A model of anonymous influence with anti-conformist agents

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Abstract. We study a stochastic model of anonymous influence with conformist and anti-conformist individuals. Each agent with a 'yes' or 'no' initial opinion on a certain issue can change his opinion due to social influence. We consider anonymous influence which depends on the number of agents having a certain opinion, but not on their identity. An individual is conformist/anti-conformist if his probability of saying 'yes' increases/decreases with the number of 'yes'- agents. In order to consider both conformists and anti-conformists in a society, we investigate a generalized aggregation mechanism. It uses the ordered weighted averages which are the only anonymous aggregation functions. Additionally, every agent has a coefficient of conformism which is a real number from -1 till 1, with the two extreme values corresponding to a pure anti-conformist and a pure conformist, respectively. We assume that both pure conformists and anti-conformists are present in a society, and we deliver a qualitative analysis of convergence in the model, i.e., find all terminal classes and conditions for their occurrence.

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1 Introduction

In this paper we study anti-conformism in a framework of anonymous social influence. Contrarily to opinion conformity which has been widely studied in various fields, settings, and using different approaches (see, e.g., Jackson (2008); Acemoglu and Ozdaglar (2011); Förster et al. (2013) for surveys), anti-conformism has received little attention in the literature. Despite the fact that anti-conformist is very natural to explain human behavior and dynamic phenomena, and plays a crucial role in many social and economic situations, there are only a few works related to this phenomenon. Grabisch and Rusinowska (2010a,b) address the problem of measuring negative influence but only in one-step (static) settings. Bramoullé et al. (2004); López-Pintado (2009); Cao et al. (2013) study network formation and anti-coordination games, i.e., games where agents prefer to choose an action different from that chosen by their partners. Anti-conformism and anti-coordination can easily be casted and detected in many frameworks. For example, the choice of a firm to go compatible or not with other firms can be seen as a problem of anti-conformism. Anti-coordination can be optimal when adopting different roles or having complementary skills are necessary for a successful interaction or realization of a task in a team.

We consider opinion formation in the framework of anonymous influence as a pure imitation process. The seminal work of DeGroot (1974) and some of its extensions (see e.g. DeMarzo et al. (2003); Jackson (2008); Golub and Jackson (2010); Büchel et al. (2014, 2015)) study a non-anonymous influence in which agents update their opinions by using a weighted average of the opinions of their neighbors. We are interested in anonymous influence, which depends only on the number of individuals having a certain opinion and is not dependent on agents' identities. Förster et al. (2013) study anonymous social influence by using the ordered weighted averages (commonly called OWA operators, Yager (1988)) which are the unique anonymous aggregation functions. More precisely, the authors departure from a general framework of influence based on aggregation functions (Grabisch and Rusinowska (2013)), where every individual updates his opinion by aggregating the agents' opinions which determines the probability that his opinion will be 'yes' in the next period. However, instead of allowing for arbitrary aggregation functions, Förster et al. (2013) consider the particular way of aggregating based on the OWA operators. Both frameworks of Grabisch and Rusinowska (2013) and Förster et al. (2013) cover only positive influence, since by definition aggregation functions are nondecreasing, and hence cannot model anti-conformism.

In order to consider societies with anti-conformists and anonymous influence, we investigate a generalized aggregation mechanism. We also use the OWA operators, i.e., every agent is assumed to have a weight vector, but is additionally characterized by a coefficient of conformism. Individuals are intrinsically either conformists or anti-conformists. The coefficient of conformism is a real number in $[-1, 1]$, where the extreme values -1 and 1 correspond to pure anti-conformists and pure conformists, respectively. Each agent has an initial yes/no opinion on a certain issue and at every time step can update his opinion by taking into account how many individuals share a given opinion. The probability of saying 'yes' is monotonic w.r.t. the number of 'yes'-agents – it increases for conformists and decreases for anti-conformists. We provide a complete qualitative analysis of convergence in this framework. The crucial information in determining all terminal classes to which a society converges is the number of yes/no agents needed to influence an individual's opinion. It is determined by the number of left/right zeroes in the agents' weight vectors. There exist several different types of terminal classes in the model, and conditions for their occurrence are determined by relations between the numbers of (anti-)conformists and the left/right zeroes in the weights vectors of conformists and anti-conformists. In the long run, the opinion of a society with conformists and anti-conformists always converges to a terminal class, but never to consensus, contrarily to a conformist society, where consensus can be reached. Moreover, a society with conformists and anti-conformists can be dichotomous, where one of the two groups say 'yes' forever. Cycles and periodic classes as well as intervals and unions of intervals are also possible terminal classes in the presence of anti-conformists.

Our framework can explain various phenomena like stable and persistent shocks, large fluctuations, stylized facts in the industry of fashion, in particular its intrinsic dynamics, booms and burst in the frequency of surnames, etc. For instance, if fashion were only a matter of conformist imitation in an anonymous framework, there would be no trends over time. Our setting can be applied to some existing models, like herd behavior and information cascades (Banerjee (1992); Bikhchandani et al. (1992)) which have been used to explain fads, investment patterns, etc.; see Anderson and Holt (2008) for a survey of experiments on cascade behavior. Although Bikhchandani et al. (1992) have already addressed the issue of fashion, the present model takes a different turn, since we assume no sequential choices and some agents are anti-conformists while others are conformists. In the model of herd behavior (Banerjee (1992)) agents play sequentially and wrong cascades can occur. Though it can be rational to follow the crowd, some anti-conformists may want to play a mixed-strategy: either following the crowd or not. This is particularly true under bounded rationality. Agents may not be able to know what is rational, for example because they lack information or do not have enough time or computational capacities. As a consequence, they may play according to rules of the thumb like counting how many people said 'yes' rather than computing bayesian probabilities. Chandrasekhar et al. (2016) show in a lab experiment that people tend to behave according to the DeGroot model rather than to Bayesian updating; see also Celen and Kariv (2004). This is also consistent with Anderson and Holt (1997) who show that counting is the most salient bias to explain departure from Bayesian updating.

The rest of the paper is structured as follows. In Section 2 we introduce the model of anonymous influence with anti-conformist agents. The convergence analysis is provided in Section 3. Section 4 contains some examples. In Section 5 we present concluding remarks. The proof of our main results on the possible terminal classes in the model is given in the Appendix.

2 The model

2.1 Basic assumptions

We consider a society N of $|N| = n$ agents, having to make a yes/no-decision on some issue. Each agent has a personal initial opinion on the issue, however, by knowing the opinion of the other agents or by some social interaction with them, the opinion of each agent may change due to mutual influence. Doing so, there is an evolution in time of the opinion of the agents, which may or may not stop at some stable state of the society.

We define the state of the society as the vector giving the opinion of each agent in N. Equivalently, the state of the society is determined by the set $S \subseteq N$ of agents whose opinion is 'yes'. Our fundamental assumption is that the evolution of the state is ruled by a homogeneous Markov chain, that is, the state evolves at discrete time steps, the state at time t depends only on the state at time $t-1$, and the transition matrix giving the probability of all possible transitions from a state S to a state T is constant over time.

These assumptions are basically those underlying (Grabisch and Rusinowska, 2013). As the number of states is 2^n, the size of the transition matrix is $2^n \times 2^n$. In order to avoid this exponential complexity, the latter reference uses a simple mechanism to generate the transition matrix, inspired by DeGroot (1974). Coding 'yes' and 'no' by 1 and 0, respectively, the probability $p_i(S)$ that an agent $i \in N$ says 'yes' at next time step, given the present state S (the set of agents saying 'yes'), is

$$p_i(S) = A_i(1_S), \tag{1}$$

where 1_S is the indicator function of S, i.e., $1_S(i) = 1$ if $i \in S$ and 0 otherwise, and A_i is a nondecreasing function from $[0, 1]^n$ to $[0, 1]$ satisfying $A_i(1_N) = 1$ and $A_i(1_\emptyset) = 0$ (called an *aggregation function*[1]). Supposing that the update of opinion is done independently, the probability of transition from a state S to a state T is

$$\lambda_{S,T} = \prod_{i \in T} p_i(S) \prod_{i \notin T} (1 - p_i(S)), \tag{2}$$

with $p_i(S)$ given by (1).

2.2 Anonymous influence

The most common example of aggregation function, used for example in DeGroot (1974), is the weighted arithmetic mean

$$A_i(x) = \sum_{j=1}^{n} w_j^i x_j,$$

where $x = (x_1, \ldots, x_n)$ and the w_j^i's are weights on the entries, satisfying $w_j^i \geq 0$ and $\sum_{j=1}^n w_j^i = 1$. Here, w_j^i represents to which extent agent i puts confidence on the opinion of agent j. It depicts a situation where every agent knows the identity of every other agent, and is able to assess to which extent he trusts or agrees with the opinion or personal tastes of others.

In many situations however, like opinions and comments given on the internet, the identity of the agents is not known, or at least, there is no clue on the reliability or kind of personality of the agents. Therefore, agents can be considered as anonymous, and influence is merely due to the *number* of agents having a certain opinion, not their identity. The natural aggregation function for this situation is the *ordered weighted average (OWA)* (Yager, 1988):

$$\mathrm{OWA}_w(x) = \sum_{j=1}^{n} w_j x_{(j)}, \tag{3}$$

where the entries $x_1, \ldots, x_n$ are rearranged in decreasing order: $x_{(1)} \geq x_{(2)} \geq \cdots \geq x_{(n)}$, and $w = (w_1, \ldots, w_n)$ is the weight vector defined as above. Hence, the weight w_j is not assigned to agent j but to rank j, and thus permits to model quantifiers. For example, taking $w_1 = 1$ and all other weights being 0 models the quantifier "there exists". Indeed, it is enough to have one of the entries being equal to 1 to get 1 as output. In our context, it means that only one agent saying 'yes' is enough to make your opinion being 'yes' for sure. Similarly, "for all" is modeled by $w_n = 1$ and all other weights being 0, and means that you need that all agents (including you) say 'yes' to be sure to continue to say 'yes'. Intermediate situations can of course be modeled as well: by letting $w_k = 1$ and $w_j = 0$ for all $k \neq j$, one obtain a model where k 'yes' among the n agents are needed to ensure that the concerned agent will say 'yes' at next time step. Moreover, soft or fuzzy quantifiers can be modeled as well: "approximately half" could be represented by the following weight vector (with $n = 10$):

$$w = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & 0 & 0 & 0 \end{pmatrix}.$$

[1] Traditionally, the domain of an aggregation is $[0, 1]^n$ or any interval of $\mathbb{R}$ to the power n. In our study, however, only the vertices $\{0, 1\}^n$ are used.

