Optimal mark up pricing without market structure consideration

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Abstract

Following Lerner 1934 a huge literature has emerged on mark-up, defining the optimal price has been a fraction of the price elasticity of demand times marginal cost. In practice, the marginal cost is difficult to evaluate and the price elasticity of demand should be less than -1 for price to be positive. This paper proposes a simple method to determine the optimal price of goods that is suitable for price elasticity greater than -1. The method that is simple to implement on market where firm buys goods at a given price and sales it at another price.[2]

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1 Introduction

This paper shows how to compute the optimal price of a good for any firm that buys a good at a given price (even if this price is bargained) and resales the good at another price.

The literature on pricing can be divided into two directions. The first one is the competitive literature where the price is given by the market, Smith 1776, Walras 1874, Arrow 1958, Debreu 1958. The argument is that atomistic consumers and atomistic firms cannot modify alone the current market price. Indeed, since every firm and every consumer is assumed to be identical, and since the information is assumed to be perfect and available at no cost, if one side of the market asks for a better price, the other side of the market refuses this price. In equilibrium, no one can deviate from the market price and in that sense this is a Nash equilibrium.

The second line of the literature is the monopolistic environment where monopolist can optimize its profit so as to determine the optimal price. Abba Lerner 1934 shows how to proceed. Suppose that a monopolist faces a demand function $q_d = q(p)$ where the demanded quantity q_d is a decreasing continuous function q of the price p. In that case, the profit $\Pi(q_d)$ of the monopolist is given by $\Pi(q_d) = p \times q(p) - TC(q(p))$, where $TC(q(p))$ is the total cost of production of the demanded quantity $q(p)$. The first order condition is known to be the Lerner's index

$$q(p) + p\frac{dq}{dp} - \frac{\partial TC}{\partial q}\frac{dq}{dp} = 0 \quad \Longleftrightarrow \quad \frac{p - \frac{\partial TC}{\partial q}}{p} = \frac{-1}{\varepsilon_{q/p}}$$

where $\varepsilon_{q/p} < 0$ is the price elasticity of the demand.

The price elasticity of the demand is a good approximation of the monopoly power. Indeed, if the market is competitive then the price elasticity tends to infinity so that the optimal pricing is such that the price equals the marginal cost. In all other situations, as long as the market deviates from competition to monopoly the price goes up until the monopoly price. This theory is appealing and very intuitive.

Things become more complex if we extracts the optimal pricing, from the Lerner's index

$$p^\star = \left[\frac{\varepsilon_{q/p}}{1 + \varepsilon_{q/p}}\right]\frac{\partial TC}{\partial q}.$$

Indeed, for the price to be positive the price elasticity of demand must be less than -1, which means that the Lerner's index is not an appropriate concept for monopolistic competition where generally the price elasticity of demand is more than -1.

Moreover, from a practical point of view, it is difficult to estimate the marginal cost, and most of the time it is unknown to the manager. In the real world, practitioners frequently use either the average cost instead of the marginal cost involving huge deviation from optimality. By definition, the markup percentage calculation is cost C markup percentage, and then add that to the original unit cost to arrive at the sales price. For example, the markup formula is as follows : if a product costs US \$ 100, the selling price with a 25% markup would be US \$ 125. Markups are normally used in retail or wholesale business as it is an easy way to price items when a store contains several different goods. For example, on fast food, the multiplier is about 6, on orange juice 15, pizza and pan makes about 12, Compact Disks 1.89, on housing furniture about 5, on meat it is about 3, etc.

Such deviations from the Lerner's index have huge consequences on the value added and more importantly on profit, as it will be shown hereafter because the mark up price is based on an exogenous multiplier on the buying price.

The objective of this paper is propose a very simple way to determine the optimal pricing in situations where the firm buys a particular good and resales it at a higher price using a given coefficient that multiplies the buying price to get the selling price. The method consists in determining an optimal multiplier which maximizes the profit of a sale. It is suitable in many profit activities. A very important example is retailing.

This paper is organized as follows. Section 2 presents the habits in retailing where an exogenous multiplier is applied. Section 3 presents a method that optimizes the multiplier. Section 4 considers the case of unsold stock goods and proposes and optimal multiplier that sold out goods. Section 5 concludes.

Contents

2 The habits

This section presents the habits on markets where a firm buys a good to a provider and sales its to the final consumer. Usually on real markets, a manager determines the selling price p_s of a given good by multiplying its buying price p_b by a given coefficient, say $k > 1$, so that the selling price is $p_s = kp_b$. Most of the time, k is the result of good practices by experts on the market and/or comes from habits. There is no particular scientific procedure to determine such a multiplier.

This section studies various cases where the multiplier is exogenous or endogenous. The first subsection presents the application of simple rules or habits in pricing, without any optimal considerations. The second subsection is devoted to optimal determination of a multiplier that maximizes profit. The third subsection studies welfare.

In general, since the marginal cost is unknown, managers use a simple rule to price a good, by applying an exogenous multiplier $\overline{k}$ to the buying price, $p_s = \overline{k}p_b$. The effective value added *VA* for the exogenous already bought quantity $\overline{Q}$ is $VA = p_s\overline{Q} - p_b\overline{Q} = (p_s - p_b)\overline{Q}$.

2.1 The habits with an exogenous demand and exogenous multiplier

In certain cases, the manager does not consider the demand $\overline{Q}$ as a function of price. The rule he uses to price the good can simply be expressed as follows: Sales Price = (Cost $\times$ Markup Percentage) + Cost. He anticipates the value added as being a number $\overline{VA}(\overline{k}) = (\overline{k} - 1)p_b\overline{Q}$. If for some reasons $\overline{VA}(\overline{k})$ differs from the anticipated number $\overline{VA}(\overline{k}) < \overline{VA}^a$, then manager considers that the money is missing on the market.

2.2 The habits with an endogenous demand and exogenous multiplier

In certain cases, the manager understands that he faces a falling in price demand, $Q(p_s)$ where $dQ(p_s)/dp_s < 0$. Replacing this quantity into the value added function and obtain

$$VA(\overline{k}) = (\overline{k} - 1)p_b Q(p_s) \tag{1}$$

3 The static model: endogenous demand and optimal multiplier

This section is devoted to a static model where the information is perfect and all individuals (the social planner, the representative consumer and the firm) are rational. The social planner sets the tax that leads the economy to its social welfare optimum. Knowing the tax, the consumer maximizes its surplus with respect to quantities and the firm maximizes its profit with respect to the multiplier.

3.1 The representative consumer's behavior

Consider $S(q)$ is an increasing quasi-concave $\mathcal{C}^2$ surplus function. $S(q)$ is defined as the difference between the utility function $U(q)$ and the expenditure $p_s q$, where U is a concave $\mathcal{C}^2$ utility function which has a one-to-one derivative function denoted $Q^\star$. The selling price is the after tax price, $p_s = p + \tau$, where τ is the tax on value added. A rational representative consumer maximizes his surplus

$$\max_q S(q) \quad \Longleftrightarrow \quad \max_q U(q) - p_s q$$

The first order condition is

$$\frac{dS(q)}{dq} = 0 \tag{2}$$

$$\Longleftrightarrow \quad \frac{dU(q)}{dq} = p + \tau. \tag{3}$$

Under the set of assumptions, $p_s = kp_b + \tau$ relation (6) can be rewritten

$$q^\star = Q^\star(k, p_b, \tau) \text{ and } \frac{\partial Q^\star(k, p_b, \tau)}{\partial k} < 0, \tag{4}$$

where $Q^\star(k, p_b, \tau)$ is the optimal consumer's demand function which is decreasing in price and in tax, given the multiplier k. The next subsection is devoted to the determination of the optimal k by the firm.

