Intensity valence*

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Abstract

This paper studies a continuous one-dimensional spatial model of electoral competition with two office-motivated candidates differentiated by their "intensity" valence. All voters agree that one candidate will implement more intensively his announced policy than his opponent. However, and contrary to existing models, the intensity valence has a different impact on the utility of voters according to their position in the policy space. The assumption that voters have utility functions with intensity valence, an assumption which has been found to be grounded empirically, generates very different results than those obtained with traditional utility functions with additive valence. First, the candidate with low intensity valence is supported by voters whose ideal points are on both extremes of the policy space. Second, there exist pure strategy Nash equilibria in which the winner is the candidate with high intensity if the distribution of voters in the policy space is sufficiently homogeneous. On the contrary, if the distribution of voters in the policy space is very heterogeneous, there are pure strategy Nash equilibria in which the candidate with low intensity wins. For moderate heterogeneity of the distribution of voters, there is no pure strategy Nash equilibrium.

JEL Classification: D72.

Keywords: valence, voter's utility functions, Downsian model, spatial voting.

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1 Introduction

Spatial models of voting have dominated formal political theory since the seminal work of Downs (1957). This work, and the literature stemming from it, has considered that candidates adopt positions in a space of possible policies; and each elector votes on the basis of his "Downsian" utility function, which depends only on the distance between his ideal point in the policy space and the ones proposed by candidates. Following Stokes (1963), a recurrent criticism of the spatial model of elections has been the absence of "valence" issues in the analysis, i.e., candidates' characteristics which are independent of the platforms they propose and which are unanimously evaluated by voters (e.g., charisma, competence). Over the last decade, various authors have tried to understand the consequences of including valence issues in the spatial model (e.g., Aragones and Palfrey, 2002; Dix and Santore, 2002; Hummel, 2010). They usually incorporate valence in an additively separable form, meaning that the valence parameter adds the same amount of utility to all voters whatever their ideal point in the policy space. However, even if the term "valence" defines a characteristic of a candidate that is unanimously evaluated, it may not be unanimously desired.

In this article, we formally study the consequences of considering an alternative valence, the "intensity" valence, in a one-dimensional model of voting with two candidates. The intensity valence supposes that candidates differ in their ability or will to implement a policy, i.e., to turn campaign promises into policy. All voters agree that one candidate shows more potential in that respect than the other. But the key feature of the intensity valence is that it has a different impact on the utility of voters according to their position in the policy space. A voter with an ideal point close to a policy proposed by a candidate, i.e., a supporter of this policy, will have a higher utility the more intensively this policy is implemented. On the contrary, a voter with an ideal point far from the policy proposed by this candidate, i.e., an opponent to this policy, will have a higher disutility the more intensively this policy is implemented. For instance, if a right-wing policy is implemented, it makes sense to assume that a left-wing voter does not want this policy to be implemented intensively.

The intensity valence utility function was initially proposed by Gouret et al. (2011) who confronted, on an empirical basis, several ways of incorporating valence issues into the Downsian utility function. Their objective was to find a parsimonious extension of this utility function that is empirically founded and simple enough to be tractable at a theoretical level. Using a survey run by the *Société Française d'Etudes par Sondages* prior to the 2007 French presidential election, they tested four utility functions: (1) the basic Downsian utility function, (2) an additive valence utility function wherein a valence is added to the Downsian utility function, (3) a multiplicative valence utility function which introduces in the Downsian utility function an interaction between the candidate's valence and the distance, and lastly (4) the intensity valence utility function. They showed that all these utility functions imply different testable restrictions on a general utility function which includes free additive and multiplicative valence parameters at the same time. They found that the intensity valence utility function is the sole function which is not rejected by the data. However, they did not try to understand the implications of intensity valence utility functions in a strategic model of voting. The current paper fills this gap.

A theory with more empirically founded assumptions (i.e., the shape of the voters' utility function here) is appealing because it can generate not only better predictions of political phenomena but also theoretical insights. We believe that it is the case here. We consider a setting with two purely-office motivated candidates in which the distribution of voters over the one-dimensional policy space is public information. We first find that the set of voters who prefer the policy implemented by the candidate with higher intensity valence is a bounded interval, while the set of voters who prefer the candidate with a lower intensity valence is a non-convex set: the candidate with lower intensity valence is supported by voters whose ideal points are on both sides of the policy space. This result differs widely from Downsian and additive valence models which predict a split into two intervals.

Second, and contrary to the Downsian model or the additive valence model, preference heterogeneity among voters does matter. We are able to show that if the distribution of voters in the policy space is relatively homogeneous, then there are pure strategy Nash equilibria in which the candidate with the highest intensity valence wins the election. On the contrary, if the distribution of voters is too heterogeneous, then there are pure strategy Nash equilibria in which the candidate with the lowest intensity valence wins the election. This is in sharp contrast with models with additive valence, which always predict a positive relationship between valence and the probability of winning the election. When the median voter is public information (Ansolabehere and Snyder, 2000; Dix and Santore, 2002), there are usually pure strategy equilibria such that if the additive-valence-advantaged candidate chooses a moderate policy, he wins with certainty. When the median voter position is unknown (but candidates share a common subjective probability on it), Groseclose (2001, p.866) observes that a pure strategy equilibrium does not exist if candidates are motivated strictly by office. However, the additive-valence-advantaged candidate has a mixed strategy (with a distribution of policies closer to the expected median voter) which makes him more likely to win the election (Aragones and Palfrey, 2002; Hummel, 2010). Groseclose (2001) shows that if in addition to seeking office, candidates have policy preferences as in Wittman (1977), then a pure strategy equilibrium may exist such that the advantaged candidate is again more likely to win.

The rest of the paper proceeds as follows. Section 2 recalls additional literature on valence advantage and presents the intensity valence utility function. Section 3 presents the model while Section 4 presents the results. Finally, Section 5 makes concluding remarks about the broader implications of this work and the role of preference heterogeneity among voters for the efficient implementation of policy.

2 Related literature and the intensity valence utility

Additive valence. A vast literature has examined the role of valence in politics since the seminal paper of Stokes (1963); Evrenk (forth.) reviews the literature. The term "valence" is used to represent a non-policy attribute of a candidate to an election, i.e., an attribute unanimously evaluated by the electorate, which is independent of policy choices (charisma, rhetorical skills, competence, etc...). It is usually assumed that this valence is exogenous and observed by voters prior to an election, an assumption that we will follow in this paper.[1] This literature usually considers an additive valence (Ansolabehere and Snyder, 2000; Dix and Santore, 2002; Aragones and Palfrey, 2002; Aragonès and Xefteris, 2012; Hummel, 2010; Groseclose, 2001). That is, if the policy space is unidimensional, a_i is the ideal point (or ideal policy) of voter i in this policy space, x_j is the policy proposed by a given candidate j, and $\theta_j > 0$ is the valence associated to candidate j, then the utility function of voter i if candidate j is elected is:

$$u(a_i, x_j, \theta_j) = \theta_j - |x_j - a_i| \tag{1}$$

[1] Note that several papers have considered that the valence is endogenous and/or private information. Various papers consider that campaign expenditures or a costly effort from the part of a candidate may improve his valence, and the outcome of the election (e.g., Ashworth and Bueno de Mesquita, 2009; Carillo and Castanheria, 2008; Meirowitz, 2008; Prat, 2002). Bernhardt et al. (2011) introduce an exogenous valence in a repeated election model à la Duggan (2000). When a candidate is elected, he holds office and his valence is revealed to the electorate. At the end of a period, the office holder may retire according to an exogenous probability. If he does not and decides to run for re-election, voters know his valence; they do not know the valence of the challenger however, and higher valence incumbents are more likely to win re-election. In these models, the valence remains additive and increases the utility of all voters.

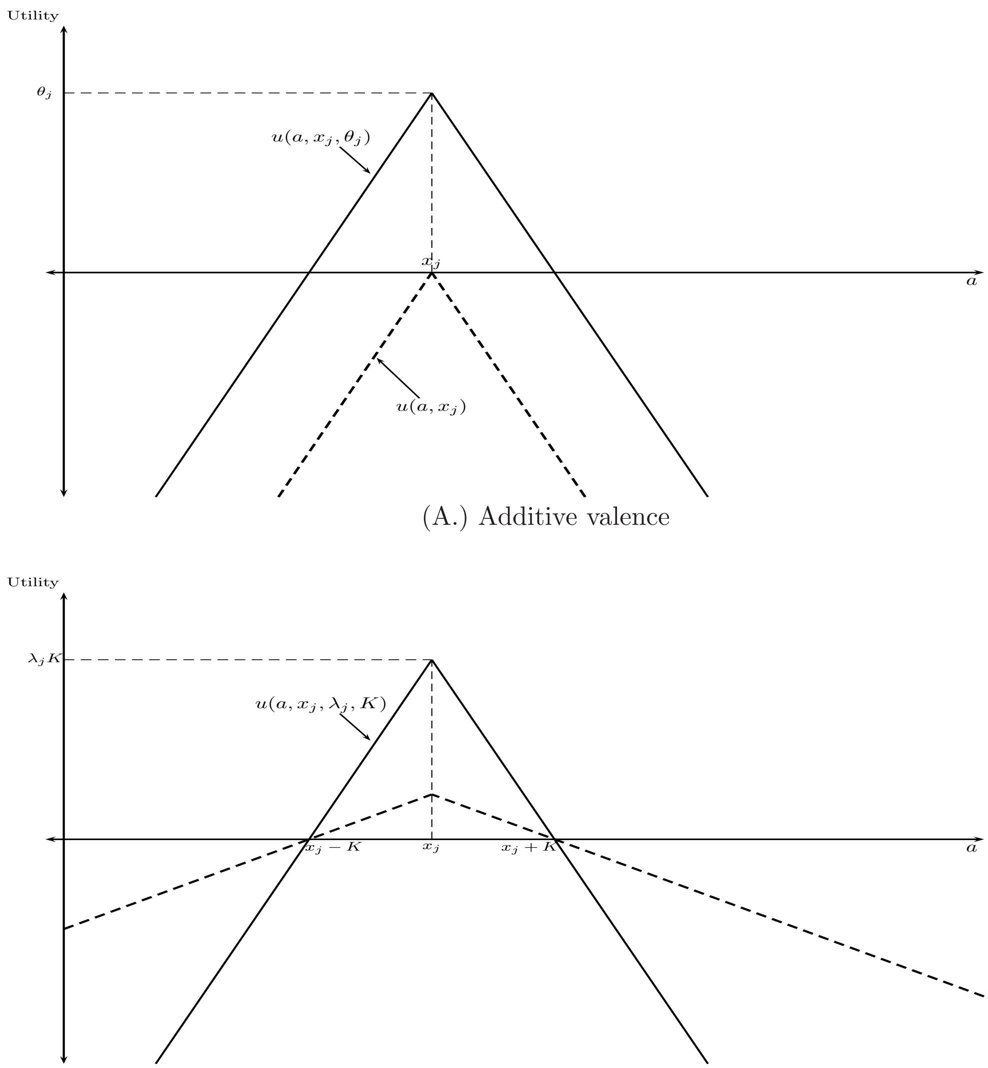
Compared to a simple Downsian utility function $u(a_i, x_j) = -|x_j - a_i|$, the additive valence utility function adds a constant θ_j which is candidate-specific. Note that instead of choosing an absolute loss function, it would have been possible to choose a quadratic loss function as in Dix and Santore (2002). It does not change however the main message: a higher valence θ_j implies a higher level of utility for all voters as shown in Panel (A.) of Figure 1. And the highest possible level of utility occurs for the voter i whose ideal point is $a_i = x_j$.

Intensity valence and its empirical foundation. Gouret et al. (2011) highlight that if the valence represents the ability or will of a candidate for implementing a policy, all voters may agree that one candidate will implement more intensively a policy than an opponent, but may be affected differently. The supporters of a candidate will be better off if their candidate is able to implement intensively his policy, while others may consider that it will decrease their utility even more if he is elected. In other words, the ability of a candidate to implement a policy is a bad thing for a voter who is too far from this policy. Gouret et al. (2011) call this valence the *intensity valence*.

More formally, if voter i's ideal policy a_i is close to the candidate j's platform x_j, i.e., $|x_j - a_i| < K$, and candidate j is elected, then the higher the intensity valence $\lambda_j > 0$ of candidate j, the higher the utility of voter i. However, if a_i is too far from x_j, i.e., $|x_j - a_i| > K$, then the higher the intensity valence λ_j, the lower the utility of voter i. Here λ_j is the intensity valence index, and K measures the size of the set of voters who will have an increase of their utility if the policy x_j is implemented. The intensity valence utility function takes the following form:

$$u(a_i, x_j, \lambda_j, K) = \lambda_j(K - |x_j - a_i|) \tag{2}$$

Panel (B.) in Figure 1 depicts the effect of a variation of the intensity valence index:



(A.) Additive valence

(B.) Intensity valence

Figure 1: Additive and intensity valence

an increase of λ_j can either increase or decrease the utility, depending on the sign of $(K - |x_j - a_i|)$.

As noticed in the Introduction, using data from a French 2007 pre-electoral survey, Gouret et al. (2011) have found that this utility function is not rejected by the data, contrary to the Downsian and additive valence utility functions. The objective of our paper is thus to learn the implications of this empirically founded intensity valence utility function in a one-dimensional model of voting with two candidates. Note that Gouret et al. (2011, p.325) consider various robustness checks. In particular, they consider utility functions which are not linear in the distance $|a_i - x_j|$, i.e., they consider that the distance is $|a_i - x_j|^\gamma$, where γ is an exponent parameter. The estimated coefficient $\widehat{\gamma}$ is not significantly different from one in all their specifications. That is why we will consider an absolute loss function for the distance as in Equation (2).

Note that contrary to the traditional additive valence utility function, the intensity valence utility function is not additively separable in distance and valence. Groseclose (2001, Appendix B, p.882) also notes that the separability could be violated if candidate j's valence represents candidate j's "competency for implementing the policy position that he or she announces". He highlights that "it is reasonable to believe that the voter appreciates a candidate's competency more when the candidate has adopted a policy that he or she likes".

