firms. Each period, firms devote a certain amount of resources to conform their technology to the most efficient one. To decide on their investment, they form expectations based on private and public information that they aggregate in a Bayesian way. Firms may thus undershoot or overshoot the optimal technology index.[3] The imperfect assessment firms have about their future economic environment result in an industrial sector composed of firms with heterogeneous technological processes. This explains why along any equilibrium path different technologies coexist at the same time, some more largely adopted than others.

Because markets do not reflect the environmental footprint of the econ-

[2]Firms using old and backward machines loose profit opportunities because their productivity is low. But on the other hand, too advanced machines can be detrimental as well. They could require high skilled workforce to operate them, material employed may be difficult and costly to acquire, and they may involve large maintenance costs.

[3]The literature on global games and information aggregation (see, e.g. Morris & Shin, 2002, Angeletos & Werning, 2006) also considers agents whose payoffs from their investments are interdependent (i.e., they have a coordination motive) and have imperfect information an economic fundamentals.

omy, the social planner designs a policy that redirects the firms' technology indexes toward greener mixes. The policy implemented by the social planner internalizes the marginal benefit of AGT and thus of a better EQ on social welfare. However, since business cycles and uncertainty affecting AGT make the path of the economy stochastic, the economy may at some point hit or even pass the tipping point while transiting toward a desired long-term value. A sustainable path thus corresponds to an appropriate valuation of EQ, a "social cost of carbon" (SCC) that offers a safety cushion to society: it must be such that, at a given confidence level, the environmental catastrophe is avoided.

Our model allows us to estimate confidence intervals for the realized paths of AGT and EQ indexes. We can therefore assess the risk of failure of the environmental policy (because of the occurrence of an environmental disaster) under various ranges of parameters. It also allows us to estimate the value of EQ that ensures that an environmental disaster is avoided with at a certain level of confidence. We illustrate these results with a simple quantitative example. Our numerical simulations show a positive relationship between the strength of demand spillovers and this minimum SCC value. The more spillover effects are at work, the more incentives should be given to direct firms' choices toward a sustainable (carbon neutral) path, but also the fastest this aim is reached under the optimal policy.

Our model is related to the large literature on growth and sustainability. In the few integrated assessment models encompassing economic risk such as Golosov et al. (2014) or Traeger (2015), the optimal policy is derived given an exogenous marginal rate of substitution between consumption and environment. Moreover, there is no absorbing lower bound in the dynamics of the environmental quality and even though there is a risk that the environmental policy is not optimal given the realized shocks, there will be no irreversible consequences. Our model instead gives us a simple tool to derive the carbon price such that given the optimal green technology subsidy, EQ avoids hitting a tipping point with a large enough probability.

Most of the literature on sustainable growth focuses on environmental uncertainty, that is, uncertainty on the frequency of catastrophic environmental events, on the size of the damage, or on the ability of the environment to recover from pollution (see, e.g. Tsur & Zemel, 1998, or De Zeeuw & Zemel, 2012, for analysis without AGT, and Bretschger & Vinogradova, 2014, and Soretz, 2007, for studies on the sharing of investment between productive capital and abatement measures in a AK setting.) Instead, we focus on shocks that affect the economy (like changes in international prices, demands shocks, etc.) and generate the so-called business cycle. The few papers that describe the responsiveness of optimal abatement policy to business cycles (Jensen & Traeger, 2014), or assess the optimal policy instrument in stochastic environments (Heutel, 2012; Fischer & Heutel, 2013) do not consider AGT.

The literature displaying endogenous green growth focuses on productivity improvements and frontier innovation. This is the case in the AK paradigm where capital-knowledge accumulates with learning by doing (Stockey, 1998), in Lucas-like extensions (Bovenberg & Smulders, 1995), in a product variety framework (Gerlagh & Kuik, 2007) or in the Schumpetarian growth paradigm of destructive creation and directed technical changes (Acemoglu et al., 2012), where new innovations are adopted by markets as soon as they are invented. While frontier innovation is needed to make green technologies competitive compared to 'brown' ones, our focus is on adoption of existing technologies that operates through spillover effects causing a gradual replacement of the stock of old and polluting machines by greener ones. Our approach is close to the literature on endogenous growth viewed as a process of adoption of existing ideas and mutual imitation between firms, as exposed by Eaton & Kortum (1999); Lucas Jr & Moll (2014); Lucas (2009); Perla & Tonetti (2014). In these papers, it is assumed that each agent in the economy is endowed with a certain amount of knowledge ("ideas") and this knowledge evolves through contact with the rest of the population. We adopt a similar approach to describe AGT: while there is no explicit R&D sector in our model, there is a pool of existing technologies with potentials that are more or less exploited depending on the proportion of firms that use them. The distribution of technology used among firms shifts over time according to the firms' incentives to adopt new techniques.

The remaining of the paper is organized as follows: In section 2, we describe the economy and the dynamics of its main variables. In section 3, we derive the optimal policy of investment subsidy that should be implemented by a social planner. In section 4, we derive the economic forecasts allowed by our model. Section 5 is devoted to numerical simulations. The last section concludes.

2 Green technology dynamics

The economy is composed of a continuum of firms, of total mass equal to one, that collectively produce at date t an amount q_t of output, taken as the numeraire, which corresponds to the GDP of the economy. In the following, we abstract from the problem of production per se (in particular, the demand and supply of labor) to focus on the cleanliness of the production processes, i.e. the extent to which firms harm the environment while producing. We take an agnostic stance on the relative productivity of 'brown' and 'green' technologies by supposing that the GDP follows the same dynamic whatever the economy's technological mix of productive capital and more specifically that it is given by the following first-order autoregressive dynamic

$$q_t = gq_{t-1} + \hat{g} + \kappa_t, \tag{1}$$

with $g \geq 0$, $\hat{g} \geq 0$ and where κ_t corresponds to exogenous shocks that affect the economy at date t and is the realization of random variable $\tilde{\kappa}_t \sim \mathcal{N}(0, \sigma_\kappa^2)$.[4] We assume that $g < 1$, which implies that the per period expected increase in GDP, $\hat{g} - (1-g)q_{t-1}$, diminishes over time and converges to $q_S = \hat{g}/(1-g)$.[5] The production processes used by firms are diverse, and in the following we suppose that the adequation of firm i's industrial production process to its economic environment at date t is captured by a real valued parameter x_{it} dubbed 'technology mix' or 'green technology' index, which on average leads to an AGT index of the industrial sector given by $\mu_t \equiv \int x_{it} di$. The mix level x_{it} allows firm i to earn the date-t profit

$$\pi(x_{it}, I_{it}; x_t^\star) = \Pi(x_t^\star) - (x_{it} - x_t^\star)^2/2 - I_{it} \tag{2}$$

where $x_t^\star$ corresponds to the technology mix that is the most efficient at time t, leading to profit level $\Pi(x_t^\star)$, $(x_{it} - x_t^\star)^2/2$ the firm's relative profit loss due to a less effective mix, and I_{it} the investment level that allows it to modify its technology mix according to

$$x_{it+1} = x_{it} + I_{it}. \tag{3}$$

[4]Throughout the paper, a random variable is topped with tilde symbol ($\tilde{\ }$) to distinguish it from its possible values.

[5]The case $g > 1$ corresponds to a steady growth in economic wealth. Sustainable states in that case are not stationary: the economy should instead follow a sustainable balanced path.

A technology mix different from the optimal one is costly because of the specific knowledge and skills required to operate and maintain technologies that are not widely used or the relative scarcity of inputs employed. Firms would ideally being equipped with this most efficient technology mix, but this mix evolves with the diffusion of knowledge and the know-how of workers (the so-called knowledge spillover: the more firms invest, the better the workers' knowledge of new technologies in general), network externality (the easiness to find specific inputs and parts to service the technology) and the engineering and research effort exerted by the machine industry firms that compete to satisfy the demand for means of production in the economy. To grasp these various effects in a parsimonious way, it is assumed that the dynamic of the optimal technology mix is given by

$$x^{\star}_{t+1} = \mu_t + \lambda \int I_{it} di, \tag{4}$$

where parameter $\lambda \in [0, 1)$ captures the machine demand spillover. Using (3) to obtain $\int I_{it} di = \mu_{t+1} - \mu_t$, we get from (4) that

$$x^{\star}_{t+1} = \lambda \mu_{t+1} + (1 - \lambda)\mu_t. \tag{5}$$

Observe that even if investment in green technology was positive in period t ($\mu_{t+1} > \mu_t$), the expected next period ideal mix is lower than the corresponding average mix μ_{t+1}. As shown below, demand spillovers embodied in (4) are not sufficient to spur investment in green technology under laissez-faire.

Investment I_{it} allows firm i to adapt its technology to the economic environment that will prevail the next period by buying new equipment or, for large investment, by replacing an entire production line. The firm chooses this investment level according to its beliefs on the future values of the optimal technology parameter. As firms make their investment decisions simultaneously each period, they must somehow anticipate the extent of the resulting total investment. This leads to an intertemporal coordination problem that is formalized as a succession of global games taking place each period: firms form their anticipations on the efficient technology mix thanks to a public and firms' private (idiosyncratic) signals, $\tilde{\omega}_t = x^{\star}_{t+1} + \tilde{\eta}_t$, and $\tilde{w}_{it} = x^{\star}_{t+1} + \tilde{\varepsilon}_{it}$ respectively, where $\tilde{\eta}_t$ and $\tilde{\varepsilon}_{it}$ are independently and normally distributed noises with zero mean and precision $\tau_\eta = \sigma_\eta^{-2}$ and $\tau_\varepsilon = \sigma_\varepsilon^{-2}$ verifying $\mathbb{E}[\tilde{\varepsilon}_{it}\tilde{\varepsilon}_{jt}] = 0$ for all i, j and $\int \tilde{\varepsilon}_{it} di = 0$. These signals allow firms to (imperfectly) coordinate their investment levels each period although their decisions are taken independently (in particular, investment plans are not best-reply functions). This dynamic global game setup is solved sequentially: our focus is on Markov perfect equilibria where $x_t^\star$ is a state variable that firms must anticipate each period to decide on their investment levels. More specifically, Bayesian updating implies that firm i's posterior beliefs about $x_{t+1}^\star$ given its signals are normally distributed with mean

$$\hat{x}_{it+1} \equiv \mathbb{E}\left[\tilde{x}_{t+1}^\star | \omega_t, w_{it}\right] = \frac{\tau_\eta \omega_t + \tau_\varepsilon w_{it}}{\tau_\eta + \tau_\varepsilon} = x_{t+1}^\star + \tau \eta_t + (1-\tau)\varepsilon_{it}, \qquad (6)$$

where $\tau = \tau_\eta/(\tau_\eta + \tau_\varepsilon)$ is the relative precision of the public signal, and variance

$$\hat{\sigma}_{it}^2 = (\tau_\eta + \tau_\varepsilon)^{-1}, \qquad (7)$$

for all i and t. The firm chooses its investment plan to maximize the discounted sum of per period profit (2), which is equivalent to minimizing the sum of expected intertemporal loss of profit compared with the optimal mix. Applying the principle of optimality, the firm's optimal investment plan is derived by using the Bellman equation which at date t is given by

$$\mathcal{V}(x_{it}; x_t^\star) = \min_{I_{it}} (x_{it} - x_t^\star)^2/2 + I_{it} + \delta_t \mathbb{E}_t[\mathcal{V}(x_{it} + I_{it}; x_{t+1}^\star)|\omega_t, w_{it}] \qquad (8)$$

where δ_t is the discount factor corresponding to interest rate r_t, i.e. $\delta_t = (1 + r_t)^{-1}$. It is shown in the appendix that

Lemma 1 *Firm i's equilibrium investment at time t is given by*

$$I_{it} = \hat{x}_{it+1} - x_{it} - r_t. \qquad (9)$$

The firms' technology levels at $t+1$ are normally distributed with mean

$$\mu_{t+1} = \mu_t - \frac{r_t - \tau \eta_t}{1 - \lambda} \qquad (10)$$

corresponding to the date-$t+1$ AGT level of the economy, and variance

$$\sigma_x^2 \equiv (1-\tau)^2 \sigma_\varepsilon^2.$$

According to (9), firms' investment strategy is to adapt their production process to their estimate of the most efficient technology diminished by the price of capital r_t, which leads firms to disinvest. For firms with a low technology level, this strategy corresponds to buying more environmentally friendly equipments. For the others, their investment is directed in the opposite direction: because they have technologies more environmentally efficient than required next period according to their estimate of $x^\star_{t+1}$, they can save on new equipment spending by buying less expensive 'brown' technologies. On average, this heterogeneity in investment policies should somehow be cancelled out, but while this is true for idiosyncratic noises, the public signal η_t distorts firms choices in the same direction to an extent that depends on its reliability τ: the better the reliability, the larger the distortion. These distortions modify stochastically the dynamic of the AGT index as given by (10), which otherwise decreases with the cost of capital r_t. The coordination problem that affects firms' investment choices translates in the dynamic of the AGT index (10) through a 'magnifying' factor $(1-\lambda)^{-1}$: the larger λ, the larger the effects of the public signal and of the cost of capital r_t. As $\mathbb{E}[\mu_{t+1}] = \mu_t - r_t/(1-\lambda)$, the larger λ, the lower the AGT index. Indeed, as indicated by (5), the ideal technology mix in that case is close to the next period AGT index when λ is close to 1. As investment is costly, each firm anticipates that other firms investments will be low, which leads to an equilibrium that tends to the lower bound of the AGT index.

Due to the idiosyncratic shocks, firms have different expectations on the optimal mix level. These discrepancies lead to a Gaussian distribution of firms' green indexes around the AGT level with a dispersion that is larger the better the relative precision of the idiosyncratic shocks $1-\tau$. Hence, the industrial sector can be thought of as a 'cloud' of firms with a technology level that is drawn each period from a normal distribution centered on the AGT index μ_t with standard deviation σ_x.

To counteract the negative effect of the interest rate on green investment, we assume in the following that the government implements at date t_0 a subsidy policy plan $\{z_t\}_{t\geq t_0}$ leading to a per period investment cost $r_t - z_t$. The dynamic of the AGT index (10) then becomes

$$\mu_{t+1} = \mu_t + \frac{z_t - r_t + \tau\eta_t}{1-\lambda} \tag{11}$$

and this dynamic is positive in expectation if $z_t > r_t$, i.e. if the net cost of capital is negative.

The environment

Production generates pollution which harms the environment, but this detrimental effect can be reduced if firms improve their technology parameter. This mechanism is embodied in the following dynamic of the EQ index

$$e_{t+1} = \theta e_t + \hat{e} - \iota_t q_t \tag{12}$$

where $\theta < 1$, $\hat{e} \geq 0$ and where ι_t is the emission intensity of the technology mix at date t which measures the damage to the environment coming from the human activities per unit of GDP. $\hat{e}$ is the per period maximum regeneration capacity of the environment, the actual regeneration level reached at period t being $\hat{e} - (1-\theta)e_{t-1}$, which depends (positively) on the environmental inertia rate θ. Without human interferences ($\iota_t = 0$), the EQ index is at its pristine level $e_N = \hat{e}/(1-\theta)$. Emission intensity is related to the AGT index by

$$\iota_t = \varphi - \xi \mu_t / q_t \tag{13}$$

where φ is the maximum emission intensity and $\xi > \sqrt{(1-\theta)\varphi}$ so that green technologies are sufficiently effective. Substituting for ι_t in (12), we obtain a dynamic of the EQ index that follows the linear first-order recursive equation

$$e_{t+1} = \theta e_t + \xi \mu_t - \varphi q_t + \hat{e}. \tag{14}$$

We suppose that $\varphi q_S > \hat{e}$, implying that without AGT, the environment will collapse once production is sufficiently large and will eventually reach the "tipping point" $\bar{e}$ that should not be passed permanently: if pollution is too large too often, the resilience of the environment is at stake, i.e. abrupt shifts in ecosystems may happen with dire and irreversible consequences for society. On the other hand,

Definition 1 (Carbon Neutral Path) *The economy has reached at date T a Carbon Neutral Path (CNP) if for all $t \geq T$, $\mathbb{E}[\tilde{\iota}_t] = 0$.*

Therefore, a CNP is a sustainable situation in which the emission intensity of the economy is null in expectation. The AGT subsidy that is required once the economy as reached a CNP should allow for μ_t to stay at $\varphi q_t / \xi$ in expectations. CNPs are of most interest because we consider green technologies aiming at reducing emissions (i.e. they do not allow for direct improvement of EQ) and thus carbon neutrality is the best society can hope for to improve the environment. Along a CNP, thanks to the natural regeneration capacity of the environment, the average EQ increases and tends toward its pristine level e_N.

However, reaching a CNP may prove to be too demanding and society may end up stabilizing around an expected EQ level $e_T < e_N$. This sustainable situation corresponds to the (weaker) notion of stable environment path:

Definition 2 (Stable Environment Path) *The economy has reached at date T a Stable Environment Path (SEP) at level e_S if for all $t \geq T$, $\mathbb{E}[\tilde{\iota}_t \tilde{q}_t] = (1-\theta)(e_N - e_S)$.*

Under an SEP, technological improvements must fill the gap between the environmental damages due to economic growth and the regenerative capacity of the environment to maintain EQ at the desired level over time. Compared to its CNP levels, the expected AGT index is lower by a constant proportional to the difference between the pristine level of EQ e_N and the stabilized level e_S : $(1-\theta)(e_N - e_S)$ correspond to the per-period loss of EQ compared to the pristine level that society does not compensate. Nevertheless, to stay at EQ level e_S, the AGT index has to increase at the same pace as the emissions coming from aggregate production.

Consumers

Although EQ has an impact on consumers' welfare, we suppose that they do not try to modify the environment through their consumption and saving plans. This could be the case because they consider that they are too numerous for their individual behavior to have a significant impact on the environmental path.[6] Accordingly, we model their behavior by considering a representative consumer who maximizes at each date t her intertemporal utility arbitraging between consumption and savings every periods so that her saving and consumption plans solve

$$\max \mathbb{E}_t \left\{ \sum_{\tau=t}^{+\infty} \beta^{\tau-t} u(\tilde{c}_\tau, \tilde{e}_\tau) : \tilde{c}_\tau = \tilde{R}_\tau + \tilde{r}_{\tau-1}\tilde{S}_{\tau-1} - \tilde{s}_\tau, \tilde{s}_\tau = \tilde{S}_\tau - \tilde{S}_{\tau-1} \right\}$$

where c_t and R_t are her date-t consumption and revenue, S_{t-1} her savings from the previous period, $r_{t-1}S_{t-1}$ the corresponding date-t capital earnings,

[6] The environmental quality being a public good, this reasoning is grounded by the standard free rider argument that results in the underprovision of public goods.

s_t the savings adjustment of period t, and β the psychological discount factor. At each date t, the Bellman equation corresponding to the consumer's problem can be written as

$$v(S_{t-1}; e_t) = \max_{s_t} u(r_{t-1}S_{t-1} + R_t - s_t, e_t) + \beta \mathbb{E}_t v(S_{t-1} + s_t; \tilde{e}_{t+1})$$

where S_t and s_t are the state and the control variables respectively. The first-order equation is given by

$$\frac{\partial u(c_t, e_t)}{\partial c} = \beta \mathbb{E}_t \left[\frac{\partial v(S_t; \tilde{e}_{t+1})}{\partial S} \right] \tag{15}$$

and the envelope theorem gives

$$\frac{\partial v(S_{t-1}; e_t)}{\partial S} = r_{t-1} \frac{\partial u(c_t, e_t)}{\partial c} + \beta \mathbb{E}_t \left[\frac{\partial v(S_t; \tilde{e}_{t+1})}{\partial S} \right].$$

Replacing the last term using (15), we get

$$\frac{\partial v(S_{t-1}; e_t)}{\partial S} = (1 + r_{t-1}) \frac{\partial u(c_t, e_t)}{\partial c}.$$

Taking the expectation and replacing in (15) yields

$$\frac{\partial u(c_t, e_t)}{\partial c} = (1 + r_t) \beta \mathbb{E}_t \left[\frac{\partial u(\tilde{c}_{t+1}, \tilde{e}_{t+1})}{\partial c} \right] \tag{16}$$

where $1+r_t$ on the RHS is factorized out of the expected value since the date-t interest rate is a known parameter. Expectation is taken over all possible date-$t+1$ consumption/net-production levels that depend on the consumer's expectation about the firms' investment decisions at that date. Equation (16) corresponds to the supply function of capital, while (11) is the demand side coming from firms. At the date-t equilibrium on the capital market, the interest rates embodied in (16) and (11) are equal. Moreover, at the good market equilibrium, aggregate production net of investment must be equal to total consumption, i.e.[7]

$$c_t = q_t - \int I_{it} di = q_t - \mu_{t+1} + \mu_t. \tag{17}$$

[7] Observe that the firm's relative loss $(x_{it} - x_t^\star)^2/2$ does not appear in (17). Indeed, this cost corresponds to supplementary expenses like payroll outlays, external services, etc., that are revenues for the other agents in the economy.