3.2 The optimal profit

Consider that the profit of the firm can be written as follows $\Pi(k,\tau) = \mathit{VA}(Q^\star(k,p_b,\tau)) - C(Q^\star(k,p_b,\tau))$, where $C(Q^\star(k,p_b,\tau))$ is a $\mathcal{C}^2$ increasing convex cost function that captures all costs except the buying cost of $Q^\star(k,p_b,\tau)$, which is already taken into account into $VA(Q^\star(k,p_b,\tau))$. Under the set of assumption, $\Pi(k,\tau)$ is concave and its first derivative is a one-to-one function. A rational firm solves the following problem

$$\max_k \Pi(k,p_b,\tau) = \mathit{VA}(Q^\star(k,p_b,\tau)) - C(Q^\star(k,p_b,\tau)).$$

Replace $\mathit{VA}(Q^\star(k,p_b,\tau))$ by its expression and obtain

$$\max_k \Pi(k,p_b,\tau) = (k-1)p_b Q^\star(k,p_b,\tau) - C(Q^\star(k,p_b,\tau))$$

Denote by $k^\star$ the optimal solution. The first order condition is

$$p_b\left[Q^\star(k^\star,p_b,\tau) + (k^\star-1)\frac{dQ^\star(k^\star,p_b,\tau)}{dk}\right] = \frac{\partial C(Q^\star(k^\star,p_b,\tau))}{\partial q}\frac{\partial Q^\star(k^\star,p_b,\tau)}{\partial k}.$$

Dividing by $p_b dQ^\star(k^\star,p_b,\tau)/dk \neq 0$ and isolating the profit-maximizing multiplier $k^\star$,

$$k^\star = 1 - \frac{Q^\star(k^\star,p_b,\tau)}{\frac{dQ^\star(k^\star,p_b,\tau)}{dk}} + \frac{1}{p_b}\frac{\partial C(Q^\star(k^\star,p_b,\tau))}{\partial Q^\star} \tag{5}$$

Note that $k^\star > 1$ since $-\frac{Q^\star(k^\star,p_b,\tau)}{\frac{dQ^\star(k^\star,p_b,\tau)}{dk}} > 0$ and $\frac{\partial C(Q^\star(k^\star,p_b,\tau))}{\partial Q^\star} > 0$.

3.3 The welfare

The objective of this subsection is to determine the welfare maximizing tax τ_w that leads the economy to the welfare maximizing consumption q_w and the welfare maximizing multiplier k_w. The benevolent social planner chooses the optimal quantities q_w that maximizes the social welfare defined as $W(q_w) = S(q_w) + \Pi(q_w)$. Since the consumer's surplus and the profit are two concave $\mathcal{C}^2$ functions which have a one-to-one first derivative, the welfare objective function is a $\mathcal{C}^2$ concave which has a one-to-one first derivative $\mathcal{Q}^\star$. The social welfare can also be rewritten as $W(q_w) = S(q_w) + \mathit{VA}(q_w) - C(q_w)$ and after replacing surplus function by its expression in term of utility and the value added function by its expression, as above, the multipliers cancels again, so that the problem becomes

$$\max_{q_w} U(q_w) - p_b q_w - C(q_w)$$

The first order condition is

$$\frac{\partial U(q_w)}{\partial q_w} - p_b - \frac{\partial C(q_w)}{\partial q_w} = 0$$

The welfare maximizing quantity is

$$q_w^\star \quad = \quad \mathcal{Q}^\star(p_b) \tag{6}$$

The social planner chooses the multiplier k_w that make the consumer's solution (4) compatible with the welfare maximizing quantity $q_w^\star$ of relation (6).

$$Q^\star(k_w, \tau) \quad = \quad \mathcal{Q}^\star(p_b). \tag{7}$$

By the implicit function theorem, $k_w^\star = K^\star(p_b, \tau)$. Note that is by definition $K^\star$ is the marginal utility (since the inverse of the inverse is the function itself). Consequently, we can rewrite the previous relation as

$$k_w^\star \quad = \quad \frac{dU}{dq_w}(\mathcal{Q}^\star(p_b), \tau) \tag{8}$$

$$\frac{\partial U^\star(K^\star(p_b, \tau_w^\star), \tau_w^\star)}{\partial Q^\star} - p_b - \frac{\partial C^\star(K^\star(p_b, \tau_w^\star), \tau_w^\star)}{\partial Q^\star} = 0.$$

The solution is obtained by equating the welfare maximizing multiplier of relation (8) with the firm's optimal multiplier of relation (5)

$$\frac{dU}{dq_w}(\mathcal{Q}^\star(p_b), \tau_w) = 1 - \frac{Q^\star(k_w^\star, \tau_w)}{\frac{dQ^\star(k_w^\star, \tau_w)}{dk}} + \frac{1}{p_b}\frac{\partial C(Q^\star(k_w^\star, \tau_w))}{\partial Q^\star}$$

$$\tau_w^\star \quad = \quad \tau(p_b). \tag{9}$$

4 The two period model

This section is devoted to the intertemporal model. There are two periods and the information is perfect over time. As in the static model, the social planner sets a tax that leads the economy to its social welfare optimum. The problem is more complex than the previous one. Knowing the tax and knowing that the good is not reproduced between periods, the consumer determines how much to consumer during the first period and how much to consume during the second period. The firm determines the two multipliers, one per period. It is demonstrated under which conditions the second period multiplier is less or equal to the first period multiplier. High (low) time preference consumer's behavior involves a lower (higher) second period multiplier than the first period multiplier.

4.1 The inter-temporal behavior of the representative consumer

Denote by $Q \in \,]0, \infty[$ the total consumption of good that the representative consumer chooses to consume over the two periods. Hereafter, β denotes the discount rate. In first period he chooses to consume $q \in \,]0, \infty[$ and in second period $Q - q \in \,]0, \infty[$. Consider the surplus function $S(q, Q, \beta)$ which is an increasing quasi-concave $\mathcal{C}^2$ function in q and Q. The surplus $S(q, Q, \beta)$ is defined as the difference between the utility function $U(q, Q, \beta)$ and the expenditure $p_{1s}q$ and $\beta p_{2s}(Q - q)$, where U is a concave $\mathcal{C}^2$ utility function in Q and q. The selling price is the after tax price, $p_{ts} = p_t + \tau_t, t = 1, 2$, where τ is the per period tax on value added over the two periods. A rational representative consumer maximizes his surplus

$$\max_{q} S(q, Q, \beta) \quad \Longleftrightarrow \quad \max_{q,Q} U(q, Q, \beta) - p_{1s}q - \beta p_{2s}(Q - q)$$

Denote the solutions by $Q^{\star\star}$ and $q^{\star\star}$.