Although they do not provide an empirical foundation for them, Kartik and McAfee (2007) and Miller (2011) have considered some utility functions that share some feature with the intensity valence one. Kartik and McAfee (2007) investigate the effect of "character". A candidate with character is formally nonstrategic because he suffers disutility from proposing a policy platform which is not his ideology. If such a character, unobservable to voters, is also similar to an additive valence in most of their article, they highlight at the end of it (Subsection IV.C., p.863) that a preference weight on character may depend on both the platform and a voter ideal point. Their argument, quite similar to ours, is that "a voter with ideal point [$a_i = 1$] may prefer a candidate with platform [$x_j = 0$] *not* to have character [...] [T]he same voter may prefer a candidate with [$x_j = 1$] to in fact have character, thus guaranteeing that he will take the same policy position on the unobservable dimension." Nevertheless, their model remains different from ours because the intensity valence is observable. Furthermore, in their model, only the candidate without character is purely office motivated and locates strategically. As a result, the candidate without character is more likely to win. In our model, the two candidates will locate strategically, and the most intensive candidate has some strategies which insure him to win for sure if the distribution of voters in the policy space is relatively homogenous.

Miller (2011) combines an additive valence with the "effectiveness" of candidate j (p_j), which captures his likelihood of changing policy from an exogenous status quo (S). Such expected utility function ($u(a_i, x_j, p_j, S, \theta_j) = -p_j|x_j - a_i| - (1 - p_j)|a_i - S| + \theta_j$) differs from the intensity valence utility function which does not incorporate statu quo. Note also that effectiveness and the additive valence are independent in Miller, so the additive-valence-advantaged candidate may differ from the high-effectiveness candidate. On the contrary, in our valence utility function, the additive term $\lambda_j K$ and the slope λ_j are not independent, so a candidate cannot have at the same time a better additive term and a lower slope (in absolute value) than the other candidate. This is the reason why the homogeneity/heterogeneity of the distribution of voters in the policy space will be important for determining the policies proposed by the candidates in our model. On the contrary, preference heterogeneity among voters does not matter in Miller; it is the distance between the statu quo and the median voter which determines the policy platforms of the candidates. However, the expected utility function of Miller may generate situations in which the set of voters who prefer the most effective candidate is a bounded interval, like the set of voters who prefer the candidate with higher intensity valence in our model.[2]

[2]More precisely, Miller (2011, pp.60-61, Proposition 3-Part 5, and Figure 3) shows that, in his model, if the less effective candidate is additive-valence-advantaged (i.e., $p_1 > p_2$ and $\theta_2 > \theta_1$), the set of voters who prefer the most effective candidate (candidate 1) may be a bounded interval, and those on the extremes of the policy space may prefer the less effective candidate (candidate 2); in such a case, the candidates ties. Note that without any additive valence advantage but only effectiveness and statu quo, Miller (2001, p.59, Proposition 2-Part 2, and Figure 1) shows that the set of voters who prefer the most effective candidate may be a bounded interval, but the other voters are indifferent; in such a case, the strategy of the most effective candidate permits to win for sure, given that the voters who strictly prefer this candidate vote for him, while those who are indifferent randomize.

Additional empirical evidence for the intensity valence. One can naturally ask if there are other empirical evidences for the intensity valence utility function. To our knowledge, Gouret et al. (2011) is the sole paper whose aim is to determine the best way to model valence, i.e., how to introduce the valence parameter in a utility function. However, various empirical papers have considered some proxies for valence advantage. Office-holding has been a standard (Ansolabehere et al., 2001; Burden, 2004; Feld and Grofman, 1991). Recently, Stone and Simas (2010) have noted that incumbency can reflect campaign skills (e.g., fundraising ability) or qualities that voters value for their own sake (e.g., competence, integrity). Using district expert informants in the 2006 U.S. House elections, they first find that incumbents with higher qualities are closer to the average district preferences while disadvantaged challengers diverge. Second, challengers obtain a greater share of the vote and higher probability to win when they diverge.

This last result contradicts theoretical models with additive valence. If these models predict that the valence-advantaged candidate chooses a moderate policy, they also predict that this candidate wins the election when the median is public information (Ansolabehere and Snyder, 2000; Dix and Santore, 2002), or are more likely to win the election when the candidates do not know the median location (Aragones and Palfrey, 2002; Hummel, 2010; Groseclose, 2001). On the contrary, and as already noticed, our model with intensity valence utility functions predicts that the disadvantaged candidate may win when the distribution of preferences among voters is too heterogeneous. Although heterogeneity in voters' preferences does not appear in the estimates of Stone and Simas (2010, p.380), the fact that an a priori disadvantaged candidate may win fits with our model. We are also able to show that there is an interval of policies (which includes the median voter) which are dominant strategies for the winning candidate. Although our model does not say what policy platform the winning candidate will choose in this interval, it is possible that he chooses one policy platform which diverges from the median. When the winning candidate is the one with high intensity valence, this result is compatible with Burden (2004) who finds empirically that a candidate with a valence advantage (proxied by incumbency) is freed to adopt a position closer to his view. When the winning candidate is the one with low intensity valence, this result is compatible with the finding of Stone and Simas (2010, p.380). However, our model presented in the next sections is compatible with moderate extremism, i.e., "the mildly but not extremely divergent policy platforms that appear, empirically, to be characteristic of two-party and multiparty competition" (Merrill and Grofman, 1999, p.4).

3 The model

There is an election between two candidates indexed by $j = 1, 2$. Each Candidate j chooses a policy platform x_j in the policy space $\mathbb{R}$. Each voter i has an ideal policy $a_i \in \mathbb{R}$. The utility function of voter i if Candidate j is elected is given by:

$$u(a_i, x_j, \lambda_j, K) = \lambda_j(K - |x_j - a_i|) \tag{3}$$

We will consider that Candidate 1 has a higher intensity valence than Candidate 2, i.e., Candidate 1 has more ability to implement his policy platform: $\lambda_1 > \lambda_2$. Without loss of generality, we normalize $\lambda_2 = 1$ to simplify.

Let $\Omega_1(x_1, x_2)$ be the set of voters who strictly prefer Candidate 1 to Candidate 2, $\Omega_2(x_1, x_2)$ the set of voters who strictly prefer Candidate 2 to Candidate 1, and $I(x_1, x_2)$

the set of voters who are indifferent between the two candidates. We thus have:

$$\Omega_1(x_1,x_2) = \{a \in \mathbb{R}; \lambda_1(K - |x_1 - a|) > K - |x_2 - a|\} \tag{4}$$

$$\Omega_2(x_1,x_2) = \{a \in \mathbb{R}; \lambda_1(K - |x_1 - a|) < K - |x_2 - a|\} \tag{5}$$

$$I(x_1,x_2) = \{a \in \mathbb{R}; \lambda_1(K - |x_1 - a|) = K - |x_2 - a|\} \tag{6}$$

Voters are distributed on $\mathbb{R}$ according to their ideal policy. We assume that the distribution of voters has a probability density function f which is an even function on $\mathbb{R}$, strictly increasing on $\mathbb{R}_-$ and strictly decreasing on $\mathbb{R}_+$. The corresponding distribution admits a second order moment. The fraction of voters who strictly prefer Candidate 1 is denoted $S_1(x_1,x_2) = \int_{\Omega_1} f(a)da$; the fraction of voters who strictly prefer Candidate 2 is $S_2(x_1,x_2) = \int_{\Omega_2} f(a)da$. If a probability distribution admits a density, then the probability of every one-point set is zero. The set I in (6) is then negligible; hence, $S_2(x_1,x_2) = 1 - S_1(x_1,x_2)$.

A candidate's goal is solely to win office as it is often assumed in the literature (e.g., Ansolabehere and Snyder, 2000; Aragones and Palfrey, 2002; Hummel, 2010). Thus, a candidate obtains a payoff equal to his probability of winning the election. We denote by $\pi_j(x_1,x_2)$ the probability that Candidate j wins the election. We have:

$$\pi_1(x_1,x_2) = \begin{cases} 1 & \text{if} \quad S_1(x_1,x_2) > \frac{1}{2} \\ \frac{1}{2} & \text{if} \quad S_1(x_1,x_2) = \frac{1}{2} \\ 0 & \text{if} \quad S_1(x_1,x_2) < \frac{1}{2} \end{cases} \tag{7}$$

and $\pi_2(x_1,x_2) = 1 - \pi_1(x_1,x_2)$.

The main elements of the game are the triple (λ_1, K, f). The game proceeds as follows. Both Candidates 1 and 2 simultaneously choose their policy x_1 and x_2 from the policy space $\mathbb{R}$. Then, each voter i observes these policy choices and vote for whichever candidate affords him a higher utility. We present the results in Section 4, solving the game backwards.

4 Results

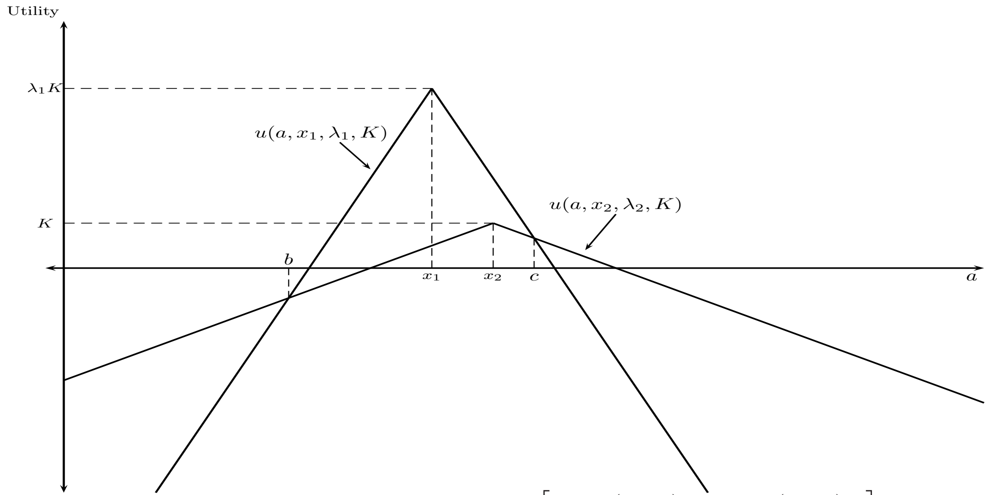
Subsection 4.1 provides some properties of the sets $\Omega_1(x_1, x_2)$ and $\Omega_2(x_1, x_2)$. We then study in Subsection 4.2 the existence of political equilibria, i.e., pure strategy Nash equilibria in the game played by the two candidates.

4.1 The sets $\Omega_1(x_1, x_2)$ and $\Omega_2(x_1, x_2)$

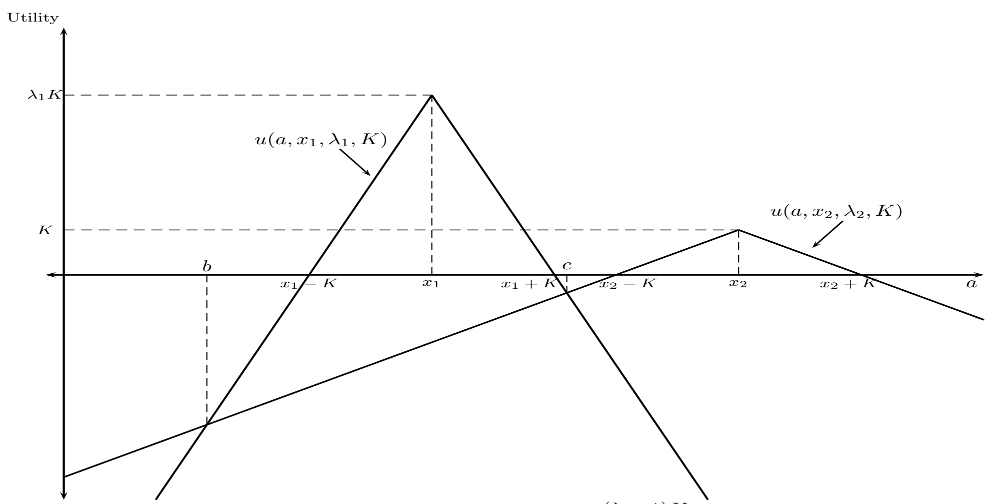
Given that Candidate 2 has a lower valence ($\lambda_2 = 1 < \lambda_1$), a voter whose ideal policy is $a_i = x_2$ does not always vote for Candidate 2. Indeed, if $a_i = x_2$, we obtain from Equation (4) that Candidate 1 can offer to this voter a higher level of utility if $\lambda_1(K - |x_1 - x_2|) > K$, i.e., if $x_1 - \frac{(\lambda_1-1)K}{\lambda_1} < x_2 < x_1 + \frac{(\lambda_1-1)K}{\lambda_1}$. Thus, we have to distinguish three cases. In Case (A.), described in Panel (A.) of Figure 2, Candidate 2 proposes a policy x_2 relatively close to x_1, the one proposed by Candidate 1, i.e., $x_2 \in \left[x_1 - \frac{(\lambda_1-1)K}{\lambda_1}, x_1 + \frac{(\lambda_1-1)K}{\lambda_1}\right]$. This case corresponds to the situation in which a voter with ideal policy $a_i = x_2$ prefers Candidate 1 to Candidate 2.[3] In Case (B.), described in Panel (B.) of Figure 2, Candidate 2 proposes a policy x_2 which is far to the right of x_1, i.e., $x_2 > x_1 + \frac{(\lambda_1-1)K}{\lambda_1}$. In this case, a voter whose ideal policy is $a_i = x_2$ strictly prefers Candidate 2 to Candidate 1. The last case, Case (B'.), described in Panel (B'.) of Figure 2, is symmetric to Case (B.): Candidate 2 proposes a policy x_2 which is far to the left of x_1, i.e., $x_2 < x_1 - \frac{(\lambda_1-1)K}{\lambda_1}$, and, again, a voter whose ideal policy is $a_i = x_2$ strictly prefers Candidate 2 to Candidate 1.

