The growth model with public capital as the dynamical system

Adam Krawiec
Institute of Economics, Finance and Management, Jagiellonian University,
Łojasiewicza 4, 30-348 Kraków, Poland,
adam.krawiec@uj.edu.pl

Marek Szydłowski
Mark Kac Complex Systems Research Centre, Jagiellonian University,
Łojasiewicza 11, 30-348 Kraków, Poland
marek.szydlowski@uj.edu.pl

April 2, 2017

Abstract

We analyse the simple model of economic growth with two kinds of physical capitals: private and public capital. We assume that a producer's decision to increase stock of private capital congest the public capital available for other producers. And the households optimize their utility from consumption in infinite time horizon. This economic growth model with congestion of public capital has the form of a 3-dimensional dynamical system with the rates of private capital, public capital as the model variables. We use the mathematical methods of the dynamical systems theory for analysing dynamics of the model. We find a unique steady-state solution as a saddle point with two-dimensional unstable subspace and one-dimensional stable subspace. The latter contains the saddle trajectory along which the system evolves to the steady-state. The economic system moves on the saddle trajectory towards the saddle where the rates of growth of these state variables are constant and equal $a + n$. We also obtain the optimal rate of tax which maximizes the households consumption in the steady state which is equal of the public capital share in income. The numerical analysis allows us to visualize the phase space of the model in the phase portraits.

1 Introduction

The problem of the existence and character of the connection between economic growth and economic policy arises because of the increasing role of government within market economies. In 1969 Arrow and Kurz [1] put the question whether aggregate production in an economy benefits from public capital services which are financed from income tax. Since public spending is not only made for consumptive or distributional purposes but also used for productive aims, the public sector should be taken into consideration in models of economic growth. However, problem is very complex especially from the point of view of the link between theoretical and empirical findings [2]. In this note we concentrate on the theoretical aspect of the problem and discuss a certain simple model of economic growth with public capital.

Models of endogenous economic growth apart the physical capital utilize other forms of capital especially human capital. To discuss the effects of public policy on short and long economic growth public capital was introduced to models of economic growth. One of the first analysis of economic policy in the endogenous growth model was presented by Barro and Sala-i-Martin where the government makes public expenditures which raise the marginal productivity of private capital. This framework was further developed with one important modification. Public capital was treated as capital stock rather than government spending [3, 4]. In this framework it was possible to analyse of the effects of policy on economic growth. Public capital is pure public goods which can have different character. In general it can be non-rival and non-excludable. It is also possible to consider public goods can be subject to congestion and then are non-rival up to some point [5].

One of the earliest models which discussed the public expenditures in the context of economic growth was presented by Barro [6]. He assumed the Cobb-Douglas production function with two inputs capital k and government spending g

$$y = f(k, g) = Ak^{1-\alpha}g^{\alpha}. \tag{1}$$

In this model the size of the government was determined as well as consumption dynamics was obtained in the market economy and in the command economy. The main result was that the competitive growth rate is smaller than the command rate growth rate.

Next Barro and Sala-i-Martin [7] considered two further extensions of the above model. The first model considered non-rival public capital, and the second one public goods subject to congestion. In the first case public services are treated as non-rival, non-excludable public goods. In the second modification the given state of congestion is maintained as long as the government keeps the ratio of G to Y.

The different class of models which assumes that public inputs have the character of capital goods and are accumulated in the similar manner as private physical capital were presented by Greiner [3] who considered the following production function

$$Y(K, G) = AK^{\alpha}G^{1-\alpha}. \tag{2}$$

The tax rate affects the disposable income of households as well as the tax revenue in equations for both private and public capital accumulation. In this model Greiner analysed and discussed in a detailed way the dynamics of model variables in different budgetary regimes.

Carboni and Medda [8] considered two categories of public capital and analysed the model of capital accumulation with optimizing agents to identify optimal government size and composition of public expenditures which maximize the rate of growth in the dynamics to the steady state and the long-run level of per capita income.

In this paper we consider the model of economic growth where the public physical capital is a subject of congestion. It means that the amount of the publicly provided capital available to producers depends on an amount of private capital. We study the dynamics of the optimal consumption of households in this model and determine the value of tax rate for the maximum consumption.