Assume that $\partial^2 S(q^{\star\star}, Q^{\star\star}, \beta)/\partial Q^2 \neq 0$ and $\partial^2 S(q^{\star\star}, Q^{\star\star}, \beta)/\partial q^2 \neq 0$. The first order condition is

$$\frac{dS(q^{\star\star}, Q^{\star\star}, \beta)}{dQ} = 0 \tag{10}$$

$$\frac{dS(q^{\star\star}, Q^{\star\star}, \beta)}{dq} = 0 \tag{11}$$

Under the set of assumptions, by the implicit function theorem, there exists neighborhoods $\mathcal{V}_Q$ and $\mathcal{W}_Q$ of $Q^{\star\star}$ in $\mathbb{R}$ and $\mathcal{V}_q$ and $\mathcal{W}_q$ of $q^{\star\star}$ in $\mathbb{R}$ and two $\mathcal{C}^2$ applications $\phi : \mathcal{V}_Q \to \mathcal{W}_Q$ and $\psi : \mathcal{V}_q \to \mathcal{W}_q$ such that $\mathcal{V}_Q \times \mathcal{W}_Q \subset \,]0, \infty[^2$ and $\mathcal{V}_q \times \mathcal{W}_q \subset \,]0, \infty[^2$ and knowing $p_{ts} = k_t p_b + \tau$

$$\forall q \in \mathcal{V}_Q, \forall Q \in \mathcal{W}_Q, \ \frac{dS(q^{\star\star}, Q^{\star\star}, \beta)}{dQ} = 0 \quad \Longleftrightarrow \quad Q = \phi(q(k_1, k_2, p_b, \tau_1, \beta)) \tag{12}$$

$$\forall q \in \mathcal{V}_q, \forall Q \in \mathcal{W}_q, \ \frac{dS(q^{\star\star}, Q^{\star\star}, \beta)}{dq} = 0 \quad \Longleftrightarrow \quad q = \psi(Q(k_1, k_2, p_b, \tau_2, \beta)) \tag{13}$$

the system of relations (18) and (19) can be rewritten as

$$S_0 : \begin{cases} q^{\star\star} = Q^{\star\star}(k_1, k_2, p_b, \tau_1, \beta) \text{ and } \dfrac{\partial Q^{\star\star}(k_1, k_2, p_b, \tau_1, \beta)}{\partial k_t} < 0, \quad t = 1, 2 & (14) \\ Q^{\star\star} = \mathcal{Q}^{\star\star}(k_1, k_2, p_b, \tau_2, \beta) \text{ and } \dfrac{\partial \mathcal{Q}^{\star\star}((k_1, k_2, p_b, \tau_2, \beta)}{\partial k_t} < 0, \quad t = 1, 2 & (15) \end{cases}$$

where $Q^{\star\star}(k_1, k_2, p_b, \tau_1, \beta)$ and $\mathcal{Q}^{\star\star}(k_1, k_2, p_b, \tau_2, \beta)$ are the optimal consumer's demand functions at each period which are decreasing in price and in tax, given the multipliers (k_1, k_2). The next subsection is devoted to the determination of the optimal (k_1, k_2) by the firm.

4.2 The inter-temporal behavior of the firm

The inter-temporal profit of the firm is

$$\Pi(k_1, k_2, p_b, \tau_1, \tau_2, \beta) = \Pi_1(k_1, k_2, p_b, \tau_1, \beta) + \Pi_2(k_1, k_2, p_b, \tau_2, \beta),$$

where subscript corresponds to period, $t = 1, 2$. There are two main cases, hereafter denoted by superscript $c = 1, 2$. The first case captures real situations where the firm buys the total demand for the two periods in the first period and supports a cost in the first period to sell it and another cost to transfer the good to the second period. In that case, the profit is $\Pi^1 = p_{1s}^1 q^1 + p_{2s}^1(Q^1 - q^1) - p_0 Q^1 - C^1(q^1) - C^1((Q^1 - q^1))$. Replace each term by its expression knowing that $Q = q + Q - q$, we have $\Pi^1 = p_{1s}^1 q^1 - p_0 q^1 - C^1(q^1) + \beta\left[p_{2s}^1(Q^1 - q^1) - p_0(Q^1 - q^1) - C^1((Q^1 - q^1))\right]$, which can be generally written as

$$\Pi^1(k_1, k_2, p_b, \tau_1, \tau_2, \beta) = \mathit{VA}_1(Q^{\star\star}(k_1, k_2, p_b, \tau_1, \beta)) - C_1(Q^{\star\star}(k_1, k_2, p_b, \tau_2, \beta))$$
$$+\mathit{VA}_2(\mathcal{Q}^{\star\star}(k_1, k_2, p_b, \tau_1, \beta)) - C_2(\mathcal{Q}^{\star\star}(k_1, k_2, p_b, \tau_2, \beta)),$$

where $C_1(Q^{\star\star}(k_1, k_2, p_b, \tau_1, \beta))$ and $C_2(\mathcal{Q}^{\star\star}(k_1, k_2, p_b, \tau_2, \beta))$ are two $\mathcal{C}^2$ increasing convex cost functions that captures all costs except the buying cost of $Q^{\star\star}$, which is already taken into account into each expression of the value added VA_1 and VA_2.

The other case corresponds to real cases where there are no particular cost to transfer the good to the second period. The profit is $\Pi^2 = p_{1s}^2 q^2 + \beta p_{2s}^1(Q^1 - q^1) - p_0 Q^2 - \beta C^2(Q^2)$. Replace each term by its expression, we have $\Pi^2 = p_{1s}^2 q^2 - p_0 q^2 + \beta p_{2s}^1(Q^1 - q^1) - p_0(Q^2 - q^2) - \beta C^2(q^2 + Q^2 - q^2)$.

$$\Pi^2(k_1, k_2, p_b, \tau_1, \tau_2, \beta) = \mathit{VA}_1(Q^{\star\star}(k_1, k_2, p_b, \tau_1, \beta)) + \mathit{VA}_2(\mathcal{Q}^{\star\star}(k_1, k_2, p_b, \tau_2, \beta))$$
$$-C_1(Q^{\star\star}(k_1, k_2, p_b, \tau_1, \beta) + \mathcal{Q}^{\star\star}(k_1, k_2, p_b, \tau_2, \beta)),$$

where $C_1(Q^{\star\star}(k_1, k_2, p_b, \tau_1, \beta))$ and $C_2(\mathcal{Q}^{\star\star}(k_1, k_2, p_b, \tau_2, \beta))$ are two $\mathcal{C}^2$ increasing convex cost functions that captures all costs except the buying cost of $Q^{\star\star}$, which is already taken into account into each expression of the value added VA_1 and VA_2.

Whatever the case, under the set of assumptions, $\Pi(k_1, k_2, p_b, \tau, \beta)$ is concave as the sum of two concave functions. A rational firm solves the following problem

$$\max_k \Pi(k_1, k_2, p_b, \tau_1, \tau_2, \beta)$$

Denote by $k_1^{\star\star}$ and $k_2^{\star\star}$ the optimal solution of the problem.

Assume that $\partial^2\Pi(k_1^{\star\star}, k_2^{\star\star}, p_b, \tau_1, \beta)/\partial k_1^2 \neq 0$ and $\partial^2\Pi(k_1^{\star\star}, k_2^{\star\star}, p_b, \tau_2, \beta)/\partial k_2^2 \neq 0$. The first order condition is

$$\frac{d\Pi(k_1^{\star\star}, k_2^{\star\star}, p_b, \tau_1, \beta)}{dk_1} = 0 \tag{16}$$

$$\frac{d\Pi(k_1^{\star\star}, k_2^{\star\star}, p_b, \tau_2, \beta)}{dk_2} = 0 \tag{17}$$

Under the set of assumptions, by the implicit function theorem, there exists neighborhoods $\mathcal{V}_{k_1}$ and $\mathcal{W}_{k_1}$ of $k_1^{\star\star}$ in $\mathbb{R}$ and $\mathcal{V}_{k_2}$ and $\mathcal{W}_{k_2}$ of $k_2^{\star\star}$ in $\mathbb{R}$ and two $\mathcal{C}^2$ applications ϕ_{k_1} : $\mathcal{V}_{k_1} \to \mathcal{W}_{k_1}$ and $\psi_{k_2} : \mathcal{V}_{k_2} \to \mathcal{W}_{k_2}$ such that $\mathcal{V}_{k_1} \times \mathcal{W}_{k_1} \subset \,]\,0, \infty\,[^2$ and $\mathcal{V}_{k_2} \times \mathcal{W}_{k_2} \subset \,]\,0, \infty\,[^2$

$$\forall k_2 \in \mathcal{V}_{k_1}, \forall_{k_1} \in \mathcal{W}_{k_1}, \ \frac{d\Pi(k_1^{\star\star}, k_2^{\star\star}, p_b, \tau_1, \beta)}{dk_1} = 0 \iff k_1 = \phi_{k_1}(k_2(p_b, \tau_1, \beta)) \tag{18}$$

$$\forall k_2 \in \mathcal{V}_{k_2}, \forall k_1 \in \mathcal{W}_{k_2}, \ \frac{d\Pi(k_1^{\star\star}, k_2^{\star\star}, p_b, \tau_2, \beta)}{dk_2} = 0 \iff k_2 = \psi_{k_2}(k_1(p_b, \tau_2, \beta)) \tag{19}$$

the system of relations (18) and (19) can be rewritten as

$$S_1 : \begin{cases} k_1^{\star\star} = K^{\star\star}(p_b, \tau_1, \beta) & (20) \\ k_2^{\star\star} = \mathcal{K}^{\star\star}(p_b, \tau_2, \beta) & (21) \end{cases}$$

where $Q^{\star\star}(p_b, \tau_1, \beta)$ and $\mathcal{Q}^{\star\star}(p_b, \tau_2, \beta)$ are the optimal firm's multipliers at each period. The next subsection is devoted to the determination of the optimal τ by the social planner.