One can easily see on the different panels of Figure 2 that the set of voters who strictly prefer Candidate 1 is a bounded open interval (b, c), while the set of voters who strictly

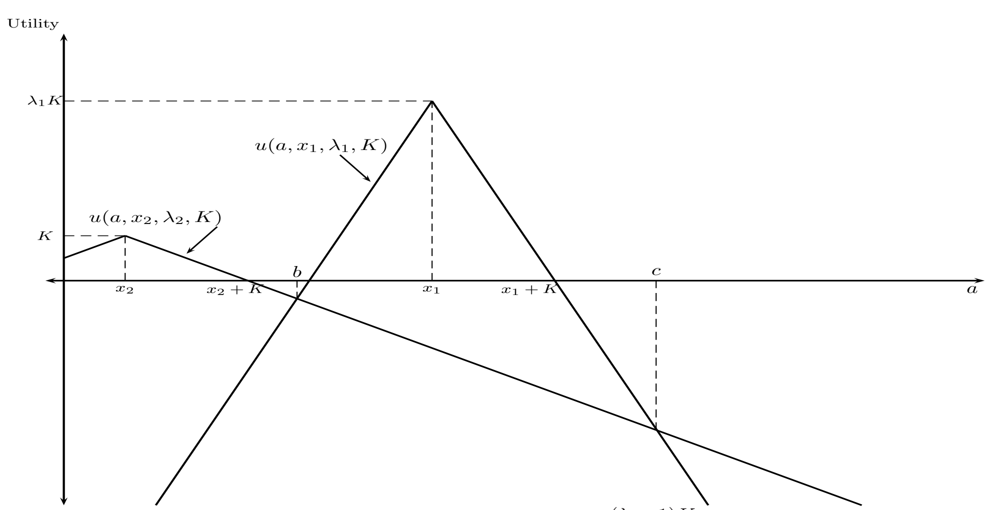
[3]Remark: to fully understand when $x_2 = x_1 + \frac{(\lambda_1-1)K}{\lambda_1}$ and when $x_2 = x_1 - \frac{(\lambda_1-1)K}{\lambda_1}$, Panels (C.) and (D.) of Figure 2 depict these two situations.



(A.) $x_2 \in \left[x_1 - \frac{(\lambda_1-1)K}{\lambda_1}, x_1 + \frac{(\lambda_1-1)K}{\lambda_1}\right]$

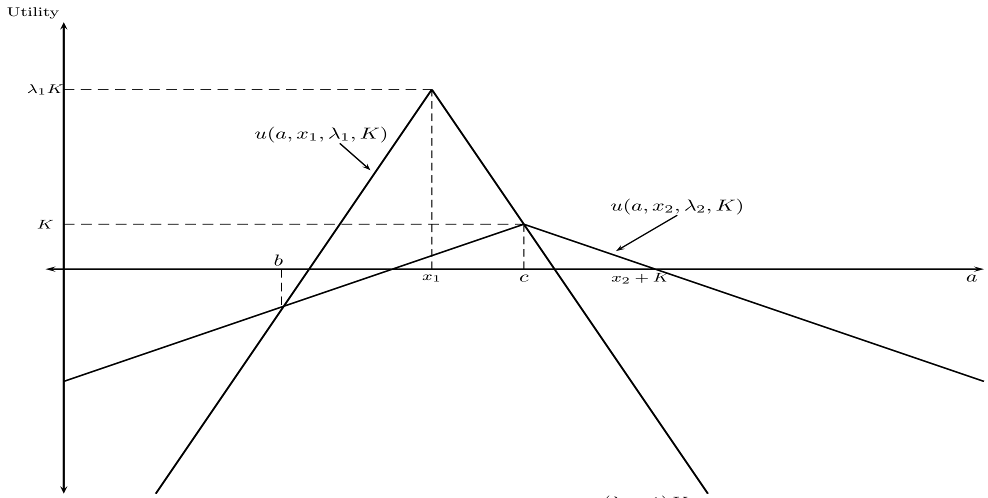


(B.) $x_2 > x_1 + \frac{(\lambda_1-1)K}{\lambda_1}$

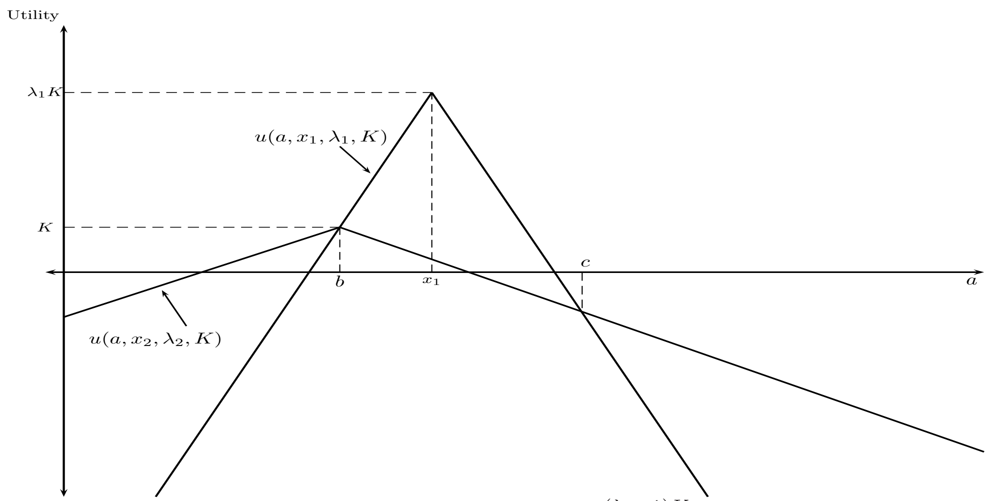


(B'.) $x_2 < x_1 - \frac{(\lambda_1-1)K}{\lambda_1}$

Figure 2: Intensity valence (*concluded on next page*)



(C.) $x_2 = x_1 + \frac{(\lambda_1 - 1)K}{\lambda_1} (= c)$



(D.) $x_2 = x_1 - \frac{(\lambda_1 - 1)K}{\lambda_1} (= b)$

Figure 2: Intensity valence (*continued from previous page*)

prefer Candidate 2 is a non-convex set. Proposition 1 gives the precise form of the sets $\Omega_1(x_1, x_2)$ and $\Omega_2(x_1, x_2)$ in the different cases.

Proposition 1. $\Omega_1(x_1, x_2) = (b, c)$ and $\Omega_2(x_1, x_2) = (-\infty, b) \cup (c, +\infty)$ where

$$\begin{cases} \text{Case (A.):} & b = \frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} - K \text{ and } c = \frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} + K \text{ if } x_2 \in \left[x_1 - \frac{(\lambda_1 - 1)K}{\lambda_1}, x_1 + \frac{(\lambda_1 - 1)K}{\lambda_1}\right] \\ \text{Case (B.):} & b = \frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} - K \text{ and } c = \frac{(\lambda_1 - 1)K + \lambda_1 x_1 + x_2}{\lambda_1 + 1} \text{ if } x_2 > x_1 + \frac{(\lambda_1 - 1)K}{\lambda_1} \\ \text{Case (B'.):} & b = \frac{(1 - \lambda_1)K + \lambda_1 x_1 + x_2}{\lambda_1 + 1} \text{ and } c = \frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} + K \text{ if } x_2 < x_1 - \frac{(\lambda_1 - 1)K}{\lambda_1} \end{cases}$$

Proof: see Appendix A.1.

Proposition 1 gives the location of the supporters of Candidate 1 and Candidate 2 in the policy space, for x_1 and x_2 given. The next section deals with the choice of the policy platforms x_1 and x_2 by the candidates, and particularly the existence of political equilibria.

4.2 Political equilibria

We can now deal with political equilibria. A political equilibrium is a pure strategy Nash equilibrium in the game played by the two candidates.

Definition 1. A political equilibrium is a policy pair (x_1^*, x_2^*) such that these two conditions are met: (i.) $\forall x_1 \in \mathbb{R}$, $\pi_1(x_1^*, x_2^*) \geq \pi_1(x_1, x_2^*)$, and (ii.) $\forall x_2 \in \mathbb{R}$, $\pi_2(x_1^*, x_2^*) \geq \pi_2(x_1^*, x_2)$.

We will see in Proposition 2 that there is a political equilibrium here only when a candidate has a dominant strategy which insures him to win against any strategy of his opponent.

Since f is an even function, $\int af(a)da = 0$. Hence, the variance is $\sigma^2 = \int a^2 f(a)da > 0$. Denote by f_1 the standardized distribution, i.e., $\int a^2 f_1(a)da = 1$. So $f = f_\sigma$, where f_σ is defined by $f_\sigma(a) = \frac{1}{\sigma} f_1\left(\frac{a}{\sigma}\right)$.

Now remark the following intermediate results described in Lemmata 1 and 2.

Lemma 1.

$$0 < \int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da = \int_{-\frac{(\lambda_1+1)K}{\lambda_1}}^{\frac{(\lambda_1-1)K}{\lambda_1}} f_\sigma(a)da < \int_{-K}^{K} f_\sigma(a)da$$

Proof: see Appendix A.2.

Lemma 2. There exist σ^* and σ^{**}, with $0 < \sigma^* < \sigma^{**}$, such that:

(i.) If $\sigma < \sigma^*$, then $\frac{1}{2} < \int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da < \int_{-K}^{K} f_\sigma(a)da$.

(ii.) If $\sigma^{**} < \sigma$, then $\int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da < \int_{-K}^{K} f_\sigma(a)da < \frac{1}{2}$.

(iii.) If $\sigma^* \leq \sigma \leq \sigma^{**}$, then $\int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da \leq \frac{1}{2} \leq \int_{-K}^{K} f_\sigma(a)da$.

Proof: see Appendix A.3.

Using Lemmata 1 and 2, we then obtain Proposition 2, which gives conditions on σ to have at least one political equilibrium.

Proposition 2. (i.) If $\sigma < \sigma^*$, then, $\forall x_2 \in \mathbb{R}$, Candidate 1 wins with certainty if he chooses $x_1^* = 0$.

(ii.) If $\sigma^{**} < \sigma$, then, $\forall x_1 \in \mathbb{R}$, Candidate 2 wins with certainty if he chooses $x_2^* = 0$.

(iii.) If $\sigma^* \leq \sigma \leq \sigma^{**}$, then there is no political equilibrium.

Proof: see Appendix A.4.

Part (i.) of Proposition 2 states that if the distribution of voters' preferences is sufficiently homogeneous in the policy space, i.e., when $\sigma < \sigma^*$, then choosing as a policy the median ideal policy $x_1^* = 0$ is a dominant strategy for Candidate 1 to be elected for sure. As shown in Appendix A.4, the reason is that $S_1(0, x_2)$ has a minimum at $x_2 = \pm\frac{(\lambda_1-1)K}{\lambda_1}$, and $S_1\left(0, -\frac{(\lambda_1-1)K}{\lambda_1}\right) = \int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da$. Since $\sigma < \sigma^*$, we know from Lemma 2 that $S_1\left(0, -\frac{(\lambda_1-1)K}{\lambda_1}\right) > \frac{1}{2}$. Thus, $S_1(0, x_2) > \frac{1}{2}$, $\forall x_2 \in \mathbb{R}$.

On the contrary, Part (ii.) of Proposition 2 says that if the distribution is too heterogeneous, i.e., when $\sigma > \sigma^{**}$, then choosing as a policy the median ideal policy $x_2^* = 0$ is a dominant strategy for Candidate 2 to be elected for sure. As shown in Appendix A.4, the reason is that if $\sigma > \sigma^{**}$, $S_1(x_1, 0) < \frac{1}{2}$, $\forall x_1 \in \mathbb{R}$.

Lastly, Part (iii.) of Proposition 2 says that for intermediate level of heterogeneity, i.e., when $\sigma \in [\sigma^*, \sigma^{**}]$, there is no political equilibrium. As shown in Appendix A.4, the reason is that for any policy pair $(\overline{x}_1, \overline{x}_2) \in \mathbb{R}^2$, each candidate can deviate unilaterally and win; hence, $(\overline{x}_1, \overline{x}_2)$ cannot be a political equilibrium.

When $\sigma < \sigma^*$ or when $\sigma > \sigma^{**}$ choosing the median ideal policy is a dominant strategy which insures one candidate to win, but it is possible that there are other strategies that also permit to do so. For instance, it is possible that when the distribution is sufficiently homogeneous, choosing a policy which is slightly different from the median voter's ideal point is still a dominant strategy which insures Candidate 1 to win. We thus now consider the set of strategies that Candidate j can play in order to win with certainty.

Definition 2. Let X_j^* be the set of strategies that Candidate j can play to be elected with certainty:

$$X_j^* = \{x_j^* \in \mathbb{R};\ \pi_j(x_j^*, x_{-j}) = 1\ ,\ \forall x_{-j} \in \mathbb{R}\}$$

Proposition 3 characterizes the sets X_1^* and X_2^*.

Proposition 3. (i.) If $\sigma < \sigma^*$, then $X_1^* = (-\alpha, \alpha)$ where α is the unique positive real number which satisfies:

$$\int_{\alpha+\frac{(1-\lambda_1)K}{\lambda_1}}^{\alpha+\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da = \frac{1}{2}$$

Moreover, $\alpha \in \left(0, \frac{(\lambda_1-1)K}{\lambda_1}\right)$, and $\sigma \mapsto \alpha(\sigma)$ is a decreasing function on $(0, \sigma^*]$, with $\alpha(\sigma^*) = 0$ and $\lim_{\sigma\to 0^+} \alpha(\sigma) = \frac{(\lambda_1-1)K}{\lambda_1}$.

(ii.) If $\sigma^{**} < \sigma$, then $X_2^* = (-\beta, \beta)$ where β is the unique positive real number which satisfies:

$$\sup_{x_1 \leq \beta - \frac{(\lambda_1-1)K}{\lambda_1}} S_1(x_1, \beta) = \frac{1}{2}$$

.

Proof: see Appendix A.5.

Part (i.) of Proposition 3 highlights that when the distribution of voters is sufficiently homogeneous in the policy space, i.e., when $\sigma < \sigma^*$, there is an interval of moderate policies $X_1^* = (-\alpha, \alpha)$ which allow Candidate 1 to win. This interval tends to $\left(\frac{(1-\lambda_1)K}{\lambda_1}, \frac{(\lambda_1-1)K}{\lambda_1}\right)$ when $\sigma \to 0^+$. When σ increases, the lower bound increases while the upper bound decreases; and this interval tends to the singleton $\{0\}$ when $\sigma \to \sigma^{*^-}$. Part (ii.) of Proposition 3 highlights that when the distribution of voters is sufficiently heterogeneous, i.e., when $\sigma > \sigma^{**}$, there is an interval of moderate policies $X_2^* = (-\beta, \beta)$ which allow Candidate 2 to win.

In these two cases, the interval always includes the median voter's ideal point. Although we do not know what is the policy that the winning candidate will choose in his range of optimal policies, this result can explain the moderate divergence from the median voter's ideal policy which is in practice a salient characteristic of many elections (e.g., Merrill and Grofman, 1999; Ansolabehere et al. 2001).

Finally, it is interesting to note how the model behaves when the valence of the high intensity candidate λ_1 varies.

Proposition 4. $\lambda_1 \mapsto \sigma^*(\lambda_1)$ is an increasing function on $(1, +\infty)$, with $\lim_{\lambda_1 \to 1^+} \sigma^*(\lambda_1) = 0$ and $\lim_{\lambda_1 \to +\infty} \sigma^*(\lambda_1) = \sigma^{**}$, while σ^{**} is independent of λ_1.