The above model using OWA as an aggregation function has been fully studied in Förster et al. (2013), in particular concerning convergence of opinion in the long run. Generally speaking (see, e.g. Kemeny and Snell (1976); Seneta (2006)), we recall that for a Markov chain with set of states E and transition matrix Λ and its associated digraph Γ, a *class* is a subset C of states such that for all states $e, f \in C$, there is a path in Γ from e to f, and C is maximal w.r.t. inclusion for this property. A class is *terminal* or *absorbing* if for every $e \in C$ there is no arc in Γ from e to a state outside C. A terminal class C is *periodic of period* k if it can be partitioned in blocks $C_1, \ldots, C_k$ such that for $i = 1, \ldots, k$, every outgoing arc of every state $e \in C_i$ goes to some state in C_{i+1}, with the convention $k + 1 = 1$. When each $C_1, \ldots, C_k$ reduces to a single state, one may speak of *cycle of length* k, by analogy with graph theory.

In our case, states are subsets of agents and therefore classes are collections of sets, which we denote by calligraphic letters, like $\mathcal{C}, \mathcal{B}$, etc. By definition, a terminal class indicates the final state of opinion of the society. If a terminal class reduces to a single state S, it means that in the long run, the society is dichotomous (unless $S = N$ or $S = \emptyset$, in which case consensus is reached): there is a set of agents S who say 'yes' forever, while the other ones say 'no' forever. Otherwise, there are endless transitions with some probability from one set $S \in \mathcal{C}$ to another one $S' \in \mathcal{C}$.

It is an obvious fact that for any type of aggregation function, $\emptyset$ and N are terminal classes. Indeed, the conditions $A_i(1_N) = 1$ and $A_i(1_\emptyset) = 0$ for all $i \in N$ imply that $\lambda_{N,N} = 1$ and $\lambda_{\emptyset,\emptyset} = 1$, that is, once these states are reached, there is no possibility to escape from them. However, many other terminal classes are possible. For the anonymous model, they are of two types:

(i) any single state $S \in 2^N$;
(ii) union of intervals of the type $[S, S \cup K]$, where $S, K \neq \emptyset, N$, with $[S, S \cup K] = \{T \in 2^N \mid S \subseteq T \subseteq S \cup K\}$.

For the second case, when the terminal class is reduced to a single interval $[S, S \cup K]$, it depicts a situation in the long run where agents in S say 'yes', those outside $S \cup K$ say 'no', and those in K oscillate between 'yes' and 'no' forever. Interestingly, no periodic class can occur, although in general for arbitrary aggregation functions cycles can occur (Grabisch and Rusinowska, 2013).

2.3 Anti-conformism and conformism

As aggregation functions are nondecreasing in each argument, models of influence based on them are necessarily *conformist*: if more agents say 'yes', your probability of saying 'yes' cannot decrease, i.e., you are more or less inclined to follow the trend. However, it is often observed that some individuals are inclined to go against the trend by some reactive behavior, which can be modeled by an "anti"-aggregation function, i.e., being nonincreasing in each argument.

In this paper, we introduce such functions, but limit our study to anonymous models. In order to consider both conformist and anti-conformist agents in a society, we propose a generalization of the above mechanism defined by (1) and (3). To this end, we find more convenient to replace 1 and 0 by 1 and -1, respectively, for the coding of 'yes' and 'no'. As usual, cardinalities of sets are denoted by the corresponding lower case, e.g., $s = |S|$.

The probability that agent i says 'yes' at next time step, given that S is the set of agents saying 'yes' at present time is now given by

$$p_i(S) = \frac{1}{2}\Big(1 + \alpha_i \mathrm{OWA}_{w^i}(1_S)\Big) = \frac{1}{2}\Big(1 + \alpha_i\Big(\sum_{j=1}^{s} w_j^i - \sum_{j=s+1}^{n} w_j^i\Big)\Big), \tag{4}$$

with $\alpha_i \in [-1, 1]$, w^i is the weight vector of agent i, and the OWA operator is given by (3). The coefficient α_i is called the *coefficient of conformism*. We easily observe the following.

(i) The values taken by p_i are comprised between $1/2(1 - \alpha_i)$ and $1/2(1 + \alpha_i)$.
(ii) If $\alpha_i > 0$, then p_i is a monotone function w.r.t. set inclusion, i.e., the bigger the set S, the higher the probability to say 'yes', which indicates a conformist attitude for agent i. This effect is maximum when $\alpha_i = 1$, and we say then that the agent is *purely conformist*. Note that in the latter case (4) is identical to (1) with the OWA operator. Hence, if $\alpha_i = 1$ for all $i \in N$, we recover the classical (conformist) anonymous model studied in Förster et al. (2013).
(iii) If $\alpha_i < 0$, then p_i is antimonotone w.r.t. set inclusion, i.e., the smaller S, the higher the probability to say 'yes', which means that the agent is anti-conformist. If $\alpha_i = -1$, then we call i a *purely anti-conformist* agent.
(iv) If $\alpha_i = 0$, then $p_i(S) = \frac{1}{2}$ for every S, that is, the agent tosses a coin whatever the situation is.

Example 1 Let us take $n = 4$ and the weight vector $(0, 0, \frac{1}{2}, \frac{1}{2})$, which can be interpreted as the soft quantifier "most of". The probability $p_i(S)$ is computed for various α_i and sizes of S in the table below.

$p_i(S)$	$S = \emptyset$	$\lvert S\rvert = 1$	$\lvert S\rvert = 2$	$\lvert S\rvert = 3$	$S = N$
$\alpha_i = 1$	0	0	0	0.5	1
$\alpha_i = 0.5$	0.25	0.25	0.25	0.5	0.75
$\alpha_i = 0$	0.5	0.5	0.5	0.5	0.5
$\alpha_i = -1$	1	1	1	0.5	0

One can see that conformist agents tend to say 'yes' if most of people do so, while anti-conformist agents tend to say 'no' in this situation.

To facilitate the analysis of the model, we distinguish between three types of agents and introduce some assumption and notation. We partition the society of agents into

$$N = N^c \cup N^a \cup N^m$$

where N^c is the set of (purely) conformist agents with $\alpha_i = 1$, N^a the set of (purely) anti-conformist agents ($\alpha_i = -1$), and N^m is the set of "mixed" agents with $\alpha_i \in]-1, 1[$. To avoid trivialities we consider that $N^c, N^a \neq \emptyset$, however in the present study we will bear on the case $N^m = \emptyset$ (called the *pure case*).

Consider a weight vector $w = \big(w_1 \cdots w_n\big)$ of an OWA operator. As our analysis will reveal, the relevant information in w is merely the number of right and left zeroes, not the precise value of the weights. We denote by l and r the number of left zeroes and right zeroes in w, respectively. Formally, $0 = w_1 = \cdots = w_l \neq w_{l+1}$ and $w_{n-r} \neq w_{n-r+1} = \cdots = w_n = 0$. For example, taking $w = \begin{pmatrix}0 & 0.1 & 0.5 & 0 & 0.2 & 0 & 0.3 & 0 & 0\end{pmatrix}$, we have $l = 1$ and $r = 2$. Observe that due to the normalization condition of w, we have $0 \leq l + r < n$.

We make the following assumption: agents in N^c may have different weight vectors, however the number of left and right zeroes is the same for all of them. We denote them by l^c, r^c. Similarly, l^a, r^a (respectively, l^m, r^m) denote the number of left and right zeroes in the weight vector of agents in N^a (respectively, N^m).

We end this section by giving an interpretation of the left and right zeroes. Let us consider a weight vector with l left zeroes and r right zeroes. We can see from the definition of OWA that these zeroes eliminate the first l 'yes' and the first r 'no'. Therefore, the decision of an agent with such a weight vector is based on the number of people saying 'yes' and 'no', after having eliminated the first l 'yes' and r 'no'. The number of left/right zeroes indicates how many people the agent needs in order to start being influenced towards the yes/no opinion. In particular, a non symmetrical weight vector w.r.t. the number of left and right zeroes means that the agent is biased towards the 'yes' or 'no' answer, i.e., he needs a different number of people to start being convinced to say 'yes' or 'no'.

2.4 Basic properties of transitions

We study in this section the properties of the transition matrix Λ, with entries $\lambda_{S,T}$, $S, T \in 2^N$. We recall that $\lambda_{S,T}$ is given by (2), with $p_i(S)$ given by (4).

Our aim is to find under which conditions one has a possible transition from S to T, i.e., $\lambda_{S,T} > 0$. From (2), we have:

$$\lambda_{S,T} > 0 \Leftrightarrow [p_i(S) > 0 \forall i \in T] \& [p_i(S) < 1 \forall i \notin T].$$

The pure case We start with the case $N^m = \emptyset$. We first observe that $p_i(\emptyset) = 1$ if $i \in N^a$ and 0 otherwise, and $p_i(N) = 1$ if $i \in N^c$ and 0 otherwise. Therefore we have in any case the sure transitions

$$\lambda_{\emptyset, N^a} = 1, \quad \lambda_{N, N^c} = 1.$$

Using (4), we find, for any $S \neq \emptyset, N$,

$$(i \in N^c) \qquad p_i(S) > 0 \quad \Leftrightarrow \quad \sum_{j=1}^{s} w_j^c > 0 \quad \Leftrightarrow \quad s > l^c \tag{5}$$

$$p_i(S) < 1 \quad \Leftrightarrow \quad \sum_{j=s+1}^{n} w_j^c > 0 \quad \Leftrightarrow \quad n - s > r^c \tag{6}$$

$$(i \in N^a) \qquad p_i(S) > 0 \quad \Leftrightarrow \quad \sum_{j=s+1}^{n} w_j^a > 0 \quad \Leftrightarrow \quad n - s > r^a \tag{7}$$

$$p_i(S) < 1 \quad \Leftrightarrow \quad \sum_{j=1}^{s} w_j^a > 0 \quad \Leftrightarrow \quad s > l^a. \tag{8}$$

Clearly, the above conditions depend only on the number of left and right zeroes of the weight vector. Therefore, as announced, the sole knowledge of the number of left and right zeroes is suffices for the analysis of transitions, and thus of convergence, as far as we are not interested in computing the precise values of the transition matrix.