5 The welfare

The benevolent social planner chooses the optimal quantities q_{1w} for the first period and q_{2w} for the second period by maximizing the social welfare function defined as the sum of the inter-temporal consumer's surplus and the inter-temporal profit. Since there are two cases, the problem is

$$\max_{q_{1w}, q_{2w}} W^c(q_{1w}, q_{2w}) = \sum_{t=1}^{2} S_t^c(q_{1w}, q_{2w}) + \sum_{t=1}^{2} \Pi_t^c(q_{1w}, q_{2w}), \text{ where } c = 1, 2.$$

5.1 The welfare in the first case

In the first case, the welfare function is obtained by replacing each function by its expression with superscript 1, that we omit hereafter since there are no confusions. The benevolent social planner solves the following problem

$$\max_{q_{1w},q_{2w}} U(q_{1w}) - p_{1w}q_{1w} + p_{1w}q_{1w} - p_b q_{1w} - C(q_{1w}) \\ + \beta\left[U(q_{2w}) - p_{2w}q_{2w} + p_{2w}q_{2w} - p_b q_{2w} - C(q_{2w})\right]$$

which becomes after simplifications

$$\max_{q_{1w},q_{2w}} U(q_{1w}) - p_b q_{1w} - C(q_{1w}) + \beta\left[U(q_{2w}) - p_b q_{2w} - C(q_{2w})\right]$$

The first order condition is

$$\left\{\begin{array}{rcl} \dfrac{\partial U(q_{1w})}{\partial q_{1w}} - p_b - \dfrac{\partial C(q_{1w})}{\partial q_{1w}} & = & 0, \\ \dfrac{\partial U(q_{2w})}{\partial q_{2w}} - p_b - \dfrac{\partial C(q_{2w})}{\partial q_{2w}} & = & 0. \end{array}\right.$$

Using the same type of argument than above, by the implicit function theorem solutions are

$$S_w : \left\{\begin{array}{l} q^\star_{1w} = \phi_w(p_b), \quad (22) \\ q^\star_{2w} = \Phi_w(p_b). \quad (23) \end{array}\right.$$

The multipliers must be compatible with the optimal consumer's solution. There exist $k^\star_{1w} \geq 1$ and $k^\star_{2w} \geq 1$ such that system S_w is compatible with S_0.

$$S_q : \left\{\begin{array}{l} Q^{\star\star}(k_1, k_2, p_b, \tau_1, \beta) = \phi_w(p_b), \quad (24) \\ \mathcal{Q}^{\star\star}(k_1, k_2, p_b, \tau_2, \beta) = \Phi_w(p_b). \quad (25) \end{array}\right.$$

By the implicit function theorem, the solutions are

$$S_k : \left\{\begin{array}{l} k^\star_{1w} = h_w(p_b, \tau_1, \beta), \quad (26) \\ k^\star_{2w} = H_w(p_b, \tau_2, \beta). \quad (27) \end{array}\right.$$

The previous multipliers of System S_k should be compatible with the optimal solutions of the firm described by system S_1. There exist an optimal tax $\tau^\star_{tw} \in [0,1]\,, t = 1, 2$ such that

$$\left\{\begin{array}{l} K^{\star\star}(p_b, \tau^\star_{1w}, \beta) = h_w(p_b, \tau^\star_{1w}, \beta), \\ \mathcal{K}^{\star\star}(p_b, \tau^\star_{2w}, \beta) = h_w(p_b, \tau^\star_{2w}, \beta). \end{array}\right.$$

$$S_\tau : \left\{\begin{array}{l} \tau^\star_{1w} = \theta_w(p_b, \beta), \quad (28) \\ \tau^\star_{2w} = \Theta_w(p_b, \beta). \quad (29) \end{array}\right.$$

Note that there are cases where θ_w is the same as Θ_w, so that the tax is unique, as it will be shown in Section 6 Example.

6 Examples

6.1 The static example

6.1.1 The representative consumer and the firm

Assume that the representative consumer's preferences are perfectly represented by the following utility function $U(q) = (\frac{b}{a} - \frac{1}{2a}q)q$. The utility function has been chosen because it generates an affine demand function. Indeed, relation (4) is $q^\star = -ap_s + b$. Note that for $q^\star > 0$ any feasible price must satisfy $p_s < b/a$ so that $kp_b + \tau < b/a$. Recall that the selling price is $p_s = kp_b + \tau$. The profit of the firm is $\Pi(k, p_b, \tau) = (k-1)p_b\left[-ap_s + b\right] - \frac{1}{2}\left[-ap_s + b\right]^2$. Relation (5) is

$$k^\star = \frac{(1+a)(b-a\tau) + ap_b}{a(2+a)p_b} > 1 \quad \Longleftrightarrow \quad p_b + \tau < \frac{b}{a} \text{ always true.}$$

6.2 The welfare

The welfare maximizing quantity of relation (6) is $q_w^\star = \frac{b-a(p_b+\tau)}{1+a}$. Relation (7) is $k_w^\star = \frac{a\tau+p_b}{(1-a)p_b}$ and finally the welfare maximizing tax is $\tau_w^\star = \frac{(1-a^2)b-a(1+2a)p_b}{1+2a}$.

6.3 A complete simulation of the static example

The following numerical simulation has been chosen to illustrate cases where the price elasticity of demand ε is negative and greater than -1, because for those cases, the Lerner index provides negative price.

The first part of the simulation is devoted to the market economy, without any social planner intervention. Tax is set to zero. The second part of the simulation is devoted to the social planner intervention, who implements the welfare maximizing tax. Comparison without and with public intervention is provided.

Assume $a = 0.5$, $b = 8, p_b = 3$. The price elasticity $\varepsilon = -0.54 > -1$. Solutions are $q^\star = 5.825,$, $k^\star = 2,1$. The selling price is $p_s = 13.34$. The consumer's surplus is $S = 5.74$ and the profit is $\Pi = 13.57$.

In order to correctly simulate the model, the tax is needed. For that reason, the resolution is made backward, starting by the welfare, and then simulating the consumer and the firm.

The welfare maximizing tax is $\tau_w^\star = 0.2$.The price elasticity $\varepsilon = -0.55 > -1$. Solutions are $q^\star = 5.72,$, $k^\star = 2,08$. The selling price is $p_s = 13.42$. The consumer's surplus is $S = 5.27$

and the profit is $\Pi = 13.11$.