Together, these two conditions allow us to determine the global equilibrium that the economy reaches each period.

This is done in the following by assuming that the representative consumer forms rational expectations over the future states of the economy and, whenever it is necessary to complete the analysis, that her preferences are CARA and that consumption and environmental quality can be subsumed in a 'global wealth index' denoted $y_t \equiv c_t + \hat{\rho} e_t$. Here, $\hat{\rho}$ is the implicit price of the environment, assumed to be the same whatever the GDP of the economy.[8] Under these assumptions, we have $u(c_t, e_t) = -e^{-\gamma y_t}$ where γ corresponds to the coefficient of absolute risk aversion. Assuming that the global wealth index y_{t+1} is normally distributed (which is the case at equilibrium), we get from (16) using $\mathbb{E}[e^{-\gamma \tilde{y}}] = e^{-\gamma(\mathbb{E}[\tilde{y}] - \gamma \mathbb{V}[\tilde{y}]/2)}$, $\beta = e^{-\psi}$ where ψ is the intrinsic discount factor, and $1 + r_t \approx e^{r_t}$, a supply function of capital satisfying[9]

$$r_t = \psi + \gamma(\mathbb{E}_t[\tilde{y}_{t+1}] - y_t) - \gamma^2 \mathbb{V}_t[\tilde{y}_{t+1}]/2 \tag{18}$$

which exhibits the familiar effects that determine the rental price of capital: the intrinsic preference for an immediate consumption ψ, the economic trend of the global wealth index that also encourages immediate consumption if it is positive, and as revealed by the last term, a precautionary effect that operates in the opposite direction and corresponds to a risk premium due to the uncertainty affecting the economy. Under these assumptions, it is possible to derive the Rational Expectation Equilibrium (REE) of this economy at each period anticipating the environmental policy that will be implemented by the government $\{z_{t+h}\}_{h \geq 0}$. More specifically, it is shown in the appendix that

Lemma 2 *Under an REE with preferences over global wealth and CARA utility, the dynamic of the technological parameter can be approximated by*

$$\mu_{t+1} = a_1 \mu_t + a_2 e_t + a_3 q_t + a_4 + a_5 \tau \eta_t + Z_t \tag{19}$$

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[8] The marginal rate of substitution between economic wealth and the environment is thus constant in that case. It is a reasonable approximation as long as $g < 1$ (i.e. wealth is bounded by q_S) and the environment has not incurred dramatic changes (e_t is above $\bar{e}$). Assuming steady growth ($g > 1$), as the environmental quality is bounded upward, this price should increase at the same pace as GDP along the balanced growth path: $\rho_t = g^t \rho_0$.

[9] While (18) is derived using the approximation $1 + r_t \approx e^{r_t}$, it should be noted that the discrepancy between the exact formulae for the interest rate and (18) is only an artifact of the discrete time setup: the shorter the time period, the better the approximation.

where

$$Z_t = a_5 \sum_{i=0}^{+\infty} (\gamma a_5)^i z_{t+i}, \tag{20}$$

and the distribution of the wealth index $\tilde{y}_{t+1}$ *by a normal distribution with variance*

$$\sigma_y^2 \equiv \mathbb{V}_t[\tilde{y}_{t+1}] = (1-a_3)^2\sigma_\kappa^2 + a_5^2\tau^2\sigma_\eta^2. \tag{21}$$

The coefficient a_1 *is given by*

$$a_1 = 1 - \frac{\sqrt{[1-\theta+(1-\lambda)/\gamma]^2 + 4\hat{\rho}\xi} - [1-\theta+(1-\lambda)/\gamma]}{2}, \tag{22}$$

and the other coefficients are deduced from

$$\begin{aligned}
a_2 &= \frac{(1-a_1)(1-\theta)}{\xi}, a_3 = \frac{\gamma[1-g+\varphi(\hat{\rho}-a_2)]}{1-\lambda+\gamma(2-a_1-g)}, \\
a_4 &= -\frac{\gamma[\hat{e}(\hat{\rho}-a_2)+\hat{g}(1-a_3)]+r_S}{1-\lambda+\gamma(1-a_1)}, \\
a_5 &= 1/[1-\lambda+\gamma(2-a_1)],
\end{aligned} \tag{23}$$

with $r_S \equiv \psi - \gamma^2\sigma_y^2/2$. *Moreover, we have* $a_1 < 1$ *and* $a_1 > 0$ *if* $2 + (1-\lambda)/\gamma - \theta > \hat{\rho}\xi$, $0 < a_2 < \hat{\rho}$, $a_3 > 0$, $0 < a_5 < 1/\gamma$ *and* $a_4 < 0$ *whenever* $r_S \geq 0$, $da_1/d\lambda < 0$, $da_2/d\lambda > 0$, $da_3/d\lambda > 0$ *and* $da_5/d\lambda > 0$.

The equilibrium dynamic of the AGT index follows a linear first-order recursive equation of the state variables (μ_t, e_t, q_t) and of a forward looking term Z_t given by (20) that accounts for the anticipated effects of the environmental policy. This policy index is an exponential smoothing of the policy measures to come. This dynamic is thus consistent only if the policy plan is known.[10] Assuming that $\hat{\rho}$ is not too large, all coefficients are positive except the constant a_4 which is negative ($r_S \geq 0$ is a sufficient condition only for

[10]As detailed in the following section, the optimal policy supposes a constant revision of the schedule $\{z_{t+h}\}_{h>0}$ at each date t to account for the shocks that have affected the economy. That the original policy plan is likely to evolve should be anticipated by the consumers. We suppose however that the consumers consider Z_t as a weighted sum of the policy scheme as announced by the regulator at date t rather than the weighted sum of random variables. This is not a too strong assumption since these weights are decreasing exponentially.

a_4 to be negative).[11] Hence, one may thus expect that under laissez-faire, even if the AGT and EQ indexes have reached very low levels, the increase in the GDP could reverse a negative trend: this is indeed the case when $q_t > -a_4/a_3$ (assuming that a catastrophe is avoided). As expected, the stronger the spillover effects captured by λ, the less the dynamic of the AGT index is dependent on its previous value and the more it is on the GDP and the economic shocks. But as a_5 determines the exponential smoothing coefficient of Z_t, the dynamic is also more reactive to the public policy when λ is large.

Because of the linearity of (19), the AGT index $\tilde{\mu}_{t+2}$ as estimated at date t is Gaussian (while the date-t realizations of the shocks are known, $\tilde{\mu}_{t+2}$ depends on the next period shocks $\tilde{\kappa}_{t+1}$ and $\tilde{\eta}_{t+1}$). From (17), $\tilde{c}_{t+1}$ is thus normally distributed, resulting in global wealth index $\tilde{y}_{t+1}$ which is also Gaussian with variance given by (21). Knowing $\{z_{t+h}\}_{h\geq 0}$, it is possible to infer statistically the state of the economy at horizon h from an initial state at date t by applying recursively (19) together with (1), (14), thanks to the normal distribution of the random shocks.

Using (11) or (18) and (19), it is possible at the beginning of each period to specify the distributions of the equilibrium interest rate that will prevail at that date and of the next period wealth index. More precisely, we have

Lemma 3 *Under an REE with preferences over global wealth and CARA utility, before the current period shock and signals are known, the current interest rate and the next period wealth index are normally distributed. Their variances are given by*

$$\sigma_r^2 = (1-\lambda)^2 a_3^2 \sigma_\kappa^2 + [1-(1-\lambda)a_5]^2 \tau^2 \sigma_\eta^2 \tag{24}$$

and

$$\sigma_{y_{+1}}^2 = \sigma_y^2 + [(1-a_3)g + (1-a_1)a_3 + (a_2-\hat{\rho})\varphi]^2 \sigma_\kappa^2 + [(1-a_1)a_5\tau\sigma_\eta]^2$$

respectively.

Observe that the variance of the interest rate is different from $\gamma^2\sigma_y^2$. This is due to the fact that the two bracketed terms in (18) are correlated random

[11] If $\hat{\rho}$ is very large, coefficient a_1 is negative which generates oscillations is the AGT dynamic (19). We suppose that it is not the case and thus that fluctuations in the AGT index are only the consequences of the shocks affecting the economy.

variables when the current shocks are unknown: the expectation of $\tilde{y}_{t+1}$ without knowing the realization of $\tilde{y}_t$ is a random variable.[12] Similarly, the variance of the next period wealth index differs form σ_y^2 by terms accounting for the correlation between $\tilde{y}_{t+1}$ and $\tilde{y}_t$.

3 Environmental policy

We consider a benevolent social planner who decides on a policy that modifies the dynamics of the AGT level (11) through setting z_t for all $t \geq t_0$ where t_0 is the first period the policy is designed and implemented. The aim of the policy is to correct for the fact that the social value of the environment is not reflected in the interest rate that prevails at equilibrium on the capital market.

Preliminary to specifying the social planer's problem, the question of the social value of the environment should be discussed. At the citizen level, the marginal rate of substitution between environment and consumption allows individuals to link the EQ with the damages they encounter from climate change (or any other kind of pollution effects).[13] Hence, from a social viewpoint, it seems that these evaluations are the relevant values that should guide the social planner in defining the optimal policy. However, these evaluations are likely to be underestimations of the actual contribution of EQ to global welfare. Indeed, a measure of the social value of EQ should encompass all the services provided by the environment and in particular, the fact that pass the tipping point $\bar{e}$, the welfare consequences of environmental degradation are dire and irreversible. Since there is always a non-zero probability that the actual environmental path hits $\bar{e}$ under the optimal policy, the policy must ensure that an environmental disaster is avoided with at least a certain level of confidence, i.e. that $\Pr\{\tilde{e}_t \leq \bar{e}\} \leq \alpha$ for all $t \geq t_0$, where α

[12] Indeed, denoting $\mathbb{E}_{t-1}[\tilde{y}_{t+1}|\tilde{y}_t] = m(\tilde{y}_t)$, it comes from (18) that

$$\tilde{r}_t - \mathbb{E}_{t-1}[\tilde{r}_t] = \gamma\{m(\tilde{y}_t) - \mathbb{E}_{t-1}[m(\tilde{y}_t)] - (\tilde{y}_t - \mathbb{E}_{t-1}[\tilde{y}_t])\},$$

which gives

$$\sigma_r^2 = \gamma^2(\sigma_y^2 + \sigma_{y_{+1}}^2) - 2\gamma cov(m(\tilde{y}_t), \tilde{y}_t).$$

[13] This gives a monetary measure of disutility that may come from a direct impact on welfare, due for example to frequent extreme weather events, or a more indirect one, due to private awareness and concern for environmental issues.

corresponds to the confidence level. The social planner's program must thus solve[14]

$$\max \mathbb{E}_{t_0} \left\{ \sum_{t=t_0}^{+\infty} \beta^{t-t_0} u(\tilde{c}_t, \tilde{e}_t) : \mathbb{E}[\tilde{\iota}_t] \geq 0, \Pr\{\tilde{e}_t \leq \bar{e}\} \leq \alpha \right\} \tag{25}$$

where the first constraint corresponds to the CNP constraint (emission intensity cannot be negative) and the second to the environmental safety constraint. If the individual valuation of EQ is very low, it could be the case that the latter constraint is binding at some date T and that the economy follows an SEP afterward at expected level $e_S > \bar{e}$, the edge depending on α. Or, for larger individual valuations of EQ, that this constraint is only temporarily binding while the economy is on its way toward a CNP.

The dynamic of the economy is affected by the public policy in the following way. From (11), the path of the AGT index evolves stochastically according to[15]

$$\tilde{\mu}_{t+1} = \mu_t + \frac{z_t - \tilde{r}_t + \tau\tilde{\eta}}{1-\lambda}. \tag{26}$$

In addition to modifying directly the AGT path, the policy also indirectly affects the equilibrium on the good market since we have, from (26) and (17),

$$\tilde{c}_t = q_t - \frac{z_t - \tilde{r}_t + \tau\tilde{\eta}}{1-\lambda}. \tag{27}$$

Finally, as we suppose that the social planner does not intervene directly on the capital market, the interest rate at equilibrium must satisfy the Euler equation

$$\mathbb{E}\left[\frac{\partial u(\tilde{c}_t, e_t)}{\partial c}\right] = \beta\mathbb{E}\left[(1+\tilde{r}_t)\frac{\partial u(\tilde{c}_{t+1}, \tilde{e}_{t+1})}{\partial c}\right] \tag{28}$$

[14] Alternatively, we may consider a state dependent utility approach (see, e.g., Karni, 1983). Assuming that pass threshold $\bar{e}$ the utility is null (because human beings are wiped out), the public objective becomes

$$\max \mathbb{E}_{t_0} \left\{ \sum_{t=t_0}^{+\infty} \beta^{t-t_0} \Pr\{e_t > \bar{e}\} u(c_t, e_t) : \mu_t \leq \varphi q_t \right\}.$$

In addition to be much simpler to solve, program (25) does not need an explicit scenario about the catastrophe that is triggered once the tipping point is passed.

[15] The optimal policy is thus a scheme that should be revised each period to account for the current state of the economy.

which is the ex ante equivalent of (16).

We first derive the policy that allows the economy to follow either a CNP or an SEP.

Lemma 4 *The economy follows a CNP or an SEP reached at date T if the government implements the policy $\{z_t\}_{t\geq T}$ given by*

$$z_t = R_t + (1-\lambda)(1-g)g^{t-T}(q_S - q_T)\varphi/\xi \tag{29}$$

where

$$R_t = r_S + \gamma(1-g)g^{t-T}(q_S - q_T)[1+(1-g)\varphi/\xi] + \gamma\hat{\rho}(1-\theta)\theta^{t-T}(e_S - e_T) \tag{30}$$

converges toward r_S as t goes to infinity, with $e_T = e_S < e_N$ under an SEP and $e_T < e_S = e_N$ under a CNP. Under an REE with preferences over global wealth and CARA utility, the corresponding policy index $\{Z_t\}_{t\geq T}$ is given by

$$\begin{aligned} Z_t = \frac{a_5 r_S}{1-a_5\gamma} + \frac{(1-g)g^{t-T}(q_S - q_T)[\gamma + (1-\lambda+\gamma(1-g))\varphi/\xi]}{1-a_5\gamma g} \\ + \frac{\gamma\hat{\rho}(1-\theta)\theta^{t-T}(e_S - e_T)}{1-a_5\gamma g\theta}. \end{aligned} \tag{31}$$

In case of an SEP, denoting by Φ the CDF of the standardized Gaussian variable, $e_S = \bar{e} + \sigma_e\Phi^{-1}(1-\alpha)$ where σ_e is independent of the initial state of the economy.

Sequence $\{R_t\}_{t\geq T}$ given by (30) corresponds to the path of the expected interest rate $\mathbb{E}[\tilde{r}_t]$ along a CNP or an SEP. From (29), it follows that in both cases the gap between the governmental subsidy and the expected interest rate shrinks exponentially. The expected interest rate also diminishes steadily toward its stationary state value r_S. The last term of (30) which involves the difference between the state of the environment at date T and its long term level, is positive in the case of a CNP since the environment keeps improving, while it is equal to zero for an SEP since, by definition, the environment is stabilized at level $e_S < e_N$.

Reaching a CNP or an SEP is a potential long term objective for society. From the initial state of the economy, the optimal policy also entails a transitory path before taking such routes that is derived as follows. Consider the problem of the social planer at date t when the state of the economy is far from both the CNP and the environmental safety constraints. The corresponding Bellman equation is given by

$$\mathcal{W}(\mu_t, e_t, q_t) = \max_{z_t} \mathbb{E}_t[u(\tilde{c}_t, e_t)] + \beta \mathbb{E}_t[\mathcal{W}(\tilde{\mu}_{t+1}, \tilde{e}_{t+1}, \tilde{q}_{t+1})] : (1), (14), (26), (27), (28)]$$

where μ_t, e_t and q_t are state variables and z_t the control variable. It is shown in the appendix that:

Proposition 1 *The production/environment state that solves the relaxed program satisfies*

$$r^e_{t+1} + r^e_t r^e_{t+1} - r^e_t \theta = \xi \frac{\mathbb{E}\left[\partial u(\tilde{c}_{t+2}, \tilde{e}_{t+2})/\partial e\right]}{\mathbb{E}\left[\partial u(\tilde{c}_{t+2}, \tilde{e}_{t+2})/\partial c\right]}, \tag{32}$$

where

$$r^e_t \equiv \frac{\mathbb{E}\left[\tilde{r}_t \partial u(\tilde{c}_{t+1}, \tilde{e}_{t+1})/\partial c\right]}{\mathbb{E}\left[\partial u(\tilde{c}_{t+1}, \tilde{e}_{t+1})/\partial c\right]} = \mathbb{E}[\tilde{r}^\star_t] + \frac{cov\left(\tilde{r}_t, \partial u(\tilde{c}_{t+1}, \tilde{e}_{t+1})/\partial c\right)}{\mathbb{E}\left[\partial u(\tilde{c}_{t+1}, \tilde{e}_{t+1})/\partial c\right]}. \tag{33}$$

The unconstrained dynamic of the economy is thus determined through the sequence of the "corrected" expected optimal interest rate r^e_t that solves (32). Compared to $\mathbb{E}[\tilde{r}^\star_t]$, it takes into account the (positive) correlation between the marginal utility of consumption and the interest rate as shown in (33).