By combining these conditions and their negation in various ways, one can see that we can have transitions to $\emptyset, N, N^a, N^c$ and any of their subset or superset. Table 1 summarizes the possible transitions, adding also those from $S = \emptyset$ and $S = N$. Let

	$0 \leq s \leq l^c$	$l^c < s < n - r^c$	$n - r^c \leq s \leq n$
$0 \leq s \leq l^a$	N^a	$T \in [N^a, N]$	N
$l^a < s < n - r^a$	$T \in [\emptyset, N^a]$	$T \in 2^N$	$T \in [N^c, N]$
$n - r^a \leq s \leq n$	$\emptyset$	$T \in [\emptyset, N^c]$	N^c

Table 1. Possible transitions from $S \in 2^N$ in the pure case

us introduce $Z = (l^c, r^c, l^a, r^a)$ the vector giving the number of left and right zeroes in the weight vectors of conformist and anti-conformist agents (in this order), and let us write p_i^Z to emphasize the dependency of p_i on these parameters (and similarly for $\lambda_{S,T}$). Equations (5) to (8) show striking symmetries when interchanging conformists and anti-conformists, as well as when interchanging left and right zeroes. Z being given, we introduce the reversal of Z, $Z^\partial := (r^c, l^c, r^a, l^a)$, which amounts to reversing the weight vectors, and the interchange of Z, $Z' = (l^a, r^a, l^c, r^c)$, which amounts to interchanging conformists with anti-conformists. Considering these operations, we observe the following symmetries:

(i) Interchange:

$$p_i^Z(S) > 0 \text{ for } i \in N^c \quad \Leftrightarrow \quad p_i^{Z'}(S) < 1 \text{ for } i \in N^a$$
$$p_i^Z(S) < 1 \text{ for } i \in N^c \quad \Leftrightarrow \quad p_i^{Z'}(S) > 0 \text{ for } i \in N^a$$

(idem with N^a, N^c exchanged)

(ii) Reversal:

$$p_i^Z(S) > 0 \text{ for } i \in N^c \quad \Leftrightarrow \quad p_i^{Z^\partial}(N \setminus S) < 1 \text{ for } i \in N^c$$
$$p_i^Z(S) < 1 \text{ for } i \in N^c \quad \Leftrightarrow \quad p_i^{Z^\partial}(N \setminus S) > 0 \text{ for } i \in N^c$$

(idem with N^a, N^c exchanged)

(iii) Interchange and reversal:

$$p_i^Z(S) > 0 \text{ for } i \in N^c \quad \Leftrightarrow \quad p_i^{(Z^\partial)'}(N \setminus S) > 0 \text{ for } i \in N^a$$
$$p_i^Z(S) < 1 \text{ for } i \in N^c \quad \Leftrightarrow \quad p_i^{(Z^\partial)'}(N \setminus S) < 1 \text{ for } i \in N^a$$

(idem with N^a, N^c exchanged)

The second case is of particular interest and leads to the following lemma.

Lemma 1 (symmetry principle) *Let $S, T \in 2^N$, and $Z = (l^c, r^c, l^a, r^a)$. The following equivalence holds:*

$$\lambda_{S,T}^Z > 0 \Leftrightarrow \lambda_{N \setminus S, N \setminus T}^{Z^\partial} > 0.$$

Proof. Letting $\lambda^Z_{S,T} > 0$ means that for every $i \in N \setminus T$, $0 \leq p^Z_i(S) < 1$, and for every $i \in T$, $0 < p^Z_i(S) \leq 1$. Using the equivalences in (ii), we find that for every $i \in N \setminus T$, $0 < p^{Z^\partial}_i(N \setminus S) \leq 1$ and for every $i \in T$, $0 \leq p^{Z^\partial}_i(N \setminus S) < 1$. But this means that $\lambda^{Z^\partial}_{N\setminus S, N\setminus T} > 0$.

The mixed case The mixed case can be easily analyzed provided the weight vector of mixed agents is a suitable convex combination of the weight vectors of conformist and anti-conformist agents. Suppose that every conformist agent has weight vector w^c and every anti-conformist agent has weight vector w^a. Consider a mixed agent $i \in N^m$ with $\alpha_i \in]-1, 1[$, with weight vector w^m given by

$$w^m = \frac{\alpha_i + 1}{2} w^c + \frac{1 - \alpha_i}{2} w^a. \tag{9}$$

Then one can check from (4) that

$$p_i(S) = \frac{\alpha_i + 1}{2} p_c(S) + \frac{1 - \alpha_i}{2} p_a(S),$$

where c, a are any conformist and anti-conformist agents, respectively. This can be interpreted as: a mixed player i plays randomly either as a conformist or an anti-conformist, with probability $\frac{1+\alpha_i}{2}$ for conformist. Under this assumption, we can easily derive the conditions for $p_i(S)$ to be 0 or 1, using (5) to (8):

$$(i \in N^m) \qquad p_i(S) = 0 \quad \Leftrightarrow \quad n - r^a \leq s \leq l^c \tag{10}$$
$$p_i(S) = 1 \quad \Leftrightarrow \quad n - r^c \leq s \leq l^a, \tag{11}$$

For all other cases, $0 < p_i(S) < 1$.

An important remark is that these conditions do not depend on the particular α of i: it means that under the assumption (9), the (qualitative) analysis of transitions can be done without knowing the α of each mixed player, and they can also be different for each player.

Now, from the above conditions it is easy to rewrite Table 1 for the mixed case. Finally,

	$0 \leq s \leq l^c$	$l^c < s < n - r^c$	$n - r^c \leq s \leq n$
$0 \leq s \leq l^a$	$T \in [N^a, N^a \cup N^m]$	$T \in [N^a, N]$	N
$l^a < s < n - r^a$	$T \in [\emptyset, N^a \cup N^m]$	$T \in 2^N$	$T \in [N^c, N]$
$n - r^a \leq s \leq n$	$\emptyset$	$T \in [\emptyset, N^c \cup N^m]$	$T \in [N^c, N^c \cup N^m]$

Table 2. Possible transitions from $S \in 2^N$ in the mixed case

if $S = \emptyset$, then $\lambda_{S,T} > 0$ for every $T \in [N^a, N^a \cup N^m]$, and if $S = N$, then $\lambda_{S,T} > 0$ for every $T \in [N^c, N^c \cup N^m]$. Table 2 presents possible transitions from $S \in 2^N$ in the mixed case.

3 Convergence of the model in the pure case ($N^m = \emptyset$)

This section is devoted to the study of terminal classes. Unlike the case of a model with only conformist agents, their study appears to be extremely complex, even in the pure case. We introduce some useful notation. We write $S \to T$ if a transition from S to T is possible, i.e., $\lambda_{S,T} > 0$, and $S \xrightarrow{1} T$ if $\lambda_{S,T} = 1$ (sure transition). We extend the latter notation to collections of sets: letting $\mathcal{S}, \mathcal{T}$ be two nonempty collections of sets in 2^N, we write

$$\mathcal{S} \xrightarrow{1} \mathcal{T} \quad \Leftrightarrow \quad \forall T \in \mathcal{T}, \exists S \in \mathcal{S} \text{ s.t. } \lambda_{S,T} > 0 \text{ and } \forall S \in \mathcal{S}, \forall T \notin \mathcal{T}, \lambda_{S,T} = 0.$$

We observe the following basic facts:

(F0) $\emptyset \xrightarrow{1} N^a$, $N \xrightarrow{1} N^c$ (as already observed).

(F1) If $\mathcal{S} \xrightarrow{1} \mathcal{T}$, $\mathcal{S}' \xrightarrow{1} \mathcal{T}'$ and $\mathcal{S} \subset \mathcal{S}'$, then $\mathcal{T} \subseteq \mathcal{T}'$.

(F2) Applying (F0) and (F1), we find that in a transition $\mathcal{S} \to \mathcal{T}$, $\emptyset \in \mathcal{S}$ implies $N^a \in \mathcal{T}$ and $N \in \mathcal{S}$ implies $N^c \in \mathcal{T}$.

(F3) Consider $\mathcal{S} \xrightarrow{1} \mathcal{T}_1 \xrightarrow{1} \cdots \xrightarrow{1} \mathcal{T}_p$, with $p \geq 2$. If $\mathcal{S} \subseteq \mathcal{T}_1$, then $\mathcal{S} \subseteq \mathcal{T}_1 \subseteq \cdots \subseteq \mathcal{T}_p$.

(F4) 2^N is a possible terminal class. Indeed, take $l^c = r^c = l^a = r^a = 0$. From Table 1 we immediately see that for any $S \neq \emptyset, N$ we have $S \xrightarrow{1} 2^N$. Since the power set of the set of states is the "default" terminal class when no other can exist, we exclude it from our study and do not consider transitions to 2^N.

(F5) From Table 1, we see that we have to deal only with the sets $\emptyset, N^a, N^c, N$ and the intervals $[\emptyset, N^c], [\emptyset, N^a], [N^a, N], [N^c, N]$ (2^N being excluded by (F4)), i.e., only these can be constituents of a terminal class. We put

$$\mathbb{B} = \{\{\emptyset\}, \{N^a\}, \{N^c\}, \{N\}, [\emptyset, N^c], [\emptyset, N^a], [N^a, N], [N^c, N]\}$$

the set of collections relevant to our study. Intervals not reduced to a singleton are called *nontrivial intervals.*

(F6) $\mathcal{S} \subseteq 2^N$ is a terminal class if and only if $\mathcal{S} \xrightarrow{1} \mathcal{S}$ and $\mathcal{S}$ is *connected* (i.e., there is a path (chain of transitions) from S to T for any $S, T \in \mathcal{S}$).

(F6) will be our unique tool to find aperiodic terminal classes, while periodic classes are of the form $\mathcal{S}_1 \xrightarrow{1} \cdots \xrightarrow{1} \mathcal{S}_p$, with $\mathcal{S}_1, \ldots, \mathcal{S}_p \subset 2^N$ and being pairwise disjoint (no common set between $\mathcal{S}_i, \mathcal{S}_j$), and $\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_p$ must be connected.

Since $N^m = \emptyset$, we have $n^a = n - n^c$, where $n^a = |N^a|$ and $n^c = |N^c|$. Hence, the model is entirely determined by $l^c, r^c, l^a, r^a, n^c, n$. We recall that these parameters must satisfy the following constraints:

$$\begin{aligned} 0 &\leq l^a + r^a < n \\ 0 &\leq l^c + r^c < n \\ 0 &< n^c < n. \end{aligned}$$

Based on these facts, we can show the main result of this section.

Theorem 1 *Assume that $N^m = \emptyset$, $N^a \neq \emptyset$ and $N^c \neq \emptyset$. There are nineteen possible terminal classes which are*[2]*:*

(i) Either one of the following singletons:

(1) N^a if and only if $n^c \geq (n - l^c) \vee (n - l^a)$;

[2] We use the standard notation $\vee$ and $\wedge$ to denote max and min, respectively.