6.4 The inter-temporal example

6.4.1 The representative consumer and the firm

The utility function is unchanged compared with the static model, $U(q) = (\frac{b}{a} - \frac{1}{2a}q)q$. System S_0 is $q^{\star\star} = b - ap_{1s} = b - a(k_1 p_b + \tau_1)$, and $Q^{\star\star} = 2b - a(p_{1s} + p_{2s}) = 2b - a((k_1 + k_2)p_b + \tau_1 + \tau_2)$. System S_1 provides the following solutions

$$k_1^{\star\star} = \frac{(1+a)(b-a\tau_1)+ap_b}{a(2+a)p_b} > 1 \;\; k_2^{\star\star} = \frac{(1+a)(b-a\tau_2)+ap_b}{a(2+a)p_b} > 1$$
$$\iff \quad p_b + \tau_t < \frac{b}{a}, \;\; t = 1,2 \;\text{ always true and } \tau_1 < \tau_2 \Rightarrow k_1 > k_2. \quad (30)$$

6.5 The welfare

System S_w leads to the following solutions

$$q_{1w} = \frac{b - ap_b}{1+a}, \;\; q_{2w} = \frac{2(b-ap_b)}{1+a}.$$

System S_k provides the following solutions

$$k_{1w}^{\star} = \frac{b + p_b - (1+a)\tau_1}{(1+a)p_b}, \;\; k_{2w}^{\star} = \frac{ap_b - b - a(1+a)\tau_2))}{a(1+a)p_b}$$

System S_τ gives

$$\begin{cases} \dfrac{b + p_b - (1+a)\tau_1}{(1+a)p_b} = \dfrac{(1+a)(b-a\tau_1)+ap_b}{a(2+a)p_b} \\[2ex] \dfrac{ap_b - b - a(1+a)\tau_2))}{a(1+a)p_b} = \dfrac{(1+a)(b-a\tau_2)+ap_b}{a(2+a)p_b} \end{cases}$$

which leads to

$$\tau_1^{\star} = \frac{(2+a)b + p_b}{1+a}$$
$$\tau_2^{\star} = \frac{ap_b - \left[3(1+a)+a^2\right]b}{a(1+a)}$$

6.6 A complete simulation of the inter-temporal example

In this subsection, b has been chosen in order to have

7 The optimal multipliers in the presence of unsold goods

As discussed in the introduction, there particular markets on which stock of unsold goods are persistent. This section analyzes in a very simple way the behavior of consumers and firm in the presence of stock of unsold good. The static approach constitutes a benchmark model. It is shown under which conditions a stock of unsold goods emerges at the optimum. A two period approach is proposed in order to show under which conditions an optimal stock of unsold goods still remains at the end of the second period.

7.1 The static model

Subsubsection 7.1 is devoted to the presentation of the habits, the modified consumer's behavior and the modified monopoly's behavior.

7.1.1 The habits

As above, the manager does not understand the determinants of the demand and only observes the existence of a permanent stock of unsold good which is thrown out at the end of the period. For what he integrates this stock of unsold goods into the determination of the value added by considering that only a fraction θ is sold out, and it remains the complementary fraction $1-\theta$ as unsold good. The demand is $q=\theta\overline{Q}$ is not considered neither as a function of price nor a function of the display of goods. The rule he uses to price the good is the same as in Subsection 2.1. He anticipates the value added as being a number that integrates the stock of unsold good in the following way $\overline{VA}(\overline{k})=(\theta\overline{k}-1)p_b\overline{Q}$.

7.1.2 The habits with an endogenous demand and exogenous multiplier

In certain cases, the manager understands that the demand is falling in price p_s and that some consumers are sensitive to the display of good Q, see introduction for more details. The demand is now $q(p_s,Q)=\theta\overline{Q}(p_s)$ where $dq(p_s,Q)/dp_s<0$ and $dq(p_s,Q)/dQ>0$. Replacing this quantity into the value added function and obtain

$$VA(k) \quad = \quad (\theta k-1)p_bQ(p_s) \tag{31}$$

7.1.3 The consumer's behavior: the preference for quantity principle

For the rest of the paper, we will use the following principle, properties and definitions.

PRINCIPLE **1** *The preference for quantity captures the consumer's valuation in terms of utility of the available quantity of the displayed goods he faces before to buy.*

The following definitions captures some empirical observations mentioned in the introduction. They are extracted from the management literature and also from the real world.

DEFINITION **1** *A sophisticated consumer has a preference for quantity.*

DEFINITION **2** *A picky consumer is any consumer with an increasing utility function in the display.*

PROPERTY **1** *A picky consumer buys if the display of goods is large enough.*

It is important to underline that the preference for quantity does not necessarily result in a stock of unsold goods at the end of the market period. Indeed, as Theorems 4 and 5 prove, the stock of unsold goods depends on the shape of the utility function. The importance of the introduction of the preference for quantity principle in this paper is to make a link between the management literature and the economic literature.

As mentioned above, the modified monopoly chooses the price p_s (or equivalently the multiplier k) and the display Q to answer the demand q. The display of goods is a parameter for the sophisticated consumer and an additional instrument for the monopoly. We consider utility functions $U(q,Q)$ that satisfy the following assumption:

TECHNICAL ASSUMPTION 1 *$u \in C^3[0,\infty)$; $u''_{qq}(q,Q) < 0$, for $q \in [0,Q)$. Moreover, $u'_Q(q,Q) > 0$, for $Q \in [0,\infty)$ and $u''_{QQ}(q,Q) < 0$, for $Q \in [0,\infty)$.*

Note that here, Q is a parameter for the consumer. Therefore $u(q,Q)$ is strictly increasing and strictly concave in q. As it will become clearer below, a bell shape utility function is necessary for the social planner solution. Note that we can always find a constant to obtain $u(0,Q) = 0$ without any loss of generality.

The consumer's preferences are represented by the following surplus function $S(q,p,Q) = u(q,Q) - p_s q$, where V — the surplus function — and $u(q,Q)$ — the utility function — are two one-to-one $\mathcal{C}^3$ concave functions in q. Note that for $0 \leq q \leq Q$, $u(q,Q)$ also satisfies Assumption 1. The consumer does not choose the quantity Q, and takes it as given for choosing

q. A consumer maximizes his surplus function, given Q and the monopoly price of good $p_s = p + \tau$ or equivalently $p_s = kp_b + \tau$:

$$\text{Problem } \mathcal{P} \quad \max_{q} \; u(q, Q) - p_s q, \tag{32}$$

The first order condition is

$$P_s(q, Q) \quad = \quad \frac{\partial u(q, Q)}{\partial q} \quad \Longleftrightarrow \quad q^{\star} = \mathcal{Q}(k, Q, p_b, \tau) \tag{33}$$

7.1.4 The modified monopoly

This Subsection considers a modified monopoly that operates under certainty. His behavior is modified (compared to the traditional monopoly) since the monopoly not only chooses the price but also chooses the display while selling goods to a sophisticated consumer. At the end of the market, due to some properties of the utility functions (see below), the monopoly can end up with no stock of unsold goods, or a stock of unsold goods.

