Public Debt, Endogenous Growth Cycles and Indeterminacy

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Abstract

This paper presents a theoretical setup for studying nonlinear effects of pubic debt in an endogenous growth model with cycles. Our results are threefold. (i) From a long-run perspective, our model exhibits multiplicity, i.e. a high-growth and a low-growth balanced growth path (BGP), due to the interaction between the government's budget constraint and households optimal saving behavior. (ii) Turning to local dynamics, while the high equilibrium is saddle-path stable, the topological behavior of the low equilibrium is more complex. Indeed, the low BGP can be locally determined, over-determined, or under-determined. In the latter case, a supercritical Hopf bifurcation occurs, leading to limit-cycles. (iii) As regards global dynamics, three typical configurations arise: local and global determinacy; local determinacy and global indeterminacy; local and global indeterminacy. Specifically, global bifurcations can emerge, in relation with the degree of social acceptance to reduce non-distorsive components of government budget.

Keywords: Growth, Cycles, Public debt, Multiplicity, Indeterminacy, Bifurcations

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1. Introduction

Assessing the effects of public debt and fiscal deficits on economic growth is a major issue in public debates. Starting the 1970s, the ratio of public debt to GDP presents an upward trend in many developed or developing countries. This trend was amplified by fiscal stimuli in response to the Great Recession, but the high level of budget deficit and debt is a persistent and structural characteristic of industrialized economies for forty years. This context raises two major questions. First, how does the relationship between economic and public debt change at high public debt levels? Second, can we take into account the diversity of historical experiences of highly-indebted countries (see, for example, Eberhardt and Presbitero, 2015)?

Although there has been a multitude of empirical paper focusing on the nonlinear effects of public debt on economic growth,[2] only few theoretical literature addresses these issues. Effectively (i) traditional neoclassical of endogenous growth model abstract

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[2]See, e.g., Reinhart and Rogoff (2010); Checherita and Rother (2012); Baum et al. (2013); Pescatori et al. (2014); Eberhardt and Presbitero (2015), among others.

from public debt by assuming a balanced-budget rule in the long-run;[3] (ii) the possible nonlinear effect of fiscal policy is most often studied in the context of short-run business cycle models, without dealing with its implications for long-run economic growth.

The goal of this paper is precisely to provide a simple theoretical setup to study the effects of public debt, in a model exhibiting both endogenous growth and cycles. In this setup, starting from a high debt ratio, we unveil different configurations: an economy can remain blocked in a growth trap, experience large debt and growth cycles, or escape the trap and converge to a high balanced growth path (hereafter, BGP), thus reproducing the diversity of historical experiences.

To this end, we built an endogenous growth model based on Barro (1990)'s archetype, with productive public expenditures modeled as flows of productive services in a constant return-to-scale production function. Minea and Villieu (2010, 2012), Nishimura et al. (2015b), have shown that such a configuration is compatible with the existence of a growing public debt in the long-run. In this paper, we extend this analysis by considering the way productive or unproductive expenditures adjust to the debt burden. A number of empirical findings have emphasized that the success of fiscal adjustment crucially depends on the ability of cutting-back unproductive spending, such that wages, transfers or government consumption, as advocated by, e.g., Alesina and Perotti (1997), and Alesina and Ardagna (2010), among others. Similarly, according to proponents of "fiscal space" (see Ostry et al., 2010, 2015), public debt sustainability is ensured only if the primary budget surplus positively reacts to the debt burden, which may not be the case at high public debt ratios, particularly if there is some resistance to changes in the current fiscal stance. Clearly, this resistance may depend on the public debt ratio itself, because, from households point of view, the marginal cost of cuts in public spending increases as their level declines, especially for "unproductive" spending, such as wages, transfers or public consumption, possibly welfare-enhancing, expenditures.