(2) N^c *if and only if* $n^c \geq (n-r^c) \vee (n-r^a)$*;*

(ii) or one of the following cycles and periodic classes:

(3) $N^a \xrightarrow{1} \emptyset \xrightarrow{1} N^a$ *if and only* $n - l^c \leq n^c \leq r^a$*;*

(4) $N^c \xrightarrow{1} N \xrightarrow{1} N^c$ *if and only if* $n - r^c \leq n^c \leq l^a$*;*

(5) $N^a \xrightarrow{1} N^c \xrightarrow{1} N^a$ *if and only if* $n^c \leq l^c \wedge l^a \wedge r^c \wedge r^a$*;*

(6) $N^a \xrightarrow{1} [\emptyset, N^c] \xrightarrow{1} N^a$ *if and only if* $n^c \leq l^c \wedge l^a \wedge r^a$ *and* $r^c < n^c < n - l^c$*;*

(7) $N^c \xrightarrow{1} [N^a, N] \xrightarrow{1} N^c$ *if and only if* $n^c \leq r^c \wedge r^a \wedge l^a$ *and* $l^c < n^c < n - r^c$*;*

(8) $[\emptyset, N^c] \xrightarrow{1} [N^a, N] \xrightarrow{1} [\emptyset, N^c]$ *if and only if* $r^c \vee l^c < n^c \leq r^a \wedge l^a \wedge (n - l^c - 1) \wedge (n - r^c - 1)$*;*

(9) $\emptyset \xrightarrow{1} N^a \xrightarrow{1} N^c \xrightarrow{1} \emptyset$ *if and only if* $n^c \leq r^c \wedge r^a \wedge l^c$ *and* $n^c \geq n - r^a$*;*

(10) $N^a \xrightarrow{1} N \xrightarrow{1} N^c \xrightarrow{1} N^a$ *if and only if* $n^c \leq l^c \wedge l^a \wedge r^c$ *and* $n^c \geq n - l^a$*;*

(iii) or one of the following intervals or union of intervals:

(11) $[\emptyset, N^a]$ *if and only if* $(n - l^c) \vee (r^a + 1) \leq n^c < n - l^a$*;*

(12) $[N^c, N]$ *if and only if* $(n - r^c) \vee (l^a + 1) \leq n^c < n - r^a$*;*

(13) $[\emptyset, N^a] \cup [\emptyset, N^c]$ *if and only if* $l^c \geq n - r^a$ *and* $n^c \in \big(]r^c, n - l^c[\cap]l^a, n - r^c[\big) \cup \Big(\big(]l^a, n - r^a[\cup]l^c, n - r^c[\big) \cap]0, r^c]\Big)$*;*

(14) $[N^a, N] \cup [N^c, N]$ *if and only if* $l^a \geq n - r^c$ *and* $n^c \in \big(]l^c, n - r^c[\cap]r^a, n - l^c[\big) \cup \Big(\big(]r^a, n - l^a[\cup]r^c, n - l^c[\big) \cap]0, l^c]\Big)$*;*

(15) $[\emptyset, N^a] \cup \{N^c\}$ *if and only if* $l^c + r^c = n - 1$, $r^a \geq r^c$ *and* $l^a < n^c \leq r^c \wedge l^c$*;*

(16) $[N^c, N] \cup \{N^a\}$ *if and only if* $l^c + r^c = n - 1$, $l^a \geq l^c$ *and* $r^a < n^c \leq r^c \wedge l^c$*.*

(17) $[\emptyset, N^c] \cup \{N^a\}$ *if and only if* $l^a + r^a = n - 1$, $l^c \geq l^a$, $n^c < n - r^c$ *and* $n^c \in]r^c, n - l^c[\cup]l^c, r^c]$*;*

(18) $[N^a, N] \cup \{N^c\}$ *if and only if* $l^a + r^a = n - 1$, $r^c \geq r^a$, $n^c < n - l^c$ *and* $n^c \in]l^c, n - r^c[\cup]r^c, l^c]$*.*

(19) 2^N *otherwise.*

Moreover, if $l^c + r^c = n - 1$*, then cases (6), (7), (8), (13) and (14) become impossible while (15) and (16) are specific to this case, and if* $l^a + r^a = n - 1$*, then cases (11) and (12) become impossible, while (17) and (18) are specific to this case.*

The proof can be found in the appendix.

According to Theorem 1, nineteen terminal classes representing three different types are possible in the model. The first one is a singleton which means that in the long run the society is dichotomous: all anti-conformists end up in saying 'yes' (and all conformists in saying 'no', case (1)) or these are all conformists who say 'yes' forever (case (2)).

Note that cases (1) till (19) are not exclusive, which can be already seen when considering cases (1) and (2). Indeed, for a given society, which is represented by the set of parameters n, n^c, l^a, r^a, l^c and r^c, under some conditions both cases (1) and (2) are possible, and therefore two different terminal classes might occur, N^a and N^c. However, the process will end up in only one of them, depending on the initial conditions.

The second type of possible terminal classes corresponds to cycles and periodic classes which have also a natural interpretation. For instance, case (3) means that in the long run, anti-conformists say 'yes' at time t, then at time $t+1$ nobody says 'yes', at time $t+2$ again anti-conformists say 'yes', etc. Under case (6), at some step in the long run all anti-conformists say 'yes', in the following step they all say 'no' but a fraction of conformists might say ‘yes’, then in the next step again all anti-conformists say ‘yes’, etc. There exist also longer cycles, like the ones described in cases (9) and (10).

The terminal class can also be an interval or a union of intervals. Case (11) corresponds to the situation where conformists say ‘no’ and anti-conformists oscillate between the two opinions. Notice that $[\emptyset, N^a]$ (case (11)) and $[N^c, N]$ (case (12)) can occur, but $[\emptyset, N^a] \cup [N^c, N]$ is never a terminal class. Under the terminal class (13), in one time step the process might be in $[\emptyset, N^a]$ (conformists say ‘no’ and anti-conformists oscillate) and in another step the process might be in $[\emptyset, N^c]$ (anti-conformists say ‘no’ and conformists oscillate). The cases (15) – (16) ((17) – (18), respectively) correspond to the situations in which conformists (anti-conformists, respectively) are not much influenceable, i.e., they need in total a maximal number of agents $(n-1)$ to start being influenced towards ‘yes’ or ‘no’ opinion. Under the terminal class (15), in one time step the process might be in $[\emptyset, N^a]$ and in another step the process might be in N^c (all conformists say ‘yes’). The terminal class (19) means that at any time step the yes-coalition can be any subset of agents.

The analysis for conformists and anti-conformists is not symmetric. However, while there is no symmetry between “a” and “c” in this framework, there exists symmetry between S and $N \setminus S$ as pointed out in Lemma 1.

4 Examples

First, we consider two particular “symmetric” cases of the weight vectors. The first one concerns a kind of one-side symmetry across conformists and anti-conformists, in the sense that they ignore the same number of yes/no answers. In this case we have $l^a = l^c$ and $r^a = r^c$ (see Example 2 below). The second case is related to symmetry within the population of conformists or anti-conformists, i.e., when weight vectors are symmetrical w.r.t. the number of left and right zeros. Formally, this means that $l^a = r^a$ and $l^c = r^c$ (see Example 3). As already mentioned before, an interpretation of such symmetrical weight vectors is that agents are not biased towards the answer ‘yes’ or ‘no’. This assumption might be relevant for instance when voting for two candidates. However, it might not be relevant when saying ‘yes’ means ‘adopting a new technology’, where a bias towards a status-quo or a bias towards technology adoption makes sense. The following examples follow directly from the results of the previous section.

Example 2 Assume that $N^a \neq \emptyset$, $N^c \neq \emptyset$, $N^m = \emptyset$, $l^a = l^c$ and $r^a = r^c$. If $l^a + r^a \neq n-1$ then the possible terminal classes are:

- N^a if and only if $n^a \leq l^a$;
- N^c if and only if $n^a \leq r^a$;
- $N^a \xrightarrow{1} N^c \xrightarrow{1} N^a$ if and only if $n^c \leq l^a \wedge r^a$.
- 2^N otherwise.

If $l^a + r^a = n-1$ then besides the terminal classes listed above, two more terminal classes are possible:

- $[\emptyset, N^c] \cup \{N^a\}$ if and only $n^a > r^a$ and $n^c \in]r^a, n - l^a[\cup]l^a, r^a]$;
- $[N^a, N] \cup \{N^c\}$ if and only if $n^a > l^a$ and $n^c \in]l^a, n - r^a[\cup]r^a, l^a]$.

Hence, if the number of anti-conformists does not exceed the number of left (right) zeroes in their weight vector, then in the long run the society might be dichotomous with all anti-conformists (conformists) saying 'yes'. Note that both N^a and N^c might occur. For instance, depending on the initial conditions, if $n = 5$, $n^a = 1$, $n^c = 4$, $l^a = l^c = 2$, $r^a = r^c \in \{1, 2\}$, then either N^a or N^c will occur. These two terminal classes exclude, however, the existence of $N^a \xrightarrow{1} N^c \xrightarrow{1} N^a$ which might be the terminal class if the number of conformists does not exceed the number of left zeroes nor the number of right zeroes, and means that in one time step all anti-conformists say 'yes' and in the following step all conformists say 'yes', etc. Moreover, under the condition $l^a + r^a = n - 1$, N^a and N^c also exclude the existence of $[\emptyset, N^c] \cup \{N^a\}$ and $[N^a, N] \cup \{N^c\}$.

Example 3 Assume that $N^a \neq \emptyset$, $N^c \neq \emptyset$, $N^m = \emptyset$, $l^a = r^a$ and $l^c = r^c$. If $2l^a \neq n - 1$ and $2l^c \neq n - 1$ then the possible terminal classes are:

- N^a and N^c if and only if $n^a \leq l^a \wedge l^c$;
- $N^a \xrightarrow{1} N^c \xrightarrow{1} N^a$ if and only if $n^c \leq l^a \wedge l^c$;
- $[\emptyset, N^c] \xrightarrow{1} [N^a, N] \xrightarrow{1} [\emptyset, N^c]$ if and only if $l^c < n^c \leq l^a \wedge (n - l^c - 1)$;
- $[\emptyset, N^a]$ and $[N^c, N]$ if and only if $l^a < n^a \leq l^c$ and $n^c \geq l^a + 1$;
- 2^N otherwise.

If $2l^a = n - 1$ or $2l^c = n - 1$, then some of the terminal classes listed above become impossible, but new possibilities appear. In particular, if $2l^a = 2l^c = n - 1$ then the possible terminal classes are:

- N^a and N^c if and only if $n^a \leq l^a$;
- $N^a \xrightarrow{1} N^c \xrightarrow{1} N^a$ if and only if $n^c \leq l^a$;
- $[\emptyset, N^c] \cup \{N^a\}$ and $[N^a, N] \cup \{N^c\}$ if and only if $n^a > l^a$ and $n^c \in]l^a, n - l^a[$;
- 2^N otherwise.

Note that while both N^a and N^c might occur, they exclude the existence of the remaining terminal classes. Similarly, while both $[\emptyset, N^c] \cup \{N^a\}$ and $[N^a, N] \cup \{N^c\}$ might occur, other cases become then impossible.

In the following example we analyze the impact of the relative number of conformists/anti-conformists on the possible terminal classes, in particular, when combining it with the two kinds of symmetry considered in the previous examples.

Example 4 In a society with at least as many conformists as anti-conformists, the terminal classes from (5) till (10), (15) and (16) listed in Theorem 1 become impossible. More precisely, cycles when the opinion of conformists might change and some unions of intervals are excluded.

Assume that there are strictly more conformists than anti-conformists and that the two kinds of symmetry hold, i.e., $l^a = l^c = r^a = r^c$. Then the possible terminal classes are only N^a and N^c if $n^a \leq l^a$, and 2^N if $n^a > l^a$. The latter condition means that the agents are more easily influenceable, since they do not need many agents to start being influenced.

If n is even and $n^a = n^c$, and the two kinds of symmetry hold, then 2^N becomes the only possible terminal class.

Consider now a society with more anti-conformists than conformists. Without imposing additional conditions, we do not exclude any terminal class. In other words, anti-conformists might make the society more 'unstable' in the sense that the opinion of any of these two groups of society might change over time. Under the two kinds of symmetry, the society cannot be dichotomous anymore and only two terminal classes become possible: $N^a \xrightarrow{1} N^c \xrightarrow{1} N^a$ if $n^c \leq l^a$, and 2^N if $n^c > l^a$. Hence, in a society with a majority of anti-conformists it is possible that at some step in the long run all anti-conformists say 'yes' and in the following step all conformists say 'yes', etc. Such a situation would be impossible in a society with a majority of conformists.