To completely specify this dynamic, we assume in the following a constant implicit price of the environmental quality, a CARA utility function and rational expectations. Using $\partial u(c,e)/\partial e = \hat{\rho} \partial u(c,e)/\partial c$, the right hand term of (32) becomes $\hat{\rho}\xi$. As the individual marginal rate of substitution between environmental quality and consumption may be different from the optimal one as discussed above, denote by ρ the value that the social planer considers in defining the policy. Substituting ρ for $\hat{\rho}$ in (32) yields

$$r^e_{t+1} + r^e_t r^e_{t+1} - r^e_t \theta = \rho \xi \tag{34}$$

which prescribes the evolution of the interest rate from one period to the next along an optimal path when neither the CNP constraint nor the environmental safety constraint are binding. Observe that this dynamic does not depend on the state of the economy (none of the state variables q_t, e_t and μ_t is involved). From its initial value $r^e_{t_0}$, it may converge to a long run level that solves

$$(r^e_\sharp + 1 - \theta) r^e_\sharp = \rho \xi. \tag{35}$$

The left-hand side of (35) is a quadratic equation which is positive for either $r_\sharp^e \le -(1-\theta)$ or $r_\sharp^e \ge 0$, while the right-hand side is strictly positive. Hence $r_\sharp^e$ may take two values, one being positive and the other negative and lower than $-(1-\theta)$. Fig. 1 depicts the situation: the parabola corresponds to the left-hand side of (35) which crosses the x-axis at 0 and $-(1-\theta)$. The horizontal line corresponds to the right-hand side of (35). The negative root of (35) is lower than $-(1-\theta)$, and increasingly so the larger $\rho\xi$. As r_t^e is greater than the optimal expected interest rate that must prevail in the long term on the capital market, only the positive root of (35) is relevant. The following proposition describes the transition of the economy toward this longterm interest rate.

Proposition 2 *Under an REE with preferences over global wealth and CARA utility, the solution of (34) is given by*

$$r_t^e = r_\sharp^e + \frac{A(r_{t_0}^e - r_\sharp^e)k^{t-t_0}}{A + (1-k^{t-t_0})(r_{t_0}^e - r_\sharp^e)} \tag{36}$$

for all $t \ge t_0$, where $A = \sqrt{(1-\theta)^2 + 4\rho\xi}$ is the square root of the discriminant of (35),

$$r_\sharp^e = (A - 1 + \theta)/2 \tag{37}$$

the positive root of (35), and

$$k = \frac{1+\theta-A}{1+\theta+A}. \tag{38}$$

r_t^e converges to $r_\sharp^e$ if $\rho\xi \ne \theta$ and $r_{t_0}^e > -(A+1-\theta)/2$. Convergence is monotonic if $\rho\xi < \theta$ and oscillatory if $\rho\xi > \theta$. The corresponding optimal expected interest rate is given by

$$\mathbb{E}_t[\tilde{r}_t^\star] = r_t^e - (1-\lambda)a_3\gamma\left(\sigma_{y+1}^2 - \sigma_y^2\right) + \gamma[1-(1-\lambda)(a_5-a_3)](1-a_1)^2 a_5^2\tau^2\sigma_\eta^2 \tag{39}$$

The sequence of rates given by (36) allows the social planer to estimate at time t_0 the expected path of the interest rate that should prevail on the capital market using (39) and thus to decide on the policy measure z_t to be implemented. It depends on an initial value $r_{t_0}^e$ deduced from the current state of the economy that should not be too negative. This sequence converges to (37) which depends on the social value of the environment ρ and on the parameters that govern the dynamic of the environment: the environmental inertia θ and the parameter that captures the impact of the technology index on the environment ξ. It should be noted that Prop. 2 describes the dynamic of the economy when neither the CNP constraint nor the environmental safety constraint is binding and that the long-run level $r^\star_\sharp$ (toward which the expected interest rate converges during this phase) is different from r_S.

To detail the convergence of the interest rate given by (36), it is convenient to use the normalized gap between the expected interest rate at horizon h and its long run level which is defined as

$$d_{t_0+h} \equiv (r^e_{t_0+h} - r^e_\sharp)/A = (\mathbb{E}[r^\star_{t_0+h}] - r^\star_\sharp)/A. \tag{40}$$

It is shown in the appendix that

Proposition 3 *Under an REE with preferences over global wealth and CARA utility, the normalized gap at horizon h from t_0 can be derived recursively using $d_{t_0+h} = f(d_{t_0+h-1})$ where*

$$f(x) \equiv kx/[1 + (1-k)x] \tag{41}$$

with k defined by (38) in Prop. 2. This gap converges to 0 provided $d_{t_0} > -1$.

A phase diagram of this recursion is given fig. 2 which depicts the relationship $d_{t+1} = f(d_t)$ in the (d_t, d_{t+1}) plane and where the bisector is used as a "mirror" to project the value d_{t+1} back on the d_t axis. Function $f(\cdot)$ is concave, with a vertical asymptote at $d_t = -1/(1-k)$ and an horizontal one for large values of d_t. We have $f(0) = 0$ and $f'(0) = k < 1$, hence the bisector is located above the graph of function $f(\cdot)$ in the positive quadrant and since $f(-1) = -1$, below it in the negative quadrant for all $d_t \leq -1$. From an initial gap d_{t_0} the arrows describe the recursion toward 0 which occurs for all $d_{t_0} > -1$.

Applying recursively $d_{t_0+h} = f(d_{t_0+h-1})$ gives the expected value of the optimal interest rate from one period to the next and thus allows the government to define the entire policy at time t_0. To initiate the recursion, we suppose that the government is able to determine the expected value of the interest rate at date $t_0 - 1$, the period before the policy is implemented, which allows it to derive the initial normalized gap: $d_{t_0-1} = (r_{t_0-1} - r^\star_\sharp)/A$. The normalized gap of the first period is deduced from this value using $d_{t_0} = f(d_{t_0-1})$. The first period expected optimal interest rate is deduced from $\mathbb{E}_{t_0}[\tilde{r}^\star_{t_0}] = r^\star_\sharp + Ad_{t_0}$, and the sequence of all future ones from applying $\mathbb{E}_{t_0}[\tilde{r}^\star_t] = r^\star_\sharp + Ad_t$ with $d_t = f(d_{t-1})$ for all $t > t_0$. This allows us to analyze the commitment (also dubbed open-loop) strategy that supposes that the government sticks to this plan whatever the state of the economy in the subsequent periods.[16]

4 Policy implementation and Environmental forecast

Assuming that the social planer implements a policy that results in a sequence of expected interest rates $\mathbb{E}_{t_0}[\tilde{r}^\star_t]$ given by (39), eventually followed by a sequence satisfying (30) once the economy has reached either an SEP or a CNP, the expected paths of the AGT and EQ indexes can be anticipated using the system of equations (1), (14) and (19) which gives the recursion

$$\tilde{Y}_t = B_t \tilde{Y}_{t-1} + H\tilde{\nu}_t \tag{42}$$

where $\tilde{Y}_t = (\tilde{\mu}_t, \tilde{e}_t, \tilde{q}_t, 1)'$ is the column vector of state values (with the constant), B_t is the time-dependent transition matrix

$$B_t = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 + Z_t \\ \xi & \theta & -\varphi & \hat{e} \\ 0 & 0 & g & \hat{g} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; H = \begin{bmatrix} a_5 \tau \sigma_\eta & 0 \\ 0 & 0 \\ 0 & \sigma_\kappa \\ 0 & 0 \end{bmatrix},$$

and $\tilde{\nu}_t = (\tilde{\nu}_{1t}, \tilde{\nu}_{2t})'$ is a column vector of independent standardized Gaussian variables. The transition matrix B_t is time-dependent because of the policy index Z_t defined by (20) which is an exponential smoothing of the policy measures that will be implemented in the following periods. This index is given by (31) after the economy has reached either an SEP or a CNP. For the transitory period where the state of the economy is not constrained, using (11) taken in expectation at date t_0, we obtain that the optimal subsidy scheme satisfies

$$z_{t+i} = \mathbb{E}[\tilde{r}^\star_{t+i}] + (1-\lambda)(\mathbb{E}[\tilde{\mu}_{t+i+1}] - \mathbb{E}[\tilde{\mu}_{t+i}]). \tag{43}$$

[16] As indicated in note 15, the optimal policy should be revised each period (feed-back or close-loop strategy). This can be easily done using (41) which can be used as a tool to update the environmental policy each period taking as initial value the observed current gap.

Multiplying each side of (43) by $a_5(\gamma a_5)^i$ and summing over all $i \geq 0$ gives

$$\begin{aligned} Z_t = a_5 \sum_{i=0}^{T-t-1} (\gamma a_5)^i \mathbb{E}[\tilde{r}^\star_{t+i}] + a_5 \sum_{i=T-t}^{+\infty} (\gamma a_5)^i R_{t+i} \\ + a_5(1-\lambda)(1-\gamma a_5) \sum_{i=0}^{+\infty} (\gamma a_5)^i \left(\mathbb{E}[\tilde{\mu}_{t+i+1}] - \mathbb{E}[\tilde{\mu}_{t+i}]\right) \end{aligned} \tag{44}$$

for all $t < T$. Because the last term of (44) involves values of the AGT index that depend on the public policy, it is not possible to obtain the optimal public policy schedule explicitly and a numerical (recursive) procedure is necessary. The recursion proceeds as follows: given a first set of values $\{\mathbb{E}[\tilde{\mu}_t]^{(1)}\}_{t\geq t_0}$, it is possible to estimate $\{Z_t^{(1)}\}_{t\geq t_0}$ from (44) to obtain an initial set of transition matrices $\{B_t^{(1)}\}_{t\geq t_0}$.[17] Then, using (42), a second set of estimated values $\{\mathbb{E}[\tilde{\mu}_t]^{(2)}\}_{t\geq t_0}$ is derived which, once plugged in (44), gives a second estimated set $\{Z_t^{(2)}\}_{t\geq t_0}$ and thus $\{B_t^{(2)}\}_{t\geq t_0}$. One may iterate this procedure until iteration i such that $\{Z_t^{(i)}\}_{t\geq t_0}$ is sufficiently close to $\{Z_t^{(i-1)}\}_{t\geq t_0}$.

As the recursive dynamic (42) is linear, the path it generates follow Gaussian random walks with expected values and variances at time t given by $\mathbb{E}[\tilde{Y}_t] = (\Pi_{i=0}^{t-t_0} B_i) X_{t_0}$ and $\mathbb{V}[\tilde{Y}_{t_0}] = \sum_{i=0}^{t-t_0} (\Pi_{j=0}^{i} B_j) H H' (\Pi_{j=0}^{i} B_j)'$.

5 Numerical simulations

The expected trends of the main economic variables and the confidence intervals in which they are lying can be derived under various assumptions using numerical simulations.[18] Of particular interest is the derivation of the so-called social cost of carbon (SCC) which in our context corresponds to the social value of the environment ρ. A general feature of the model is the convexity of the path of EQ under an optimal policy that leads to a CNP: once the policy is implemented, the EQ index decreases for a few years, then

[17] Of course, date T at which one of the constraints is binding should be derived in the process. We give in the appendix an approximation of dynamic (42) that allows us to determine a first set of values $\{\mathbb{E}[\tilde{\mu}_t]^{(1)}\}_{t\geq t_0}$.

[18] These simulations were realized using Mathematica. The corresponding programs are available from the authors upon request.

reaches a minimum before turning up toward the CNP (see Fig. 5). It is during this phase that the EQ is the most likely to hit the tipping point. In the following simulations, we determine for each set of parameters the minimum value of ρ such that the environmental safety constraint is never binding (SEPs are not considered). More precisely, the SCC that we derive is defined as the minimum value of ρ that allows society to find its way to a CNP without passing the environmental tipping point with confidence level α that is set to 0.01%.[19]

Calibration of the model

In the simulations, the EQ index e_t is defined as the difference between a tipping point level of CO_2 in the atmosphere ℓ_M (i.e. a threshold above which the dynamic of the climate is irreversibly changed and no return to the pre-industrial state of the atmosphere is possible even if green technologies allows us to completely abate GHG emissions) and the level of GHG at date t, ℓ_t, expressed in CO_2 equivalent: $e_t = \ell_M - \ell_t$.[20] Hence, e_t can be thought of as a global "carbon budget" at date t that the economy should not deplete entirely to avoid to be environmentally bankrupt. Parameter θ is determined from the IPCC Fifth Assessment Report which estimates that 100 years after a 100 Gt CO_2 pulse in the preindustrial atmosphere, there would remain 40% of the 100 Gt CO_2 emitted, while after 1000 years 25% would remain. Accordingly, denoting by $\hat{\ell}$ the preindustrial level of CO_2, after an initial period $\ell_0 = \hat{\ell} + 100$, we have $\ell_{100} = \hat{\ell} + 40$ and $\ell_{1000} = \hat{\ell} + 25$. Using (14) without industrial interferences and solving the recursion gives

$$\ell_t = \theta^t(\hat{\ell} + 100) + (1 - \theta^t)\ell_M - \frac{1 - \theta^t}{1 - \theta}\hat{e}.$$

Using the IPCC's estimates for ℓ_{100} and ℓ_{1000} we obtain

$$\ell_M - \hat{\ell} - \frac{\hat{e}}{1 - \theta} = \frac{40 - 100\theta^{100}}{1 - \theta^{100}} = \frac{25 - 100\theta^{1000}}{1 - \theta^{1000}}.$$

[19] While very different from the usual interpretation of the SCC, this definition is in the spirit of approaches that account for catastrophic damages (see, e.g., Weitzman, 2013).

[20] The unit used in the following is the gigaton or Gt shorthand, i.e. 10^9 metric tons. Theses levels are also commonly expressed in atmospheric concentration, the unit being the part per million or ppm shorthand, i.e. 0.01%. Each ppm represents approximately 2.13 Gt of carbon in the atmosphere as a whole, equivalent to 7.77 Gt of CO_2. Conversion values can be found on the dedicated US department of energy website http://cdiac.ornl.gov/pns/convert.html#3.

The last equality can be expressed as $1 - 5x + 4x^{10} = 0$ where $x = \theta^{100}$ which gives $\theta \approx (1/5)^{1/100} \approx 0.984$ independently of the choice of $\hat{\ell}$ and ℓ_M.[21] We deduce parameter $\hat{e}$ by assuming that the EQ index has reached its long term equilibrium $e_N = \ell_M - \hat{\ell}$ in the preindustrial period which gives $\hat{e} = (1 - \theta)e_N$. Considering that $\hat{\ell} = 2176$ Gt CO_2 (280 ppm) and $\ell_M = 5439$ Gt CO_2 (700 ppm), we obtain $\hat{e} \approx 52.1$.[22] Our reference year is 2005 which corresponds to a GHG level equal to 2945 Gt CO_2 (379 ppm) hence an initial EQ index $e_0 = 2494.17$ Gt CO_2.

The AGT index in the reference year is deduced from the World Bank's estimates of the world CO_2 intensity which in our framework is given by function (13).[23] The data show that the world CO_2 intensity has sharply decreased over the second half of the 20th century and has been plateauing since 2000 (the period covered by the data is 1960 - 2013) with a value in the reference year ι_{2005} =0.51kg/US$. We assume in our baseline setup (Table 1) that the maximum emission intensity φ is equal to 6.5kg/US$ (for static comparative exercises, we also consider the cases $\varphi = 7.5$kg/US$ and $\varphi = 5$kg/US$, see Tables 6 and 7), so that the level of the AGT index in the reference year is given by

$$\mu_{2005} = q_{2005}(\varphi - \iota_{2005})/\xi$$

where $q_{2005} = 47$ trillions US$. The effectiveness of the green technologies is captured by the ratio φ/ξ. It is set to 1/2 is the baseline setup, and we also consider the strong effectiveness ($\varphi/\xi = 1$) and the weak effectiveness ($\varphi/\xi = 1/3$) cases (cf. Tables 4 and 5).

Parameters of the GDP dynamics (1) are set to $g = 0.99$ and $\hat{g} = 0.2$. These values correspond to a growth rate of 3.1% for the first year (2005) and a growth rate of 0.9% in year 2055 ($t = 50$) and then 0.4% in year 2105

[21] An obvious root of this equation is $x = 1$. The other roots are complex numbers.

[22] According to the IPCC Fifth Assessment Report, 700 ppm lead to a temperature increase of approximately 4°C, a situation where "many global risks are high to very high." However, several tipping points are considered by climatologists. Candidates include levels corresponding to an irreversible melt of the Greenland ice sheet, the dieback of the Amazon rainforest and the shift of the West African monsoon. Acemoglu et al. (2012) use the atmospheric CO_2 concentration that would lead to an approximate 6°C increase in temperatures. Stern (2007) reports that increases in temperature of more than 5°C will lead among other things to the melting of the Greenland Ice Sheet.

[23] The data can be found on the world bank website: http://data.worldbank.org/indicator/EN.ATM.CO2E.KD.GD;

($t = 100$).[24] Risk aversion coefficient γ is set to 3 and ψ at 10% which corresponds to a stationary state value of the interest rate r_S of approximately 6% in the baseline scenario. The standard deviation of the shocks affecting the GDP is set to $\sigma_\kappa = 1.5$ (trillions US$), which corresponds to 3.19% of the 2005 GDP value. We also consider small and large GDP shocks, these cases corresponding to $\sigma_\kappa = 2$ (Table 2) and $\sigma_\kappa = 1$ (Table 3) respectively. The shocks affecting firms' beliefs are set to $\sigma_\varepsilon = \sigma_\eta = 0.01$ (trillions US$), hence 0.21% of the 2005 GDP value in the baseline. Table 8 reports values assuming $\sigma_\eta = 1$ which is awfully large, but small variations around the baseline proved insignificant (we examined the cases where they are tenfold that of the baseline values and found no significant effects). For each set of parameters, the consumer's marginal rate of substitution between the environment and consumption $\hat{\rho}$ is derived from (18) so that the interest rate matches its 2005 value, equal to 6%.[25] As discussed above, the SCC value ρ corresponds to the minimal value that allows EQ to stay above 0 along the optimal path with a confidence level equal to 99.9%. All tables show the resulting estimates when λ is ranging from 0 to 0.9 by decimal steps.

Results

Table 1 presents the results of our baseline scenario. For each value of λ indicated in the first column, the second column indicates the implied consumer's valuation of the environment $\hat{\rho}$ that leads to an interest rate equal to 6% in 2005 given the other parameter values. The following columns σ_y, σ_μ and r_S give the corresponding values of the (one period) standard deviation of the total wealth, $y_t = c_t + \hat{\rho} e_t$, and of the AGT index μ_t respectively, and the longterm stationary level of the interest rate. The rest of the table reports the environmental policy parameters. Column ρ gives the social cost of CO_2 necessary to avoid an environmental catastrophe with a confidence level of 99.9%, column $r^\star_\sharp$ the corresponding long term expected interest rate that the public policy aims at bringing about on the capital market during the transitory phase toward a CNP, column T indicates the date at which the expected EQ level under this policy merges with the expected CNP. A

[24] In Acemoglu et al. (2012), the innovation function is calibrated so as to obtain a 2% long run growth. In DICE 2010, the economy grows at a rate equal to 1.9% from 2010 to 2100 and 0.9% from 2100 to 2200.

[25] See King & Low (2014), for a measure of world interest rates. According to this study, the 2005 quarterly values of the interest rate are 1.479%, 1.456%, 1.449% and 1.542%.

value equal to 199 indicates that the expected EQ does not reach the CNP in that time horizon (the maximum considered in the simulations). Finally, column e_T indicates the expected EQ level reached a date T if this policy is implemented.