The originality of our paper is to introduce such a mechanism into a standard endogenous growth model. To this end, we suppose that a share of the debt burden can be offset by adjustments in unproductive public spending, which decreases as the public debt-to-GDP ratio increases, reflecting the contraction of the fiscal space. The social acceptance to finance the debt burden by adjustments in unproductive spending is made endogenous through a "tenseness" function defining the share of the debt burden that must be financed either by new debt, or a cut in productive spending.

The main value of our approach is to develop an original method that allows fully characterizing the short and long-run behaviour of the economy. Based on a very simple small-scale (two dimensional) dynamic system, we show that quite rich dynamics appear, depending, in particular, on the elasticity of the tenseness function. Effectively, small changes in this elasticity generate radical shifts in the dynamics properties of the economy, i.e., bifurcations.

Our results are threefold.

(i) From a long-run perspective, our model exhibits multiplicity, i.e. a high-growth and a low-growth equilibria, due to the interaction between the government's budget constraint and households' optimal saving behavior. We notably exhibit a threshold effect of public debt on economic growth along the high BGP. Indeed, fiscal deficits

[3] With the notable exception of Minea and Villieu (2012), Boucekkine et al. (2015), Menuet et al. (2015), Nishimura et al. (2015a), Nishimura et al. (2015b).

bring additional resources for productive expenditures but also increase the debt burden. As long as the public debt remains sufficiently low, the debt burden can be mainly absorbed by a downward adjustment of unproductive expenditures, and the positive effect of debt on productive public spending and economic growth prevails. As government debt increases, however, so does the resistance of unproductive spending to downward pressures; and, from a certain threshold, the negative effect of the debt burden overcomes the positive effect of deficits.

(ii) Turning to local dynamics, while the high equilibrium is always saddle-path stable, the topological behavior of the low equilibrium is more complex. According to the elasticity of the tenseness function (which serves as a bifurcation parameter), the low BGP can be locally determined (stable saddle-path), over-determined (unstable), or under-determined (stable). In this last case, a supercritical Hopf bifurcation occurs, leading to limit-cycles.

(iii) The simplicity of our framework allows fully characterizing the global dynamics of the system. We highlight three typical configurations: local and global determinacy; local determinacy and global indeterminacy; local and global indeterminacy.[4] These configurations crucially depend on the initial level of public debt, the response of economic growth to an increase in productive expenditures along the low BGP, and the elasticity of the tenseness function. Specifically, global bifurcations can emerge, depending on the value of the latter parameter. Indeed, if there is strong resistance to adjust non-distorsive spending (namely, a narrow fiscal space), the economy is trapped in a region with high public debt and low growth, which can lead either to a stagnation equilibrium, or a low-growth limit-cycle with large oscillating public debt. In the case of a large acceptance to cut-back unproductive spending, in contrast, the fiscal adjustment can preserve productive expenditures, and the economy can escape from the low growth (unstable) equilibrium and converge towards the high BGP.

Although principally oriented to methodological issues, our model addresses major long-lasting topics in macroeconomics.

First, our work echoes old theoretical studies (see, e.g., Blinder and Solow, 1973; Sargent and Wallace, 1981; Liviatan, 1984) showing that the way the debt burden is accommodated may have crucial implications for the stability of the BGPs. Our setup also stresses the importance of introducing the concept of fiscal space in a simple macroeconomic framework. Thanks to local and global bifurcations (depending on the elasticity of the tenseness function), our findings can also replicate the diversity of empirical findings that exhibit contradictory results on the effect of fiscal adjustment in OECD countries, in line with the discussions on the so-called "expansionary austerity" (see, e.g., Guajardo et al., 2014).

Second, our framework gives rise to a theoretical mechanism that generate an endogenous threshold in the relationship between public debt and long-run economic growth.

[4] As usual, local indeterminacy is associated with the existence of a continuum of equilibrium paths, which, starting from different initial conditions in the neighborhood of a specific BGP, converge towards this BGP. Global indeterminacy is defined by the existence of multiple equilibrium paths starting from a given initial condition and converging towards different stationary BGPs. In a context of multiple BGPs, global indeterminacy leads to a selection problem: starting from the same initial values of the public debt ratio, either the low BGP or the high BGP can be reached.