We also examine the impact of more influenceable agents (conformists and anti-conformists) on possible terminal classes. In the next example we assume that either anti-conformists or conformists, or all agents, have the weight vectors without zeroes. This means that these agents start being influenced immediately, i.e., when meeting the first individual, as they do not ignore any yes/no answer.

Example 5 Assume that $N^a \neq \emptyset$, $N^c \neq \emptyset$, $N^m = \emptyset$. If $l^a = r^a = 0$, then the possible terminal classes are:

- $[\emptyset, N^a]$ if and only if $n - l^c \leq n^c < n$;
- $[N^c, N]$ if and only if $n - r^c \leq n^c < n$;
- 2^N otherwise.

This means that, roughly speaking, anti-conformists who are more influenceable do not have a fixed opinion (a fraction of them can say 'yes'), contrarily to conformists who always say either 'no' or 'yes'.
On the other hand, if $l^c = r^c = 0$, then the possible terminal classes are:

- $[\emptyset, N^c] \xrightarrow{1} [N^a, N] \xrightarrow{1} [\emptyset, N^c]$ if and only if $n^c \leq r^a \wedge l^a$;
- $[\emptyset, N^c] \cup \{N^a\}$ if and only if $l^a = 0$, $r^a = n - 1$, $n^c \in]0, n[$;
- $[N^a, N] \cup \{N^c\}$ if and only if $l^a = n - 1$, $r^a = 0$, $n^c \in]0, n[$;
- 2^N otherwise.

Finally, we can conclude that if all agents have the weight vector without zeroes, i.e., $l^a = r^a = l^c = r^c = 0$, then the only possible terminal class is 2^N.

5 Concluding remarks

In the paper we analyzed a process of opinion formation in a society with conformists and anti-conformists. We focused on anonymous influence meaning that an individual can change his initial opinion but the change depends on the number of agents with a certain opinion and not on their identities. If the number of yes-agents increases, then the probability that a given agent says 'yes' increases if the agent is conformist and decreases if he is anti-conformist. Every individual has a coefficient of conformism which is a real number between -1 and 1. We assumed that pure conformists (agents with the coefficient of conformism being 1) as well as pure anti-conformists (the ones with the coefficient of conformism being -1) exist. So far we focused on a society without 'mixed' individuals (i.e., agents with the coefficient of conformism belonging to $]-1, 1[$), determined all possible terminal classes and conditions for their occurrence.

We observe natural but essential differences between a society formed entirely by conformists and a society with both conformists and anti-conformists. First of all, while under anonymous influence a society of conformists can reach consensus as shown in Förster et al. (2013), no consensus is possible under anonymous influence in a society of conformists and anti-conformists. On the other hand, while no periodic terminal class can exists in a conformist society, the presence of anti-conformists makes cycles possible, and even a number of different cycles and periodic classes might exist in such a society.

Another interesting observation is that the analysis for conformists and anti-conformists is not symmetric, in the sense that we cannot simply replace "a" by "c" in our study, but what does hold is the symmetry principe as shown in Lemma 1.

We notice that there always exists a terminal class, since by virtue of Theorem 1 if none of cases (1) till (14) occurs, the terminal class 2^N (case (15)) is possible. However, some of the conditions stated in Theorem 1 are not exclusive, and as a consequence, sometimes several different terminal classes can be possible. For instance, if $l^a = l^c = r^a = r^c$, then the possible terminal classes are N^a, N^c and $N^a \xrightarrow{1} N^c \xrightarrow{1} N^a$, and which of them will occur in the society will depend on the initial conditions. On the other hand, not all natural situations can be present in a society with both conformists and anti-conformists. For instance, as already mentioned, consensus between all society members is impossible under the coexistence of conformists and anti-conformists. Another example of a final state of opinion of the society which cannot appear in this framework is the situation when all anti-conformists say 'no' forever and conformists oscillate between 'yes' and 'no'.

In our follow-up research on anti-conformism, first of all, we intend to relax the assumption that a society consists of only pure conformists and pure anti-conformists, and to allow for the presence of 'mixed' individuals in the society. Furthermore, we would like to relax the anonymity assumption and to study a more general framework of anti-conformism.

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A Proof of Theorem 1

Our strategy is based on (F6): aperiodic terminal classes are connected collections $\mathcal{S}$ such that $\mathcal{S} \xrightarrow{1} \mathcal{S}$. Periodic terminal classes are of the form $\mathcal{S}_1 \xrightarrow{1} \cdots \xrightarrow{1} \mathcal{S}_p$ with all $\mathcal{S}_i$ pairwise incomparable, and $\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_p$ is connected. Consequently, we study all possible kinds of transition $\mathcal{S} \xrightarrow{1} \mathcal{T}$, and check connectedness for each candidate. We distinguish between "simple" transitions of the type $\mathcal{B} \xrightarrow{1} \mathcal{B}'$ with $\mathcal{B}, \mathcal{B}' \in \mathbb{B}$, and "multiple" transitions $\mathcal{S} \xrightarrow{1} \mathcal{T}$, where $\mathcal{S}, \mathcal{T}$ are composed with several elements of $\mathbb{B}$, e.g., $[\emptyset, N^a] \cup [\emptyset, N^c]$.

A.1 Simple transitions

We focus on transitions of the type $\mathcal{B} \xrightarrow{1} \mathcal{B}'$, with $\mathcal{B}, \mathcal{B}' \in \mathbb{B}$, and look for conditions on the parameters of the model to obtain such transitions.

Observe that if $\mathcal{B}'$ is a nontrivial interval, it cannot be the union of other elements of $\mathcal{B}$. Therefore, $\mathcal{B} \xrightarrow{1} \mathcal{B}'$ if and only if for any $S \in \mathcal{B}$, $S \xrightarrow{1} \mathcal{B}''$ with $\mathcal{B}'' \in \mathbb{B}$ and $\mathcal{B}'' \subseteq \mathcal{B}'$, and there is at least one $S \in \mathcal{B}$ s.t. $S \xrightarrow{1} \mathcal{B}'$. Let us denote by $\mathcal{C}[\mathcal{B}]$ the conditions on $s = |S|$ to have a sure transition from S to $\mathcal{B}$, as given in Table 1. All these conditions are intervals.

Observe that all $\mathcal{B} \in \mathbb{B}$ are either singletons $\{B\}$ or nontrivial intervals $[\underline{B}, \overline{B}]$, and $\mathcal{B} \subset \mathcal{B}'$ if and only if $\mathcal{B} = \{\underline{B'}\}$ or $\{\overline{B'}\}$, with $\mathcal{B}' = [\underline{B'}, \overline{B'}]$. Hence:

$$\mathcal{B} \xrightarrow{1} \mathcal{B}' \Leftrightarrow \begin{cases} [\underline{b}, \overline{b}] \subseteq \mathcal{C}[\mathcal{B}'] \cup \mathcal{C}[\{\underline{B'}\}] \cup \mathcal{C}[\{\overline{B'}\}] \\ [\underline{b}, \overline{b}] \cap \mathcal{C}[\mathcal{B}'] \neq \emptyset, \end{cases} \tag{12}$$

with $\underline{b}, \overline{b}$ the cardinalities of $\underline{B}, \overline{B}$. Let us apply (12) to all possibilities. When $\{\mathcal{B}'\}$ is a singleton, the above condition reduces to $[\underline{b}, \overline{b}] \subseteq \mathcal{C}[\mathcal{B}']$, as given in Table 1. Otherwise,

(i) with $\mathcal{B}' = [\emptyset, N^a]$, we obtain $[\underline{b}, \overline{b}] \subseteq [0, l^c]$ and $[\underline{b}, \overline{b}] \cap]l^a, n - r^a[\cap [0, l^c] \neq \emptyset$, which simplifies to

$$[\underline{b}, \overline{b}] \subseteq [0, l^c] \text{ and } [\underline{b}, \overline{b}] \cap]l^a, n - r^a[\neq \emptyset; \tag{13}$$

(ii) with $\mathcal{B}' = [\emptyset, N^c]$, we obtain

$$[\underline{b}, \overline{b}] \subseteq [n - r^a, n] \text{ and } [\underline{b}, \overline{b}] \cap]l^c, n - r^c[\neq \emptyset; \tag{14}$$

(iii) with $\mathcal{B}' = [N^c, N]$, we obtain

$$[\underline{b}, \overline{b}] \subseteq [n - r^c, n] \text{ and } [\underline{b}, \overline{b}] \cap]l^a, n - r^a[\neq \emptyset; \tag{15}$$

(iv) with $\mathcal{B}' = [N^a, N]$, we obtain

$$[\underline{b}, \overline{b}] \subseteq [0, l^a] \text{ and } [\underline{b}, \overline{b}] \cap]l^c, n - r^c[\neq \emptyset. \tag{16}$$

This yields Table 3. Observe that the table is symmetric w.r.t. its center by the symmetry principle (Lemma 1): just exchange r with l. The transitions being sure, all cases on each line are exclusive.

From Table 3, we can deduce terminal classes reduced to singletons or intervals: they correspond to transitions $\mathcal{S} \xrightarrow{1} \mathcal{S}$ in the table, provided they are connected. We obtain:

(i) N^a, under the condition $n^c \geq (n - l^c) \vee (n - l^a)$;
(ii) N^c, under the condition $n^c \geq (n - r^c) \vee (n - r^a)$;
(iii) $[\emptyset, N^a]$, under the condition $n - l^c \leq n^c < n - l^a$;
(iv) $[N^c, N]$, under the condition $n - r^c \leq n^c < n - r^a$.

We check connectedness for (iii) ((iv) follows by symmetry). We see from Table 1 that every $S \in [\emptyset, N^a]$ with $s \leq l^a$ has a sure transition to N^a, while the other ones go to every set in the interval. Therefore, the interval is connected if and only if N^a has a possible transition to every set in the interval, i.e., we need $l^a < n^a < n - r^a$ and $n^a \leq l^c$, so the additional condition $n^a < n - r^a$ is needed. In summary:

(i) N^a is a terminal class if and only if $n^c \geq (n - l^c) \vee (n - l^a)$;
(ii) N^c is a terminal class if and only if $n^c \geq (n - r^c) \vee (n - r^a)$;
(iii) $[\emptyset, N^a]$ is a terminal class if and only if $(n - l^c) \vee (r^a + 1) \leq n^c < n - l^a$;
(iv) $[N^c, N]$ is a terminal class if and only if $(n - r^c) \vee (l^a + 1) \leq n^c < n - r^a$.

In order to get (terminal) cycles and periodic classes, we study chains of sure transitions of length 2: $\mathcal{S}_1 \xrightarrow{1} \mathcal{S}_2 \xrightarrow{1} \mathcal{S}_3$, with $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ being pairwise disjoint, except possibly $\mathcal{S}_1 = \mathcal{S}_3$. An inspection of Table 3 yields all such possible chains of length 2, summarized in Table 4. A second table can be obtained by symmetry.