We suppose a strictly increasing convex total cost $TC(Q)$ of class C^2, $TC(0) = 0, TC'(Q) > 0$, for all $Q > 0, TC'(0) = 0$, and $TC''(Q) \geq 0, \forall Q \geq 0$. Knowing $P_s(Q, q)$ the price the monopoly charges to the sophisticated consumer, which is consistent with problem $\mathcal{P}$, the maximization problem $\mathcal{P}_{M1}$ of the monopoly is:

$$\max_{k,\, Q,} \Pi_1(Q,\, q) = kp_b q - p_b Q - C(Q) \tag{34}$$

$$\text{subject} \quad \text{to: } 0 \leq q \leq Q, \tag{35}$$

Replacing the solution of the sophisticated consumer (33) into the profit function of the monopoly, Problem $\mathcal{P}_{M1}$ becomes

$$\max_{k,\, Q} \Pi_1(Q,\, q) = kp_b \mathcal{Q}(k, Q, p_b, \tau) - p_b Q - C(Q) \tag{36}$$

$$\text{subject} \quad \text{to: } 0 \leq q \leq Q. \tag{37}$$

The Lagrangian is $\mathcal{L}(k, Q, p_b, \tau, \lambda, \mu) = kp_b \mathcal{Q}(k, Q, p_b, \tau) - p_b Q - C(Q) + \lambda q + \mu(Q - q)$. The problem becomes

$$\max_{k,\, Q,} \mathcal{L}(k, Q, p_b, \tau, \lambda, \mu)$$

The first order condition for a maximum is

$$S_2 : \begin{cases} S(q^\star, Q^\star) + \lambda^\star - \mu^\star = 0, & (38)\\ T(q^\star, Q^\star) + \mu^\star = 0, & (39)\\ q^\star \geq 0, & (40)\\ Q^\star - q^\star \geq 0, & (41)\\ \lambda^\star q^\star = 0, & (42)\\ \mu^\star (Q^\star - q^\star) = 0, & (43)\end{cases}$$

Note that from relation (42) cases where $\lambda > 0 \to q = 0$ are not interesting. Consequently, there are only two cases : $\lambda = 0$ and $\mu > 0$ which corresponds to the case where the display matches the demand. This case has already been studied in Section 3. For that reason, we only concentrate on case $\lambda = 0$ and $\mu = 0$. The problem is now

$$\max_{k,\,Q} \Pi_1(Q,\, q) = kp_b \mathcal{Q}(k, Q, p_b, \tau) - p_b Q - C(Q)$$

The profit function must be concave or equivalently the Hessian matrix must be semi definite negative. Define $\mathcal{S}(k, Q, p_b, \tau) := \frac{\partial \mathcal{Q}(k,Q,p_b,\tau)}{\partial k} kp_b + \mathcal{Q}(k, Q, p_b, \tau), T(k, Q, p_b, \tau) = \frac{\partial \mathcal{Q}(k,Q,p_b,\tau)}{\partial Q} kp_b - p_b - \frac{\partial TC(Q)}{\partial Q}$ and $\mathcal{S}'_k(k, Q, p_b, \tau) := \frac{\partial^2 \mathcal{Q}(k,Q,p_b,\tau)}{\partial k^2} kp_b + 2\frac{\partial \mathcal{Q}(k,Q,p_b,\tau)}{\partial k}$, $T'_Q(k, Q, p_b, \tau) = \frac{\partial^2 \mathcal{Q}(k,Q,p_b,\tau)}{\partial Q^2} kp_b - \frac{\partial^2 TC(Q)}{\partial Q^2}$. The first order condition implies that the first order derivative of the profit are nil.

$$S_1 : \begin{cases} \mathcal{S}(k^\star, Q^\star, p_b, \tau) = 0, & (44)\\ T(k^\star, Q^\star, p_b, \tau) = 0. & (45)\end{cases}$$

Assume system (44)-(45) has a solution $(k^\star, Q^\star)$. We have to show that it is actually a local maximum. The determinant of the Hessian matrix at this point is:

$$\begin{vmatrix} \mathcal{S}'_k(k^\star, Q^\star, p_b, \tau) & S'_Q(k^\star, Q^\star, p_b, \tau) \\ T'_k(k^\star, Q^\star, p_b, \tau) & T'_Q(k^\star, Q^\star, p_b, \tau) \end{vmatrix}$$

Note that by relation (38) the mixed derivative are nil. For concavity, the condition is either

$$\begin{cases} \mathcal{S}'_k(k^\star, Q^\star, p_b, \tau) \leq 0 \\ T'_Q(k^\star, Q^\star, p_b, \tau) \leq 0 \end{cases}$$

Define

$$\alpha := \frac{-\left[\frac{\partial^2 \mathcal{Q}(k,Q,p_b,\tau)}{\partial k^2}\right] kp_b}{2\frac{\partial \mathcal{Q}(k,Q,p_b,\tau)}{\partial k}} \qquad \beta := \frac{\frac{\partial^2 TC(Q)}{\partial Q^2}}{\frac{\partial^2 \mathcal{Q}(k,Q,p_b,\tau)}{\partial Q^2} kp_b}$$

$$\text{The profit is concave} \quad \Longleftrightarrow \quad : \begin{cases} \alpha \leq 1 \\ \beta \leq 1 \end{cases}$$

Therefore $(k^\star, Q^\star)$ is a local maximum. The solution exists if the first order condition is satisfied.

$$S_1 : \begin{cases} \dfrac{\partial \mathcal{Q}(k, Q, p_b, \tau)}{\partial k} k p_b + \mathcal{Q}(k, Q, p_b, \tau) = 0, & (46) \\ \dfrac{\partial \mathcal{Q}(k, Q, p_b, \tau)}{\partial Q} k p_b - p_b - \dfrac{\partial TC(Q)}{\partial Q} = 0. & (47) \end{cases}$$

Solution must satisfy system S_1. Note that $\mathcal{Q}(k, Q, p_b, \tau)$ is positive decreasing in price as a demand function, consequently decreasing in k so that $\mathcal{Q}'_k < 0$. As long as $\mathcal{Q}'_k$ is not homothetic to $\mathcal{Q}$ relation (46) has a solution[3]. Moreover, by the preference for quantity principle, the demand is increasing in Q so that $\mathcal{Q}'_Q k p_b > 0$ and there is a solution

$$k^\star = \kappa(p_b, \tau) \tag{48}$$

$$Q^\star = \Omega^\star(p_b, \tau). \tag{49}$$

It is interesting to note that the solution is such that $k^\star(p_b, \tau) > 1$. Indeed, from relation (47) we have

$$k^\star = 1 + \frac{\frac{\partial TC(Q)}{\partial Q}}{\frac{\partial \mathcal{Q}(k, Q, p_b, \tau)}{\partial Q}}$$

7.1.5 Welfare

Subsection 7.1.5 determines the welfare maximizing tax τ_w that leads the economy to the welfare maximizing consumption q_w and the welfare maximizing multiplier k_w. As above, the social welfare is defined as $W(q_w, Q_w) = S(q_w, Q_w) + \Pi(q_w, Q_w)$ which is concave for the same reason as above, which can be rewritten as $W(q_w, Q_w) = S(q_w, Q_w) + VA(q_w, Q_w) - C(Q_w)$. Replacing the surplus function by its expression in term of utility and the value added function by its expression, as above, the multipliers cancels again, so that the problem becomes

$$\max_{q_w, Q_w} U(q_w, Q_w) - p_b Q_w - C(Q_w)$$

The first order condition is

$$\frac{\partial U(q_w, Q_w)}{\partial q_w} = 0,$$

[3]For example, if $S(q, Q) = Q \ln q - k p_b q$ the demand function is $Q/(k p_b)$ and relation (46) has no solution.

$$\frac{\partial U(q_w, Q_w)}{\partial Q_w} - p_b - \frac{\partial C(Q_w)}{\partial Q^\star} = 0.$$

Since the utility function reveals a bell shape, there is a solution to the first equation. Replace the solution $q_w^\star$ into the second relation and find $Q_w^\star$. The welfare maximizing quantity is

$$q_w^\star = \mathcal{Q}^\star \quad (50)$$

$$Q_w^\star = Q_w(p_b) \quad (51)$$

The social planner chooses the multiplier k_w that make the consumer's solution (33) compatible with the welfare maximizing quantity $q_w^\star$ of relation (50).

$$\mathcal{Q}(k_w, Q_w, p_b, \tau_w) = \mathcal{Q}^\star, \quad (52)$$

$$k_w^\star = \zeta(\mathcal{Q}^\star, p_b, \tau). \quad (53)$$

The welfare maximizing tax is obtained by equating (48) with (53)

$$\kappa(p_b, \tau) = \zeta(\mathcal{Q}^\star, p_b, \tau) \quad \Longleftrightarrow \quad \tau_w^\star = \tau_w(\mathcal{Q}^\star, p_b).$$

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7.2 The two period model: endogenous demand and optimal multiplier

This section is devoted to a 2 period model where the information is perfect and all individuals (the social planner, the representative consumer and the firm) are rational. In many countries, firms first operate with a high margin price and try to sold as many as possible goods. They then operate at a low margin price in order to sold out products. For that reason, we divide time into two periods. These periods are of various length in the real world; depending on the market. For example, in France, there are street market every two days in town. At 6am prices are very high and shelves are full, but at the end of the market, around noon, prices are very low and shelves are not necessarily full. Consumers that like choosing their food arrive at the earlier market period, while others, who prefer prices to display show up at noon. The social planner sets the tax that leads the economy to its social welfare optimum. Knowing the tax, the consumer maximizes its surplus with respect to quantities and the firm maximizes its profit with respect to the multiplier.