We observe that standard deviations do not evolve monotonically with λ : σ_y first decreases and then increases, while the reverse is true for σ_μ. This is due to the changes in coefficients $\{a_k\}$ that are triggered by the variations of λ and $\hat{\rho}$. The stationary level of the interest rate r_S is also non-monotonic: it first increases and then decreases. Still, it stays relatively close to $r_0 = 6\%$. The expected optimal interest rate $r_\sharp^\star$, that is, the market interest rate that the social planner targets while implementing the optimal policy, is particularly high compared to r_S: while r_S stays around 6%, $r_\sharp^\star$ ranges from 18% to almost 33%.

The consumer's valuation $\hat{\rho}$ is very large for low values of λ and decreases as λ increases, while the trend for the SCC ρ is the opposite: it increases with λ. One can observe that in this baseline case, ρ is lower than $\hat{\rho}$ as long as λ is lower than 0.7. Graphic 3 depicts the evolutions of theses values and shows that they cross approximately around $\lambda = 0.65$. Hence, while the capital market does not value correctly the environment, the government can safely implement a policy that associate a low SCC when λ is low (for example, if $\lambda = 0.5$, the SCC is 15.93\$/t, while the implicit consumers' valuation is 23.79\$/t).[26]

We observe that $r_\sharp^\star$ increases with λ. This can be easily understood: when λ is large, the economy is less dependent on past values of the AGT index and firms' investment choices have a stronger impact on their future profits. Thus, the economy is likely to react stronger to economic incentives. Therefore, the growth rate of y_t is expected to be larger, and the interest rate is as well. The change in $\hat{\rho}$ also drives the way $r_\sharp^\star$ reacts to an increase in λ: market interest rates tend to decrease when $\hat{\rho}$ is low, and because $\hat{\rho}$ decreases with λ, the targeted optimal interest rate $r_\sharp^\star$ should increase in order to reach the same investment level. Contrary to $\hat{\rho}$, the SCC ρ increases with λ. When λ increases, the actual cost of AGT, measured by $r_t - z_t$, increases, which means that given the impact of the policy on the market interest rate, the net subsidy is lower.

[26] We can compare these values to the social costs of carbon found in the literature. Nordhaus (2007), finds a SCC of 11.5 \$/t CO_2 for 2015. More recently, with similar calibration for the rate of pure time preference, Traeger (2015) and Golosov et al. (2014), find SCC of respectively 15.5 and 16.4 \$/t CO_2.

Finally, under the baseline scenario, carbon neutrality is a very long term objective: only when λ is larger than 0.7 is carbon neutrality reached within two centuries (but less than a century if λ is very large).

Tables 2 and 3 allow to assess the effects of the magnitude of shocks affecting the GDP. Not surprisingly, the standard deviations of $\tilde{y}_t$ and $\tilde{\mu}_t$ are increasing with σ_κ, whereas the implied consumers' valuation $\hat{\rho}$ and the long term interest rate r_S are decreasing. The policy parameters are also negatively affected by a change in σ_κ, but to a lower extent: Large shocks slightly lower the optimal interest rate $r_\sharp^\star$ and the SCC, while the CNP is reached faster (and conversely for small shocks). Large GDP shocks lower the market interest rates, by increasing the precautionary motive for savings. This compensates increased variances and explains a lower SCC. Variations in the effectiveness of the green technologies, reported in Tables 4 and 5, have a intuitive impact on the policy parameters: they are larger the less the green technologies are effective. Similarly, the variations in the environmental damages due to industrial emissions reported in Tables 6 and 7 show that the more they impact the environment, the larger the policy parameters are.

Finally, Table 8 shows the impact of a variation in the public signal. It should be noted that the change in σ_η is 100 times larger than the baseline value. Smaller values (a tenfold increase of either the public or the private signal standard deviation) did not show any significant effect. Nevertheless, the standard deviations of $\tilde{y}_t$ and $\tilde{\mu}_t$ are not much affected. Interestingly, the range of ρ is larger than in the baseline (its smaller value is reduced while its larger value is increased) while $r_\sharp^\star$ is reduced (by approximately 3US\$/t).

Using our baseline calibration and assuming $\lambda = .8$, Fig. 4 and Fig. 5 illustrate the dynamics of the AGT and the EQ indexes respectively, starting from the first period of the implementation of the environmental policy (2006). In these figures, the expected paths and the upper and lower limits of a 95% confidence interval around these paths are depicted. Stochastically generated shocks allow us to illustrate the difference between a laissez-faire situation and the corresponding "realized" paths of the indexes under the environmental policy.

Also depicted Fig. 4 is the carbon neutral path (the dashed blue curve indicated μ_t^{CN}) that the expected value of the AGT index under the policy joins at date $T = 147$. The laissez-faire curve shows a positive trend, but significantly below the one under the policy (actually, the curve stays below the lower-bound of the 95% interval). As a result, Fig. 5 shows that the EQ is rapidly deteriorating under laissez-faire (the carbon budget is entirely

Table 1: Carbon price evaluation -Baseline

λ	$\hat{\rho}$	σ_y	σ_μ	r_S	ρ	$r_\sharp^\star$	T	e_T
0	31.88	$9.08 \cdot 10^{-2}$	$5.92 \cdot 10^{-2}$	6.29	11.51	17.96	199	775.08
.1	30.64	$8.97 \cdot 10^{-2}$	$6.03 \cdot 10^{-2}$	6.38	12.08	18.48	199	799.46
.2	29.26	$8.87 \cdot 10^{-2}$	$6.13 \cdot 10^{-2}$	6.46	12.75	19.08	199	835.86
.3	27.7	$8.79 \cdot 10^{-2}$	$6.21 \cdot 10^{-2}$	6.52	13.57	19.79	199	894.33
.4	25.91	$8.73 \cdot 10^{-2}$	$6.27 \cdot 10^{-2}$	6.57	14.6	20.58	199	971.57
.5	23.79	$8.72 \cdot 10^{-2}$	$6.28 \cdot 10^{-2}$	6.58	15.93	21.59	199	1,110.62
.6	21.2	$8.79 \cdot 10^{-2}$	$6.21 \cdot 10^{-2}$	6.52	17.76	22.91	199	1,356.75
.7	17.88	$8.99 \cdot 10^{-2}$	$6.01 \cdot 10^{-2}$	6.36	20.47	24.73	199	1,855.88
.8	13.35	$9.47 \cdot 10^{-2}$	$5.53 \cdot 10^{-2}$	5.96	25.08	27.58	147	1,541.66
.9	6.76	0.11	$4.44 \cdot 10^{-2}$	4.98	34.85	32.76	82	1,046.54

Parameters: $\sigma_\eta = 0.01, \sigma_\kappa = 0.15, \sigma_\epsilon = 0.01, \xi = 6.5/2, \varphi = 6.5, \gamma = 3, \psi = 0.1, r_0 = 6\%, \alpha = 0.1\%, \mu_0 = 8.66, \sigma_\kappa/q_0 = 3.19\%$.

Table 2: Carbon price evaluation -Large GDP shocks

λ	$\hat{\rho}$	σ_y	σ_μ	r_S	ρ	$r_\sharp^\star$	T	e_T
0	29.31	0.13	$7.44 \cdot 10^{-2}$	2.9	11.18	17.15	199	1,189.55
.1	28.2	0.12	$7.60 \cdot 10^{-2}$	3.08	11.68	17.65	199	1,216.78
.2	26.95	0.12	$7.74 \cdot 10^{-2}$	3.23	12.27	18.24	199	1,262.76
.3	25.52	0.12	$7.86 \cdot 10^{-2}$	3.36	13	18.92	199	1,336.38
.4	23.85	0.12	$7.94 \cdot 10^{-2}$	3.45	13.9	19.68	199	1,439.4
.5	21.86	0.12	$7.96 \cdot 10^{-2}$	3.47	15.06	20.65	199	1,621.95
.6	19.39	0.12	$7.87 \cdot 10^{-2}$	3.38	16.63	21.87	199	1,946.53
.7	16.21	0.12	$7.60 \cdot 10^{-2}$	3.07	18.91	23.52	166	1,768.81
.8	11.86	0.13	$6.95 \cdot 10^{-2}$	2.34	22.64	26.01	123	1,492.76
.9	5.72	0.14	$5.54 \cdot 10^{-2}$	0.59	29.94	30.24	68	1,057.43

Parameters: $\sigma_\eta = 0.01, \sigma_\kappa = 0.2, \sigma_\epsilon = 0.01, \xi = 6.5/2, \varphi = 6.5, \gamma = 3, \psi = 0.1, r_0 = 6\%, \alpha = 0.1\%, \mu_0 = 8.66, \sigma_\kappa/q_0 = 4.26\%$.

Table 3: Carbon price evaluation - Small GDP shocks

λ	$\hat{\rho}$	σ_y	σ_μ	r_S	ρ	$r^\star_\sharp$	T	e_T
0	33.51	$5.92 \cdot 10^{-2}$	$4.09 \cdot 10^{-2}$	8.43	11.37	18.18	199	491.92
.1	32.19	$5.85 \cdot 10^{-2}$	$4.16 \cdot 10^{-2}$	8.46	11.97	18.68	199	504.08
.2	30.73	$5.78 \cdot 10^{-2}$	$4.22 \cdot 10^{-2}$	8.5	12.69	19.32	199	539.44
.3	29.09	$5.73 \cdot 10^{-2}$	$4.27 \cdot 10^{-2}$	8.52	13.56	20.01	199	578.9
.4	27.22	$5.69 \cdot 10^{-2}$	$4.31 \cdot 10^{-2}$	8.54	14.66	20.86	199	645.85
.5	25.02	$5.69 \cdot 10^{-2}$	$4.32 \cdot 10^{-2}$	8.54	16.08	21.91	199	756.72
.6	22.36	$5.73 \cdot 10^{-2}$	$4.27 \cdot 10^{-2}$	8.52	18.06	23.29	199	954.69
.7	18.95	$5.87 \cdot 10^{-2}$	$4.14 \cdot 10^{-2}$	8.45	21.04	25.23	199	1,360.06
.8	14.31	$6.18 \cdot 10^{-2}$	$3.82 \cdot 10^{-2}$	8.28	26.22	28.3	169	1,596.58
.9	7.47	$6.92 \cdot 10^{-2}$	$3.08 \cdot 10^{-2}$	7.84	37.74	34.19	94	1,036.41

Parameters: $\sigma_\eta = 0.01, \sigma_\kappa = 0.1, \sigma_\epsilon = 0.01, \xi = 6.5/2, \varphi = 6.5, \gamma = 3, \psi = 0.1, r_0 = 6\%, \alpha = 0.1\%, \mu_0 = 8.66, \sigma_\kappa/q_0 = 2.13\%$.

Table 4: Carbon price evaluation - Highly effective GT

λ	$\hat{\rho}$	σ_y	σ_μ	r_S	ρ	$r^\star_\sharp$	T	e_T
0	28.29	0.11	$4.50 \cdot 10^{-2}$	5.03	5.35	17.12	199	685.81
.1	27.01	0.1	$4.53 \cdot 10^{-2}$	5.06	5.68	17.75	199	742.45
.2	25.59	0.1	$4.54 \cdot 10^{-2}$	5.08	6.06	18.46	199	821.93
.3	23.97	0.1	$4.54 \cdot 10^{-2}$	5.08	6.54	19.27	199	927.89
.4	22.1	0.1	$4.51 \cdot 10^{-2}$	5.05	7.14	20.26	199	1,091.92
.5	19.91	0.11	$4.44 \cdot 10^{-2}$	4.98	7.93	21.48	199	1,355.49
.6	17.25	0.11	$4.29 \cdot 10^{-2}$	4.84	9.03	23.07	199	1,821.79
.7	13.94	0.11	$4.04 \cdot 10^{-2}$	4.6	10.68	25.29	158	1,569.13
.8	9.68	0.11	$3.60 \cdot 10^{-2}$	4.15	13.48	28.63	111	1,218.52
.9	4.3	0.12	$2.85 \cdot 10^{-2}$	3.35	19.16	34.41	59	790.66

Parameters: $\sigma_\eta = 0.01, \sigma_\kappa = 0.15, \sigma_\epsilon = 0.01, \xi = 6.5, \varphi = 6.5, \gamma = 3, \psi = 0.1, r_0 = 6.\%, \alpha = 0.1\%, \mu_0 = 4.3, \sigma_\kappa/q_0 = 3.19\%$.

Table 5: Carbon price evaluation - Less effective GT

λ	$\hat{\rho}$	σ_y	σ_μ	r_S	ρ	$r_\sharp^\star$	T	e_T
0	32.88	$8.33 \cdot 10^{-2}$	$6.67 \cdot 10^{-2}$	6.88	20.8	19.96	199	1,063.11
.1	31.63	$8.17 \cdot 10^{-2}$	$6.83 \cdot 10^{-2}$	7	21.6	20.39	199	1,075.53
.2	30.25	$8.01 \cdot 10^{-2}$	$6.99 \cdot 10^{-2}$	7.11	22.57	20.92	199	1,103.45
.3	28.71	$7.86 \cdot 10^{-2}$	$7.14 \cdot 10^{-2}$	7.22	23.75	21.55	199	1,150.22
.4	26.96	$7.73 \cdot 10^{-2}$	$7.27 \cdot 10^{-2}$	7.31	25.24	22.25	199	1,213.49
.5	24.91	$7.64 \cdot 10^{-2}$	$7.36 \cdot 10^{-2}$	7.37	27.15	23.16	199	1,329.64
.6	22.44	$7.62 \cdot 10^{-2}$	$7.38 \cdot 10^{-2}$	7.39	29.72	24.32	199	1,533.17
.7	19.3	$7.74 \cdot 10^{-2}$	$7.26 \cdot 10^{-2}$	7.3	33.48	25.91	199	1,932.57
.8	14.95	$8.16 \cdot 10^{-2}$	$6.84 \cdot 10^{-2}$	7	39.72	28.39	154	1,689.05
.9	8.25	$9.35 \cdot 10^{-2}$	$5.66 \cdot 10^{-2}$	6.07	52.79	32.94	92	1,258.7

Parameters: $\sigma_\eta = 0.01, \sigma_\kappa = 0.15, \sigma_\epsilon = 0.01, \xi = 6.5/3, \varphi = 6.5, \gamma = 3, \psi = 0.1, r_0 = 6.\%, \alpha = 0.1\%, \mu_0 = 12.99, \sigma_\kappa/q_0 = 3.19\%$.

Table 6: Carbon price evaluation - Large emissions potential

λ	$\hat{\rho}$	σ_y	σ_μ	r_S	ρ	$r_\sharp^\star$	T	e_T
0	28.69	$8.92 \cdot 10^{-2}$	$6.08 \cdot 10^{-2}$	6.42	12.2	20.02	199	1,007.51
.1	27.54	$8.82 \cdot 10^{-2}$	$6.18 \cdot 10^{-2}$	6.5	12.73	20.52	199	1,046.48
.2	26.27	$8.73 \cdot 10^{-2}$	$6.27 \cdot 10^{-2}$	6.57	13.35	21.11	199	1,104.98
.3	24.83	$8.65 \cdot 10^{-2}$	$6.35 \cdot 10^{-2}$	6.63	14.11	21.8	199	1,191.06
.4	23.19	$8.61 \cdot 10^{-2}$	$6.39 \cdot 10^{-2}$	6.67	15.07	22.57	199	1,303.89
.5	21.25	$8.61 \cdot 10^{-2}$	$6.39 \cdot 10^{-2}$	6.67	16.29	23.56	199	1,497.03
.6	18.89	$8.69 \cdot 10^{-2}$	$6.31 \cdot 10^{-2}$	6.6	17.96	24.83	199	1,828.48
.7	15.88	$8.91 \cdot 10^{-2}$	$6.09 \cdot 10^{-2}$	6.43	20.41	26.59	170	1,736.22
.8	11.79	$9.42 \cdot 10^{-2}$	$5.58 \cdot 10^{-2}$	6.01	24.5	29.34	127	1,456.48
.9	5.93	0.11	$4.46 \cdot 10^{-2}$	4.99	32.94	34.26	71	1,001.47

Parameters: $\sigma_\eta = 0.01, \sigma_\kappa = 0.15, \sigma_\epsilon = 0.01, \xi = 7.5/2, \varphi = 7.5, \gamma = 3, \psi = 0.1, r_0 = 6.\%, \alpha = 0.1\%, \mu_0 = 8.76, \sigma_\kappa/q_0 = 3.19\%$.

Table 7: Carbon price evaluation - Low emissions potential

λ	$\hat{\rho}$	σ_y	σ_μ	r_S	ρ	$r_\sharp^\star$	T	e_T
0	38.28	$9.40 \cdot 10^{-2}$	$5.60 \cdot 10^{-2}$	6.02	10.17	14.53	199	546.22
.1	36.87	$9.28 \cdot 10^{-2}$	$5.72 \cdot 10^{-2}$	6.12	10.79	15.06	199	548.52
.2	35.3	$9.17 \cdot 10^{-2}$	$5.83 \cdot 10^{-2}$	6.22	11.53	15.67	199	558.9
.3	33.52	$9.07 \cdot 10^{-2}$	$5.93 \cdot 10^{-2}$	6.3	12.43	16.39	199	580.72
.4	31.45	$9.00 \cdot 10^{-2}$	$6.01 \cdot 10^{-2}$	6.36	13.57	17.2	199	610.94
.5	28.99	$8.96 \cdot 10^{-2}$	$6.04 \cdot 10^{-2}$	6.39	15.06	18.23	199	677.41
.6	25.97	$9.00 \cdot 10^{-2}$	$6.00 \cdot 10^{-2}$	6.36	17.13	19.58	199	808.42
.7	22.06	$9.16 \cdot 10^{-2}$	$5.84 \cdot 10^{-2}$	6.22	20.29	21.46	199	1,098.21
.8	16.63	$9.59 \cdot 10^{-2}$	$5.42 \cdot 10^{-2}$	5.86	25.85	24.44	194	1,807.6
.9	8.59	0.11	$4.39 \cdot 10^{-2}$	4.94	38.4	30.1	104	1,162.06

Parameters: $\sigma_\eta = 0.01, \sigma_\kappa = 0.15, \sigma_\epsilon = 0.01, \xi = 2.5, \varphi = 5, \gamma = 3, \psi = 0.1, r_0 = 6.\%, \alpha = 0.1\%, \mu_0 = 8.44, \sigma_\kappa / q_0 = 3.19\%$.

Table 8: Carbon price evaluation - Very Large public signal shocks

λ	$\hat{\rho}$	σ_y	σ_μ	r_S	ρ	$r_\sharp^\star$	T	e_T
0	38.28	$9.40 \cdot 10^{-2}$	$5.60 \cdot 10^{-2}$	6.02	10.17	14.53	199	546.3
.1	36.87	$9.28 \cdot 10^{-2}$	$5.72 \cdot 10^{-2}$	6.12	10.79	15.06	199	548.61
.2	35.3	$9.17 \cdot 10^{-2}$	$5.84 \cdot 10^{-2}$	6.21	11.53	15.67	199	559
.3	33.52	$9.07 \cdot 10^{-2}$	$5.93 \cdot 10^{-2}$	6.3	12.43	16.39	199	580.83
.4	31.45	$9.00 \cdot 10^{-2}$	$6.01 \cdot 10^{-2}$	6.36	13.57	17.2	199	611.07
.5	28.99	$8.96 \cdot 10^{-2}$	$6.04 \cdot 10^{-2}$	6.38	15.06	18.23	199	677.55
.6	25.97	$9.00 \cdot 10^{-2}$	$6.01 \cdot 10^{-2}$	6.35	17.13	19.58	199	808.61
.7	22.05	$9.16 \cdot 10^{-2}$	$5.84 \cdot 10^{-2}$	6.22	20.29	21.46	199	1,098.46
.8	16.63	$9.59 \cdot 10^{-2}$	$5.42 \cdot 10^{-2}$	5.86	25.85	24.44	194	1,808.01
.9	8.58	0.11	$4.40 \cdot 10^{-2}$	4.93	38.39	30.09	104	1,162.39

Parameters: $\sigma_\eta = 1, \sigma_\kappa = 0.15, \sigma_\epsilon = 0.01, \xi = 6.5/2, \varphi = 6.5, \gamma = 3, \psi = 0.1, r_0 = 6\%, \alpha = 0.1\%, \mu_0 = 8.66, \sigma_\kappa / q_0 = 3.19\%$.

depleted in 25 years). As mentioned above, the EQ is also deteriorating under the optimal policy at first, reaching its minimum after approximately 80 years, but still sufficiently high to safely reduce the odds of reaching 0. It then increases, but the level reached at the time the economy is carbon neutral ($e_T = 1541$Gt CO_2) is considerably lower than the one in 2005 ($e_0 = 2494$Gt CO_2).