2 The growth model with congestion

One of possible ways to extend the neoclassical Solow model of economic growth [9] is to add public capital. Bajo-Rubio [4] considered additional forms of capital, such as human capital Z, public physical capital G, and transfer payments T, provided by government and obtained the following production function

$$Y = K^{\alpha} Z_1^{\beta_1} \cdots Z_m^{\beta_m} (AL)^{1-\alpha-\beta_1-\cdots-\beta_m} \left(\frac{G}{K}\right)^{\gamma} \left(\frac{T}{K}\right)^{\theta} \tag{3}$$

where $\alpha > \gamma + \theta$, and Y denotes output, K is private physical capital, Z_i are other inputs such as, e.g., human capital, L is labour, and A is a labour-augmenting factor. The most interesting feature of this model is the congestion of the public capital. Public capital is necessary in production but for a given level of public capital, the increase of private capital decreases the quantity of public capital available to every producer.

In this paper we use this approach and study the dynamics of optimal consumption. We simplify the form of the production function and neglect the human capital and government transfers. Now the production function has the form

$$Y = K^{\alpha}(AL)^{1-\alpha} \left(\frac{G}{K}\right)^{\beta}, \qquad 0 < \beta < \alpha < 1. \tag{4}$$

We assume that labour L and knowledge A are exogenous factors which grow in constant rates n and a, respectively. The accumulation of private physical capital depends on saving of households and the accumulation of public physical capital depends on the tax revenue of government. If we use variables per units of effective labour denoted by a small letter ($k = K/AL$ and $g = G/AL$) and assume the tax rate τ then the capital accumulation equations have the form

$$\dot{k} = (1-\tau)k^{\alpha} \left(\frac{g}{k}\right)^{\beta} - c - (a+n)k \tag{5}$$

$$\dot{g} = \tau k^{\alpha} \left(\frac{g}{k}\right)^{\beta} - (a+n)g. \tag{6}$$

We assume that the households choose their consumption c over infinite time to maximize their utility function $u(c(t))$ subject to dynamic constraints. It means that they face the following optimization problem

$$\max u(0) = \int_0^{\infty} e^{-\rho t} u(c(t))dt \tag{7}$$

$$\text{subject to} \quad \dot{k} = (1-\tau)k^{\alpha} \left(\frac{g}{k}\right)^{\beta} - c - (a+n)k$$

$$\dot{g} = \tau k^{\alpha} \left(\frac{g}{k}\right)^{\beta} - (a+n)g$$

where the parameter ρ is the positive rate of time preference of households.

If we assume the constant elasticity of substitution utility function in the form

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma} \tag{8}$$

and use the Pontryagin maximum principle we derive the differential equation for consumption c

$$\dot{c} = \frac{1}{\sigma}\left[(1-\tau)(\alpha-\beta)k^{\alpha-1}\left(\frac{g}{k}\right)^{\beta} - (a+n) - \rho\right]c \tag{9}$$

where $-\sigma$ is the constant elasticity of marginal utility.

The equations for the accumulation of capital and for dynamics of consumption form the three-dimensional dynamical system

$$\frac{\dot{k}}{k} = (1-\tau)k^{\alpha-1}\left(\frac{g}{k}\right)^{\beta} - (a+n) - \frac{c}{k} \tag{10}$$

$$\frac{\dot{g}}{g} = \tau k^{\alpha-1}\left(\frac{g}{k}\right)^{\beta-1} - (a+n) \tag{11}$$

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}\left[(1-\tau)(\alpha-\beta)k^{\alpha-1}\left(\frac{g}{k}\right)^{\beta} - (a+n) - \rho\right]. \tag{12}$$

Let find the critical points of this system. First, we calculate

$$\frac{g^k}{k^*} = \frac{\tau}{1-\tau}\frac{a+n+\rho}{(\alpha-\beta)(a+n)}, \tag{13}$$

$$\frac{c^*}{k^*} = \frac{(1-\alpha+\beta)(a+n)-\rho}{\alpha-\beta}. \tag{14}$$

Hence

$$k^* = \left[\frac{a+n+\rho}{(1-\tau)(\alpha-\beta)}\right]^{\frac{1}{\alpha-1}}\left[\frac{1-\tau}{\tau}\frac{(\alpha-\beta)(a+n)}{a+n+\rho}\right]^{\frac{\beta}{\alpha-1}} \tag{15}$$

$$g^* = \frac{\tau}{1-\tau}\frac{a+n+\rho}{(\alpha-\beta)(a+n)}k^* \tag{16}$$

$$c^* = \frac{(1-\alpha+\beta)(a+n)+\rho}{\alpha-\beta}k^*. \tag{17}$$

The critical point (15)-(17) is the only critical point which lies inside the positive octant $k>0$, $g>0$, $c>0$. The condition of the existence of this critical point is

$$\alpha - \beta > 0. \tag{18}$$

Therefore, the share of private capital in income is required to be greater than the share of public capital in income.