This threshold rests in particular on the form of the tenseness function, and highlights the important role of social acceptance to unproductive expenditures adjustments. This is an important finding because, to the best of our knowledge, this is the first paper that provides a theoretical basis to the famous and large empirical literature emphasizing thresholds in the effect of debt on economic growth.

Third, following the pioneer work of Schmitt-Grohé and Uribe (2004) a number of contributions illustrate that a balanced-budget rule can lead to aggregate instability (see, e.g., Giannitsarou, 2007; Linnemann, 2008; Ghilardi and Rossi, 2014). Typically, these model focus on time-varying tax-rates in neoclassical growth models. In the present model, the government budget constraint also is a source of aggregate instability, but we consider a general set of deficit rules (including the balanced-budget rule) in an endogenous growth model. More fundamentally, we show that aggregates fluctuations can arise with constant tax-rate but time-varying productive expenditures.

Fourth, our paper joins an important strand of literature that addresses the question of indeterminacy in endogenous growth models. Although the literature on indeterminacy is almost exclusively based on local analysis,[5] some major contributions have stressed the relevance of global analysis. In this respect, Matsuyama (1991), in a very stimulating work, presents a comprehensive two-sector model to study global bifurcations, according to the value of time-preference. This pioneer setup has been notably developed by Benhabib and Nishimura (1998), Matsuyama (1999), Benhabib et al. (2008), Boldrin et al. (2001) and, in models with public spending, Raurich (2001), Brito and Venditti (2010). In the existing literature, global indeterminacy comes from positive externalities, associated to increasing returns or two-sector frameworks. In contrast, in our model, global indeterminacy is established in a simple one-sector model and does not fundamentally rest on increasing returns in production. Indeed, it rather resorts to the nonlinearity between public debt and productive public spending in the Government's budget constraint. Outstandingly, our results exemplify the famous "History versus Expectations" story (Krugman, 1991; Matsuyama, 1991), according to which equilibrium selection depends alternatively on the set of initial conditions or self-fulfilling expectations. In our model, both scenarios can happen, depending on the propensity of the society to accept a non-distorsive-based fiscal adjustment.

The rest of the paper is organized as follows. Section 2 presents the model, Section 3 computes the steady-state, Section 4 focuses on local dynamics while Section 5 addresses global analysis and Section 6 concludes.

2. The Model

We consider a simple continuous-time endogenous-growth model with three infinitely-lived agents: a representative consumer, a competitive firm and a government. All agents have perfect foresight.

[5] There is a rich literature focusing on local indeterminacy in endogenous growth models, see the surveys of Benhabib and Farmer (1999), chap. 6, or Mino et al. (2008).

2.1. The household

Each period (we assimilate a period to a unit of time), the representative household is endowed with a fixed amount of labor that is normalized to unity. Preferences are described by the following utility function depending on the flow of per-period consumption (C_t)

$$U = \int_0^\infty u\left(C_t\right) \exp\left(-\rho t\right) dt, \tag{1}$$

where U defines the expected intertemporal welfare, which is the discounted sum of utility per period, with ρ the subjective discount rate. To obtain endogenous growth solutions, we define a constant-elasticity of substitution (CES) utility function

$$u\left(C_t\right) = \begin{cases} \frac{S}{S-1}\left\{\left(C_t\right)^{\frac{S-1}{S}} - 1\right\}, & \text{if } S \neq 1 \\ \log\left(C_t\right), & \text{if } S = 1, \end{cases} \tag{2}$$

with $S := -u_{cc}C_t/u_c > 0$ (using $u_c := du\left(C_t\right)/dC_t$) the elasticity of intertemporal substitution in consumption.