Proof: see Appendix A.6.

Figure 3 illustrates Proposition 4. In Region (1.) of the (λ_1, σ) plane, there is a set of strategies which insure that Candidate 1 wins for sure. In Region (2.), there is a set of strategies which insure that Candidate 2 wins for sure. In Region (3.), i.e., when $\sigma \in [\sigma^*(\lambda_1), \sigma^{**}]$, there is no political equilibrium. The interval $[\sigma^*(\lambda_1), \sigma^{**}]$ reduces when λ_1 increases, given that σ^* is increasing in λ_1, and tends to the singleton interval $\{\sigma^{**}\}$ when $\lambda_1 \to +\infty$.

5 Conclusion

This paper has studied the implications of considering a utility function which has some empirical foundation in a strategic model of voting. This utility function, called the intensity valence, assumes that all voters agree on a candidate's ability or will to implement a policy. However, and contrary to the additive valence, the intensity valence implies that voters are affected in different ways depending on their proximity to the policy implemented by a candidate. In a strategic model of voting with two candidates, there are two important implications of considering intensity valence utility functions for the voters.

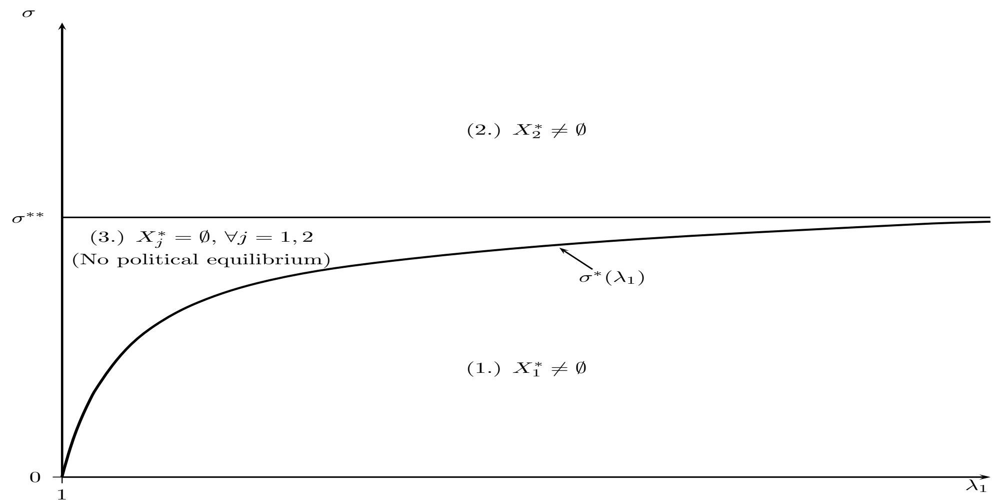


Figure 3: Parameter configuration and political equilibria

First, voters who are far from the median voter prefer the less intensive candidate. Second, we show that pure strategy Nash equilibria exist in two situations: the candidate with high ability wins with certainty if the preferences of voters are sufficiently homogeneous, while the candidate with low ability wins with certainty if the preferences of voters are too heterogeneous. In both situations, we show that there is an interval of moderate policies which are optimal for the winning candidate.

It is important to stress how preference heterogeneity/homogeneity in a society may affect the implementation of the winning policy in our model. Heterogeneity is not important in the Downsian model and the additive valence model. On the contrary, a model with intensity valence requires specific attention to heterogeneity in voters' preferences. It echoes (partially) Gerber and Lewis (2004) who have found empirically that preference heterogeneity (measured by the variance) is extremely important to understand divergence from the median voter. Using data from 55 Los Angeles County districts, they have found that in districts wherein voters have heterogeneous preferences, legislators are less constrained by the preferences of the median voter. In line with their result, our model says that conditional to the fact that preferences are relatively heterogeneous (i.e.,

$\sigma > \sigma^{**}$), then Candidate 2 is less constrained by the median voter to win. But our model also says that if preferences remain relatively homogeneous (i.e., $\sigma < \sigma^*$), then Candidate 1 is more and more constrained by the median voter when heterogeneity increases. If our model predicts non-monotonicity, note that Gerber and Lewis (2004, Table 5, p.1375) assume linearity between the variance and the legislator's location relative to the median in their econometric specification. Our model may thus provide a road map for empirical analyses between heterogeneity in voters' preferences, location of the winning candidate, and valence.

Lastly, our model does not study the issue of political recruitment but it suggests that if there is too much heterogeneity among voters, then a political party may deliberately choose not to recruit the best candidate in terms of intensity valence. On the contrary if there is enough homogeneity, then a political party will choose to recruit the best candidate. This argument echoes Mattozzi and Merlo (2014) who distinguish "mediocracy" (if a party does not recruit a good politician) and "aristocracy" (if a party does so).[4] In their model, recruiting the best possible candidate (i.e., the one with the best political ability) may improve the probability to win the election but recruiting a relatively mediocre candidate may maximize the collective effort of the other recruits of the party because "the presence of "superstars" may discourage other party members and induce them to shirk" (p.3).[5] Our model is complementary to their theory in the sense that it shows that a mediocre candidate, i.e., a candidate who is less able to implement a policy, may win if preferences are too heterogeneous among citizens.

[4]Mattozzi and Merlo (2014) note that the term "aristocracy" comes from the Greek "aristokratía" meaning "the government of the best" while "mediocracy" is defined as the "rule by the mediocre".

[5]To be more precise about their paper, Mattozzi and Merlo (2014) compare majoritarian and proportional systems. The majoritarian system is more competitive because it is a winner-takes-all system while the proportional system implies that the probability that each candidate wins the elections is proportional to his effort. The winner-takes-all nature of the majoritarian system makes the electoral return to candidate's ability higher; thus it is less likely to generate a mediocracy than the proportional system.

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A Proofs

A.1 Proof of Proposition 1

We prove first that $\Omega_1(x_1, x_2) = (b, c)$, i.e., the set of voters who strictly prefer Candidate 1, is a bounded open interval, and find the bounds b and c. Let $u_1 = u(a, x_1, \lambda_1, K) = \lambda_1(K - |x_1 - a|)$ and $u_2 = u(a, x_2, \lambda_2, K) = K - |x_2 - a|$. We thus have:

$$\begin{aligned} a \in \Omega_1(x_1, x_2) &\Leftrightarrow u_1 > u_2 \\ &\Leftrightarrow \lambda_1(K - |x_1 - a|) > K - |x_2 - a| \\ &\Leftrightarrow (\lambda_1 - 1)K + |x_2 - a| > \lambda_1|x_1 - a| \end{aligned} \quad \text{(A1)}$$

Note also the three preliminary results in (A2), (A3) and (A4):

$$\text{If } a = x_2, \text{ then } u_1 > u_2 \;\Leftrightarrow\; \frac{(\lambda_1 - 1)K}{\lambda_1} > |x_1 - x_2| \tag{A2}$$

$$\begin{aligned}\text{If } a \leq \min(x_1, x_2), \text{ then } u_1 > u_2 \;&\Leftrightarrow\; (\lambda_1 - 1)K + (x_2 - a) > \lambda_1(x_1 - a) \\ &\Leftrightarrow\; a > \frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} - K \end{aligned}\tag{A3}$$

$$\begin{aligned}\text{If } a \geq \max(x_1, x_2), \text{ then } u_1 > u_2 \;&\Leftrightarrow\; (\lambda_1 - 1)K + (a - x_2) > \lambda_1(a - x_1) \\ &\Leftrightarrow\; a < \frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} + K \end{aligned}\tag{A4}$$

Because of Inequality (A2), consider the three following cases: Case (A.) $|x_1 - x_2| \leq \frac{(\lambda_1-1)K}{\lambda_1}$, Case (B.) $x_2 > x_1 + \frac{(\lambda_1-1)K}{\lambda_1}$, and Case (B'.) $x_2 < x_1 - \frac{(\lambda_1-1)K}{\lambda_1}$.

Case (A.)

- If $|x_1 - x_2| \leq \frac{(\lambda_1-1)K}{\lambda_1}$ and $a \in (\min\{x_1, x_2\}, \max\{x_1, x_2\})$, then $\lambda_1|x_1 - a| < \lambda_1|x_1 - x_2| \leq (\lambda_1 - 1)K < (\lambda_1 - 1)K + |x_2 - a|$. Inequality (A1) is thus satisfied, so $u_1 > u_2$.
- If $|x_1 - x_2| \leq \frac{(\lambda_1-1)K}{\lambda_1}$ and $a \leq \min\{x_1, x_2\}$, then $u_1 > u_2 \Leftrightarrow a > \frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} - K$ according to Inequality (A3).
- If $|x_1 - x_2| \leq \frac{(\lambda_1-1)K}{\lambda_1}$ and $a \geq \max\{x_1, x_2\}$, then $u_1 > u_2 \Leftrightarrow a < \frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} + K$ according to Inequality (A4).

Conclusion: If $|x_1 - x_2| \leq \frac{(\lambda_1-1)K}{\lambda_1}$, then $\Omega_1(x_1, x_2) = \left(\frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} - K, \frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} + K\right)$.

Case (B.)

- If $x_2 > x_1 + \frac{(\lambda_1-1)K}{\lambda_1}$ and $x_1 < a < x_2$, then, according to Inequality (A1), $u_1 > u_2 \Leftrightarrow (\lambda_1 - 1)K + (x_2 - a) > \lambda_1(a - x_1) \Leftrightarrow a < \frac{(\lambda_1-1)K + \lambda_1 x_1 + x_2}{\lambda_1 + 1}$.
- If $x_2 > x_1 + \frac{(\lambda_1-1)K}{\lambda_1}$ and $a \leq x_1$, then $u_1 > u_2 \Leftrightarrow a > \frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} - K$ according to Inequality (A3).

– If $x_2 > x_1 + \frac{(\lambda_1-1)K}{\lambda_1}$ and $a \geq x_2$, then $u_1 > u_2 \Leftrightarrow a < \frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} + K$ according to Inequality (A4). However, since $x_2 > x_1 + \frac{(\lambda_1-1)K}{\lambda_1}$ and $a \geq x_2$, note that $\frac{\lambda_1 x_1 - x_2}{\lambda_1-1} + K = \frac{\lambda_1\left[x_1 + \frac{(\lambda_1-1)K}{\lambda_1}\right] - x_2}{\lambda_1 - 1} < \frac{(\lambda_1-1)x_2}{\lambda_1-1} = x_2 \leq a$. Hence, $u_1 \leq u_2$ if $a \geq x_2$ and $x_2 > x_1 + \frac{(\lambda_1-1)K}{\lambda_1}$.

<u>Conclusion</u>: If $x_2 > x_1 + \frac{(\lambda_1-1)K}{\lambda_1}$, then $\Omega_1(x_1, x_2) = \left(\frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} - K, \frac{(\lambda_1-1)K + \lambda_1 x_1 + x_2}{\lambda_1+1}\right)$.

Case (B'.) – If $x_2 < x_1 - \frac{(\lambda_1-1)K}{\lambda_1}$ and $x_2 < a < x_1$, then, according to Inequality (A1), $u_1 > u_2 \Leftrightarrow (\lambda_1 - 1)K + (a - x_2) > \lambda_1(x_1 - a) \Leftrightarrow a > \frac{(1-\lambda_1)K + \lambda_1 x_1 + x_2}{\lambda_1 + 1}$.

– If $x_2 < x_1 - \frac{(\lambda_1-1)K}{\lambda_1}$ and $a \leq x_2$, then $u_1 > u_2 \Leftrightarrow a > \frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} - K$ according to Inequality (A3). However, since $x_2 < x_1 - \frac{(\lambda_1-1)K}{\lambda_1}$ and $a \leq x_2$, note that $\frac{\lambda_1 x_1 - x_2}{\lambda_1-1} - K = \frac{\lambda_1\left[x_1 - \frac{(\lambda_1-1)K}{\lambda_1}\right] - x_2}{\lambda_1 - 1} > \frac{(\lambda_1-1)x_2}{\lambda_1-1} = x_2 \geq a$. Hence $u_1 \leq u_2$ if $a \leq x_2$ and $x_2 < x_1 - \frac{(\lambda_1-1)K}{\lambda_1}$.

– If $x_2 < x_1 - \frac{(\lambda_1-1)K}{\lambda_1}$ and $a \geq x_1$, then $u_1 > u_2 \Leftrightarrow a < \frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} + K$ according to Inequality (A4).

<u>Conclusion</u>: If $x_2 < x_1 - \frac{(\lambda_1-1)K}{\lambda_1}$, then $\Omega_1(x_1, x_2) = \left(\frac{(1-\lambda_1)K + \lambda_1 x_1 + x_2}{\lambda_1+1}, \frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} + K\right)$.

<u>**Now we prove**</u> that $\Omega_2(x_1, x_2) = (-\infty, b) \cup (c, +\infty)$, i.e., the set of voters who strictly prefer Candidate 2, is a non-convex set. In the three cases (A.), (B.) and (B'.), it is easily shown (replacing $u_1 > u_2$ by $u_1 = u_2$) that $I(x_1, x_2) = \{b, c\}$. Given that $\Omega_2(x_1, x_2) = \mathbb{R}\setminus[\Omega_1(x_1, x_2) \cup I(x_1, x_2)]$, we obtain that $\Omega_2(x_1, x_2) = (-\infty, b) \cup (c, +\infty)$. ■

A.2 Proof of Lemma 1

We first show that (i.) $\int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da = \int_{-\frac{(\lambda_1+1)K}{\lambda_1}}^{\frac{(\lambda_1-1)K}{\lambda_1}} f_\sigma(a)da$, and then that (ii.) $0 < \int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da < \int_{-K}^{K} f_\sigma(a)da$, i.e., that $\int_{-K}^{K} f_\sigma(a)da - \int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da > 0$.

(i.) Making the substitution $u = -a$, $du = -da$, we get:

$$\int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da = -\int_{\frac{(\lambda_1-1)K}{\lambda_1}}^{\frac{-(\lambda_1+1)K}{\lambda_1}} f_\sigma(-u)du = \int_{\frac{-(\lambda_1+1)K}{\lambda_1}}^{\frac{(\lambda_1-1)K}{\lambda_1}} f_\sigma(-u)du$$

Given that f_σ is an even function, we have $\int_{\frac{-(\lambda_1+1)K}{\lambda_1}}^{\frac{(\lambda_1-1)K}{\lambda_1}} f_\sigma(-u)du = \int_{\frac{-(\lambda_1+1)K}{\lambda_1}}^{\frac{(\lambda_1-1)K}{\lambda_1}} f_\sigma(u)du$.