7.2.1 The two period model

Consider that there are two periods $t = 1, 2$ under perfect information. The rate of unsold goods during the market period $t = 1$ is τ. The firm prices $p_b(\tau)$ the good knowing that the consumer preferences are sensitive to display as well as inventory. From a theoretical point of view we consider the two following possibilities : more display or/and more inventory may induce sales on some markets, but less display or/and less inventory may create scarcity that induces sales on other markets. During the second period, the firm prices the stock of unsold goods in other to clear it by choosing another value of the optimal multiplier. Consequently, there are two multipliers k for the normal sales (during the first market period) and k_i for the selling off market period. As above, the first subsection is devoted to the analysis of the value added with habits, then the second one to the analysis of the optimal value added, the third one to profit with the particularity that there are two multipliers, the fourth studies the benevolent social planner's behavior.

7.2.2 The habits during the market period and the selling off period

Consider the habit case during the market period $t = 1$ and the selling off period $t = 2$ in the presence of a stock of unsold goods. The discount rate is β.

7.2.2.1 The exogenous demand

The manager considers inventory as exogenous, $Q = \overline{Q}$, ignoring that consumer's preferences are sensitive to both display and inventory. The value added is the sum of the first period and the second period discounted value added

$$VA(\tau) = VA_1(\tau) + \beta A_2(\tau) \iff VA(\tau) = (1-\tau)p_s\overline{Q} - p_b\overline{Q} + \beta\tau p_i\overline{Q} = ((1-\tau)p_s - p_b + \beta\tau p_i)\overline{Q}.$$

Most of the time, the true value of τ is unknown to the manager, or not correctly anticipated, then the realized value added differs from the one of Section 2. The manager uses the traditional multiplier $\overline{k}$ to determine the first period sale price $p_s = \overline{k}p_b$, as well as the second period sale price $p_i = \overline{k}_i p_b$, so that

$$VA_1(\tau) \quad = \quad ((1-\tau)\overline{k} - 1 + \beta\tau\overline{k}_i)p_b\overline{Q}. \tag{54}$$

7.2.2.2 The endogenous demand

Consider that the manager understands that the demand is falling in price, $Q1 = Q_1(p_s) = Q_1(k)$ and $Q_2 = Q_2(p_i) = Q_2(k_i)$. He buys $\overline{Q} = Q_1(k) + Q_2(k_i)$.

$$VA_1(\tau) \quad = \quad ((1-\tau)k - 1)\, p_b Q(k) + \beta\tau\overline{k}_i p_b Q(k_i). \tag{55}$$

For concavity the hessian matrix imposes that

$$\frac{\partial^2 Q}{\partial k^2} \leq 0, \quad \frac{\partial^2 Q}{\partial k_i^2} \leq 0, \quad \frac{\partial^2 Q}{\partial k^2}\frac{\partial^2 Q}{\partial k_i^2} \geq \left[\frac{\partial^2 Q}{\partial k \partial k_i}\right]^2.$$

In general the new price does not sold out the available quantity of unsold goods.

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7.2.3 The role of display and inventory on the multipliers in the presence of stock of unsold goods

It is commonly observed that sales are display and/or inventories depending, as shown in the introduction of the paper. In some cases, more stocks induce sales, in some other case, scarcity induce sales. We are neutral from a theoretical point of view and take both cases as possible, knowing that these effects are non linear according to the literature in management science. For doing that, let us consider that the slope of the demand function is display or inventory dependent. A simple way to model such an empirical observation is to consider that the rational representative consumer takes the display d and the inventory Q as parameters into his utility function. The representative consumer preferences are perfectly described by the following utility function: $V(q) = (\alpha d + \beta Q + \gamma)v(q)$, where v is a continuous differentiable and concave utility function with respect to the demanded quantity of good q, where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}_+$ and we assume for the rest of the paper that $\alpha d + \beta Q + \gamma > 0$.

A rational representative consumer maximizes its surplus given the multiplier k

$$\max_q \; (\alpha d + \beta Q + \gamma)u(q) - p_s q.$$

The first order condition is

$$(\alpha d + \beta Q + \gamma)u'(q) = kp_b.$$

Note that with $u(q) = (A - Bq)q$ the slope of the demand function is display and inventory dependent, and is

$$q^\star \quad = \quad \frac{-1}{2B(\alpha d + \beta Q + \gamma)} k p_b + \frac{A}{2B} \tag{56}$$

If α or β are negative less display, or less inventory induces sales. On the contrary, if they are positive, more display and more inventory induce sales. Replace the previous solution into the expression of the value added to obtain

$$\iff \quad VA(k) \quad = \quad ((1-\tau)k - 1)p_b \left[\frac{-1}{2B(\alpha d + \beta Q + \gamma)} k p_b + \frac{A}{2B} \right]. \tag{57}$$

This expression captures the habit case for k exogenous, under the condition that the demanded quantity is positive, i.e., $k < (\alpha d + \beta Q + \gamma) A / p_b$.

7.2.4 The optimal value added in the presence of stock of unsold goods

7.2.4.1 The optimal value added during the market period

Note that the previous expression (4) is concave in k. The rational manager solves the following problem

$$\max_k \; VA(k)$$

The optimal solution is

$$k^\star(\tau) = \frac{(1-\tau)A(\alpha d + \beta Q + \gamma) + p_b}{2(1-\tau)p_b}$$

$$\forall \alpha > 0, \beta > 0, \frac{\partial k^\star(\tau)}{\partial d} > 0, \frac{\partial k^\star(\tau)}{\partial Q} > 0, \forall \tau \in [\,0, 1\,[\;\; \frac{\partial k^\star(\tau)}{\partial \tau} > 0$$

Define $a_1 = -1/(2B(\alpha d + \beta Q + \gamma))$ and $b = A/2B$.

$$k^\star(\tau) \quad = \quad \frac{(1-\tau)b + a_1 p_b}{2a_1(1-\tau)p_b} \tag{58}$$

The optimal multiplier $k^\star(\tau)$ is increasing in the display and in the inventory if the consumer appreciates display and inventory or reciprocally decreasing if not. Moreover, the higher the rate of unsold goods, the higher the multiplier. The expression of the optimal value added is

$$VA(k^\star(\tau)) = \frac{(p_b - (1-\tau)A(\alpha d + \beta Q + \gamma))^2}{8(1-\tau)B(\alpha d + \beta Q + \gamma)}.$$

Using the new notations, we have

$$VA(k^\star(\tau)) = \frac{(a_1 p_b - b(1-\tau))^2}{4a_1(1-\tau)} > 0. \tag{59}$$

It is interesting to note that the previous expression is convex for $\tau \in [\,0, 1\,[$.