From Table 4, we obtain the following candidates for terminal cycles and periodic classes, after eliminating doublons and using symmetry:

(i) $N^a \xrightarrow{1} \emptyset \xrightarrow{1} N^a$, under the condition $n - l^c \leq n^c \leq r^a$;
(ii) $N^c \xrightarrow{1} N \xrightarrow{1} N^c$, under the condition $n - r^c \leq n^c \leq l^a$;
(iii) $N^c \xrightarrow{1} N^a \xrightarrow{1} N^c$, under the condition $n^c \leq l^c \wedge l^a \wedge r^c \wedge r^a$;
(iv) $[\emptyset, N^c] \xrightarrow{1} N^a \xrightarrow{1} [\emptyset, N^c]$, under the condition $n^c \leq l^c \wedge l^a \wedge r^a, r^c < n^c < n - l^c$
(v) $[N^a, N] \xrightarrow{1} N^c \xrightarrow{1} [N^a, N]$, under the condition $n^c \leq r^c \wedge r^a \wedge l^a, l^c < n^c < n - r^c$
(vi) $[N^a, N] \xrightarrow{1} [\emptyset, N^c] \xrightarrow{1} [N^a, N]$, under the condition $r^c \vee l^c < n^c \leq r^a \wedge l^a$.

It remains to check connectedness of (iv) and (vi) ((v) is obtained by symmetry). For (iv), we must check that N^a has a possible transition to every set in $[\emptyset, N^c]$. By Table 1, we must have $n^a \geq n - r^a$ and $l^c < n^a < n - r^c$, which is true by the conditions in (iv). We address (vi). We claim that under the conditions in (vi) $[N^a, N] \cup [\emptyset, N^c]$ is connected if and only if $N^a \xrightarrow{1} [\emptyset, N^c]$ and $N^c \xrightarrow{1} [N^a, N]$. Take any $S \in [\emptyset, N^c]$. Then S goes either to any set T in $[N^a, N]$ or only to N^a or only to N. In the first case, similarly, T goes either to any set $S' \in [\emptyset, N^c]$ (and we are done) or only to $\emptyset$ or only to N^c. If $T \xrightarrow{1} \emptyset$, then we have $T \xrightarrow{1} \emptyset \xrightarrow{1} N^a \xrightarrow{1} [\emptyset, N^c]$ and we are done. Otherwise we have $T \xrightarrow{1} N^c \rightarrow N^a \xrightarrow{1} [\emptyset, N^c]$. Suppose now that $S \xrightarrow{1} N^a$, then N^a goes to any $S' \in [\emptyset, N^c]$ and we are done. Otherwise, $S \xrightarrow{1} N \xrightarrow{1} N^c \rightarrow N^a \xrightarrow{1} [\emptyset, N^c]$ and we are done. This proves sufficiency. Now suppose the condition is not fulfilled. This means that N^a goes to either

$\emptyset$ or N^c (or similar condition for N^c). In fact, due to the conditions in (vi) and Table 1, we have that $N^a \xrightarrow{1} \emptyset$, but this yields the cycle $N^a \xrightarrow{1} \emptyset \xrightarrow{1} N^a$.

So in summary, candidates from (i) to (v) are all periodic classes under the specified conditions, and for (vi), the additional condition that $N^a \xrightarrow{1} [\emptyset, N^c]$ and $N^c \xrightarrow{1} [N^a, N]$ yields:

(vi') $[N^a, N] \xrightarrow{1} [\emptyset, N^c] \xrightarrow{1} [N^a, N]$ under the condition $r^c \vee l^c < n^c \leq r^a \wedge l^a \wedge (n - l^c - 1) \wedge (n - r^c - 1)$.

For cycles and periodic classes of length 3, by combining the possible chains of length 2 of Table 4 with possible transitions of Table 3, we have only one candidate, all other being eliminated because the collections are not disjoint:

$$N^c \xrightarrow{1} \emptyset \xrightarrow{1} N^a \xrightarrow{1} N^c.$$

Hence we find, taking into account the symmetry, two additional cycles:

(i) $N^c \xrightarrow{1} \emptyset \xrightarrow{1} N^a \xrightarrow{1} N^c$, under the condition $n^c \leq r^c \wedge r^a \wedge l^c, n^c \geq n - r^a$;

(ii) $N^a \xrightarrow{1} N \xrightarrow{1} N^c \xrightarrow{1} N^a$, under the condition $n^c \leq l^c \wedge l^a \wedge r^c, n^c \geq n - l^a$.

We now show that periodic classes of period greater than three cannot exist, which finishes the study of simple transitions.

Lemma 1. *There exists no periodic class of period $k \geq 4$.*

Proof. Let $\mathcal{S}$ be a periodic class. First, observe that if $\emptyset, N$ are not elements of $\mathcal{S}$, it is not possible to choose four distinct elements of $\mathbb{B} \setminus \{\{\emptyset\}, \{N\}\}$ such that these elements are pairwise disjoint. Hence, we suppose that there are transitions $\mathcal{B} \xrightarrow{1} \emptyset$ and/or $\mathcal{B} \xrightarrow{1} N$ in $\mathcal{S}$. From Table 3, we see that $\mathcal{B}$ is necessarily $\{N^a\}$ or $\{N^c\}$.

We claim that the cycle $\emptyset \xrightarrow{1} N^a \xrightarrow{1} N \xrightarrow{1} N^c \xrightarrow{1} \emptyset$ is impossible. Indeed, by Table 4, we have $\emptyset \xrightarrow{1} N^a \xrightarrow{1} N$ iff $n - l^a \leq n^c \leq r^c$ and $N \xrightarrow{1} N^c \xrightarrow{1} \emptyset$ (its symmetric) iff $n - r^a \leq n^c \leq l^c$. This yields, respectively,

$$2n^c \geq 2n - l^a - r^a > n$$
$$2n^c \leq r^c + l^c < n,$$

a contradiction.

Assume that we have a transition to $\emptyset$ (the case for N is obtained by symmetry). We have either $N^a \xrightarrow{1} \emptyset$ (which is discarded because it leads to the cycle $N^a \xrightarrow{1} \emptyset \xrightarrow{1} N^a$) or $N^c \xrightarrow{1} \emptyset$. Then, the only possible terminal class of the form $N^c \xrightarrow{1} \emptyset \xrightarrow{1} N^a \xrightarrow{1} \mathcal{B}_1 \xrightarrow{1} \cdots \xrightarrow{1} \mathcal{B}_p \xrightarrow{1} N^c$ is the cycle $\emptyset \xrightarrow{1} N^a \xrightarrow{1} N^c \xrightarrow{1} \emptyset$, for, either $\mathcal{B}_1 = N$, and we obtain the impossible cycle in the claim above, or $\mathcal{B}_1$ contains N^a or N^c, which is impossible since elements in $\mathcal{S}$ should be pairwise disjoint.

A.2 Multiple transitions

We examine the case of transitions of the form $\mathcal{S} \xrightarrow{1} \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_p$, with $p \geq 2$, $\mathcal{S} \in 2^N$ and formed only from sets in $\mathbb{B}$, $\mathcal{B}_1, \ldots, \mathcal{B}_p \in \mathbb{B}$, and all $\mathcal{B}_1, \ldots, \mathcal{B}_p$ are pairwise incomparable by inclusion[3]. The analysis is done in the same way as for simple transitions: the above transition exists if and only if for every $S \in \mathcal{S}$, $S \xrightarrow{1} \mathcal{B}'$ with $\mathcal{B}' \in \mathbb{B}$ and $\mathcal{B}' \subseteq \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_p$ and there exist distinct $S_1, \ldots, S_p \in \mathcal{S}$ such that $S_j \xrightarrow{1} \mathcal{B}_j$ for $j = 1, \ldots, p$, which readily shows that $\mathcal{S}$ cannot be a singleton. More explicitly, using previous notation and denoting by $\mathrm{supp}(\mathcal{S}) = \{|S| \ : \ S \in \mathcal{S}\}$ the support of $\mathcal{S}$, we get:

$$\mathcal{S} \xrightarrow{1} \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_p \Leftrightarrow \begin{cases} \mathrm{supp}(\mathcal{S}) \subseteq \bigcup_{j=1}^{p} \mathcal{C}[\mathcal{B}_j] \cup \bigcup_{j=1}^{p} \mathcal{C}[\{\underline{B}_j\}] \cup \bigcup_{j=1}^{p} \mathcal{C}[\{\overline{B}_j\}] \\ \mathrm{supp}(\mathcal{S}) \cap \mathcal{C}[\mathcal{B}_j] \neq \emptyset, \quad j = 1, \ldots, p. \end{cases} \tag{17}$$

Let us investigate what the possible candidates for $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_p$ are. We begin by restricting to nontrivial intervals and $p = 2$. From Table 1, we find:

(i) $[\emptyset, N^a] \cup [\emptyset, N^c]$ if and only if

$$\mathrm{supp}(\mathcal{S}) \subseteq [0, l^c] \cup [n - r^a, n] \text{ and } \begin{cases} \mathrm{supp}(\mathcal{S}) \cap]l^a, n - r^a[\cap [0, l^c] \neq \emptyset \\ \mathrm{supp}(\mathcal{S}) \cap]l^c, n - r^c[\cap [n - r^a, n] \neq \emptyset \end{cases} ; \tag{18}$$

(ii) $[\emptyset, N^a] \cup [N^c, N]$ if and only if

$$\mathrm{supp}(\mathcal{S}) \subseteq [0, l^c] \cup [n - r^c, n] \text{ and } \begin{cases} \mathrm{supp}(\mathcal{S}) \cap]l^a, n - r^a[\cap [0, l^c] \neq \emptyset \\ \mathrm{supp}(\mathcal{S}) \cap]l^a, n - r^a[\cap [n - r^c, n] \neq \emptyset \end{cases} ; \tag{19}$$

(iii) $[N^a, N] \cup [\emptyset, N^c]$ if and only if

$$\mathrm{supp}(\mathcal{S}) \subseteq [0, l^a] \cup [n - r^a, n] \text{ and } \begin{cases} \mathrm{supp}(\mathcal{S}) \cap]l^c, n - r^c[\cap [0, l^a] \neq \emptyset \\ \mathrm{supp}(\mathcal{S}) \cap]l^c, n - r^c[\cap [n - r^a, n] \neq \emptyset \end{cases} ; \tag{20}$$

(iv) $[N^a, N] \cup [N^c, N]$ if and only if

$$\mathrm{supp}(\mathcal{S}) \subseteq [0, l^a] \cup [n - r^c, n] \text{ and } \begin{cases} \mathrm{supp}(\mathcal{S}) \cap]l^c, n - r^c[\cap [0, l^a] \neq \emptyset \\ \mathrm{supp}(\mathcal{S}) \cap]l^a, n - r^a[\cap [n - r^c, n] \neq \emptyset \end{cases} , \tag{21}$$

the other combinations $[\emptyset, N^a] \cup [N^a, N]$ and $[\emptyset, N^c] \cup [N^c, N]$ being impossible as it can be checked. This readily shows that $p > 2$ with nontrivial intervals is impossible since a forbidden combination would appear in the list.