(ii.) Remark that

$$\int_{-K}^{K} f_\sigma(a)da = \int_{-K}^{\frac{(1-\lambda_1)K}{\lambda_1}} f_\sigma(a)da + \int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{K} f_\sigma(a)da \tag{A5}$$

and

$$\int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da = \int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{K} f_\sigma(a)da + \int_{K}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da \tag{A6}$$

Subtracting Equations (A5) and (A6), we get

$$\int_{-K}^{K} f_\sigma(a)da - \int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da = \int_{-K}^{\frac{(1-\lambda_1)K}{\lambda_1}} f_\sigma(a)da - \int_{K}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da \tag{A7}$$

f_σ is an even function, so $\int_{-K}^{\frac{(1-\lambda_1)K}{\lambda_1}} f_\sigma(a)da = \int_{\frac{(\lambda_1-1)K}{\lambda_1}}^{K} f_\sigma(a)da$. Equation (A7) is thus equivalent to:

$$\begin{aligned}
\int_{-K}^{K} f_\sigma(a)da - \int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da &= \int_{\frac{(\lambda_1-1)K}{\lambda_1}}^{K} f_\sigma(a)da - \int_{K}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da \\
&= \int_{\frac{(\lambda_1-1)K}{\lambda_1}}^{K} \left[f_\sigma(a) - f_\sigma\left(a + \frac{K}{\lambda_1}\right)\right] da
\end{aligned} \tag{A8}$$

The limits of integration in the right hand side of (A8) are positive, and $f_\sigma(a)$ is strictly decreasing on $\mathbb{R}_+$, so $\left[f_\sigma(a) - f_\sigma\left(a + \frac{K}{\lambda_1}\right)\right] > 0$, and (A8) is strictly positive. The result in Lemma 1 follows.■

A.3 Proof of Lemma 2

Let $g_1(\sigma) = \int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da$ and $g_2(\sigma) = \int_{-K}^{K} f_\sigma(a)da$. According to Lemma 1, we know that $0 < g_1(\sigma) < g_2(\sigma)$, $\forall \sigma > 0$.

Recall the definition of f_σ and note that $g_1(\sigma) = \int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} \frac{1}{\sigma} f_1(\frac{a}{\sigma})da$. Let $z = \frac{a}{\sigma}$. Then $g_1(\sigma) = \int_{\frac{(1-\lambda_1)K}{\lambda_1\sigma}}^{\frac{(\lambda_1+1)K}{\lambda_1\sigma}} f_1(z)dz$: one can see that when σ increases, $g_1(\sigma)$ is an integral of f_1 on a smaller interval. Thus, $\sigma \mapsto g_1(\sigma)$ is a continuous and strictly decreasing function on $\mathbb{R}_+^*$, with

$$\lim_{\sigma\to 0^+} g_1(\sigma) = \int_{-\infty}^{+\infty} f_1(z)dz = 1 \text{ and } \lim_{\sigma\to+\infty} g_1(\sigma) = 0$$

Hence, there is a unique σ^* such that $g_1(\sigma^*) = \frac{1}{2}$; moreover, $g_1(\sigma) > \frac{1}{2} \Leftrightarrow \sigma < \sigma^*$.

Now consider $g_2(\sigma) = \int_{-K}^{K} \frac{1}{\sigma} f_1(\frac{a}{\sigma})da$. Let $z = \frac{a}{\sigma}$. Then $g_2(\sigma) = \int_{-\frac{K}{\sigma}}^{\frac{K}{\sigma}} f_1(z)dz$. When σ increases, $g_2(\sigma)$ is an integral of f_1 on a smaller interval. Thus, $\sigma \mapsto g_2(\sigma)$ is a continuous and strictly decreasing function on $\mathbb{R}_+^*$, with

$$\lim_{\sigma\to 0^+} g_2(\sigma) = \int_{-\infty}^{+\infty} f_1(z)dz = 1 \text{ and } \lim_{\sigma\to+\infty} g_2(\sigma) = 0$$

Hence, there is a unique σ^{**} such that $g_2(\sigma^{**}) = \frac{1}{2}$. Moreover, $g_2(\sigma) > \frac{1}{2} \Leftrightarrow \sigma < \sigma^{**}$.

According to Lemma 1, $g_1(\sigma) < g_2(\sigma)$, $\forall \sigma > 0$, which implies $g_1(\sigma^*) = \frac{1}{2} < g_2(\sigma^*)$. Given that $g_2(\sigma^{**}) = \frac{1}{2}$, then $g_2(\sigma^{**}) < g_2(\sigma^*)$. Thus, $\sigma^{**} > \sigma^*$ since g_2 is a strictly decreasing function. Results (i.), (ii.) and (iii.) in Lemma 2 follow.■

A.4 Proof of Proposition 2

We prove (i.) first, i.e., if $\sigma < \sigma^*$, Candidate 1 wins with certainty if he chooses $x_1^* = 0$, $\forall x_2 \in \mathbb{R}$. We thus have to show that if $\sigma < \sigma^*$, then $S_1(0, x_2) = \int_b^c f_\sigma(a)da > \frac{1}{2}$, $\forall x_2 \in \mathbb{R}$. Since $\sigma < \sigma^*$, we know from Lemma 2 that $\frac{1}{2} < \int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da < \int_{-K}^{K} f_\sigma(a)da$. Let's consider the three possible cases: Case (A.) $x_2 \in \left[-\frac{(\lambda_1-1)K}{\lambda_1}, \frac{(\lambda_1-1)K}{\lambda_1}\right]$, Case (B.) $x_2 > \frac{(\lambda_1-1)K}{\lambda_1}$, and Case (B'.) $x_2 < -\frac{(\lambda_1-1)K}{\lambda_1}$ which correspond to Panels (A.), (B.) and (B'.) in Figure 2 when $x_1 = 0$.

Case (A.) If $x_2 \in \left[-\frac{(\lambda_1-1)K}{\lambda_1}, \frac{(\lambda_1-1)K}{\lambda_1}\right]$ and $x_1 = x_1^* = 0$, then we know from Proposition 1 that $b = \frac{-x_2}{\lambda_1-1} - K$ and $c = \frac{-x_2}{\lambda_1-1} + K$. If so, $S_1(0, x_2) = \int_{\frac{-x_2}{\lambda_1-1}-K}^{\frac{-x_2}{\lambda_1-1}+K} f_\sigma(a)da$ and $\frac{\partial S_1}{\partial x_2}(0, x_2) = \frac{-1}{\lambda_1-1} f_\sigma\left(\frac{-x_2}{\lambda_1-1} + K\right) + \frac{1}{\lambda_1-1} f_\sigma\left(\frac{-x_2}{\lambda_1-1} - K\right)$. Given that f_σ is an even function, $f_\sigma\left(\frac{-x_2}{\lambda_1-1}-K\right) = f_\sigma\left(\frac{x_2}{\lambda_1-1}+K\right)$, so $\frac{\partial S_1}{\partial x_2} = \frac{-1}{\lambda_1-1}f_\sigma\left(\frac{-x_2}{\lambda_1-1}+K\right) + \frac{1}{\lambda_1-1}f_\sigma\left(\frac{x_2}{\lambda_1-1}+K\right)$. Furthermore, given that this even function is strictly increasing on $\mathbb{R}_-$, and strictly decreasing on $\mathbb{R}_+$, then $f_\sigma\left(\frac{x_2}{\lambda_1-1}+K\right) \geq f_\sigma\left(\frac{-x_2}{\lambda_1-1}+K\right)$ if $x_2 \leq 0$, while $f_\sigma\left(\frac{x_2}{\lambda_1-1}+K\right) \leq f_\sigma\left(\frac{-x_2}{\lambda_1-1}+K\right)$ if $x_2 \geq 0$. That is $\frac{\partial S_1}{\partial x_2} \geq 0 \Leftrightarrow x_2 \leq 0$. Thus, $x_2 \mapsto S_1(0, x_2)$ is strictly increasing on $\left[-\frac{(\lambda_1-1)K}{\lambda_1}, 0\right]$, and strictly decreasing on $\left[0, \frac{(\lambda_1-1)K}{\lambda_1}\right]$, and it has a minimum at $x_2 = \pm\frac{(\lambda_1-1)K}{\lambda_1}$. Note that $S_1\left(0, \frac{(\lambda_1-1)K}{\lambda_1}\right) = \int_{-\frac{(\lambda_1+1)K}{\lambda_1}}^{\frac{(\lambda_1-1)K}{\lambda_1}} f_\sigma(a)da$ and $S_1\left(0, -\frac{(\lambda_1-1)K}{\lambda_1}\right) = \int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da$, and according to Lemma 1, we have $S_1\left(0, \frac{(\lambda_1-1)K}{\lambda_1}\right) = S_1\left(0, -\frac{(\lambda_1-1)K}{\lambda_1}\right)$. Now since $\sigma < \sigma^*$ and given Lemma 2, we have $S_1\left(0, -\frac{(\lambda_1-1)K}{\lambda_1}\right) = \int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da > \frac{1}{2}$; we can thus conclude that $S_1(0, x_2) \geq S_1\left(0, -\frac{(\lambda_1-1)K}{\lambda_1}\right) > \frac{1}{2}$, $\forall x_2 \in \left[-\frac{(\lambda_1-1)K}{\lambda_1}, \frac{(\lambda_1-1)K}{\lambda_1}\right]$.

Case (B.) If $x_2 > \frac{(\lambda_1-1)K}{\lambda_1}$ and $x_1 = x_1^* = 0$, then we know from Proposition 1 that $b = \frac{-x_2}{\lambda_1-1} - K$ and $c = \frac{(\lambda_1-1)K+x_2}{\lambda_1+1}$. We thus have $S_1(0, x_2) = \int_{\frac{-x_2}{\lambda_1-1}-K}^{\frac{(\lambda_1-1)K+x_2}{\lambda_1+1}} f_\sigma(a)da$ and $\frac{\partial S_1}{\partial x_2}(0, x_2) = \frac{1}{\lambda_1+1}f_\sigma\left(\frac{(\lambda_1-1)K+x_2}{\lambda_1+1}\right) + \frac{1}{\lambda_1-1}f_\sigma\left(\frac{-x_2}{\lambda_1-1}-K\right) > 0$. Then, $x_2 \mapsto S_1(0, x_2)$ is strictly increasing on $\left[\frac{(\lambda_1-1)K}{\lambda_1}, +\infty\right)$, and has a minimum at $x_2 = \frac{(\lambda_1-1)K}{\lambda_1}$. Consequently, $S_1(0, x_2) > S_1\left(0, \frac{(\lambda_1-1)K}{\lambda_1}\right) > \frac{1}{2}$, $\forall x_2 > \frac{(\lambda_1-1)K}{\lambda_1}$.

Case (B'.) If $x_2 < -\frac{(\lambda_1-1)K}{\lambda_1}$ and $x_1 = x_1^* = 0$, we know from Proposition 1 that $b = \frac{(1-\lambda_1)K+x_2}{\lambda_1+1}$ and $c = \frac{-x_2}{\lambda_1-1} + K$. We thus have $S_1(0, x_2) = \int_{\frac{(1-\lambda_1)K+x_2}{\lambda_1+1}}^{\frac{-x_2}{\lambda_1-1}+K} f_\sigma(a)da$ and $\frac{\partial S_1}{\partial x_2}(0, x_2) = \frac{-1}{\lambda_1-1}f_\sigma\left(\frac{-x_2}{\lambda_1-1}+K\right) - \frac{1}{\lambda_1+1}f_\sigma\left(\frac{(1-\lambda_1)K+x_2}{\lambda_1+1}\right) < 0$. Then, $x_2 \mapsto S_1(0, x_2)$ is strictly decreasing on $\left(-\infty, -\frac{(\lambda_1-1)K}{\lambda_1}\right]$ and has a minimum at $x_2 = -\frac{(\lambda_1-1)K}{\lambda_1}$. Consequently, $S_1(0, x_2) > S_1\left(0, -\frac{(\lambda_1-1)K}{\lambda_1}\right) > \frac{1}{2}$, $\forall x_2 < -\frac{(\lambda_1-1)K}{\lambda_1}$.

Conclusion: we have shown that if $\sigma < \sigma^*$, then $S_1(0, x_2) > \frac{1}{2}$, $\forall x_2 \in \mathbb{R}$.

Now we prove (ii.), i.e., if $\sigma > \sigma^{**}$, Candidate 2 wins with certainty if he chooses $x_2^* = 0$, $\forall x_1 \in \mathbb{R}$. We thus have to show that if $\sigma > \sigma^{**}$, then $S_1(x_1, 0) = \int_b^c f_\sigma(a)da < \frac{1}{2}$, $\forall x_1 \in \mathbb{R}$. Since $\sigma > \sigma^{**}$, we know from Lemma 2 that $\int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da < \int_{-K}^{K} f_\sigma(a)da < \frac{1}{2}$.

Let's consider the three possible cases: Case (A.) $x_1 \in \left[-\frac{(\lambda_1-1)K}{\lambda_1}, \frac{(\lambda_1-1)K}{\lambda_1}\right]$, Case (B.) $x_1 < -\frac{(\lambda_1-1)K}{\lambda_1}$, and Case (B'.) $x_1 > \frac{(\lambda_1-1)K}{\lambda_1}$ which correspond to Panels (A.), (B.) and (B'.) in Figure 2 when $x_2 = 0$.