The policy scheme z_t and the resulting policy index Z_t are illustrated Fig. 6. The subsidy z_t increases rapidly the first 5 years (from 15% to 33%) then slowly decreases toward 28% during the following century. As the targeted interest rate level is 27.58%, the net expected subsidy rate is thus around 5%. The actual level is stochastic as shown Fig. 7 where the dynamics of the interest rate under the policy and laissez-faire are depicted together with the corresponding expected rates. Interest rates are similarly affected by the GDP shocks, the amplitude of the variations being larger under the policy than under laissez-faire.

Fig. 8 illustrates the per period investment rates (relative to the AGT index level) under laissez-faire and under the environmental policy. The investment is negative under laissez-faire in the first 5 periods, that is, firms tend to invest increasingly in polluting technologies, while investment in green technology is always positive under the policy. Investment rates are comparable after 15 years (but not the level as noted above), and converge to the growth rate of the economy.

Fig. 9 shows the consumption path under the optimal policy and under laissez-faire. A higher consumption differential between laissez-faire and the regulated economy means that the policy implies a higher investment level and thus a higher cost in terms of postponed welfare.

Fig. 10 shows total wealth dynamics. Because of the decrease in the environmental quality, the laissez-faire curve decreases rapidly.

There are two aspects in the welfare impact of an environmental policy that can be analyzed here. First, implementing a subsidy for AGT has the immediate and most direct effect of increasing the rate of investment in green technologies, thus introducing a trade-off between current consumption and future environmental quality. Since our model disentangles green technology investment from productivity growth, increasing investment has no stock effect on the GDP level. On the other hand, the environmental quality is a stock and reducing the size of the negative impact of production on the environment has long term effects on welfare through environmental quality accumulation.

6 Conclusion

In this paper, we analyze the AGT process when demand spillovers are a key determinant of technical efficiency and when the economy is subject to uncertainty. We consider the problem of a social planner in charge of determining an investment subsidy policy to incentivize firms to increase their AGT. Because investment choices are ultimately made by private agents who react to the economic context according to their beliefs and because the efficiency of technologies is the result of their many choices, the public policy can only imperfectly direct the economy to a sustainable path. Besides, the value that should be ascribed to the environment to guide the public policy is difficult to assess because of the lack of scientific knowledge and the uncertainty that affects the economy. While extremely stylized, our model allows us to ascertain the effects of these uncertainties on the optimal policy. It provides a tool to estimate the value of the environment considering that society should decide on a safety level in order to avoid an environmental catastrophe. Of course, the SCC values that we obtain depend on the safety level chosen, but more importantly, they are very dependent of the spillover effects at work in the AGT process. Our numerical simulations show that the sharper the demand spillovers are, the higher is the SCC and the larger should be the subsidies given to firms to direct their choices toward a carbon neutral path.

We have considered in this analysis that the productivity growth is unaffected by the technologies employed: technologies only differ in their impact on the environment. Including in the analysis the process that leads to productivity growth would permit to assess the intertemporal trade-off between growth, consumption and environmental safety when society faces both economic and environmental risks. This undertaking could also consider other policy instruments than investment subsidies, like the environmental tax, and would certainly give better estimates of the SCC.

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Appendix

A Proof of lemma 1

Maximization of (8) with respect to I_{it} leads to the first-order condition

$$-1+\delta_t\mathbb{E}\left[\left.\frac{\mathcal{V}(x_{it}+I_{it};x^\star_{t+1})}{\partial x}\right|\omega_t,w_{it}\right]=0 \tag{45}$$

while the envelop condition yields

$$\frac{\partial\mathcal{V}(x_{it};x^\star_t)}{\partial x}=x^\star_t-x_{it}+\delta_t\mathbb{E}\left[\left.\frac{\mathcal{V}(x_{it}+I_{it};x^\star_{t+1})}{\partial x}\right|\omega_t,w_{it}\right] \tag{46}$$

implying

$$\frac{\partial\mathcal{V}(x_{it};x^\star_t)}{\partial x}=x^\star_t-x_{it}+1.$$

Plugging this expression in (45) evaluated in expectation for period $t+1$ yields

$$1+r_t=\mathbb{E}\left[x^\star_{t+1}-(x_{it}+I_{it})+1|\omega_t,w_{it}\right]=\hat{x}_{it+1}-(x_{it}+I_{it})+1.$$

Reorganizing terms gives (9). Replacing into (3) and using (6), we obtain that firm i next period AGT index follows

$$x_{it+1}=x^\star_{t+1}+\tau\eta_t+(1-\tau)\varepsilon_{it}-r_t.$$

As $\int\varepsilon_{it}di=0$, we obtain

$$\mu_{t+1}=\int_i x_{it+1}di=x^\star_{t+1}+\tau\eta_t-r_t, \tag{47}$$

and thus

$$x_{it+1}=\mu_{t+1}+(1-\tau)\varepsilon_{it}.$$

As idiosyncratic investments depend on the firms' current technologies and on signals that are normally distributed, their next period technologies are also normally distributed around μ_{t+1} with variance $\mathbb{V}[x_{it+1}]=(1-\tau)^2\sigma^2_\varepsilon$.

Using (47) to substitutes for $x^\star_{t+1}$ in (5) yields

$$\mu_{t+1}-(\tau\eta_t-r_t)=\mu_t+\lambda(\mu_{t+1}-\mu_t)$$

which gives (11).

B Proof of lemma 2

Using (1), (17) and (14), we have

$$\tilde{y}_{t+1} = \tilde{c}_{t+1} + \rho e_{t+1} = g q_t + \hat{g} + \tilde{\kappa}_{t+1} - (\tilde{\mu}_{t+2} - \mu_{t+1}) + \rho(\theta e_t + \xi \mu_t - \varphi q_t + \hat{e}) \quad (48)$$

in which, using (19),

$$\tilde{\mu}_{t+2} = a_1 \mu_{t+1} + a_2(\theta e_t + \xi \mu_t - \varphi q_t + \hat{e}) + a_3(g q_t + \hat{g} + \tilde{\kappa}_{t+1}) + a_4 + a_5(\tau \tilde{\eta}_{t+1}) + Z_{t+1} \quad (49)$$

where Z_{t+1} is a function of the z_{t+h}, $h = 2, 3...$ As the resulting expression of $\tilde{y}_{t+1}$ is a linear combination of iid random shocks normally distributed, it is also normally distributed with variance σ_y given by (21) which is independent of t.

The coefficients $\{a_j\}_{j=1,\dots,6}$ and Z_t in (19) are derived as follows. Using $y_t = q_t - \mu_{t+1} + \mu_t + \rho e_t$ yields

$$\mathbb{E}_t[\tilde{y}_{t+1}] - y_t = \hat{g} - (1 - g + \hat{\rho}\varphi) q_t - \mathbb{E}_t[\tilde{\mu}_{t+2}] + 2\mu_{t+1} - (1 - \hat{\rho}\xi)\mu_t - \hat{\rho}[(1-\theta)e_t - \hat{e}]$$

which gives, using (49) and collecting terms,

$$\begin{aligned}\mathbb{E}_t[\tilde{y}_{t+1}] - y_t = {} & \hat{g}(1 - a_3) - [1 - (1 - a_3)g + (\hat{\rho} - a_2)\varphi] q_t - (a_4 + Z_{t+1}) + (2 - a_1)\mu_{t+1} \\ & - [1 - (\hat{\rho} - a_2)\xi]\mu_t - [\hat{\rho}(1-\theta) + a_2\theta] e_t + (\hat{\rho} - a_2)\hat{e}.\end{aligned}$$

Replacing into (18) gives

$$\begin{aligned} r_t = {} & \psi - \gamma^2 \sigma_y^2 / 2 + \gamma\{[\varphi a_2 - (1 - g + \varphi\hat{\rho}) - a_3 g] q_t - (a_1 - 2)\mu_{t+1} - [1 + \xi(a_2 - \hat{\rho})]\mu_t\} \\ & + \gamma\{\hat{g}(1 - a_3) - e_t[a_2\theta + \hat{\rho}(1-\theta)] - \hat{e}(a_2 - \hat{\rho}) - a_4 - Z_{t+1}\}. \end{aligned} \quad (50)$$

Equalizing with (11), which can be rewritten as

$$r_t = z_t + \tau \eta_t - (1 - \lambda)(\mu_{t+1} - \mu_t), \quad (51)$$

and collecting terms yields

$$\begin{aligned}\mu_{t+1} = {} & \frac{1 - \lambda + \gamma[1 + \xi(a_2 - \hat{\rho})]}{1 - \lambda + \gamma(2 - a_1)} \mu_t + \gamma \frac{[a_2\theta + \hat{\rho}(1-\theta)] e_t}{1 - \lambda + \gamma(2 - a_1)} + \gamma \frac{[1 - g + \varphi(\hat{\rho} - a_2) + a_3 g] q_t}{1 - \lambda + \gamma(2 - a_1)} \\ & - \frac{\psi + \gamma\hat{g}(1 - a_3) - \gamma\hat{e}(a_2 - \hat{\rho}) - \gamma a_4 - \gamma^2\sigma_y^2/2}{1 - \lambda + \gamma(2 - a_1)} + \frac{\tau\eta_t}{1 - \lambda + \gamma(2 - a_1)} \\ & + \frac{z_t + \gamma Z_{t+1}}{1 - \lambda + \gamma(2 - a_1)}.\end{aligned}$$

Identifying with (19) gives

$$\begin{aligned}
a_1 &= \frac{1-\lambda+\gamma[1+\xi(a_2-\hat{\rho})]}{1-\lambda+\gamma(2-a_1)}, a_2 = \frac{\gamma\hat{\rho}(1-\theta)}{1-\lambda+\gamma(2-a_1-\theta)}, \\
a_3 &= \gamma\frac{1-g+\varphi(\hat{\rho}-a_2)}{1-\lambda+\gamma(2-a_1-g)}, \\
a_4 &= \frac{\gamma[\hat{e}(a_2-\hat{\rho})+a_4+\gamma\sigma_y^2/2]-\psi-\gamma\hat{g}(1-a_3)}{1-\lambda+\gamma(2-a_1)}, \\
a_5 &= \frac{1}{1-\lambda+\gamma(2-a_1)}
\end{aligned} \tag{52}$$

and

$$Z_t = a_5(z_t+\gamma Z_{t+1}) = a_5\sum_{i=0}^{+\infty}(a_5\gamma)^i z_{t+i}. \tag{53}$$

The first two equations of (52) form a system involving only coefficients a_1 and a_2 that can be solved separately from the others. Observe also that $a_2 = \hat{\rho}$ and $a_1 = a_1^0 \equiv 1+(1-\lambda)/\gamma$ is a degenerate solution of this system. More precisely, using $1-\lambda+\gamma(2-a_1) = \gamma(a_1^0-a_1+1)$, we can express (52) as

$$\begin{aligned}
a_1 &= \frac{a_1^0+\xi(a_2-\hat{\rho})}{a_1^0-a_1+1}, a_2 = \frac{\hat{\rho}(1-\theta)}{a_1^0-a_1+1-\theta} \\
a_3 &= \frac{1-g+\varphi(\hat{\rho}-a_2)}{a_1^0-a_1+1-g}, a_5 = \frac{1}{\gamma(a_1^0-a_1+1)} \\
a_4 &= -\frac{\hat{e}(\hat{\rho}-a_2)+\hat{g}(1-a_3)+\psi/\gamma-\gamma\sigma_y^2/2}{a_1^0-a_1}.
\end{aligned} \tag{54}$$

and we get, assuming that the solution is $a_2 = \hat{\rho}$ and $a_1 = a_1^0$, that $a_3 = 1$, $a_5 = 1/\gamma$ but a_4 diverges unless $\psi = \gamma^2\sigma_y^2/2 = \tau^2\sigma_\eta^2/2$ in which case a_4 is indefinite. Alternative solutions can be derived as follows. From the expression of a_2, we get

$$a_2-\hat{\rho} = \frac{-\hat{\rho}(a_1^0-a_1)}{a_1^0-a_1+1-\theta}, \tag{55}$$

which, plugged into the expression for a_1, gives

$$a_1(a_1^0-a_1+1)-a_1^0 = \frac{-\hat{\rho}\xi(a_1^0-a_1)}{a_1^0-a_1+1-\theta},$$

that can also be expressed as $(a_1^0 - a_1)P(a_1 - 1) = 0$ where $P(x) \equiv x(a_1^0 - \theta - x) + \hat{\rho}\xi$ is a second degree polynomial. Non-degenerate solutions must solve $P(a_1 - 1) = 0$. As $P(0) = P(a_1^0 - \theta) = \hat{\rho}\xi$ and $P(x)$ is concave, a_1 is either lower than 1 ($a_1 - 1$ is equal to the negative root of $P(x) = 0$) or greater than $a_1^0 - \theta + 1 = 2 - \theta + (1 - \lambda)/\gamma$ ($a_1 - 1$ is then equal to the positive root of $P(x) = 0$). As z_t impacts positively μ_{t+1} for all t, it comes from (53) that $a_5 > 0$ which implies $a_1 - 1 < 1 + (1 - \lambda)/\gamma$ and thus rules out the positive root of $P(a_1 - 1) = 0$. Consequently, $a_1 - 1$ corresponds to the negative solution of $P(x) = 0$ which is given by

$$x = \frac{1}{2}\left(a_1^0 - \theta - \sqrt{(a_1^0 - \theta)^2 + 4\hat{\rho}\xi}\right).$$

After substituting $1 + (1 - \lambda)/\gamma$ for a_1^0, we get

$$a_1 = 1 - \frac{1}{2}\left(\sqrt{[1 - \theta + (1 - \lambda)/\gamma]^2 + 4\hat{\rho}\xi} - 1 + \theta - (1 - \lambda)/\gamma\right)$$

which gives (22). The condition $a_1 > 0$ can be expressed as

$$[3 - \theta + (1 - \lambda)/\gamma]^2 - [1 - \theta + (1 - \lambda)/\gamma]^2 - 4\hat{\rho}\xi > 0$$

which simplifies to $2 + (1 - \lambda)/\gamma - \theta > \hat{\rho}\xi$. Using $a_1^0 - \theta - a_1 + 1 = \hat{\rho}\xi/(1 - a_1)$ in (54) gives

$$a_2 = \hat{\rho}(1 - \theta)(1 - a_1)/\xi. \tag{56}$$

As $a_1^0 > 1 > a_1$ we have $a_2 > 0$ and from (55), $a_2 < \hat{\rho}$. We obtain from (54) that $a_3 > 0$, and we have $a_3 < 1$ if

$$a_1^0 - a_1 > \varphi(\hat{\rho} - a_2) = \frac{\varphi\hat{\rho}(a_1^0 - a_1)}{a_1^0 - a_1 + 1 - \theta}$$

hence if $a_1^0 - a_1 + 1 - \theta > \varphi\hat{\rho}$. From $P(a_1 - 1) = 0$, we have $a_1^0 - \theta - (a_1 - 1) = \hat{\rho}\xi/(1 - a_1)$. Replacing, the condition can be expressed as

$$\xi/\varphi > 1 - a_1 = \frac{1}{2}\left(\sqrt{[1 - \theta + (1 - \lambda)/\gamma]^2 + 4\hat{\rho}\xi} - 1 + \theta - (1 - \lambda)/\gamma\right).$$

As the RHS of this inequality is increasing in $\hat{\rho}$ and null when $\hat{\rho} = 0$, this is the case when $\hat{\rho}$ is not too large. Since we assume $2 + (1 - \lambda)/\gamma - \theta > \hat{\rho}\xi$ (to have $a_1 > 0$), a sufficient condition for $a_3 < 1$ is given by

$$\frac{1}{2}\left(\sqrt{[1 - \theta + (1 - \lambda)/\gamma]^2 + 4[2 + (1 - \lambda)/\gamma - \theta]} - 1 + \theta - (1 - \lambda)/\gamma\right) \leq \xi/\varphi$$

which can be written as

$$\frac{1}{2}\left(\sqrt{y^2+4y+4}-y\right) \leq K$$

where $y = 1-\theta+(1-\lambda)/\gamma$ and $K = \xi/\varphi$, which gives

$$0 \geq y^2+4y+4-(2K+y)^2 = 4y(1-K)+4(1-K^2) = 4(1-K)(y+1+K).$$

Consequently, a sufficient condition for $a_3 < 1$ is $\xi \geq \varphi$. We also have $a_5 > 0$ and $\gamma a_5 < 1$ if $\gamma < 1-\lambda+\gamma(2-a_1)$ hence if $0 < 1-\lambda+\gamma(1-a_1)$ which is always the case since $a_1 < 1$. Z_t is thus an exponential smoothing of the public policy scheme $\{z_{t+h}\}_{h\geq 0}$.