We consider now that singletons may appear. We begin by noticing that there is no terminal class of the form $\{S_1, \ldots, S_p\}$ with $S_j \in \{\emptyset, N, N^a, N^c\}$ for all j and $p \geq 2$. Indeed, Table 3 shows that transitions from a set S can only lead to a single T, with no possibility of multiple transition. Hence, such collections would never be connected.

Let us examine the case $\mathcal{S} \xrightarrow{1} \mathcal{B}_1 \cup \{S\}$, where $\mathcal{B}_1$ is a nontrivial interval. With $[\emptyset, N^a] \cup \{N\}$ we obtain:

$$\mathrm{supp}(\mathcal{S}) \subseteq [0, l^c] \cup ([0, l^a] \cap [n - r^c, n]) \text{ and } \begin{cases} \mathrm{supp}(\mathcal{S}) \cap [0, l^c] \cap]l^a, n - r^a[\neq \emptyset \\ \mathrm{supp}(\mathcal{S}) \cap [0, l^a] \cap [n - r^c, n] \neq \emptyset \end{cases} ,$$

[3] The "$\cup$" is understood at the level of collections of sets, i.e., $\mathcal{B}_1 \cup \mathcal{B}_2 = \{S \in 2^N \mid S \in \mathcal{B}_1 \text{ or } S \in \mathcal{B}_2\}$.

which is impossible. With $[\emptyset, N^a] \cup \{N^c\}$ we obtain

$$\operatorname{supp}(\mathcal{S}) \subseteq [0, l^c] \cup ([n-r^a, n] \cap [n-r^c, n]) \text{ and } \begin{cases} \operatorname{supp}(\mathcal{S}) \cap [0, l^c] \cap]l^a, n - r^a[\neq \emptyset \\ \operatorname{supp}(\mathcal{S}) \cap [n - r^c, n] \cap [n - r^a, n] \neq \emptyset \end{cases}, \tag{22}$$

which is possible. Similarly, we find that $[\emptyset, N^c] \cup \{N\}$, $[N^a, N] \cup \{\emptyset\}$ and $[N^c, N] \cup \{\emptyset\}$ are impossible, while the following are possible:

(i) $\mathcal{S} \xrightarrow{1} [\emptyset, N^c] \cup \{N^a\}$ iff

$$\operatorname{supp}(\mathcal{S}) \subseteq [n-r^a, n] \cup ([0, l^a] \cap [0, l^c]) \text{ and } \begin{cases} \operatorname{supp}(\mathcal{S}) \cap [n - r^a, n] \cap]l^c, n - r^c[\neq \emptyset \\ \operatorname{supp}(\mathcal{S}) \cap [0, l^a] \cap [0, l^c] \neq \emptyset \end{cases}, \tag{23}$$

(ii) $\mathcal{S} \xrightarrow{1} [N^a, N] \cup \{N^c\}$ iff

$$\operatorname{supp}(\mathcal{S}) \subseteq [0, l^a] \cup ([n-r^a, n] \cap [n-r^c, n]) \text{ and } \begin{cases} \operatorname{supp}(\mathcal{S}) \cap [0, l^a] \cap]l^c, n - r^c[\neq \emptyset \\ \operatorname{supp}(\mathcal{S}) \cap [n - r^a, n] \cap [n - r^c, n] \neq \emptyset \end{cases}, \tag{24}$$

(iii) $\mathcal{S} \xrightarrow{1} [N^c, N] \cup \{N^a\}$ iff

$$\operatorname{supp}(\mathcal{S}) \subseteq [n-r^c, n] \cup ([0, l^a] \cap [0, l^c]) \text{ and } \begin{cases} \operatorname{supp}(\mathcal{S}) \cap [n - r^c, n] \cap]l^a, n - r^a[\neq \emptyset \\ \operatorname{supp}(\mathcal{S}) \cap [0, l^a] \cap [0, l^c] \neq \emptyset \end{cases}. \tag{25}$$

This shows that transitions of the form $\mathcal{S} \xrightarrow{1} \mathcal{B} \cup \{S_1\} \cup \{S_2\}$ are not possible since a forbidden configuration would appear.

We are now in position to study aperiodic terminal classes.

(i) With $\mathcal{S} = [\emptyset, N^a] \cup [\emptyset, N^c]$, we find from (18) that

$$[0, n^a \vee n^c] \subseteq [0, l^c] \cup [n - r^a, n] \text{ and } \begin{cases} [0, n^a \vee n^c] \cap]l^a, n - r^a[\cap [0, l^c] \neq \emptyset \\ [0, n^a \vee n^c] \cap]l^c, n - r^c[\cap [n - r^a, n] \neq \emptyset \end{cases}$$

which is equivalent to

$$n^a \vee n^c > l^c \geq n - r^a. \tag{26}$$

We check connectedness. We begin by a simple observation. We have $\emptyset \xrightarrow{1} N^a$, therefore we must forbid the transitions $N^a \xrightarrow{1} \emptyset$ and $N^a \xrightarrow{1} N^a$. Using Table 1 and (26), we find that $n^a \in]l^a, n - r^a[\cup]l^c, n[$. Suppose that $n^a \in]l^a, n - r^a[$. From Table 1, we obtain that $N^a \xrightarrow{1} [\emptyset, N^a] \xrightarrow{1} [\emptyset, N^a]$, hence no connection to $[\emptyset, N^c]$ is obtained. Therefore we are forced to consider $n^a \in]l^c, n[$, which with (26) leads to

$$n^a > l^c \geq n - r^a. \tag{27}$$

From Table 1 again, this implies $N^a \xrightarrow{1} [\emptyset, N^c]$ when $n^a \in]l^c, n - r^c[$, or $N^a \xrightarrow{1} N^c$ when $n^a \in [n - r^c, n[$. We distinguish the two cases.

1. Suppose $n^a \in]l^c, n-r^c[$, so we have $\emptyset \xrightarrow{1} N^a \xrightarrow{1} [\emptyset, N^c]$. In order to connect $\emptyset, N^a$ to any set in $]\emptyset, N^a[$, there must exist $S \in [\emptyset, N^c]$ such that $S \xrightarrow{1} [\emptyset, N^a]$, i.e., $s \in]l^a, n-r^a[\cap [0, l^c] =]l^a, n-r^a[$ by (27). This is possible iff $n^c > l^a$. Let us check whether N^c is connected to any set in the class. From Table 1 and the condition $n^c > l^a$, we see that there is a possible transition to $\emptyset$, which suffices to prove that N^c is connected to any set in the class, except if $n^c \in [n-r^c, n]$ in which case $N^c \xrightarrow{1} N^c$. Therefore, we must ensure the following condition:

$$n^c \in]l^a, n-r^c[\,. \tag{28}$$

We check similarly whether any other set in the class is connected with the rest. Take $S \in]\emptyset, N^a[$. If $s \leq l^c$, there will be either a possible transition to $\emptyset$ or to N^a, so that S is connected to any set in the class. If $s > l^c$, S behaves like N^a and we are done. Take now $S \in]\emptyset, N^c[$. If $s \leq l^a$, then $S \xrightarrow{1} N^a$ and we are done. If $s \in]l^a, l^c]$, S has a possible transition to $\emptyset$ and we are done. Finally, if $s \in]l^c, n-r^c[$, S behaves like N^c. In conclusion, (28) summarizes the condition for connectedness in Case 1.

2. Suppose $n^a \in [n-r^c, n[$, so we have $\emptyset \xrightarrow{1} N^a \xrightarrow{1} N^c$. We must ensure that N^c is connected to any set in the class. In order to avoid $N^c \xrightarrow{1} N^c$ and the transitions $N^c \xrightarrow{1} N^a$ and $N^c \xrightarrow{1} \emptyset$ which would lead to cycles, we are left with the cases $n^c \in]l^a, n-r^a[$ (yielding $N^c \xrightarrow{1} [\emptyset, N^a]$) and $n^c \in]l^c, n-r^c[$ (yielding $N^c \xrightarrow{1} [\emptyset, N^c]$). We examine both cases.

2.1. Suppose $n^c \in]l^a, n-r^a[$, then we have $N^c \xrightarrow{1} [\emptyset, N^a]$. It remains to ensure that there exists $S \in]\emptyset, N^a[$ which is connected with $[\emptyset, N^c]$. We must have $s \in]l^c, n-r^c[$, always possible under Case 2. So we have established that $\emptyset, N^a, N^c$ are connected with the rest of the class. It remains to check if this is true for the other sets in the class. Take $S \in]\emptyset, N^a[$. If $s \leq l^c$, a transition to $\emptyset$ of N^a is possible, and so we are done. If $s \in]l^c, n[$, then $S \rightarrow N^c$, and we are done. Take now $S \in]\emptyset, N^c[$. Then $s \in]0, n-r^a[$, so that $S \rightarrow N^a$ and we are done. As a conclusion, connectedness holds when $n^c \in]l^a, n-r^a[$.

2.2. Suppose $n^c \in]l^c, n-r^c[$, then $N^c \xrightarrow{1} [\emptyset, N^c]$. It remains to connect some set S in $]\emptyset, N^c[$ to $[\emptyset, N^a]$, which is possible iff $s \in]l^a, n-r^a[$. This is possible under Case 2, so N^c is connected to any set in the class. We check for the remaining sets. Take $S \in]\emptyset, N^a[$. If $s \leq l^c$, a connection is possible to N^a or $\emptyset$ so we are done. Otherwise, a connection to N^c is possible and we are done. For $S \in]\emptyset, N^c[$, it works exactly the same.

In conclusion of Case 2, connectedness is ensured iff $n^c \in]l^a, n-r^a[\cup]l^c, n-r^c[$.

There does not seem to be a simple way to write the final condition. Here is one possible: connectedness holds iff $l^c \geq n-r^a$ and

$$n^c \in \big(]r^c, n-l^c[\cap]l^a, n-r^c[\big) \cup \Big(\big(]l^a, n-r^a[\cup]l^c, n-r^c[\big) \cap]0, r^c]\Big).$$

(ii) Similarly, using (19), $\mathcal{S} = [N^a, N] \cup [N^c, N]$ is a terminal class if and only if $l^a \geq n-r^c$ and $n^c \in \big(]l^c, n-r^c[\cap]r^a, n-l^c[\big) \cup \Big(\big(]r^a, n-l^a[\cup]r^c, n-l^c[\big) \cap]0, l^c]\Big)$.

(iii) With $\mathcal{S} = [\emptyset, N^c] \cup [N^a, N]$ we find from (20) the condition $l^c \vee r^c < n^c \leq l^a \wedge r^a$. Let us check connectedness. Starting from $\emptyset$, we have $\emptyset \xrightarrow{1} N^a$, and by Table 1 and the above condition we have $N^a \xrightarrow{1} [\emptyset, N^c]$ if $n^a > l^c$, and $N^a \xrightarrow{1} \emptyset$ otherwise.