Case (A.) If $x_1 \in \left[-\frac{(\lambda_1-1)K}{\lambda_1}, \frac{(\lambda_1-1)K}{\lambda_1}\right]$ and $x_2 = x_2^* = 0$, then we know from Proposition 1 that $b = \frac{\lambda_1 x_1}{\lambda_1-1} - K$ and $c = \frac{\lambda_1 x_1}{\lambda_1-1} + K$. If so, $S_1(x_1, 0) = \int_{\frac{\lambda_1 x_1}{\lambda_1-1}-K}^{\frac{\lambda_1 x_1}{\lambda_1-1}+K} f_\sigma(a)da$ and $\frac{\partial S_1}{\partial x_1}(x_1, 0) = \frac{\lambda_1}{\lambda_1-1} f_\sigma\left(\frac{\lambda_1 x_1}{\lambda_1-1} + K\right) - \frac{\lambda_1}{\lambda_1-1} f_\sigma\left(\frac{\lambda_1 x_1}{\lambda_1-1} - K\right)$. Given that f_σ is an even function, strictly increasing on $\mathbb{R}_-$, and strictly decreasing on $\mathbb{R}_+$, $f_\sigma\left(\frac{\lambda_1 x_1}{\lambda_1-1} + K\right) \geq f_\sigma\left(\frac{\lambda_1 x_1}{\lambda_1-1} - K\right)$ if $x_1 \leq 0$, while $f_\sigma\left(\frac{\lambda_1 x_1}{\lambda_1-1} + K\right) \leq f_\sigma\left(\frac{\lambda_1 x_1}{\lambda_1-1} + K\right)$ if $x_1 \geq 0$. That is $\frac{\partial S_1}{\partial x_1} \geq 0 \Leftrightarrow x_1 \leq 0$. Thus, $x_1 \mapsto S_1(x_1, 0)$ is strictly increasing on $\left[-\frac{(\lambda_1-1)K}{\lambda_1}, 0\right]$, and strictly decreasing on $\left[0, \frac{(\lambda_1-1)K}{\lambda_1}\right]$, and it has a maximum at $x_1 = 0$. $S_1(0,0) = \int_{-K}^{K} f_\sigma(a)da$, and since $\sigma > \sigma^{**}$, we know from Lemma 2 that $S_1(0,0) = \int_{-K}^{K} f_\sigma(a)da < \frac{1}{2}$. We can thus conclude that $S_1(x_1, 0) \leq S_1(0,0) < \frac{1}{2}$, $\forall x_1 \in \left[-\frac{(\lambda_1-1)K}{\lambda_1}, \frac{(\lambda_1-1)K}{\lambda_1}\right]$.

Case (B.) If $x_1 < -\frac{(\lambda_1-1)K}{\lambda_1}$ and $x_2 = x_2^* = 0$, then we know from Proposition 1 that $b = \frac{\lambda_1 x_1}{\lambda_1-1} - K$ and $c = \frac{(\lambda_1-1)K+\lambda_1 x_1}{\lambda_1+1}$. If so, $S_1(x_1, 0) = \int_{\frac{\lambda_1 x_1}{\lambda_1-1}-K}^{\frac{(\lambda_1-1)K+\lambda_1 x_1}{\lambda_1+1}} f_\sigma(a)da$. Now remark that if $x_1 < -\frac{(\lambda_1-1)K}{\lambda_1}$, then $\frac{(\lambda_1-1)K+\lambda_1 x_1}{\lambda_1+1} < 0$, so $S_1(x_1, 0) = \int_{\frac{\lambda_1 x_1}{\lambda_1-1}-K}^{\frac{(\lambda_1-1)K+\lambda_1 x_1}{\lambda_1+1}} f_\sigma(a)da < \int_{\frac{\lambda_1 x_1}{\lambda_1-1}-K}^{0} f_\sigma(a)da < \int_{-\infty}^{0} f_\sigma(a)da = \frac{1}{2}$, i.e., $S_1(x_1, 0) < \frac{1}{2}$, $\forall x_1 < -\frac{(\lambda_1-1)K}{\lambda_1}$.

Case (B'.) If $x_1 > \frac{(\lambda_1-1)K}{\lambda_1}$ and $x_2 = x_2^* = 0$, then we know from Proposition 1 that $b = \frac{(1-\lambda_1)K+\lambda_1 x_1}{\lambda_1+1}$ and $c = \frac{\lambda_1 x_1}{\lambda_1-1} + K$. If so, $S_1(x_1, 0) = \int_{\frac{(1-\lambda_1)K+\lambda_1 x_1}{\lambda_1+1}}^{\frac{\lambda_1 x_1}{\lambda_1-1}+K} f_\sigma(a)da$. Now remark that if $x_1 > \frac{(\lambda_1-1)K}{\lambda_1}$, then $\frac{(1-\lambda_1)K+\lambda_1 x_1}{\lambda_1+1} > 0$, so $S_1(x_1, 0) = \int_{\frac{(1-\lambda_1)K+\lambda_1 x_1}{\lambda_1+1}}^{\frac{\lambda_1 x_1}{\lambda_1-1}+K} f_\sigma(a)da < \int_{0}^{\frac{\lambda_1 x_1}{\lambda_1-1}+K} f_\sigma(a)da \leq \int_{0}^{+\infty} f_\sigma(a) = \frac{1}{2}$, $\forall x_1 > \frac{(\lambda_1-1)K}{\lambda_1}$.

<u>Conclusion</u>: we have shown that if $\sigma > \sigma^{**}$, then $S_1(x_1, 0) < \frac{1}{2}$, $\forall x_1 \in \mathbb{R}$.

<u>**Finally, we prove (iii.)**</u>, i.e., if $\sigma^* \leq \sigma \leq \sigma^{**}$, there is no political equilibrium. Since $\sigma^* \leq \sigma \leq \sigma^{**}$, we know from Lemma 2 that $\int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da \leq \frac{1}{2} \leq \int_{-K}^{K} f_\sigma(a)da$. Because

of Lemma 1, one of these inequalities must be strict, so without loss of generality, we consider that $\int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da < \frac{1}{2} \leq \int_{-K}^{K} f_\sigma(a)da$. Now let $(\overline{x}_1, \overline{x}_2) \in \mathbb{R}^2$. We want to show that $(\overline{x}_1, \overline{x}_2)$ is not a political equilibrium. It is sufficient to prove that there exist x_1 and x_2 such that $S_1(x_1, \overline{x}_2) \geq \frac{1}{2} > S_1(\overline{x}_1, x_2)$. We proceed in two steps.

First step. We first show that $S_1(x_1, \overline{x}_2) \geq \frac{1}{2}$ if $x_1 = \frac{\overline{x}_2}{\lambda_1}$, $\forall \overline{x}_2$.

- If $\overline{x}_2 \in [-K, K]$, then $|\overline{x}_2 - x_1| = \left|\frac{(\lambda_1-1)\overline{x}_2}{\lambda_1}\right| \leq \frac{(\lambda_1-1)K}{\lambda_1}$. We are thus in Case (A.), where $b = \frac{\lambda_1 x_1 - \overline{x}_2}{\lambda_1 - 1} - K$ and $c = \frac{\lambda_1 x_1 - \overline{x}_2}{\lambda_1 - 1} + K$ (see Proposition 1). Given that $x_1 = \frac{\overline{x}_2}{\lambda_1}$, then $b = -K$ and $c = K$, so $S_1\left(\frac{\overline{x}_2}{\lambda_1}, \overline{x}_2\right) = \int_{-K}^{+K} f_\sigma(a)da \geq \frac{1}{2}$.
- If $\overline{x}_2 > K$, then $\overline{x}_2 - x_1 = \frac{(\lambda_1-1)\overline{x}_2}{\lambda_1} > \frac{(\lambda_1-1)K}{\lambda_1}$. We are thus in Case (B.), where $b = \frac{\lambda_1 x_1 - \overline{x}_2}{\lambda_1 - 1} - K$ and $c = \frac{(\lambda_1-1)K + \lambda_1 x_1 + \overline{x}_2}{\lambda_1+1}$ (see Proposition 1). Given that $x_1 = \frac{\overline{x}_2}{\lambda_1}$, then $b = -K$ and $c = \frac{(\lambda_1-1)K + 2\overline{x}_2}{\lambda_1+1} > K$ (since $\overline{x}_2 > K$). Hence, $S_1\left(\frac{\overline{x}_2}{\lambda_1}, \overline{x}_2\right) = \int_{-K}^{\frac{(\lambda_1-1)K+2\overline{x}_2}{\lambda_1+1}} f_\sigma(a)da > \int_{-K}^{K} f_\sigma(a)da \geq \frac{1}{2}$.
- If $\overline{x}_2 < -K$, then $\overline{x}_2 - x_1 = \frac{(\lambda_1-1)\overline{x}_2}{\lambda_1} < -\frac{(\lambda_1-1)K}{\lambda_1}$. We are thus in Case (B'.), where $b = \frac{(1-\lambda_1)K + \lambda_1 x_1 + \overline{x}_2}{\lambda_1+1}$ and $c = \frac{\lambda_1 x_1 - \overline{x}_2}{\lambda_1 - 1} + K$ (see Proposition 1). Given that $x_1 = \frac{\overline{x}_2}{\lambda_1}$, then $b = \frac{(1-\lambda_1)K + 2\overline{x}_2}{\lambda_1+1} < -K$ (since $\overline{x}_2 < -K$) and $c = K$. Hence, $S_1\left(\frac{\overline{x}_2}{\lambda_1}, \overline{x}_2\right) = \int_{\frac{(1-\lambda_1)K+2\overline{x}_2}{\lambda_1+1}}^{K} f_\sigma(a)da > \int_{-K}^{K} f_\sigma(a)da \geq \frac{1}{2}$.

Second step. We now show that $S_1(\overline{x}_1, x_2) < \frac{1}{2}$ if x_2 is such that: $x_2 = \overline{x}_1 - \frac{(\lambda_1-1)K}{\lambda_1}$, $\forall \overline{x}_1 \geq 0$, and $x_2 = \overline{x}_1 + \frac{(\lambda_1-1)K}{\lambda_1}$, $\forall \overline{x}_1 < 0$. For any $\overline{x}_1$, we are thus in Case (A.), where $b = \frac{\lambda_1 \overline{x}_1 - x_2}{\lambda_1 - 1} - K$ and $c = \frac{\lambda_1 \overline{x}_1 - x_2}{\lambda_1 - 1} + K$ (see Proposition 1).

- If $\overline{x}_1 \geq 0$, and given that $x_2 = \overline{x}_1 - \frac{(\lambda_1-1)K}{\lambda_1}$, then $b = \overline{x}_1 + \frac{(1-\lambda_1)K}{\lambda_1}$ and $c = \overline{x}_1 + \frac{(\lambda_1+1)K}{\lambda_1}$, so $S_1\left(\overline{x}_1, \overline{x}_1 - \frac{(\lambda_1-1)K}{\lambda_1}\right) = \int_{\overline{x}_1 + \frac{(1-\lambda_1)K}{\lambda_1}}^{\overline{x}_1 + \frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da \leq \int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da < \frac{1}{2}$.

– If $\overline{x}_1 < 0$, and given that $x_2 = \overline{x}_1 + \frac{(\lambda_1-1)K}{\lambda_1}$, then $b = \overline{x}_1 - \frac{(\lambda_1+1)K}{\lambda_1}$ and $c = \overline{x}_1 + \frac{(\lambda_1-1)K}{\lambda_1}$, so $S_1\left(\overline{x}_1, \overline{x}_1 + \frac{(\lambda_1-1)K}{\lambda_1}\right) = \int_{\overline{x}_1-\frac{(\lambda_1+1)K}{\lambda_1}}^{\overline{x}_1+\frac{(\lambda_1-1)K}{\lambda_1}} f_\sigma(a)da < \int_{-\frac{(\lambda_1+1)K}{\lambda_1}}^{\frac{(\lambda_1-1)K}{\lambda_1}} f_\sigma(a)da$. Because of Lemma 1, we know that $\int_{-\frac{(\lambda_1+1)K}{\lambda_1}}^{\frac{(\lambda_1-1)K}{\lambda_1}} f_\sigma(a)da = \int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da$, so $S_1\left(\overline{x}_1, \overline{x}_1 + \frac{(\lambda_1-1)K}{\lambda_1}\right) < \int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da < \frac{1}{2}$.

Conclusion: we have shown that if $\sigma^* \leq \sigma \leq \sigma^{**}$, then there is no political equilibrium.■

A.5 Proof of Proposition 3

We prove (i.) first, i.e., if $\sigma < \sigma^*$, then $X_1^* = (-\alpha, \alpha)$ where α is the unique positive real which satisfies $\int_{\alpha+\frac{(1-\lambda_1)K}{\lambda_1}}^{\alpha+\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da = \frac{1}{2}$. Moreover, we prove that $\alpha \in \left(0, \frac{(\lambda_1-1)K}{\lambda_1}\right)$, $\sigma \mapsto \alpha(\sigma)$ is a decreasing function on $(0, \sigma^*]$, with $\alpha(\sigma^*) = 0$ and $\lim_{\sigma\to 0^+} \alpha(\sigma) = \frac{(\lambda_1-1)K}{\lambda_1}$.

We proceed in three steps.

First step. First we show that there is a unique positive real α satisfying $\int_{\alpha+\frac{(1-\lambda_1)K}{\lambda_1}}^{\alpha+\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da = \frac{1}{2}$, and $\alpha \in \left(0, \frac{(\lambda_1-1)K}{\lambda_1}\right)$. Since $\sigma < \sigma^*$, we know from Lemma 2 that $\frac{1}{2} < \int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da < \int_{-K}^{K} f_\sigma(a)da$. Now let $g(v) = \int_{v-K}^{v+K} f_\sigma(a)da$, for $v \in \mathbb{R}$, we have $g'(v) = f_\sigma(v+K) - f_\sigma(v-K)$. Given that f_σ is an even function strictly increasing on $\mathbb{R}_-$, and strictly decreasing on $\mathbb{R}_+$, $f_\sigma(v+K) \leq f_\sigma(v-K)$ if $v \geq 0$, and $f_\sigma(v+K) \geq f_\sigma(v-K)$ if $v \leq 0$. It implies that g is an even continuous function, strictly increasing on $\mathbb{R}_-$, and strictly decreasing on $\mathbb{R}_+$. Since $g\left(\frac{K}{\lambda_1}\right) = \int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da > \frac{1}{2}$ and $g(K) = \int_0^{2K} f_\sigma(a)da < \int_0^{+\infty} f_\sigma(a)da = \frac{1}{2}$, there exists a unique $v_0 > 0$ such that $g(v_0) = \frac{1}{2}$. And we must have $\frac{K}{\lambda_1} < v_0 < K$. Now let $\alpha = v_0 - \frac{K}{\lambda_1}$, then $0 < \alpha < \frac{(\lambda_1-1)K}{\lambda_1}$. Given that $g(v_0) = \int_{v_0-K}^{v_0+K} f_\sigma(a)da = \frac{1}{2}$,

$$g\left(\alpha + \frac{K}{\lambda_1}\right) = \int_{\alpha+\frac{(1-\lambda_1)K}{\lambda_1}}^{\alpha+\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da = \frac{1}{2} \tag{A9}$$

Second step. Now we show that $X_1^* = (-\alpha, \alpha)$.