Differentiating $P(a_1-1)=0$ wrt λ yields

$$\frac{da_1}{d\lambda} = \frac{-(1-a_1)}{\gamma[a_1^0-\theta+2(1-a_1)]} < 0$$

which also gives, using (56),

$$\frac{da_2}{d\lambda} = -\frac{\hat{\rho}(1-\theta)}{\xi}\frac{da_1}{d\lambda} > 0.$$

Differentiating a_5 wrt λ gives

$$\frac{da_5}{d\lambda} = a_5^2\left(1+\gamma\frac{da_1}{d\lambda}\right)$$

where, using $P(a_1-1)=0$,

$$1+\gamma\frac{da_1}{d\lambda} = \frac{a_1^0-\theta-(1-a_1)}{a_1^0-\theta+2(1-a_1)} = \frac{\hat{\rho}\xi}{(1-a_1)[a_1^0-\theta+2(1-a_1)]} = -\frac{da_1}{d\lambda}\frac{\gamma\hat{\rho}\xi}{(1-a_1)^2} > 0.$$

We thus have $da_5/d\lambda > 0$. Differentiating a_3 wrt λ yields

$$\frac{da_3}{d\lambda} = \gamma\frac{-\varphi[1-\lambda+\gamma(2-a_1-g)](da_2/d\lambda)+[1-g+\varphi(\hat{\rho}-a_2)](1+\gamma da_1/d\lambda)}{[1-\lambda+\gamma(2-a_1-g)]^2}$$

where, using $a_1^0 = 1+(1-\lambda)/\gamma$,

$$1-\lambda+\gamma(2-a_1-g) = \gamma(a_1^0-a_1+1-g)$$

and, using (55) and (56),

$$\hat{\rho} - a_2 = \frac{\hat{\rho}(a_1^0 - a_1)}{a_1^0 - a_1 + 1 - \theta} = \frac{(a_1^0 - a_1)(1 - a_1)}{\xi}.$$

Replacing, we obtain that $da_3/d\lambda > 0$ if

$$\begin{aligned} 0 &> \frac{\varphi(1-\theta)}{\xi}(a_1^0 - a_1 + 1 - g) - \frac{\xi}{(1-a_1)^2}\left[1 - g + \varphi\frac{(a_1^0 - a_1)(1-a_1)}{\xi}\right] \\ &= (1-g)\left(\frac{\varphi(1-\theta)}{\xi} - \frac{\xi}{(1-a_1)^2}\right) + \varphi(a_1^0 - a_1)\left[\frac{1-\theta}{\xi} - \frac{1}{1-a_1}\right] \\ &= \frac{(1-g)\xi}{(1-a_1)^2}\left(\frac{\varphi a_2^2}{(1-\theta)\hat{\rho}^2} - 1\right) + \frac{\varphi(a_1^0 - a_1)}{1-a_1}\left(\frac{a_2}{\hat{\rho}} - 1\right). \end{aligned}$$

As $a_2 < \hat{\rho}$, this is always the case if $a_2 < \hat{\rho}\sqrt{(1-\theta)/\varphi}$, hence if $a_1 > 1 - \xi/\sqrt{(1-\theta)\varphi}$ which yields

$$\frac{1}{2}\left(\sqrt{[1-\theta+(1-\lambda)/\gamma]^2 + 4\hat{\rho}\xi} - 1 + \theta - (1-\lambda)/\gamma\right) < \xi/\sqrt{(1-\theta)\varphi}.$$

As the LHS of this inequality is increasing in $\hat{\rho}$ and null when $\hat{\rho} = 0$, we thus have $da_3/d\lambda > 0$ when $\hat{\rho}$ is not too large. Since we assume $2 + (1-\lambda)/\gamma - \theta > \hat{\rho}\xi$ (to have $a_1 > 0$), a sufficient condition for $da_3/d\lambda > 0$ is given by

$$\frac{1}{2}\left(\sqrt{[1-\theta+(1-\lambda)/\gamma]^2 + 4[2+(1-\lambda)/\gamma-\theta]} - 1 + \theta - (1-\lambda)/\gamma\right) < \xi/\sqrt{(1-\theta)\varphi}$$

which can be written as

$$\frac{1}{2}\left(\sqrt{y^2+4y+4} - y\right) < K$$

where $y = 1 - \theta + (1-\lambda)/\gamma$ and $K = \xi/\sqrt{(1-\theta)\varphi}$, which gives

$$0 > y^2 + 4y + 4 - (2K+y)^2 = 4y(1-K) + 4(1-K^2) = 4(1-K)(y+1+K).$$

Consequently, a sufficient condition for a_3 to increase with λ is $\xi > \sqrt{(1-\theta)\varphi}$.

C Proof of lemma 3

Using (50), we get

$$\tilde{r}_t - \mathbb{E}_{t-1}[\tilde{r}_t] = \gamma\{[\varphi a_2 - (1-g+\varphi\hat{\rho}) - g a_3 - (a_1-2)a_3]\tilde{\kappa}_t - (a_1-2)a_5(\tau\tilde{\eta}_t)\}.$$

Using the expressions of a_3 and a_5 in (52), it can be expressed as

$$\tilde{r}_t - \mathbb{E}_{t-1}[\tilde{r}_t] = [1-(1-\lambda)a_5](\tau\tilde{\eta}_t) - (1-\lambda)a_3\tilde{\kappa}_t, \tag{57}$$

that can also be derived from (51), which becomes

$$\tilde{r}_t = z_t + \tau\tilde{\eta}_t - (1-\lambda)(\tilde{\mu}_{t+1} - \mu_t),$$

using (19). The two-period-ahead wealth index is deduced from

$$\tilde{y}_{t+1} = \tilde{c}_{t+1} + \hat{\rho}\tilde{e}_{t+1} = \tilde{q}_{t+1} + \tilde{\mu}_{t+1} - \tilde{\mu}_{t+2} + \hat{\rho}\tilde{e}_{t+1}$$

where

$$\tilde{\mu}_{t+2} = a_1\tilde{\mu}_{t+1} + a_2\tilde{e}_{t+1} + a_3\tilde{q}_{t+1} + a_4 + a_5(\tau\tilde{\eta}_{t+1}) + Z_{t+1}.$$

Replacing leads to

$$\tilde{y}_{t+1} = (1-a_3)\tilde{q}_{t+1} + (1-a_1)\tilde{\mu}_{t+1} + (\hat{\rho}-a_2)\tilde{e}_{t+1} - a_4 - a_5(\tau\tilde{\eta}_{t+1}) - Z_{t+1}$$

which gives

$$\begin{aligned}
\tilde{y}_{t+1} - \mathbb{E}[\tilde{y}_{t+1}] &= (1-a_3)(g\tilde{\kappa}_t + \tilde{\kappa}_{t+1}) + (1-a_1)[a_3\tilde{\kappa}_t + a_5(\tau\tilde{\eta}_t)] \\
&\quad + (a_2-\hat{\rho})\varphi\tilde{\kappa}_t - a_5(\tau\tilde{\eta}_{t+1}) \\
&= (1-a_3)\tilde{\kappa}_{t+1} - a_5(\tau\tilde{\eta}_{t+1}) + [(1-a_3)g + (1-a_1)a_3 \\
&\quad + (a_2-\hat{\rho})\varphi]\tilde{\kappa}_t + a_5(1-a_1)(\tau\tilde{\eta}_t)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{V}[\tilde{y}_{t+1}] &= (1-a_3)^2\sigma_\kappa^2 + a_5^2\tau^2\sigma_\eta^2 + [(1-a_3)g + (1-a_1)a_3 + (a_2-\hat{\rho})\varphi]^2\sigma_\kappa^2 \\
&\quad + (1-a_1)^2a_5^2\tau^2\sigma_\eta^2 \\
&= \sigma_y^2 + [(1-a_3)g + (1-a_1)a_3 + (a_2-\hat{\rho})\varphi]^2\sigma_\kappa^2 + (1-a_1)^2a_5^2\tau^2\sigma_\eta^2 \\
&\equiv \sigma_{y_{+1}}^2.
\end{aligned}$$

D Proof of Lemma 4

The expected AGT index satisfies $\mathbb{E}[\tilde{\mu}_t] = \mathbb{E}[\tilde{q}_t]\varphi/\xi$ for all $t > T$ along a CNP, so that expected wealth satisfies

$$\begin{aligned}
\mathbb{E}[\tilde{y}_t] &= \mathbb{E}[\tilde{c}_t] + \hat{\rho}\mathbb{E}[\tilde{e}_t] = \mathbb{E}[\tilde{q}_t] - \mathbb{E}[\tilde{\mu}_{t+1}] + \mathbb{E}[\tilde{\mu}_t] + \hat{\rho}\mathbb{E}[\tilde{e}_t] \\
&= \mathbb{E}[\tilde{q}_t] - (\mathbb{E}[\tilde{q}_{t+1}] - \mathbb{E}[\tilde{q}_t])\varphi/\xi + \hat{\rho}\mathbb{E}[\tilde{e}_t] \\
&= \mathbb{E}[\tilde{q}_t][1 + (1-g)\varphi/\xi] - \hat{g}\varphi/\xi + \hat{\rho}\mathbb{E}[\tilde{e}_t].
\end{aligned}$$

while it is given by

$$\mathbb{E}[\tilde{y}_t] = \mathbb{E}[\tilde{q}_t][1 + (1-g)\varphi/\xi] - \hat{g}\varphi/\xi + \hat{\rho}e_T$$

along an SEP. From (18), the expected interest rate along a CNP satisfies

$$\begin{aligned}
\mathbb{E}[\tilde{r}_t] &= r_S + \gamma(\mathbb{E}[y_{t+1}] - \mathbb{E}[y_t]) \\
&= r_S + \gamma(\mathbb{E}[\tilde{q}_{t+1}] - \mathbb{E}[\tilde{q}_t])[1 + (1-g)\varphi/\xi] + \gamma\hat{\rho}(\mathbb{E}[\tilde{e}_{t+1}] - \mathbb{E}[\tilde{e}_t]) \\
&= r_S + \gamma(1-g)(q_S - \mathbb{E}[\tilde{q}_t])[1 + (1-g)\varphi/\xi] + \gamma\hat{\rho}(1-\theta)(e_S - \mathbb{E}[\tilde{e}_t])
\end{aligned}$$

where, solving the recursion from T to $t > T$,

$$\mathbb{E}[\tilde{q}_t] = q_S - g^{t-T}(q_S - q_T)$$

and

$$\mathbb{E}[\tilde{e}_t] = e_S - \theta^{t-T}(e_S - e_T)$$

with $e_T < e_S = e_N$ along a CNP and $e_S = e_T < e_N$ in the case of an SEP. Substituting gives (30). Now, from (11) and $\mathbb{E}[\tilde{\mu}_t] = \mathbb{E}[\tilde{q}_t]\varphi/\xi + (1-\theta)(e_N - e_S)/\xi$, we get

$$z_t = \mathbb{E}[\tilde{r}_t] + (1-\lambda)(\mathbb{E}[\tilde{q}_{t+1}] - \mathbb{E}[\tilde{q}_t])\varphi/\xi = \mathbb{E}[\tilde{r}_t] + (1-\lambda)(1-g)g^{t-T}(q_S - q_T)\varphi/\xi$$

hence (29). Replacing in (20) and (30) yield

$$Z_t = a_5 \sum_{i=0}^{+\infty} (a_5\gamma)^i \left\{(1-g)g^{t+i-T}(q_S - q_T)[\gamma + (1-\lambda+\gamma(1-g))\varphi/\xi] + \gamma\hat{\rho}(1-\theta)\theta^{t+i-T}(e_S - e_T)\right\}$$

which gives (31). The SEP stationary state e_S is derived as follows. Suppose the economy has reached an SEP at date T with corresponding GDP level q_T and EQ level $e_T = e_S$. Using (31) to substitute for Z_t in (19), we obtain

$$\mu_{t+1} = a_1\mu_t + a_2 e_t + a_3 q_t + \Gamma_t + a_4 + \frac{a_5 r_S}{1 - a_5\gamma} + a_5\tau\eta_t$$

for all $t > T$, where $\Gamma_t = g\Gamma_{t-1}$ with

$$\Gamma_T \equiv \frac{(1-g)(q_S - q_T)[\gamma + (1-\lambda+\gamma(1-g))\varphi/\xi]}{1 - a_5\gamma g}.$$

We can write the dynamic of the economy along the SEP as

$$\tilde{Y}_t = B\tilde{Y}_{t-1} + H\tilde{\nu}_t \tag{58}$$

where $\tilde{Y}_t = (\tilde{\mu}_t, \tilde{e}_t, \tilde{q}_t, \Gamma_t, 1)'$ is a column vector,

$$B = \begin{bmatrix} a_1 & a_2 & a_3 & 1 & a_4 + \frac{a_5 r_S}{1-a_5\gamma} \\ \xi & \theta & -\varphi & 0 & \hat{e} \\ 0 & 0 & g & 0 & \hat{g} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & g & 0 \end{bmatrix}, \; H = \begin{bmatrix} a_5\tau\sigma_\eta & 0 \\ 0 & 0 \\ 0 & \sigma_\kappa \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and $\tilde{\nu}_t = (\tilde{\nu}_{1t}, \tilde{\nu}_{2t})'$ is a column vector of independent standardized Gaussian variables.

We have $\mathbb{E}[\tilde{Y}_t] = B\mathbb{E}[\tilde{Y}_{t-1}]$ which gives

$$\tilde{Y}_t - \mathbb{E}[\tilde{Y}_t] = B(\tilde{Y}_{t-1} - \mathbb{E}[\tilde{Y}_{t-1}]) + H\tilde{\nu}_t.$$

The covariance matrix thus satisfies

$$\mathbb{E}[(\tilde{Y}_t - \mathbb{E}[\tilde{Y}_t])(\tilde{Y}_t - \mathbb{E}[\tilde{Y}_t])'] = B[\mathbb{E}(\tilde{Y}_{t-1} - \mathbb{E}[\tilde{Y}_{t-1}])(\tilde{Y}_{t-1} - \mathbb{E}[\tilde{Y}_{t-1}])']B' + HH'.$$

By definition of an SEP, we have $\mathbb{E}[\tilde{e}_t] = e_S$ and $\mathbb{E}[\tilde{\iota}_t\tilde{q}_t] = \varphi\mathbb{E}[\tilde{q}_t] - \xi\mathbb{E}[\tilde{\mu}_t] = (1-\theta)(e_N - e_S)$ for all t. As q_t converges to its stationary value q_S, the stationary value of the AGT index is given by $\mu_S \equiv q_S\varphi/\xi + (1-\theta)(e_N - e_S)/\xi$. Hence, the stationary value of $\tilde{Y}_t$ is given by $Y_S = (\mu_S, e_S, q_S, 0, 1)'$ and satisfies $Y_S = BY_S$. The stationary covariance matrix satisfies

$$\mathbb{V}_S \equiv \mathbb{E}[(\tilde{Y}_t - Y_S)(\tilde{Y}_t - Y_S)'] = B\mathbb{E}[(\tilde{Y}_{t-1} - Y_S)(\tilde{Y}_{t-1} - Y_S)']B' + HH',$$

and thus solves the Lyapunov equation

$$\mathbb{V}_S = B\mathbb{V}_S B' + HH'.$$

As neither B nor H depend on e_S and q_T, $\mathbb{V}_S$ is independent of the date at which the SEP is reached (it only depends on the parameters of the model).

As the dynamic (58) is linear, the distribution of $\tilde{Y}_t$ follows a Gaussian distribution with mean Y_S and covariance matrix $\mathbb{V}_S$ at the stationary equilibrium. Denoting by σ_e the standard deviation corresponding to EQ, its stationary distribution must satisfy

$$\alpha = \Pr\{\tilde{e}_t \leq \bar{e}\} = \Pr\{(\tilde{e}_t - e_S)/\sigma_e \leq (\bar{e} - e_S)/\sigma_e\}$$

Denoting by $\Phi(x)$ the CDF of the standardized Gaussian variable at level x and using $\Phi(-x) = 1 - \Phi(x)$, we get

$$\alpha = \Pr\{\tilde{e}_t \leq \bar{e}\} = 1 - \Phi((e_S - \bar{e})/\sigma_e),$$

hence $e_S = \bar{e} + \sigma_e \Phi^{-1}(1 - \alpha)$.

E Proof of Proposition 1

From (26) and (27), the first-order condition with respect to z_t is given by

$$\mathbb{E}\left[\frac{\partial u_t}{\partial c}\right] = \beta \mathbb{E}\left[\frac{\partial \mathcal{W}_{t+1}}{\partial \mu}\right] \tag{59}$$

where u_t and $\mathcal{W}_t$ are abbreviated notations for $u(c_t, e_t)$ and $\mathcal{W}(\mu_t, e_t, q_t)$ respectively. The envelop theorem gives

$$\frac{\partial \mathcal{W}_t}{\partial \mu} = \xi \beta \mathbb{E}\left[\frac{\partial \mathcal{W}_{t+1}}{\partial e}\right] + \beta \mathbb{E}\left[\frac{\partial \mathcal{W}_{t+1}}{\partial \mu}\right] \tag{60}$$

and

$$\frac{\partial \mathcal{W}_t}{\partial e} = \mathbb{E}\left[\frac{\partial u_t}{\partial e}\right] + \theta \beta \mathbb{E}\left[\frac{\partial \mathcal{W}_{t+1}}{\partial e}\right]. \tag{61}$$

Using (59), (60) can be written as

$$\frac{\partial \mathcal{W}_t}{\partial \mu} = \xi \beta \mathbb{E}\left[\frac{\partial \mathcal{W}_{t+1}}{\partial e}\right] + \mathbb{E}\left[\frac{\partial u_t}{\partial c}\right]. \tag{62}$$

Evaluating (62) in expectation one period ahead gives

$$\mathbb{E}\left[\frac{\partial \mathcal{W}_{t+1}}{\partial \mu}\right] = \mathbb{E}\left[\frac{\partial u_{t+1}}{\partial c}\right] + \xi \beta \mathbb{E}\left[\frac{\partial \mathcal{W}_{t+2}}{\partial e}\right] = \frac{1}{\beta} \mathbb{E}\left[\frac{\partial u_t}{\partial c}\right]$$

using (59), hence

$$\mathbb{E}\left[\frac{\partial \mathcal{W}_{t+2}}{\partial e}\right] = \frac{1}{\xi\beta^2}\mathbb{E}\left[\frac{\partial u_t}{\partial c}\right] - \frac{1}{\xi\beta}\mathbb{E}\left[\frac{\partial u_{t+1}}{\partial c}\right].$$

Plugging this expression in (61) evaluated one period ahead yields

$$\mathbb{E}\left[\frac{\partial \mathcal{W}_{t+1}}{\partial e}\right] = \mathbb{E}\left[\frac{\partial u_{t+1}}{\partial e}\right] + \frac{\theta}{\xi\beta}\mathbb{E}\left[\frac{\partial u_t}{\partial c}\right] - \frac{\theta}{\xi}\mathbb{E}\left[\frac{\partial u_{t+1}}{\partial c}\right].$$

We can thus express (62) as

$$\frac{\partial \mathcal{W}_t}{\partial \mu} = (1+\theta)\mathbb{E}\left[\frac{\partial u_t}{\partial c}\right] + \xi\beta\mathbb{E}\left[\frac{\partial u_{t+1}}{\partial e}\right] - \theta\beta\mathbb{E}\left[\frac{\partial u_{t+1}}{\partial c}\right]$$

which, evaluated one period ahead yield gives,

$$\mathbb{E}\left[\frac{\partial \mathcal{W}_{t+1}}{\partial \mu}\right] = (1+\theta)\mathbb{E}\left[\frac{\partial u_{t+1}}{\partial c}\right] + \xi\beta\mathbb{E}\left[\frac{\partial u_{t+2}}{\partial e}\right] - \theta\beta\mathbb{E}\left[\frac{\partial u_{t+2}}{\partial c}\right] = \frac{1}{\beta}\mathbb{E}\left[\frac{\partial u_t}{\partial c}\right]$$

using (59). Reorganizing terms, we obtain

$$\mathbb{E}\left[\frac{\partial u_t}{\partial c}\right] = (1+\theta)\beta\mathbb{E}\left[\frac{\partial u_{t+1}}{\partial c}\right] - \theta\beta^2\mathbb{E}\left[\frac{\partial u_{t+2}}{\partial c}\right] + \beta^2\xi\mathbb{E}\left[\frac{\partial u_{t+2}}{\partial e}\right]. \tag{63}$$

From (28) and (33), we have

$$\mathbb{E}\left[\frac{\partial u_t}{\partial c}\right] = (1+r_t^e)\beta\mathbb{E}\left[\frac{\partial u_{t+1}}{\partial c}\right].$$