7.2.4.2 The optimal value added during the selling off period

This subsection uses the same methodology as above in order to propose an optimal stock of unsold good valuation. During the selling off period, the manager can differently price the good— using another multiplier k_i — in order to induce more sales, according to the role of display and inventory in the slope of the demand function. To do so, the manager maximizes the sales of the stock of unsold goods by solving $\max_{k_i} p_i \tau Q$ with $p_i = k_i p_b$, which is equivalent to solve the following program

$$\max_{k_i} VA(k^\star(\tau)) = k_i p_b \tau \left[\frac{-1}{2B(\alpha d + \beta Q + \gamma)} k_i p_b + \frac{A}{2B} \right]$$

The first order condition is

$$\frac{-2}{2B(\alpha d + \beta Q + \gamma)} k_i p_b^2 + \frac{A}{2B} p_b = 0$$

The optimal solution is

$$k_i^\star = \frac{A(\alpha d + \beta Q + \gamma)}{2p_b}.$$

Using the new notations

$$k_i^\star = \frac{b}{2a_1 p_b}. \tag{60}$$

Note that $k_i^\star > 1 \Rightarrow p_b < b/(2a_1)$. The optimal value added is

$$VA(k^\star(\tau)) = \frac{\tau b^2}{4a_1 p_b}$$

7.2.5 The optimal profit in the presence of stock of unsold goods

In this subsection we consider that there are two periods: the market period ($\overline{k}^{\star\star}$ captures a high margin) and the selling off period ($\underline{k}^{\star\star}$ captures a low margin). For simplicity we do not discount between the two periods, since in general on real markets these periods may take place within the same week, or within the same day (on food market for instance).

7.2.5.1 The optimal profit during the market period

As above, during the market period the rational manager solves the following programme

$$\max_k \ \Pi(k) = \left[((1-\tau)k - 1)p_b - \frac{1}{2} \left[\frac{-1}{2B(\alpha d + \beta Q + \gamma)} k p_b + \frac{A}{2B} \right] \right] \left[\frac{-1}{2B(\alpha d + \beta Q + \gamma)} k p_b + \frac{A}{2B} \right]$$

The optimal multiplier is

$$k^{\star\star} = \frac{(\alpha d + \beta Q + \gamma)\left[2AB(1-\tau)(\alpha d + \beta Q + \gamma) + A + 2Bp_b\right]}{\left(1 + 4B(1-\tau)(\alpha d + \beta Q + \gamma)\right)p_b}.$$

Using these new notations, we have

$$k^{\star\star} = \frac{b(1 + \frac{1-\tau}{a_1}) + p_b}{(a_1 + 2(1-\tau))p_b} \tag{61}$$

Let us study the properties of the optimal solution. First of all, $k^{\star\star} > 1 \Rightarrow p_b < b/a_1$ which is exactly the same condition as above for $k^\star$ with the new slope a_1. The optimal profit is

$$\Pi(k^\star) = \frac{(p_b - (1-\tau)(\alpha d + \beta Q + \gamma)A)^2}{2(1 + 4B(1-\tau)(\alpha d + \beta Q + \gamma))}.$$

Using the new notations, we have

$$\Pi(k^\star) = \frac{(a_1 p_b - b(1-\tau))^2}{2a_1(a_1 + 2(1-\tau))} > 0. \tag{62}$$

7.2.5.2 The optimal profit during the selling off period

Consider now that the stock of unsold goods is costly to manage. Assume that the cost function is convex in the stock of unsold goods. The profit function during the selling off period is defined

$$\max_{k_i} VA(k^\star(\tau)) = k_i p_b \tau \left[\frac{-1}{2B(\alpha d + \beta Q + \gamma)} k_i p_b + \frac{A}{2B}\right] - \frac{1}{2}\tau^2 \left[\frac{-1}{2B(\alpha d + \beta Q + \gamma)} k_i p_b + \frac{A}{2B}\right]^2$$

The first order condition is

$$\frac{-2}{2B(\alpha d + \beta Q + \gamma)} k_i p_b^2 + \frac{A}{2B} p_b - \frac{\tau p_b}{2B(\alpha d + \beta Q + \gamma)} \left[\frac{-1}{2B(\alpha d + \beta Q + \gamma)} k_i p_b + \frac{A}{2B}\right] = 0$$

The optimal solution is

$$k_i^\star = \frac{A(\alpha d + \beta Q + \gamma)(2B + \tau)}{2B(4B(\alpha d + \beta Q + \gamma) + \tau)p_b}.$$

Using the new notations

$$k_i^\star = \frac{b(1 + a_1\tau)}{a_1(2 + a_1\tau)p_b}. \tag{63}$$

Note that $k_i^\star > 1 \Rightarrow p_b < b/(2a_1)$. The optimal value added is

$$VA(k^\star(\tau)) = \frac{\tau b^2(2b(1 + a_1\tau)^2 - a_1^2\tau(2 + a_1\tau)^2 p_b}{2a_1^2(2 + a_1\tau)^3 p_b}$$

7.2.6 The social planner in the presence of stock of unsold goods

As above, the representative consumer preferences are perfectly defined by the utility function $V(q) = (\alpha d + \beta Q + \gamma)v(q)$. The rational consumer maximizes its surplus function. Using (56) the optimal value of its surplus is

7.2.6.1 The utility and growth maximizing multiplier

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7.2.6.2 The welfare maximizing multiplier

7.3 A complete example

8 Conclusion

A Appendix

A.1 The optimal value added in the static model

Given p_b, replace $p_s = kp_b$ into the demand function, $Q(p_s) = Q(kp_b) = Q(k)$ where $dQ/dk < 0$. Note that for relation (31) to be a concave function of the multiplier k, the second derivative of $VA(Q(k))$ should be non positive, which gives the following condition

$$2\frac{dQ(k)}{dk} + (k-1)\frac{d^2Q(k)}{dk^2} \leq 0. \tag{64}$$

Under the previous condition (64), the value-added maximizing multiplier $k^\star$ is solution of the following program

$$\max_k \ (k-1)p_b Q(k).$$

The first order condition is

$$p_b\left[Q(k^\star) + (k^\star - 1)\frac{dQ(k^\star)}{dk}\right] = 0$$

$\forall p_b \neq 0$, the optimal multiplier is

$$k^\star \ = \ 1 - \frac{Q(k^\star)}{\frac{dQ(k)}{dk}} > 1 \ \text{ since } \ \frac{dQ(k)}{dk} < 0. \tag{65}$$

The optimal sale is $Q(k^\star)$ and the optimal value added is $VA(k^\star) = (k^\star - 1)p_b Q(k^\star) > 0$. Moreover, note that $VA(k^\star) > \overline{V}A(\overline{k})$, and $VA(k^\star) > VA(\overline{k})$.

A.1.1 The surplus and growth maximizing social planner

Let us suppose that the social planner can either maximize the modified welfare constituted by the sum of the surplus and the value added, which is analog to the gross national product, defined as $G(q) = S(q) + VA(q)$. For concavity, necessary the second order condition must be satisfied

$$\frac{d^2G(q)}{dq^2} = \frac{d^2S(q)}{dq^2} + \frac{d^2VA(q)}{dq^2} \leq 0.$$

THEOREM **1** *The modified welfare $G(q(k))$ is maximized for the value-added maximizing multiplier $k^\star = 1$.*

Proof. The social planner chooses the optimal quantity $q_g^\star$ that solves the following program $\max_q G(q)$

$$\max_q S(q) + VA(q)$$

Replace the modified welfare function by its expression in term of utility and value added to obtain

$$\max_q U(q) - kp_bq + (k-1)p_bq$$

After simplification

$$\max_q U(q) - p_bq$$

Concavity of the objective function is insured by the concavity of the utility function. The first order condition is

$$\frac{\partial U(q^\star)}{\partial q^\star} = p_b \quad \Longleftrightarrow \quad q^\star = Q^\star(p_b) \text{ and } k^\star = 1.$$

□