– **First**, we show that if $x_1 \in (-\alpha, \alpha)$, then $x_1 \in X_1^*$, i.e., $S_1(x_1, x_2) > \frac{1}{2}$, $\forall x_2$. Consider the three possible cases: Case (A.) $-\frac{(\lambda_1-1)K}{\lambda_1} \leq x_1 - x_2 \leq \frac{(\lambda_1-1)K}{\lambda_1}$, Case (B.) $x_2 > x_1 + \frac{(\lambda_1-1)K}{\lambda_1}$, and Case (B'.) $x_2 < x_1 - \frac{(\lambda_1-1)K}{\lambda_1}$.

Case (A.) If $-\frac{(\lambda_1-1)K}{\lambda_1} \leq x_1 - x_2 \leq \frac{(\lambda_1-1)K}{\lambda_1}$, then $b = \frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} - K$ and $c = \frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} + K$ (see Proposition 1). So $S_1(x_1, x_2) = \int_{\frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} - K}^{\frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} + K} f_\sigma(a)da = g\left(\frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1}\right)$.
If $-\frac{(\lambda_1-1)K}{\lambda_1} \leq x_1 - x_2 \leq \frac{(\lambda_1-1)K}{\lambda_1}$ and $-\alpha < x_1 < \alpha$ (so $-(\lambda_1 - 1)\alpha < (\lambda_1 - 1)x_1 < (\lambda_1 - 1)\alpha$), then adding these two inequalities we obtain $-(\lambda_1 - 1)\left(\alpha + \frac{K}{\lambda_1}\right) < \lambda_1 x_1 - x_2 < (\lambda_1 - 1)\left(\alpha + \frac{K}{\lambda_1}\right)$. Dividing by $(\lambda_1 - 1)$, then $-\left(\alpha + \frac{K}{\lambda_1}\right) < \frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} < \alpha + \frac{K}{\lambda_1}$. Since g is an even function, strictly increasing on $\mathbb{R}_-$, and strictly decreasing on $\mathbb{R}_+$, then $g\left(\frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1}\right) > g\left(\alpha + \frac{K}{\lambda_1}\right) = \frac{1}{2}$ (see Equation (A9)). Hence, if $x_1 \in (-\alpha, \alpha)$, $S_1(x_1, x_2) > \frac{1}{2}$, $\forall x_2 \in \left[x_1 - \frac{(\lambda_1-1)K}{\lambda_1}, x_1 + \frac{(\lambda_1-1)K}{\lambda_1}\right]$.

Case (B.) If $x_2 > x_1 + \frac{(\lambda_1-1)K}{\lambda_1}$, then $b = \frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} - K$ and $c = \frac{(\lambda_1-1)K + \lambda_1 x_1 + x_2}{\lambda_1 + 1}$ (see Proposition 1), and $S_1(x_1, x_2) = \int_{\frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} - K}^{\frac{(\lambda_1-1)K + \lambda_1 x_1 + x_2}{\lambda_1 + 1}} f_\sigma(a)da$. Thus, it is easy to see that if $x_2 > x_1 + \frac{(\lambda_1-1)K}{\lambda_1}$, then $S_1(x_1, x_2)$ is an increasing function of x_2, and $S_1(x_1, x_2) > S_1\left(x_1, x_1 + \frac{(\lambda_1-1)K}{\lambda_1}\right)$, with $S_1\left(x_1, x_1 + \frac{(\lambda_1-1)K}{\lambda_1}\right) > \frac{1}{2}$ according to Case (A.). Hence, if $x_1 \in (-\alpha, \alpha)$, $S_1(x_1, x_2) > \frac{1}{2}$, $\forall x_2 > x_1 + \frac{(\lambda_1-1)K}{\lambda_1}$.

Case (B'.) If $x_2 < x_1 - \frac{(\lambda_1-1)K}{\lambda_1}$, then $b = \frac{(1-\lambda_1)K + \lambda_1 x_1 + x_2}{\lambda_1 + 1}$ and $c = \frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} + K$ (see Proposition 1), and $S_1(x_1, x_2) = \int_{\frac{(1-\lambda_1)K + \lambda_1 x_1 + x_2}{\lambda_1 + 1}}^{\frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} + K} f_\sigma(a)da$. Thus, it is easy to see that if $x_2 < x_1 - \frac{(\lambda_1-1)K}{\lambda_1}$, $S_1(x_1, x_2)$ is a decreasing function of x_2, and $S_1(x_1, x_2) > S_1\left(x_1, x_1 - \frac{(\lambda_1-1)K}{\lambda_1}\right)$, with $S_1\left(x_1, x_1 - \frac{(\lambda_1-1)K}{\lambda_1}\right) > \frac{1}{2}$ according to Case (A.). Hence, if $x_1 \in (-\alpha, \alpha)$, $S_1(x_1, x_2) > \frac{1}{2}$, $\forall x_2 < x_1 - \frac{(\lambda_1-1)K}{\lambda_1}$.

Conclusion: we have shown that if $x_1 \in (-\alpha, \alpha)$, then $x_1 \in X_1^*$.

- **Second**, we show that if $x_1 \notin (-\alpha, \alpha)$, then $x_1 \notin X_1^*$. Assume that $x_1 \geq \alpha$, and consider $x_2 = x_1 - \frac{(\lambda_1-1)K}{\lambda_1}$. We are then in Case (A.), and $S_1(x_1,x_2) = \int_{\frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} - K}^{\frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} + K} f_\sigma(a)da = \int_{\frac{\lambda_1 x_1 - x_1 + \frac{(\lambda_1-1)K}{\lambda_1}}{\lambda_1 - 1} - K}^{\frac{\lambda_1 x_1 - x_1 + \frac{(\lambda_1-1)K}{\lambda_1}}{\lambda_1 - 1} + K} f_\sigma(a)da = \int_{x_1 + \frac{(1-\lambda_1)K}{\lambda_1}}^{x_1 + \frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da = g\left(x_1 + \frac{K}{\lambda_1}\right)$. Given that $x_1 \geq \alpha$, then $x_1 + \frac{K}{\lambda_1} \geq \alpha + \frac{K}{\lambda_1}$. Since g is strictly decreasing on $\mathbb{R}_+$, then $g\left(x_1 + \frac{K}{\lambda_1}\right) \leq g\left(\alpha + \frac{K}{\lambda_1}\right) = \frac{1}{2}$ (see Equation (A9)). Thus, for each $x_1 \geq \alpha$, there exists x_2 such that $S_1(x_1,x_2) \leq \frac{1}{2}$, so $x_1 \notin X_1^*$. Similarly, if $x_1 \leq -\alpha$, consider $x_2 = x_1 + \frac{(\lambda_1-1)K}{\lambda_1}$. We are again in Case (A.), and $S_1(x_1,x_2) = \int_{\frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} - K}^{\frac{\lambda_1 x_1 - x_2}{\lambda_1 - 1} + K} f_\sigma(a)da = \int_{\frac{\lambda_1 x_1 - x_1 - \frac{(\lambda_1-1)K}{\lambda_1}}{\lambda_1 - 1} - K}^{\frac{\lambda_1 x_1 - x_1 - \frac{(\lambda_1-1)K}{\lambda_1}}{\lambda_1 - 1} + K} f_\sigma(a)da = \int_{x_1 - \frac{(\lambda_1+1)K}{\lambda_1}}^{x_1 + \frac{(\lambda_1-1)K}{\lambda_1}} f_\sigma(a)da = g\left(x_1 - \frac{K}{\lambda_1}\right)$. We have $x_1 \leq -\alpha$, so $x_1 - \frac{K}{\lambda_1} \leq -\alpha - \frac{K}{\lambda_1}$. Now recall that g is an even function so $g\left(-\alpha - \frac{K}{\lambda_1}\right) = g\left(\alpha + \frac{K}{\lambda_1}\right) = \frac{1}{2}$ (see Equation (A9)). And given that g is strictly increasing on $\mathbb{R}_-$, then $g\left(x_1 - \frac{K}{\lambda_1}\right) \leq g\left(\alpha + \frac{K}{\lambda_1}\right) = \frac{1}{2}$. Thus, for each $x_1 \leq -\alpha$, there exists x_2 such that $S_1(x_1,x_2) \leq \frac{1}{2}$, so $x_1 \notin X_1^*$.

Conclusion: we have shown that if $x_1 \notin (-\alpha, \alpha)$, then $x_1 \notin X_1^*$. Given that we have also shown that if $x_1 \in (-\alpha, \alpha)$, then $x_1 \in X_1^*$, it implies that $X_1^* = (-\alpha, \alpha)$.

Third step. It remains to show that $\sigma \mapsto \alpha(\sigma)$ is a decreasing function on $(0, \sigma^*]$, with $\alpha(\sigma^*) = 0$ and $\lim_{\sigma \to 0^+} \alpha(\sigma) = \frac{(\lambda_1-1)K}{\lambda_1}$.

- **First**, we show that $\sigma \mapsto \alpha(\sigma)$ is a decreasing function on $(0, \sigma^*]$. We know from Equation (A9) that $G(\sigma, \alpha) = g\left(\alpha + \frac{K}{\lambda_1}\right) - \frac{1}{2} = \int_{\alpha + \frac{(1-\lambda_1)K}{\lambda_1}}^{\alpha + \frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da - \frac{1}{2} = 0$. The implicit function theorem gives $\frac{d\alpha}{d\sigma} = -\frac{\frac{\partial G}{\partial \sigma}}{\frac{\partial G}{\partial \alpha}}$. Let $z = \frac{a}{\sigma}$, so $G(\sigma, \alpha) = \int_{\alpha + \frac{(1-\lambda_1)K}{\lambda_1}}^{\alpha + \frac{(\lambda_1+1)K}{\lambda_1}} \frac{1}{\sigma} f_1(\frac{a}{\sigma})da - \frac{1}{2} = \int_{\frac{\alpha + \frac{(1-\lambda_1)K}{\lambda_1}}{\sigma}}^{\frac{\alpha + \frac{(\lambda_1+1)K}{\lambda_1}}{\sigma}} f_1(z)dz - \frac{1}{2} = 0$ and $\frac{\partial G}{\partial \sigma} = -\frac{\alpha + \frac{(\lambda_1+1)K}{\lambda_1}}{\sigma^2} f_1\left(\frac{\alpha + \frac{(\lambda_1+1)K}{\lambda_1}}{\sigma}\right) + \frac{\alpha - \frac{(\lambda_1-1)K}{\lambda_1}}{\sigma^2} f_1\left(\frac{\alpha - \frac{(\lambda_1-1)K}{\lambda_1}}{\sigma}\right)$. Now recall that we have shown that $\alpha < \frac{(\lambda_1-1)K}{\lambda_1}$ (first step of this proof), so $\frac{\partial G}{\partial \sigma} < 0$. Concerning

$\frac{\partial G}{\partial \alpha} = f_\sigma\left(\alpha + \frac{(\lambda_1+1)K}{\lambda_1}\right) - f_\sigma\left(\alpha + \frac{(1-\lambda_1)K}{\lambda_1}\right)$, given that f_σ is an even function, strictly increasing on $\mathbb{R}_-$ and strictly decreasing on $\mathbb{R}_+$, $f_\sigma\left(\alpha + \frac{(\lambda_1+1)K}{\lambda_1}\right) < f_\sigma\left(\alpha + \frac{(1-\lambda_1)K}{\lambda_1}\right)$ and $\frac{\partial G}{\partial \alpha} < 0$. Since $\frac{\partial G}{\partial \sigma} < 0$ and $\frac{\partial G}{\partial \alpha} < 0$, then $\frac{d\alpha}{d\sigma} < 0$ according to the implicit function theorem. Hence $\sigma \mapsto \alpha(\sigma)$ is a decreasing function on $(0, \sigma^*]$.

- **Second**, we show that $\alpha(\sigma^*) = 0$. According to Lemma 2, σ^* is defined by $\int_{\frac{(1-\lambda_1)K}{\lambda_1}}^{\frac{(\lambda_1+1)K}{\lambda_1}} f_{\sigma^*}(a)da = \frac{1}{2}$. We also know from Equation (A9) that $\int_{\alpha+\frac{(1-\lambda_1)K}{\lambda_1}}^{\alpha+\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da = \frac{1}{2}$. It is thus obvious that if $\sigma = \sigma^*$, then we must have $\alpha = 0$.

- **Third**, we show that $\lim_{\sigma\to 0^+} \alpha(\sigma) = \frac{(\lambda_1-1)K}{\lambda_1}$. According to Equation (A9), α is defined by $\int_{\alpha+\frac{(1-\lambda_1)K}{\lambda_1}}^{\alpha+\frac{(\lambda_1+1)K}{\lambda_1}} f_\sigma(a)da = \frac{1}{2}$. Let $z = \frac{a}{\sigma}$. We then obtain that $\alpha = \alpha(\sigma)$ is defined by $\int_{\frac{1}{\sigma}\left[\alpha+\frac{(1-\lambda_1)K}{\lambda_1}\right]}^{\frac{1}{\sigma}\left[\alpha+\frac{(\lambda_1+1)K}{\lambda_1}\right]} f_1(z)dz = \frac{1}{2}$. $\sigma \mapsto \alpha(\sigma)$ is a decreasing function on $\sigma \in (0, \sigma^*)$, with $0 < \alpha < \frac{(\lambda_1-1)K}{\lambda_1}$, i.e., $\sigma \mapsto \alpha(\sigma)$ is bounded. Thus there is a limit $\alpha_0 = \lim_{\sigma\to 0^+} \alpha(\sigma)$, with $\alpha_0 \in \left[0, \frac{(\lambda_1-1)K}{\lambda_1}\right]$.
  We must have $\lim_{\sigma\to 0^+} \int_{\frac{1}{\sigma}\left[\alpha+\frac{(1-\lambda_1)K}{\lambda_1}\right]}^{\frac{1}{\sigma}\left[\alpha+\frac{(\lambda_1+1)K}{\lambda_1}\right]} f_1(z)dz = \int_{\lim_{\sigma\to 0^+} \frac{1}{\sigma}\left[\alpha+\frac{(1-\lambda_1)K}{\lambda_1}\right]}^{\lim_{\sigma\to 0^+} \frac{1}{\sigma}\left[\alpha+\frac{(\lambda_1+1)K}{\lambda_1}\right]} f_1(z)dz = \int_{\lim_{\sigma\to 0^+} \frac{1}{\sigma}\left[\alpha+\frac{(1-\lambda_1)K}{\lambda_1}\right]}^{+\infty} f_1(z)dz = \frac{1}{2}$. If $\alpha_0 < \frac{(\lambda_1-1)K}{\lambda_1}$, then $\int_{\lim_{\sigma\to 0^+} \frac{1}{\sigma}\left[\alpha+\frac{(1-\lambda_1)K}{\lambda_1}\right]}^{+\infty} f_1(z)dz = \int_{-\infty}^{+\infty} f_1(z)dz = 1 \neq \frac{1}{2}$. It is impossible, thus $\alpha_0 = \frac{(\lambda_1-1)K}{\lambda_1}$, and we have $\int_{\lim_{\sigma\to 0^+} \frac{1}{\sigma}\left[\alpha+\frac{(1-\lambda_1)K}{\lambda_1}\right]}^{+\infty} f_1(z)dz = \int_0^{+\infty} f_1(z)dz = \frac{1}{2}$.