Substituting in (63) for initial dates t and $t+1$ yields

$$\frac{\beta^2}{\delta_t^e\delta_{t+1}^e}\mathbb{E}\left[\frac{\partial u_{t+2}}{\partial c}\right] = \frac{(1+\theta)\beta^2}{\delta_{t+1}^e}\mathbb{E}\left[\frac{\partial u_{t+2}}{\partial c}\right] - \theta\beta^2\mathbb{E}\left[\frac{\partial u_{t+2}}{\partial c}\right] + \beta^2\xi\mathbb{E}\left[\frac{\partial u_{t+2}}{\partial e}\right]$$

where $\delta_t^e \equiv (1+r_t^e)^{-1}$, which upon simplifying and rearranging terms yields

$$\begin{aligned}
\xi\frac{\mathbb{E}\left[\partial u_{t+2}/\partial e\right]}{\mathbb{E}\left[\partial u_{t+2}/\partial c\right]} &= (1+r_t^e)(1+r_{t+1}^e) - (1+r_t^e)(1+\theta) + \theta \\
&= (1+r_t^e)(r_{t+1}^e - \theta) + \theta \\
&= r_{t+1}^e + r_t^e r_{t+1}^e - r_t^e\theta.
\end{aligned}$$

F Proof of Proposition 2

Expression (34) gives the recursive equation

$$r^e_{t+1} = \frac{\rho\xi + r^e_t\theta}{1 + r^e_t} \tag{64}$$

which can be solved as follows. Defining $v_t = (r^e_t + c)^{-1}$ or equivalently $r^e_t = 1/v_t - c$ where c is a constant to determine, (64) becomes

$$\begin{aligned}\frac{1}{v_{t+1}} &= c + \frac{\rho\xi + \theta/v_t - \theta c}{1 + 1/v_t - c}\\ &= c + \frac{\theta + v_t(\rho\xi - \theta c)}{v_t(1-c)+1}\\ &= \frac{v_t[c(1-c) + \rho\xi - \theta c] + c + \theta}{v_t(1-c)+1},\end{aligned}$$

which gives

$$v_{t+1} = \frac{v_t(1-c)+1}{v_t[c(1-c-\theta)+\rho\xi] + c + \theta},$$

an equation that simplifies to

$$\begin{aligned}v_{t+1} &= \frac{1-c}{c+\theta}v_t + \frac{1}{c+\theta}\\ &\equiv kv_t + k_0,\end{aligned} \tag{65}$$

under the conditions $c \neq -\theta$ and

$$c(1-c-\theta) + \rho\xi = 0. \tag{66}$$

Provided that $k \neq 1$, the solution of the recurrence equation (65) is given by

$$v_{t+1} = k^{t+1}v_0 + k_0(1-k^{t+1})/(1-k) \tag{67}$$

which converges to $v_\infty = k_0/(1-k)$ if $|k| < 1$, i.e. if $1 > |(1-c)/(c+\theta)| > 0$, albeit with oscillations along its path if $k < 0$. The corresponding solution of (34) would converge to

$$r^e_\infty = (1-k)/k_0 - c = \theta - 1 + c$$

which must be equal to the solution of (35), i.e. we must have

$$r_\sharp^e = \theta - 1 + c \tag{68}$$

where $r_\sharp^e$ is either equal to $(A-1+\theta)/2 > 0$ or $-(A+1-\theta)/2 < 0$. Using (68) to substitute $r_\sharp^e + 1 - \theta$ for c in (66) yields (35). (66) is thus satisfied if (68) is true. Using (68) and (65) yields

$$k = \frac{\theta - r_\sharp^e}{r_\sharp^e + 1}. \tag{69}$$

We have $k > 0$ if $-1 < r_\sharp^e < \theta$ (as we cannot have $r_\sharp^e > \theta$ and $r_\sharp^e < -1$). The condition $k < 1$ implies $\theta - r_\sharp^e < 1 + r_\sharp^e$, hence $(\theta-1)/2 < r_\sharp^e$. We thus have $1 > k > 0$ if $(\theta-1)/2 < r_\sharp^e < \theta$, which rules out the negative root of (35), since we cannot have $(\theta-1)/2 < -(A+1-\theta)/2$. The lower-bound condition on $r_\sharp^e$ gives $(\theta-1)/2 < (A-1+\theta)/2$, which is always true. The upper-bound condition on $r_\sharp^e$ can be written as $\theta > (A-1+\theta)/2$ which gives $A < \theta+1$. Squaring both terms, we arrive at $4\rho\xi < (1+\theta)^2 - (1-\theta)^2 = 4\theta$ hence $\rho\xi < \theta$.

We have $k < 0$ if either $r_\sharp^e < -1$ or if $r_\sharp^e > \theta$. Moreover we have $|k| < 1$ if $r_\sharp^e - \theta < 1 + r_\sharp^e$ which is always true. Hence, we have $-1 < k < 0$ if $r_\sharp^e > \theta$ or if $r_\sharp^e < -1$. Taking the positive root, this condition gives $A - 1 + \theta > 2\theta$ i.e. $A > 1 + \theta$ hence $\rho\xi > \theta$.

Substituting $(A-1+\theta)/2$ for $r_\sharp^e$ in (69) yields (38). From (68), the condition $c \neq \theta$ is equivalently stated as $r_\sharp^e \neq -1$ which is always the case with the positive root. Using (67) and as initial value $r_{t_0}^e = 1/v_{t_0} - c$, the solution of (34) at period $t \geq t_0$ satisfies

$$r_t^e = \frac{(1-k)(r_{t_0}^e + c)}{(1-k)k^{t-t_0} + k_0(1-k^{t-t_0})(r_{t_0}^e + c)} - c.$$

Using $(1-k)/k_0 = r_\sharp^e + c$, we get

$$\begin{aligned} r_t^e &= \frac{(r_\sharp^e + c)(r_{t_0}^e + c)}{(r_\sharp^e + c)k^{t-t_0} + (1-k^{t-t_0})(r_{t_0}^e + c)} - c \\ &= r_\sharp^e + \frac{(r_\sharp^e + c)(r_{t_0}^e - r_\sharp^e)k^{t-t_0}}{r_\sharp^e + c + (1-k^{t-t_0})(r_{t_0}^e - r_\sharp^e)}. \end{aligned}$$

Using (68) and (37), we obtain $r_\sharp^e + c = 2r_\sharp^e + 1 - \theta = A$. Substituting allows us to obtain (36) which converges to $r_\sharp^e$ when $r_\sharp^e > r_{t_0}^e$ if $A > r_\sharp^e - r_{t_0}^e$ hence $r_{t_0}^e > -(A+1-\theta)/2$.

The covariance term in (33) is derived using $u(y_t) = -e^{-\gamma y_t}$ and $\mathbb{E}[e^{-\gamma \tilde{y}}] = e^{-\gamma(\mathbb{E}[\tilde{y}] - \gamma \mathbb{V}[\tilde{y}]/2)}$ which gives

$$\frac{u'(\tilde{y}_{t+1})}{\mathbb{E}_{t-1}\left[u'(\tilde{y}_{t+1})\right]} = e^{-\gamma(\tilde{y}_{t+1} - \mathbb{E}_{t-1}[\tilde{y}_{t+1}] + \gamma \sigma^2_{y+1}/2)}.$$

Consequently,

$$\begin{aligned}
\frac{Cov_{t-1}\left(\tilde{r}_t, u'(\tilde{y}_{t+1})\right)}{\mathbb{E}_{t-1}\left[u'(\tilde{y}_{t+1})\right]} &= \mathbb{E}\left[(\tilde{r}_t - \mathbb{E}_{t-1}[\tilde{r}_t])\left(\frac{u'(\tilde{y}_{t+1})}{\mathbb{E}_{t-1}\left[u'(\tilde{y}_{t+1})\right]} - 1\right)\right] \\
= \mathbb{E}\left[\{[1-(1-\lambda)a_5]a_5\tau\tilde{\eta}_t - (1-\lambda)a_3\tilde{\kappa}_t\}\left(e^{-\gamma(\tilde{y}_{t+1} - \mathbb{E}_{t-1}[\tilde{y}_{t+1}] + \gamma\sigma^2_{y+1}/2)} - 1\right)\right]& \\
&= e^{-\gamma^2 \sigma^2_{y+1}/2}\{[1-(1-\lambda)a_5]\mathbb{E}\left[\tau\tilde{\eta}_t e^{-\gamma(\tilde{y}_{t+1} - \mathbb{E}[\tilde{y}_{t+1}])}\right] \\
&\qquad - (1-\lambda)a_3\mathbb{E}\left[\tilde{\kappa}_t e^{-\gamma(\tilde{y}_{t+1} - \mathbb{E}_{t-1}[\tilde{y}_{t+1}])}\right]\}
\end{aligned}$$

where the last term can be written as

$$\begin{aligned}
\mathbb{E}\left[\tilde{\kappa}_t e^{-\gamma(\tilde{y}_{t+1} - \mathbb{E}_{t-1}[\tilde{y}_{t+1}])}\right] &= \mathbb{E}[\tilde{\kappa}_t e^{-\gamma[(1-a_3)g + (1-a_1)a_3 + (a_2-\rho)\varphi]\tilde{\kappa}_t}] \\
&\times \mathbb{E}[e^{-\gamma[(1-a_3)\tilde{\kappa}_{t+1} + (1-a_1)a_5\tau\tilde{\eta}_t - a_5\tau\tilde{\eta}_{t+1}]}
\end{aligned}$$

from independence. Using $\mathbb{E}[e^{-\gamma \tilde{X}}] = e^{-\gamma(\mathbb{E}\tilde{X} - \gamma\sigma^2_X/2)}$ for a normal random variable $\tilde{X}$, it comes that the last term is equal to $e^{\gamma^2\{\sigma^2_y + (1-a_1)^2 a_5^2 \tau^2 \sigma^2_\eta\}/2}$. Moreover, using

$$\mathbb{E}\left[\tilde{X}e^{-\gamma\tilde{X}}\right] = -\frac{d}{d\gamma}\mathbb{E}[e^{-\gamma\tilde{X}}] = -\frac{d}{d\gamma}e^{-\gamma(\mathbb{E}\tilde{X} - \gamma\sigma^2_X/2)} = (\mathbb{E}\tilde{X} - \gamma\sigma^2_X)e^{-\gamma(\mathbb{E}\tilde{X} - \gamma\sigma^2_X/2)},$$

we get

$$\mathbb{E}[\tilde{\kappa}_t e^{-\gamma[(1-a_3)g + (1-a_1)a_3]\tilde{\kappa}_t}] = -\gamma[(1-a_3)g + (1-a_1)a_3 + (a_2-\rho)\varphi]^2\sigma^2_\kappa e^{\gamma^2[(1-a_3)g + (1-a_1)a_3]^2\sigma^2_\kappa/2}$$

which gives

$$\mathbb{E}\left[\tilde{\kappa}_t e^{-\gamma(\tilde{y}_{t+1} - \mathbb{E}_{t-1}[\tilde{y}_{t+1}])}\right] = -\gamma[(1-a_3)g + (1-a_1)a_3 + (a_2-\rho)\varphi]^2\sigma^2_\kappa e^{\gamma^2\sigma^2_{y_{t+1}}/2}.$$

Similarly, we have

$$\begin{aligned}
\mathbb{E}\left[(\tau\tilde{\eta}_t)e^{-\gamma(\tilde{y}_{t+1} - \mathbb{E}\tilde{y}_{t+1})}\right] &= \mathbb{E}\left[(\tau\tilde{\eta}_t)e^{-\gamma a_5(1-a_1)(\tau\tilde{\eta}_t)}\right] \\
&\times \mathbb{E}[e^{-\gamma\{(1-a_3)\tilde{\kappa}_{t+1} + [(1-a_3)g + (1-a_1)a_3]\tilde{\kappa}_t - a_5(\tau\tilde{\eta}_{t+1})\}}] \\
&= -\gamma(1-a_1)^2 a_5^2 \tau^2 \sigma^2_\eta e^{\gamma^2\sigma^2_{y_{t+1}}/2}.
\end{aligned}$$

Collecting terms, we get

$$\begin{aligned}\frac{cov\left(\tilde{r}_t, u'(\tilde{y}_{t+1})\right)}{\mathbb{E}_{t-1}\left[u'(\tilde{y}_{t+1})\right]} &= (1-\lambda)a_3\gamma[(1-a_3)g + (1-a_1)a_3 + (a_2-\rho)\varphi]^2\sigma_\kappa^2 \\ &- [1-(1-\lambda)a_5]\gamma(1-a_1)^2a_5^2\tau^2\sigma_\eta^2 \\ &= \gamma(1-\lambda)a_3(\sigma_{y+1}^2 - \sigma_y^2) - \gamma[1-(1-\lambda)(a_5-a_3)](1-a_1)^2a_5^2\tau^2\sigma_\eta^2\end{aligned}$$

which gives (39).

G Proof of Proposition 3

This result can be obtained recursively by observing that (36) can be rewritten in term of normalized gaps as $d_{t_0+h} = f^h(d_{t_0})$ with $f^0(x) = x$ and $f^h(x) \equiv f \circ f^{h-1}(x)$ for all $h \geq 1$. Indeed, it is true for $h = 1$ since $d_{t_0+1} = f(d_{t_0})$ and supposing it is true for $t+h$, it is true for $t+h+1$: we have $f(d_{t+h}) = f(f^h(d_t)) = f^{h+1}(d_t) = d_{t+h+1}$. A direct and alternative proof is obtained using $f^h(x) = k^h x/[1+(1-k^h)x]$: we get

$$\begin{aligned}f(f^h(x)) &= \frac{kf^h(x),}{1+(1-k)f^h(x)} = \frac{k^{h+1}x}{1+(1-k^h)x+(1-k)k^hx} = \frac{k^{h+1}x}{1+(1-k^{h+1})x} \\ &= f^{h+1}(x).\end{aligned}$$

H Approximation of the optimal dynamic

Using

$$\mathbb{E}_{t_0}[\tilde{r}_t^\star] = r_\sharp^\star + Ad_t = r_\sharp^\star + Af^{t-t_0}(d_{t_0})$$

for $t < T$ and, from (30),

$$\begin{aligned}\mathbb{E}_{t_0}[\tilde{R}_t] &= r_S + \gamma(1-g)g^{t-T}(q_S - \mathbb{E}_{t_0}[\tilde{q}_T])[1+(1-g)\varphi/\xi] + \gamma\rho(1-\theta)\theta^{t-T}(e_N - \mathbb{E}_{t_0}[\tilde{e}_T]) \\ &= r_\sharp^\star + (r_S - r_\sharp^\star) + \gamma(1-g)g^t(q_S - q_{t_0})[1+(1-g)\varphi/\xi] + \gamma\rho(1-\theta)\theta^{t-T}(e_S - \mathbb{E}_{t_0}[\tilde{e}_T])\end{aligned}$$

for $t \geq T$, we arrive at

$$\begin{aligned}Z_t = &\frac{a_5 r_\sharp^\star}{1-\gamma a_5} + a_5 A\sum_{i=t}^{T-1}(\gamma a_5)^{i-t}f^i(d_{t_0}) + \Gamma_t \\ &+ a_5(1-\lambda)(1-\gamma a_5)\sum_{i=0}^{+\infty}(\gamma a_5)^i\left(\mathbb{E}_{t_0}[\tilde{\mu}_{t+i+1}] - \mathbb{E}_{t_0}[\tilde{\mu}_{t+i}]\right)\end{aligned}$$

where

$$\Gamma_t \equiv a_5 \sum_{i=T-t}^{+\infty} (\gamma a_5)^i (\mathbb{E}_{t_0}[\tilde{R}_{t+i}] - r_\sharp^\star) = a_5 (\gamma a_5)^{T-t} \sum_{i=0}^{+\infty} (\gamma a_5)^i (\mathbb{E}_{t_0}[\tilde{R}_{T+i}] - r_\sharp^\star)$$
$$= a_5 (\gamma a_5)^{T-t} \left\{ \frac{r_S - r_\sharp^\star}{1-\gamma a_5} + \frac{\gamma(1-g) g^T (q_S - q_{t_0})[1+(1-g)\varphi/\xi]}{1-\gamma a_5 g} + \frac{\gamma\rho(1-\theta)(e_S - \mathbb{E}_{t_0}[\tilde{e}_T])}{1-\gamma a_5 \theta} \right\}$$
$$= \Gamma_{t_0}/(\gamma a_5)^t$$

with

$$\Gamma_{t_0} \equiv a_5 (\gamma a_5)^T \left\{ \frac{r_S - r_\sharp^\star}{1-\gamma a_5} + \frac{\gamma(1-g) g^T (q_S - q_{t_0})[1+(1-g)\varphi/\xi]}{1-\gamma a_5 g} + \frac{\gamma\rho(1-\theta)(e_S - \mathbb{E}_{t_0}[\tilde{e}_T])}{1-\gamma a_5 \theta} \right\}, \tag{70}$$

where $e_S = \mathbb{E}_{t_0}[\tilde{e}_T] < e_N$ under an SEP and $e_S = e_N > \mathbb{E}_{t_0}[\tilde{e}_T]$ under a CNP. Using a first-order Taylor expansion of $f^i(d_{t_0})$ around 0 which gives $f^i(d_{t_0}) \approx k^i d_{t_0}$, yields

$$a_5 A \sum_{i=t}^{T-1} (\gamma a_5)^{i-t} f^i(d_{t_0}) \approx \frac{a_5 A d_{t_0}}{(\gamma a_5)^t} \sum_{i=t}^{T-1} (\gamma a_5 k)^i = a_5 A d_{t_0} k^t \sum_{i=0}^{T-t-1} (\gamma a_5 k)^i$$
$$= \frac{a_5 A d_{t_0} k^t [1 - (\gamma a_5 k)^{T-t}]}{1-\gamma a_5 k}$$

which can be expressed recursively as

$$a_5 A \sum_{i=t}^{T-1} (\gamma a_5)^{i-t} f^i(d_{t_0}) \approx U_t - \frac{a_5 A d_{t_0}}{1-\gamma a_5 k} (\gamma a_5 k)^T \frac{\Gamma_t}{\Gamma_{t_0}}$$

with $U_t = k U_{t-1}$ and

$$U_{t_0} = \frac{a_5 A d_{t_0}}{1-\gamma a_5 k}.$$

Substituting in (44) we get

$$Z_t \approx \frac{a_5 r_\sharp^\star}{1-\gamma a_5} + \left(1 - \frac{a_5 A d_{t_0}}{1-\gamma a_5 k} \frac{(\gamma a_5 k)^T}{\Gamma_{t_0}}\right) \Gamma_t + U_t$$
$$+ a_5 (1-\lambda)(1-\gamma a_5) \sum_{i=0}^{+\infty} (\gamma a_5)^i \left(\mathbb{E}_{t_0}[\tilde{\mu}_{t+i+1}] - \mathbb{E}_{t_0}[\tilde{\mu}_{t+i}]\right).$$

This equation contains a forward looking term which is an exponential smoothing of the expected values of the AGT index. For all t, approximating $\mathbb{E}_{t_0}[\tilde{\mu}_{t+i+1}] - \mathbb{E}_{t_0}[\tilde{\mu}_{t+i}]$ by $\mathbb{E}[\tilde{\mu}_{t+1}] - \mathbb{E}[\tilde{\mu}_t]$ for all $i \geq 1$, we get

$$(1-\gamma a_5)\sum_{i=0}^{+\infty}(\gamma a_5)^i \left(\mathbb{E}_{t_0}[\tilde{\mu}_{t+i+1}] - \mathbb{E}_{t_0}[\tilde{\mu}_{t+i}]\right) \approx \mathbb{E}_{t_0}[\tilde{\mu}_{t+1}] - \mathbb{E}_{t_0}[\tilde{\mu}_t]$$

which gives

$$Z_t \approx \frac{a_5 r^\star_\sharp}{1-\gamma a_5} + \left(1 - \frac{a_5 A d_{t_0}}{1-\gamma a_5 k}\frac{(\gamma a_5 k)^T}{\Gamma_{t_0}}\right)\Gamma_t + U_t + a_5(1-\lambda)\left(\mathbb{E}_{t_0}[\tilde{\mu}_{t+1}] - \mathbb{E}_{t_0}[\tilde{\mu}_t]\right).$$