We will analyse the dynamics of the model (10)-(12) using the mathematical methods of dynamical systems [10]. Both analytical and numerical studies will be performed. We will analyse the critical points and their stability. First we study the dependence of model dynamics on one of model parameters, the tax rate τ.

The consumption on the steady-state path depends on the tax rate

$$c^* = \text{const}\cdot\left(\frac{1}{1-\tau}\right)^{\frac{1}{1-\alpha}}\left(\frac{1-\tau}{\tau}\right)^{\frac{\beta}{\alpha-1}}. \tag{19}$$

Hence, we can obtain the tax rate which maximizes the consumption. We get that

$$\tau = \beta. \tag{20}$$

We have that on the steady-state path the optimal tax rate should be equal to the public capital share in income.

Now, let us analyse the stability of the critical point (15)-(17). The linearization matrix A in this critical point is

$$A = \begin{bmatrix} \rho & \frac{(1-\tau)\beta a}{\tau} & -1 \\ \frac{\tau}{(1-\tau)(d+\rho)} & (\beta-1)\tau & 0 \\ -\frac{(1-\alpha+\beta)(d+\rho)[(1-\alpha+\beta)d+\rho]}{\sigma(\alpha-\beta)} & \frac{1-\tau}{\sigma\tau}[(1-\alpha+\beta)d+\rho]\beta d & 0 \end{bmatrix} \tag{21}$$

where for the notation shortening we put

$$d = a + n. \tag{22}$$

The characteristic equation for the linearization matrix A is

$$\lambda^3 + \operatorname{tr}(A)\lambda^2 + [\operatorname{tr}(A)^2 - \operatorname{tr} A^2]\lambda + \det A = 0. \tag{23}$$

where

$$\operatorname{tr} A = \rho + (\beta - 1)\tau \tag{24}$$

$$\operatorname{tr} A^2 = \rho^2 + (\beta-1)^2\tau^2 + \frac{2\beta a}{d+\rho} + \frac{2(1-\alpha+\beta)d+\rho}{\sigma(\alpha-\beta)} \tag{25}$$

$$\det A = -\det \begin{bmatrix} \frac{\tau}{(1-\tau)(d+\rho)} & (\beta-1)\tau \\ -\frac{(1-\alpha+\beta)(d+\rho)[(1-\alpha+\beta)d+\rho]}{\sigma(\alpha-\beta)} & \frac{1-\tau}{\sigma\tau}[(1-\alpha+\beta)d+\rho]\beta d \end{bmatrix} \tag{26}$$

One can check that at the critical point we have a two-dimensional unstable subspace spanned on eigenvectors for two positive eigenvalues and a one-dimensional stable subspace corresponding to a negative eigenvalue. The solution of the economic problem is therefore the incoming separatrix to the saddle critical point situated at (15)-(17). While in this critical the state variables grow as $K(t) = k_0 e^{(a+n)t}$, $G(t) = g_0 e^{(a+n)t}$ and $C(t) = c_0 e^{(a+n)t}$, the rates of growth of these state variables are constant and equal $a+n$. The initial value of consumption c is chosen on the separatrix and then consumption c and physical capital k and public capital g change along this path and reach the saddle point as its long-term behaviour.

Let us introduce the projective variables

$$u = \frac{1}{k}, \quad v = \frac{g}{k}, \quad w = \frac{c}{k}. \tag{27}$$

In this variables, system (10)-(12) has the form

$$\dot{u} = -(1-\tau)u^{2-\alpha}v^{\beta} + uw + (a+n)u, \tag{28}$$

$$\dot{v} = \tau u^{1-\alpha}v^{\beta} - (1-\tau)u^{1-\alpha}v^{\beta+1} + vw, \tag{29}$$

$$\dot{w} = \left[\frac{1}{\sigma}(\alpha-\beta) - 1\right](1-\tau)u^{1-\alpha}v^{\beta}w + w^2 + \left(a - \frac{a+\rho}{\sigma}\right)w. \tag{30}$$

The system (28)-(30) possesses three invariant submanifolds which forms the boundary of the phase space $\mathcal{M}^3 = \{(u, v, w) \colon u > 0, v > 0, w > 0\}$. They has the following form

$$\{u = 0,\ v, w \in \mathcal{M}^3\} \qquad \{v = 0,\ u, w \in \mathcal{M}^3\} \qquad \{w = 0,\ u, v \in \mathcal{M}^3\} \tag{31}$$