Clearly, the latter must be forbidden otherwise a cycle occurs. Therefore, we must have $n^a > l^c$. Moreover, we have $N^c \xrightarrow{1} [N^a, N]$ if $n^c < n - r^c$ and $N^c \xrightarrow{1} N$ otherwise. Since $N \xrightarrow{1} N^c$, the latter must be forbidden to avoid a cycle. Therefore, we must have $n^c < n - r^c$. Under these condition, from $\emptyset$ or N^a or N^c, any set can be attained. Now, taking $S \in]\emptyset, N^c[$, we have $S \xrightarrow{1} N^a$ or $S \xrightarrow{1} [N^a, N]$ so that $S \rightarrow N^a$ and we are done. Lastly, taking $S \in]N^a, N[$, we have $S \xrightarrow{1} N^c$ or $[\emptyset, N^c]$ and we are done. As a conclusion, the condition is $l^c \vee r^c < n^c \leq l^a \wedge r^a$ and $n^c < (n - l^c) \wedge (n - r^c)$, but then we obtain the periodic terminal class studied before. Indeed, we see from the proof that we have necessarily $[\emptyset, N^c] \xrightarrow{1} [N^a, N] \xrightarrow{1} [\emptyset, N^c]$.

(iv) With $\mathcal{S} = [\emptyset, N^a] \cup [N^c, N]$, using (19), we find that $l^a \vee r^a < n^a \leq l^c \wedge r^c$. However, under these conditions, $\mathcal{S}$ cannot be connected. Indeed, starting from N^a, we have from Table 1 that for any set $S \in [\emptyset, N^a]$, we have either $S \xrightarrow{1} N^a$, or $S \xrightarrow{1} [\emptyset, N^a]$ or $S \xrightarrow{1} \emptyset$. Therefore, $[\emptyset, N^a]$ is not connected with every set in $\mathcal{S}$.

(v) We show that $[\emptyset, N^a] \cup \{N^c\}$ cannot be connected when $l^c + r^c \neq n - 1$. Indeed, we must have $N^c \xrightarrow{1} [\emptyset, N^a]$, which implies by Table 1 the condition $n^c \leq l^c$. However, by (22) and the condition $l^c + r^c \neq n - 1$, $\mathrm{supp}(\mathcal{S})$ must be in two disjoint intervals, implying that $[0, n^a] \subseteq [0, l^c]$ and $n^c \in [n - r^c, n]$, a contradiction.
We suppose now $l^c + r^c = n - 1$ and $n - r^a \leq n - r^c$, so that in (22) the first condition reduces to the void condition $\mathrm{supp}(\mathcal{S}) \subseteq [0, n]$, while the second becomes: $l^c > l^a$ and either $n^a \geq n - r^c$ (case 1), or $n^c \geq n - r^c$ and $n^a \in]l^a, n - r^a[$ (case 2). We check connectedness. N^c must be connected to $[\emptyset, N^a]$ or $\emptyset$ or N^a, which implies $n^c \leq l^c$, contradicting case 2. Therefore only case 1 is possible, so that $n^a \geq n - r^c$ and $n^c \leq l^c$. Note that this implies $N^a \xrightarrow{1} N^c$, so that we must ensure $N^c \xrightarrow{1} [\emptyset, N^a]$, implying $l^a < n^c \leq l^c$. Finally, for any $S \in [\emptyset, N^a]$, either $S \xrightarrow{1} N^c$ or $S \xrightarrow{1} [\emptyset, N^a]$ or $S \xrightarrow{1} N^a$, hence connectedness holds. In summary, this class exists iff $l^c + r^c = n - 1$, $n - r^a \leq n - r^c$, $n^a \geq n - r^c$ and $l^a < n^c \leq l^c$.

(vi) With $[\emptyset, N^c] \cup \{N^a\}$, we find from (23) and the assumption $l^a + r^a \neq n - 1$ that $\mathrm{supp}(\mathcal{S})$ must be in two disjoint intervals, which forces $n - r^a \leq n^a < n - r^c$ and $n^c \leq l^a \wedge l^c$. We know already that $[\emptyset, N^c] \xrightarrow{1} N^a$ is a periodic class. Let us show that this is the only possibility. Indeed, otherwise there should exist $S \in [\emptyset, N^c]$ such that $S \xrightarrow{1} [\emptyset, N^c]$. This would imply that $l^c < s < n - r^c$, which is impossible by the condition $n^c \leq l^c$.
Let us consider now that $l^a + r^a = n - 1$ and $l^c \geq l^a$, so that in (23) the first condition simply reduces to the void condition $\mathrm{supp}(\mathcal{S}) \subseteq [0, n]$, while the second becomes: either $n^a \in]l^c, n - r^c[$ or $n^c > l^c$. Let us check connectedness. We must have $N^a \xrightarrow{1} [\emptyset, N^c]$ or $N^a \xrightarrow{1} N^c$. The first case happens iff $n^a \in]l^c, n - r^c[$. Then observe that without further condition on n^c, any set in $[\emptyset, N^c]$ is connected to either N^a, $\emptyset$, $[\emptyset, N^c]$ or N^c. It suffices then to forbid the transition $N^c \xrightarrow{1} N^c$, i.e., $n^c < n - r^c$. The second case happens iff $n^a \geq n - r^c$, which forces $n^c > l^c$. To ensure that N^c is connected to $[\emptyset, N^c]$, we must have $l^c < n^c < n - r^c$. In summary, this class exists iff $l^a + r^a = n - 1$, $l^c \geq l^a$, and either $n^a \in]l^c, n - r^c[$ and $n^c < n - r^c$, or $n^a \geq n - r^c$ and $l^c < n^c < n - r^c$.

(vii) The case of $[N^a, N] \cup \{N^c\}$ is similar to its symmetric $[\emptyset, N^c] \cup \{N^a\}$. The class exists iff $l^a + r^a = n - 1$, $n - r^c \leq n - r^a$, and either $n^c \in]l^c, n - r^c[$ and $n^a > l^c$, or $n^c \leq l^c$ and $l^c < n^a < n - r^c$.

(viii) The case of $[N^c, N] \cup \{N^a\}$ is similar to its symmetric $[\emptyset, N^a] \cup \{N^c\}$. The class exists iff $l^c + r^c = n - 1$, $l^a \geq l^c$, $n^c \leq l^c$ and $n - r^c \leq n^a < n - r^a$.

It remains to study the existence of periodic classes. Since the collections must be pairwise disjoint, the only possibility is the periodic class $[\emptyset, N^a] \cup [\emptyset, N^c] \xrightarrow{1} N \xrightarrow{1} [\emptyset, N^a] \cup [\emptyset, N^c]$. But we know that the second transition is impossible since a singleton cannot lead to a multiple transition. Hence, there is no such periodic terminal classes.

$\mathcal{S} \backslash \mathcal{T}$	$\emptyset$	N^a	$[\emptyset, N^c]$	$[\emptyset, N^a]$	$[N^c, N]$	$[N^a, N]$	N^c	N
$\emptyset$	$\times$	always	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
N^a	$n - l^c \leq n^c \leq r^a$	$n^c \geq n - l^c$ $n^c \geq n - l^a$	$n^c \leq r^a$ $r^c < n^c < n - l^c$	$n^c \geq n - l^c$ $r^a < n^c < n - l^a$	$n^c \leq r^c$ $r^a < n^c < n - l^a$	$n^c \geq n - l^a$ $r^c < n^c < n - l^c$	$n^c \leq r^c \wedge r^a$	$n - l^a \leq n^c \leq r^c$
$[\emptyset, N^c]$	$\times$	$n^c \leq l^c \wedge l^a$	$\times$	$l^a < n^c \leq l^c$	$\times$	$l^c < n^c \leq l^a$	$\times$	$\times$
$[\emptyset, N^a]$	$\times$	$n^c \geq n - l^c$ $n^c \geq n - l^a$	$\times$	$n - l^c \leq n^c$ $n^c < n - l^a$	$\times$	$n - l^a \leq n^c$ $n^c < n - l^c$	$\times$	$\times$
$[N^c, N]$	$\times$	$\times$	$n - r^a \leq n^c$ $n^c < n - r^c$	$\times$	$n - r^c \leq n^c$ $n^c < n - r^a$	$\times$	$n^c \geq n - r^c$ $n^c \geq n - r^a$	$\times$
$[N^a, N]$	$\times$	$\times$	$r^c < n^c \leq r^a$	$\times$	$r^a < n^c \leq r^c$	$\times$	$n^c \leq r^c \wedge r^a$	$\times$
N^c	$n - r^a \leq n^c \leq l^c$	$n^c \leq l^c \wedge l^a$	$n^c \geq n - r^a$ $l^c < n^c < n - r^c$	$n^c \leq l^c$ $l^a < n^c < n - r^a$	$n^c \geq n - r^c$ $l^a < n^c < n - r^a$	$n^c \leq l^a$ $l^c < n^c < n - r^c$	$n^c \geq n - r^c$ $n^c \geq n - r^a$	$n - r^c \leq n^c \leq l^a$
N	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	always	$\times$

Table 3. Conditions for sure transitions $\mathcal{S}$ to $\mathcal{T}$

$N^a \xrightarrow{1} \emptyset \xrightarrow{1} N^a$	$n - l^c \leq n^c \leq r^a$
$N^c \xrightarrow{1} \emptyset \xrightarrow{1} N^a$	$n - r^a \leq n^c \leq l^c$
$\emptyset \xrightarrow{1} N^a \xrightarrow{1} \emptyset$	$n - l^c \leq n^c \leq r^a$
$\emptyset \xrightarrow{1} N^a \xrightarrow{1} [N^c, N]$	$n^c \leq r^c$ $r^a < n^c < n - l^a$
$\emptyset \xrightarrow{1} N^a \xrightarrow{1} N^c$	$n^c \leq r^c \wedge r^a$
$\emptyset \xrightarrow{1} N^a \xrightarrow{1} N$	$n - l^a \leq n^c \leq r^c$
$N^c \xrightarrow{1} N^a \xrightarrow{1} \emptyset$	$n - l^c \leq n^c \leq l^c \wedge l^a \wedge r^a$
$[\emptyset, N^c] \xrightarrow{1} N^a \xrightarrow{1} [\emptyset, N^c]$	$n^c \leq l^c \wedge l^a \wedge r^a$ $r^c < n^c < n - l^c$
$N^c \xrightarrow{1} N^a \xrightarrow{1} N^c$	$n^c \leq l^c \wedge l^a \wedge r^c \wedge r^a$
N^c or $[\emptyset, N^c] \xrightarrow{1} N^a \xrightarrow{1} N$	$n - l^a \leq n^c \leq l^a \wedge l^c \wedge r^c$
$N^a \xrightarrow{1} [\emptyset, N^c] \xrightarrow{1} N^a$	$n^c \leq l^a \wedge l^c \wedge r^a$ $r^c < n^c < n - l^c$
$[N^a, N] \xrightarrow{1} [\emptyset, N^c] \xrightarrow{1} [N^a, N]$	$l^c \vee r^c < n^c \leq r^a \wedge l^a$

Table 4. Conditions for chains of length 2 potentially yielding periodic classes