Substituting for Z_t in (19), we obtain

$$\begin{aligned}
\mathbb{E}_{t_0}[\tilde{\mu}_{t+1}] \approx{}& \frac{a_1 - a_5(1-\lambda)}{1-a_5(1-\lambda)}\mathbb{E}_{t_0}[\tilde{\mu}_t] + \frac{a_2 \mathbb{E}_{t_0}[\tilde{e}_t]}{1-a_5(1-\lambda)} + \frac{a_3 \mathbb{E}_{t_0}[\tilde{q}_t]}{1-a_5(1-\lambda)} \\
&+ \frac{\Gamma_t}{1-a_5(1-\lambda)}\left(1 - \frac{A a_5 d_{t_0}}{1-\gamma a_5 k}\frac{(\gamma a_5 k)^T}{\Gamma_{t_0}}\right) + \frac{U_t}{1-a_5(1-\lambda)} + \frac{a_4 + a_5 r^\star_\sharp/(1-\gamma a_5)}{1-a_5(1-\lambda)} \\
&\equiv \hat{a}_1 \mathbb{E}_{t_0}[\tilde{\mu}_t] + \hat{a}_2 \mathbb{E}_{t_0}[\tilde{e}_t] + \hat{a}_3 \mathbb{E}_{t_0}[\tilde{q}_t] + \hat{a}_4 \Gamma_t + \hat{a}_5 U_t + \hat{a}_6.
\end{aligned}$$

Hence, the dynamic of the economy can be approximated by the first-order recursive linear equation

$$\tilde{X}_t = M\tilde{X}_{t-1} + C\tilde{\nu}_t, \tag{71}$$

for all $t < T$, where $\tilde{X}_t = (\tilde{\mu}_t, \tilde{e}_t, \tilde{q}_t, \Gamma_t, U_t, 1)'$,

$$M = \begin{bmatrix} \hat{a}_1 & \hat{a}_2 & \hat{a}_3 & \hat{a}_4 & \hat{a}_5 & \hat{a}_6 \\ \xi & \theta & -\varphi & 0 & 0 & \hat{e} \\ 0 & 0 & g & 0 & 0 & q_0 \\ 0 & 0 & 0 & (\gamma a_5)^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & k & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; C = \begin{bmatrix} a_5 \tau \sigma_\eta & 0 \\ 0 & 0 \\ 0 & \sigma_\kappa \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

with

$$\begin{aligned}
\hat{a}_1 &= \frac{a_1 - a_5(1-\lambda)}{1-a_5(1-\lambda)}, \hat{a}_2 = \frac{a_2}{1-a_5(1-\lambda)}, \hat{a}_3 = \frac{a_3}{1-a_5(1-\lambda)} \\
\hat{a}_4 &= \frac{1}{1-a_5(1-\lambda)}\left(1 - \frac{a_5 A d_{t_0}}{1-\gamma a_5 k}\frac{(\gamma a_5 k)^T}{\Gamma_{t_0}}\right), \hat{a}_5 = \frac{1}{1-a_5(1-\lambda)}, \\
\hat{a}_6 &= \frac{a_4 + a_5 r^\star_\sharp/(1-\gamma a_5)}{1-a_5(1-\lambda)}.
\end{aligned}$$

Observe that (71) entails terms that are exponentially decreasing with T and that for a CNP, (70) also involves $\mathbb{E}[\tilde{e}_T]$. In case of an ESP, the last term of (70) is nil. For a CNP, denoting by τ_i the 6−dimensional column vector of zeroes except in position i where there is a 1, date T corresponding to the junction with a CNP is deduced from $\max_T\{\varphi\tau_1'.M^{T-t_0}.X_{t_0} \leq \xi\tau_3'.M^{T-t_0}.X_{t_0}\}$. Applying such a procedure generates a first set of values $\{\mathbb{E}[\tilde{\mu}_t]^{(1)}\}_{t\geq t_0}$ that can be used to estimate $\{Z_t^{(1)}\}_{t\geq t_0}$ from (44) to obtain an initial set of transition matrices $\{B_t^{(1)}\}_{t\geq t_0}$.

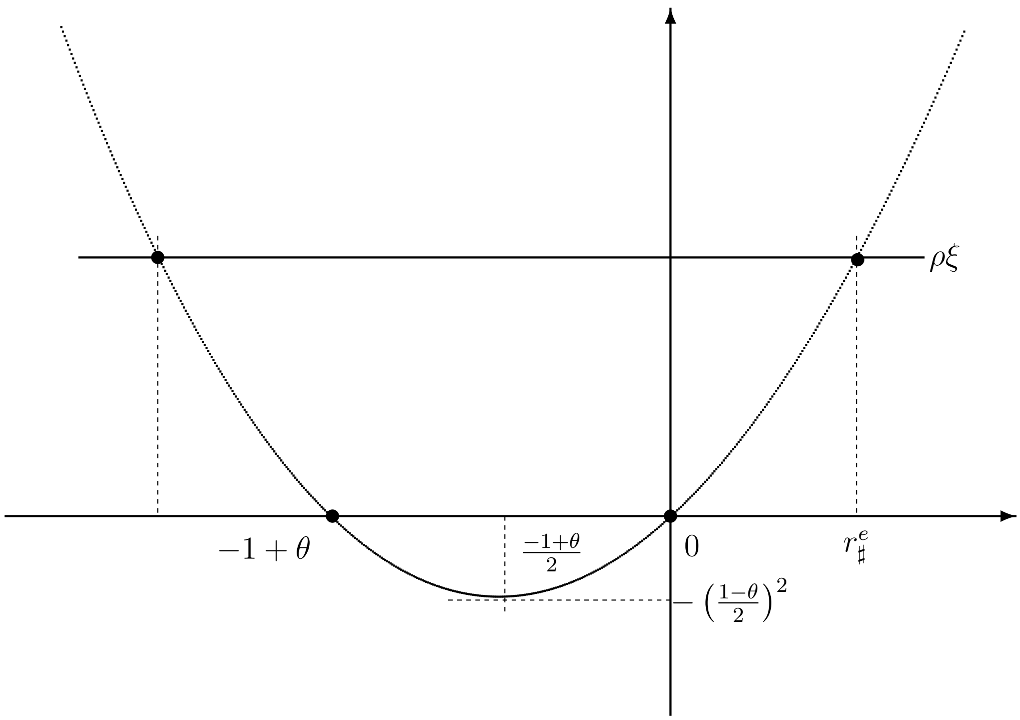

Figure 1: Long run optimal interest rate.

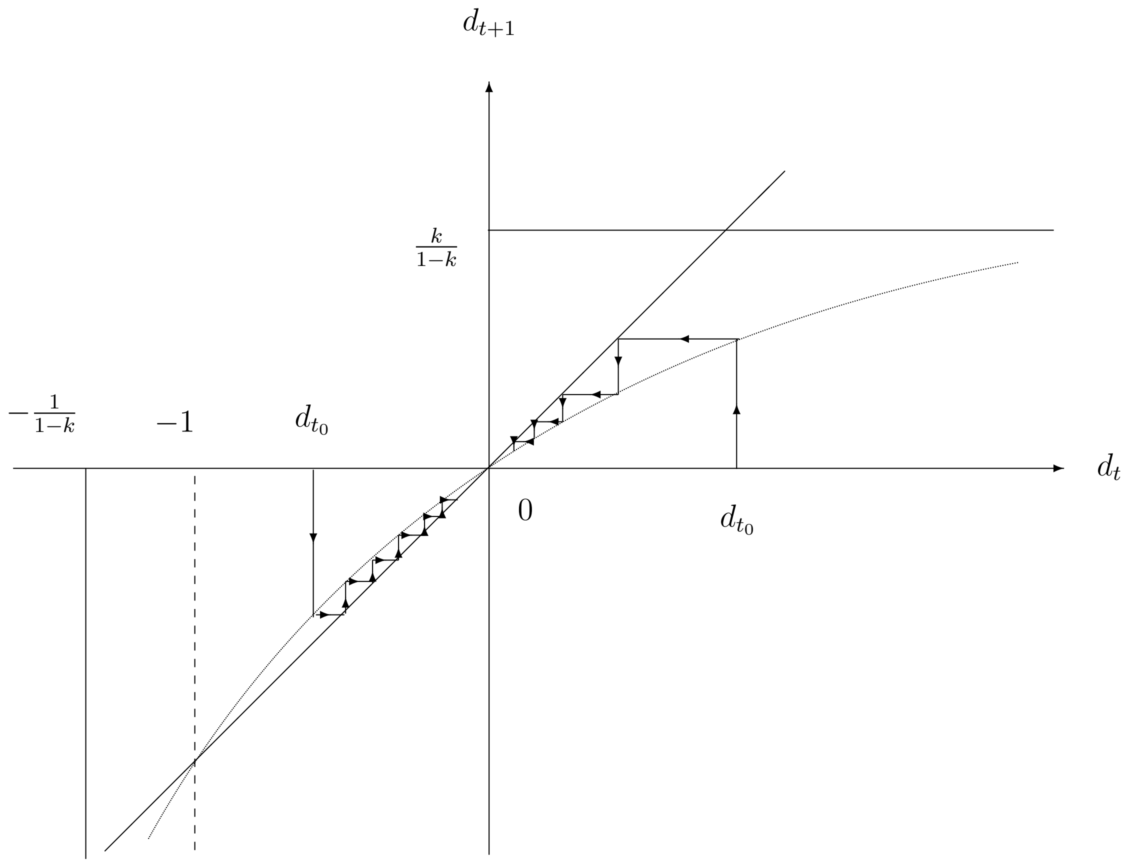

Figure 2: Convergence of the normalized gap.

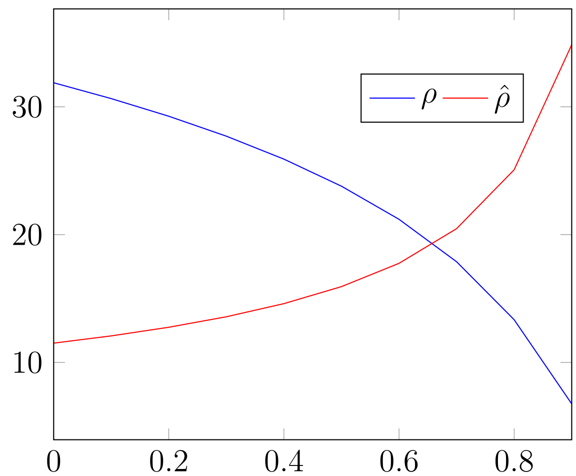

Figure 3: Carbon prices as functions of λ.

Baseline with $\lambda = .8$. Unit: US\$/t CO_2

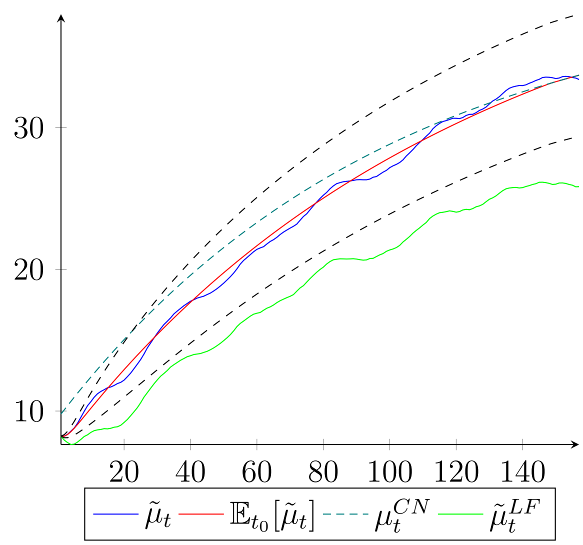

Figure 4: AGT index dynamic.

Baseline with $\lambda = .8$.

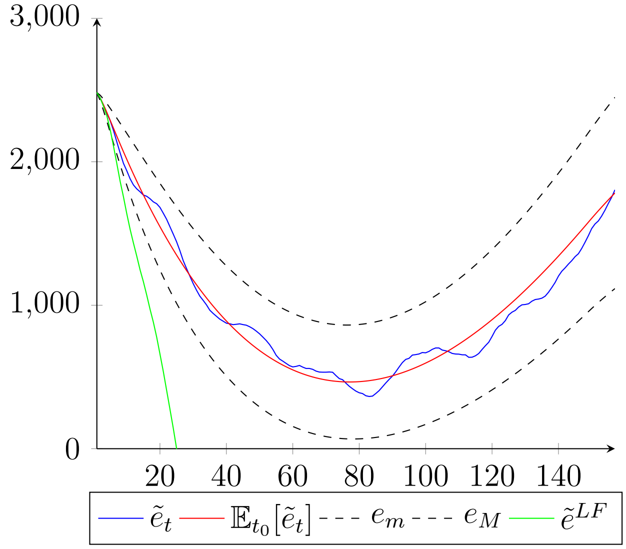

Figure 5: Environmental quality dynamic.

Baseline with $\lambda = .8$. Unit: Gt CO_2

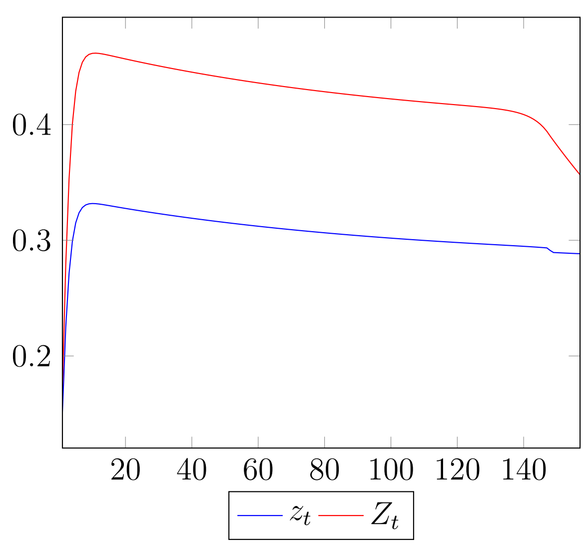

Figure 6: Optimal policy scheme.

Baseline with $\lambda = .8$.

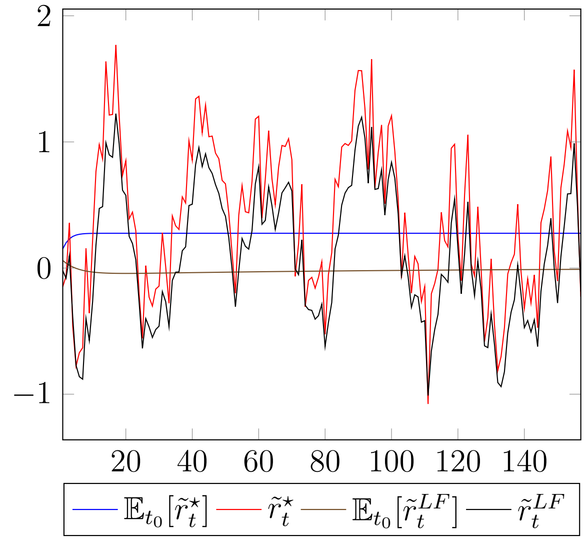

Figure 7: Interest rates.

Baseline with $\lambda = .8$.

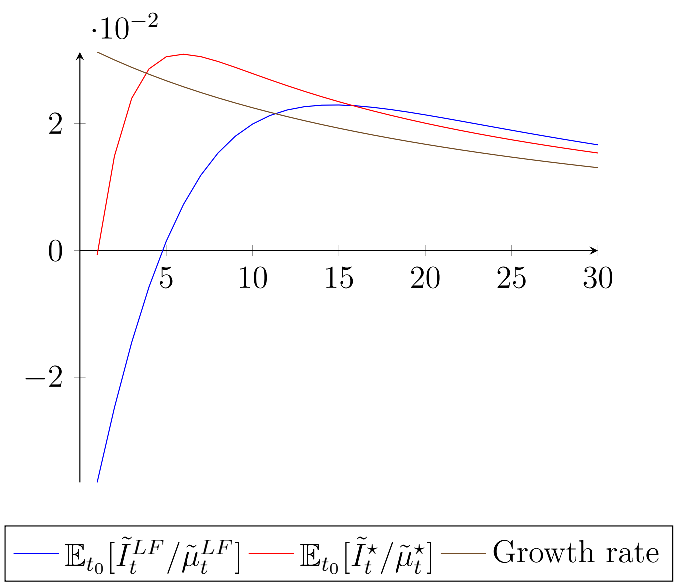

Figure 8: Investment rates.

Baseline with $\lambda = .8$.

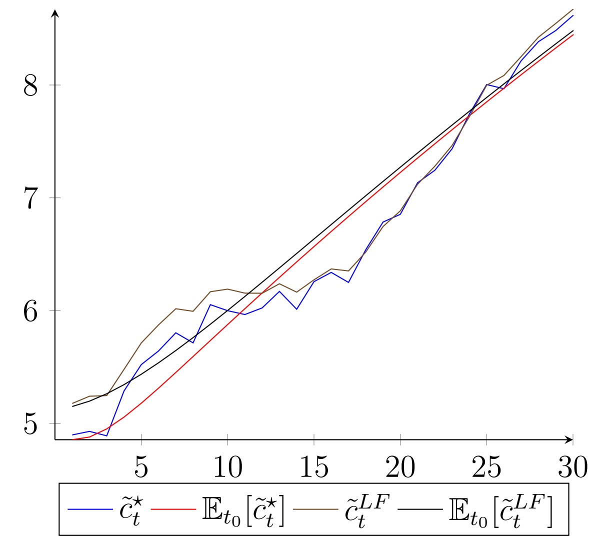

Figure 9: Consumption dynamic.

Baseline with $\lambda = .8$. Unit: 10 trillions US$

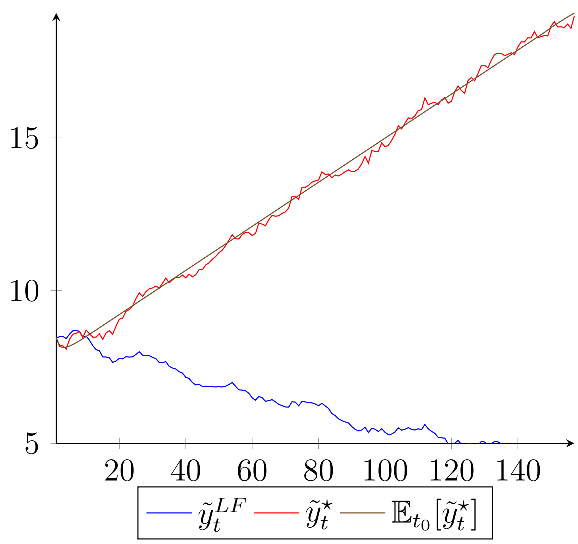

Figure 10: Total wealth dynamic.

Baseline with $\lambda = .8$. Unit: 10 trillions US$