The dynamical systems on the first invariant submanifolds

$$\dot{v} = vw, \tag{32}$$

$$\dot{w} = w^2 + \left(a - \frac{a+\rho}{\sigma}\right) w, \tag{33}$$

on the second invariant submanifold

$$\dot{u} = -(1-\tau)u^{2-\alpha}v^{\beta} + uw + Au, \tag{34}$$

$$\dot{w} = \left[\frac{1}{\sigma}(\alpha - \beta) - 1\right](1-\tau)u^{1-\alpha}v^{\beta}w + w^2 + \left(a - \frac{a}{\sigma} - \frac{\rho}{\sigma}\right) w, \tag{35}$$

on the third invariant submanifold

$$\dot{u} = -(1-\tau)u^{2-\alpha}v^{\beta} + Au, \tag{36}$$

$$\dot{v} = \tau u^{1-\alpha}v^{\beta}u^{1-\alpha}v^{\beta+1}. \tag{37}$$

3 Dynamical system in exponential coordinates

To better visualize the dynamics in the phase portraits, let us introduce a new exponential variables

$$k = e^K, \quad g = e^G, \quad c = e^C. \tag{38}$$

In this variables the system (10)-(12) assumes the form

$$\dot{K} = (1-\tau)e^{[(\alpha-\beta-1)K+\beta G]} - (a+n) - e^{C-K} \tag{39}$$

$$\dot{G} = \tau e^{[(\alpha-\beta)K+(\beta-1)G]} - (a+n) \tag{40}$$

$$\dot{C} = \frac{1}{\sigma}\left[(1-\tau)(\alpha-\beta)e^{[(\alpha-\beta-1)K+\beta+\beta G]} - (a+n) - \rho\right]. \tag{41}$$

The phase portraits of this system are presented in Fig. 1 and 2. The parameter values are chosen $\alpha = 0.3$, $\beta = 0.3$, $\rho = 0.1$, $\sigma = 2$, $a = 0.05$ and $n = 0.05$. In the phase portraits the position of the critical point is shown as well as some trajectories to visualize their behaviour in the neighbourhood of this saddle.

In Fig. 3 there are presented eigenvectors. As was mentioned in the previous section at the critical point we two-dimensional unstable subspace spanned on eigenvectors for two positive eigenvalues (blue arrows) and one-dimensional stable subspace corresponding to a negative eigenvalue (red arrow).

4 Conclusion

The aim of the paper was to analyze the model of economic growth with private physical capital and public physical capital. The specific feature of this model

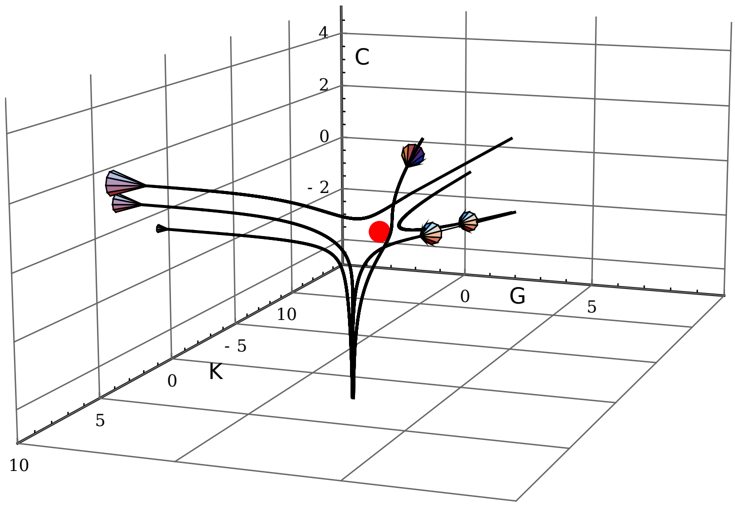

Figure 1: The phase portrait of system (39)-(41).