Conclusion: we have shown that $\sigma \mapsto \alpha(\sigma)$ is a decreasing function on $(0, \sigma^*]$, with $\alpha(\sigma^*) = 0$ and $\lim_{\sigma\to 0^+} \alpha(\sigma) = \frac{(\lambda_1-1)K}{\lambda_1}$.

Now we prove (ii.), i.e., if $\sigma > \sigma^{**}$, then $X_2^* = (-\beta, \beta)$ where β is the unique positive real which satisfies $\sup_{x_1 \le \beta - \frac{(\lambda_1-1)K}{\lambda_1}} S_1(x_1, \beta) = \frac{1}{2}$. Since $\sigma > \sigma^{**}$, we know from Lemma 2 that $\int_{-K}^{+K} f_\sigma(a)da < \frac{1}{2}$. For a given x_2, let us set:

$$M_A(x_2) = \sup_{x_1;\ |x_1-x_2| \le \frac{(\lambda_1-1)K}{\lambda_1}} S_1(x_1, x_2) = \sup_{x_1;\ \text{Case A}} S_1(x_1, x_2)$$

$$M_B(x_2) = \sup_{x_1;\ x_1 \le x_2 - \frac{(\lambda_1-1)K}{\lambda_1}} S_1(x_1,x_2) = \sup_{x_1;\ \text{Case B}} S_1(x_1,x_2)$$

$$M_{B'}(x_2) = \sup_{x_1;\ x_1 \ge x_2 + \frac{(\lambda_1-1)K}{\lambda_1}} S_1(x_1,x_2) = \sup_{x_1;\ \text{Case B'}} S_1(x_1,x_2)$$

$$M(x_2) = \sup_{x_1} S_1(x_1,x_2) = \max(M_A(x_2), M_B(x_2), M_{B'}(x_2))$$

Remark that each sup is in fact a max since S_1 is continuous and $S_1(x_1,x_2) \to 0$ if $x_1 \to \pm\infty$. Now, consider the three cases, Case (A.), Case (B.), and Case (B'.).

Case (A.) If $x_1 \in \left[x_2 - \frac{(\lambda_1-1)K}{\lambda_1}, x_2 + \frac{(\lambda_1-1)K}{\lambda_1}\right]$, then $S_1(x_1,x_2) = \int_{\frac{\lambda_1 x_1 - x_2}{\lambda_1-1}-K}^{\frac{\lambda_1 x_1 - x_2}{\lambda_1-1}+K} f_\sigma(a)da$ (see Proposition 1). Remark that the length of the interval $\Omega_1(x_1,x_2)$ is $2K$ for all $x_1 \in \left[x_2 - \frac{(\lambda_1-1)K}{\lambda_1}, x_2 + \frac{(\lambda_1-1)K}{\lambda_1}\right]$. We then have $\int_{\frac{\lambda_1 x_1 - x_2}{\lambda_1-1}-K}^{\frac{\lambda_1 x_1 - x_2}{\lambda_1-1}+K} f_\sigma(a)da \le \int_{-K}^{+K} f_\sigma(a)da$ since $[-K,+K]$ is the interval of length $2K$ on which the integral of f_σ has the highest value. Thus $S_1(x_1,x_2) \le \int_{-K}^{+K} f_\sigma(a)da < \frac{1}{2}$. It implies that $M_A(x_2) = \sup_{x_1;\ \text{Case A}} S_1(x_1,x_2) < \frac{1}{2}$.

Case (B.) If $x_1 \le x_2 - \frac{(\lambda_1-1)K}{\lambda_1}$, then $S_1(x_1,x_2) = \int_{\frac{\lambda_1 x_1 - x_2}{\lambda_1-1}-K}^{\frac{(\lambda_1-1)K+\lambda_1 x_1 + x_2}{\lambda_1+1}} f_\sigma(a)da$ (see Proposition 1). Here $\frac{\partial S_1}{\partial x_2} > 0$, thus $x_2 \mapsto M_B(x_2)$ is an increasing function. Moreover $M(0) < \frac{1}{2}$ according to Proposition 2 (ii.), thus $M_B(0) \le M(0) < \frac{1}{2}$. Since $\lim_{x_2 \to +\infty} M_B(x_2) = 1$ and M_B is a continuous strictly increasing function, thus: there exists a unique $\beta > 0$ such that $M_B(\beta) = \frac{1}{2}$.

- If $x_2 \ge \beta$, then $M_B(x_2) \ge \frac{1}{2}$, thus $M(x_2) \ge \frac{1}{2}$, which implies that $x_2 \notin X_2^*$.
- If $0 \le x_2 < \beta$, then $M_B(x_2) < \frac{1}{2}$.

Case (B'.) If $x_1 \ge x_2 + \frac{(\lambda_1-1)K}{\lambda_1}$, then $S_1(x_1,x_2) = \int_{\frac{(1-\lambda_1)K+\lambda_1 x_1 + x_2}{\lambda_1+1}}^{\frac{\lambda_1 x_1 - x_2}{\lambda_1-1}+K} f_\sigma(a)da$ (see Proposition 1). Here $\frac{\partial S_1}{\partial x_2} < 0$, thus $x_2 \mapsto M_{B'}(x_2)$ is a decreasing function. Moreover $M(0) < \frac{1}{2}$ according to Proposition 2 (ii.), thus $M_{B'}(0) \le M(0) < \frac{1}{2}$, and $\lim_{x_2 \to -\infty} M_{B'}(x_2) = 1$. By symmetry (since f_σ is an even function), for the same $\beta > 0$, we have $M_{B'}(\beta) = \frac{1}{2}$.

– If $x_2 \leq -\beta$, then $M_{B'}(x_2) \geq \frac{1}{2}$, thus $M(x_2) \geq \frac{1}{2}$, which implies that $x_2 \notin X_2^*$.

– If $-\beta < x_2 \leq 0$, then $M_{B'}(x_2) < \frac{1}{2}$.

Conclusion: To sum up, we have:

– If $x_2 \geq \beta$ or $x_2 \leq -\beta$, then $x_2 \notin X_2^*$.

– If $0 \leq x_2 < \beta$, then $M_B(x_2) < \frac{1}{2}$. Since $M_A(x_2) < \frac{1}{2}$ and $M_{B'}(x_2) \leq M_B(x_2)$ for $x_2 \geq 0$, we can conclude that $M(x_2) < \frac{1}{2}$, i.e., $x_2 \in X_2^*$.

– If $-\beta < x_2 \leq 0$, then $M_{B'}(x_2) < \frac{1}{2}$. Since $M_A(x_2) < \frac{1}{2}$ and $M_B(x_2) \leq M_{B'}(x_2)$ for $x_2 \leq 0$, we can conclude that $M(x_2) < \frac{1}{2}$, i.e., $x_2 \in X_2^*$.■

A.6 Proof of Proposition 4

We proceed in three steps.

First step. We first show that $\lambda_1 \mapsto \sigma^*(\lambda_1)$ is an increasing function on $(1, +\infty)$. Recall that according to the proof of Lemma 2 in Appendix A.3, σ^* is unique and defined by:

$$g_1(\sigma^*) = \int_{\frac{(1-\lambda_1)K}{\sigma^*\lambda_1}}^{\frac{(\lambda_1+1)K}{\sigma^*\lambda_1}} f_1(z)dz = \frac{1}{2} \tag{A10}$$

Using the implicit function theorem, we know that $\frac{d\sigma^*}{d\lambda_1} = -\frac{\frac{\partial g_1}{\partial \lambda_1}}{\frac{\partial g_1}{\partial \sigma^*}}$. We have $\frac{\partial g_1}{\partial \sigma^*} = -\frac{\frac{(\lambda_1+1)K}{\lambda_1}}{\sigma^{*2}} f_1\left(\frac{\frac{(\lambda_1+1)K}{\lambda_1}}{\sigma^*}\right) + \frac{\frac{(1-\lambda_1)K}{\lambda_1}}{\sigma^{*2}} f_1\left(\frac{\frac{(1-\lambda_1)K}{\lambda_1}}{\sigma^*}\right) < 0$. Concerning $\frac{\partial g_1}{\partial \lambda_1} = -\frac{K}{\sigma^*\lambda_1^2} f_1\left(\frac{(\lambda_1+1)K}{\sigma^*\lambda_1}\right) + \frac{K}{\sigma^*\lambda_1^2} f_1\left(\frac{(1-\lambda_1)K}{\sigma^*\lambda_1}\right)$, given that f_1 is an even function, strictly increasing on $\mathbb{R}_-$ and strictly decreasing on $\mathbb{R}_+$, $f_1\left(\frac{(\lambda_1+1)K}{\sigma^*\lambda_1}\right) < f_1\left(\frac{(1-\lambda_1)K}{\sigma^*\lambda_1}\right)$ thus $\frac{\partial g_1}{\partial \lambda_1} > 0$. Since $\frac{\partial g_1}{\partial \sigma^*} < 0$ and $\frac{\partial g_1}{\partial \lambda_1} > 0$, then $\frac{d\sigma^*}{d\lambda_1} > 0$ according to the implicit function theorem.

Second step. Now we show that $\lim_{\lambda_1 \to 1^+} \sigma^*(\lambda_1) = 0$. According to Equation (A10), $\sigma^* = \sigma^*(\lambda_1)$ is defined by $\int_{\frac{(1-\lambda_1)K}{\lambda_1\sigma^*}}^{\frac{(\lambda_1+1)K}{\lambda_1\sigma^*}} f_1(z)dz = \frac{1}{2}$, for all $\lambda_1 > 1$. $\lambda_1 \mapsto \sigma^*(\lambda_1)$ is an increasing function on $\lambda_1 > 1$, with $\sigma^*(\lambda_1) > 0$, i.e., $\lambda_1 \mapsto \sigma^*(\lambda_1)$ has a lower bound on $\lambda_1 > 1$. Thus there is a limit $\sigma_0^* = \lim_{\lambda_1 \to 1^+} \sigma^*(\lambda_1)$, with $\sigma_0^* \geq 0$. We must have $\lim_{\lambda_1 \to 1^+} \int_{\frac{(1-\lambda_1)K}{\lambda_1 \sigma^*}}^{\frac{(\lambda_1+1)K}{\lambda_1 \sigma^*}} f_1(z)dz = \int_{\lim_{\lambda_1 \to 1^+} \frac{(1-\lambda_1)K}{\lambda_1 \sigma^*}}^{\lim_{\lambda_1 \to 1^+} \frac{(\lambda_1+1)K}{\lambda_1 \sigma^*}} f_1(z)dz = \frac{1}{2}$. If $\sigma_0^* > 0$, then $\int_{\lim_{\lambda_1 \to 1^+} \frac{(1-\lambda_1)K}{\lambda_1 \sigma^*}}^{\lim_{\lambda_1 \to 1^+} \frac{(\lambda_1+1)K}{\lambda_1 \sigma^*}} f_1(z)dz = \int_0^{\frac{2K}{\sigma_0^*}} f_1(z)dz < \frac{1}{2}$. It is impossible, thus $\sigma_0^* = 0$, and we find $\int_{\lim_{\lambda_1 \to 1^+} \frac{(1-\lambda_1)K}{\lambda_1 \sigma^*}}^{\lim_{\lambda_1 \to 1^+} \frac{(\lambda_1+1)K}{\lambda_1 \sigma^*}} f_1(z)dz = \int_0^{+\infty} f_1(z)dz = \frac{1}{2}$.

Third step. We finally show that $\lim_{\lambda_1 \to +\infty} \sigma^*(\lambda_1) = \sigma^{**}$, and that σ^{**} does not depend on λ_1. First, recall that according to the proof of Lemma 2 in Appendix A.3, σ^{**} is defined by $g_2(\sigma^{**}) = \int_{-K}^{K} \frac{1}{\sigma^{**}} f_1\left(\frac{a}{\sigma^{**}}\right) da = \frac{1}{2}$; thus σ^{**} does not depend on λ_1. Second, $\lambda_1 \mapsto \sigma^*(\lambda_1)$ is an increasing function on $\lambda_1 > 1$, with $0 < \sigma^*(\lambda_1) \leq \sigma^{**}$, i.e., $\lambda_1 \mapsto \sigma^*(\lambda_1)$ has a upper bound on $\lambda_1 > 1$. Thus there is a limit $\sigma_\infty^* = \lim_{\lambda_1 \to +\infty} \sigma^*(\lambda_1)$, with $0 < \sigma_\infty^* \leq \sigma^{**}$. We must have $\lim_{\lambda_1 \to +\infty} \int_{\frac{(1-\lambda_1)K}{\lambda_1 \sigma^*}}^{\frac{(\lambda_1+1)K}{\lambda_1 \sigma^*}} f_1(z)dz = \int_{\lim_{\lambda_1 \to +\infty} \frac{(1-\lambda_1)K}{\lambda_1 \sigma^*}}^{\lim_{\lambda_1 \to +\infty} \frac{(\lambda_1+1)K}{\lambda_1 \sigma^*}} f_1(z)dz = \int_{\frac{-K}{\sigma_\infty^*}}^{\frac{K}{\sigma_\infty^*}} f_1(z)dz = \frac{1}{2}$. Since $\int_{\frac{-K}{\sigma^{**}}}^{\frac{K}{\sigma^{**}}} f_1(z)dz = \frac{1}{2}$, then $\sigma_\infty^* = \sigma^{**}$, i.e., $\lim_{\lambda_1 \to +\infty} \sigma^*(\lambda_1) = \sigma^{**}$.

<u>Conclusion</u>: We have shown that $\lambda_1 \mapsto \sigma^*(\lambda_1)$ is an increasing function on $(1, +\infty)$, with $\lim_{\lambda_1 \to 1^+} \sigma^*(\lambda_1) = 0$ and $\lim_{\lambda_1 \to +\infty} \sigma^*(\lambda_1) = \sigma^{**}$, and that σ^{**} does not depend on λ_1. ■