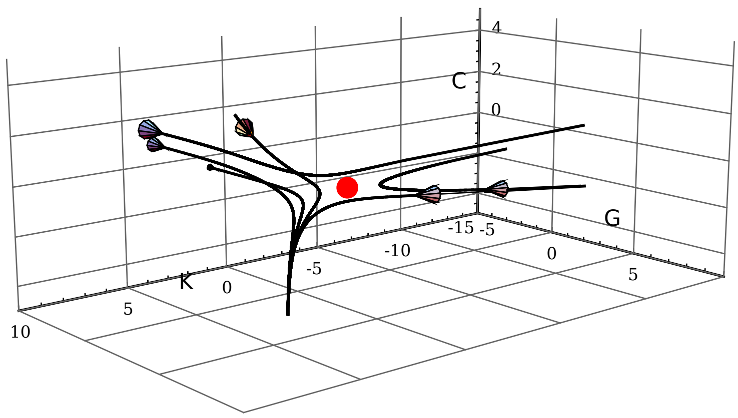

Figure 2: The phase portrait of system (39)-(41).

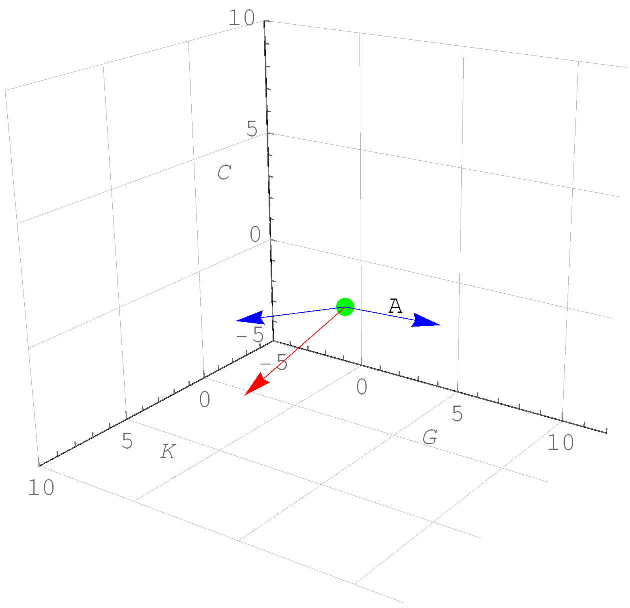

Figure 3: The critical point of system (39)-(41). The eigenvectors corresponding to eigenvalues calculated at this critical point are shown: the blue vectors (unstable subspace) and the red vector (stable subspace).

is the congestion of public capital. It means that for a given level of public capital, the increase of private capital decreases the quantity of public capital available to every producer.

The problem has the form ofthe 3-dimensional dynamical system, which possesses in the generic case the unique saddle point. The economic system moves on saddle trajectory towards to the saddle where the rates of growth of these state variables are identical and equal $(a + n)$. After reaching the saddle the long-term behaviour is expontial growth of private capital, public capital and consumption.

We found that on the steady-state path the optimal tax rate should be equal to the public capital share in income β.

We drew the phase portraits to show the localization of the critical point—the saddle. Both some trajectories going in the neighbourhood of the saddle and eigenvectors on which 2-dimensional unstable subspace and 1-dimensional stable subspace are spanned were shown to visualize the location of the saddle path in 3-dimensional phase space.

Acknowledgments

The authors acknowledge the support of the Narodowe Centrum Nauki (NCN) project 2014/15/B/HS4/04264. The authors thanks Franciszek Humieja for comments and remarks.

References

[1] K. J. Arrow, M. Kurz, Optimal public investment policy and controllability with fixed private savings ratio, Journal of Economic Theory, 1 (1969) 141–77.

[2] J. Agell, T. Lindh, H. Ohlsson, Growth and the public sector: a critical review essay, European Journal of Political Economy 13 (1997) 33–52.

[3] A. Greiner, Fiscal Policy and Economic Growth, Avebury, Aldershot, 1996.

[4] O. Bajo-Rubio, A further generalization of the Solow model: the role of the public sector, Economics Letters 68 (5) (2000) 79–84.

[5] W. H. Fisher, S. J. Turnovsky, Public investment, congestion, and private capital accumulation, Economic Journal, 108 (1998) 399–412.

[6] R. Barro, Government spending in a simple model of endogenous growth, Journal of Political Economy 98 (1990) S103–S125.

[7] R. Barro, X. Sala-i Martin, Public finance in models of economic growth, Review of Economic Studies 59 (1992) 645–661.

[8] O. A. Carboni and G. Medda, Size and composition of public spending in a neoclassical growth model, Metroeconomica 62 (2011) 150–170.

[9] R. M. Solow, A contribution to the theory of economic growth, Quarterly Journal of Economics 70 (1956) 65–94.

[10] H.-W. Lorenz, Nonlinear Dynamical Economics and Chaotic Motion, 2nd Edition, Springer-Verlag, Berlin, 1993. doi:10.1007/978-3-642-78324-1.