The household enters period t with initial (predetermined) stocks of private capital (K_t) and government indexed bonds (B_t), whose returns are respectively q_t (the rental rate of capital) and R_t (the gross real interest rate). He perceives wages (w_t) and has do decide how much to consume (C_t) and save during the period. Since we consider a closed economy, the only forms of asset accumulation are capital $(\dot{K}_t)$ and government bonds $(\dot{B}_t)$. In addition, he pays a flat-tax rate $(\tau \in (0,1))$ on total income, and receives a lump-sum transfer (X_t, positive or negative) from the government; hence the following budget constraint:

$$\dot{K}_t + \dot{B}_t = (1-\tau)\left(R_t B_t + q_t K_t + w_t\right) - C_t - \delta K_t + X_t. \tag{3}$$

The first order conditions for the maximization of the consumer's programme give rise to the familiar Keynes-Ramsey relationship

$$\frac{\dot{C}_t}{C_t} = S\left[(1-\tau)R_t - \rho\right], \tag{4}$$

while the trade-off between public debt accumulation and private capital accumulation provides the following condition

$$(1-\tau)\, R_t = (1-\tau)\, q_t - \delta, \tag{5}$$

and the optimal path has to verify the set of transversality conditions:

$$\lim_{t\to+\infty}\left\{\exp(-\rho t)\ u_c\left(C_t\right) K_t\right\} = 0 \text{ and } \lim_{t\to+\infty}\left\{\exp(-\rho t)\ u_c\left(C_t\right) B_t\right\} = 0,$$

which ensure that lifetime utility U is bounded.[6]

[6] On the BGP associated to constant growth and interest rates (γ^* and R^*, respectively), these conditions correspond to the no-Ponzi game constraint $\gamma^* < (1-\tau)R^*$. Such condition ensures that public debt will be repaid in the long run, and does not preclude the possibility that $\gamma > (1-\tau)$ in the short run.

2.2. Firms

Output (Y_t) is produced with the stock of capital and a flow of labor, using a Cobb-Douglas technology, augmented with a public good externality, namely: $Y_t = AK_t^{\alpha}\left(L_tG_t\right)^{1-\alpha}$. A is a scale parameter and G_t denotes productive public spending (i.e. public expenditures that positively affect the marginal productivity of labor, for example); and $\alpha \in (0,1)$ is the elasticity of output to private capital. With population normalized to one, this specification is the same as Barro (1990), and the production function exhibits decreasing marginal productivity of private capital and constant returns to scale, in order to generate an endogenous growth path in the long run.

First order conditions for profit maximization are usual

$$q_t = \alpha A \left(G_t/K_t\right)^{1-\alpha}, \tag{6}$$

$$w_t = (1-\alpha)\, AK_t \left(G_t/K_t\right)^{1-\alpha}. \tag{7}$$

2.3. Government

To study the impact of public deficits and debts on economic growth, we need to escape from the balanced-budget rule (hereafter BBR) used in Barro (1990). A number of recent papers have shown that a balanced endogenous growth path is feasible when allowing for a general budget constraint.[7] Effectively, endogenous growth models are compatible with permanent deficit, namely growing public debt in the long-run. With endogenous growth, output grows continuously along the BGP, and public debt also may grow continuously. Thus, contrary to exogenous growth models, the BBR is not necessarily required in the long-run. The only requirement for the transversality condition to be verified is that the long-run rate of growth of public debt must be less than the long-run real interest rate (here, the rate of return of public debt). If the BGP is such that all variables grow at the same rate, as it will be the case below, an additional requirement is that the debt-to-output ratio must be constant in the long run. Thus we introduce the following government budget constraint

$$\dot{B}_t = R_tB_t + G_t + X_t - \tau\left(R_tB_t + q_tK_t + w_t\right).$$

Defining by $r_t := (1-\tau)\, R_t$ the net real interest rate, we obtain, in equilibrium ($Y_t = q_tK_t + w_t$)

$$\dot{B}_t = r_tB_t + G_t + X_t - \tau Y_t, \tag{8}$$

where r_tB_t is the debt burden.

In this constraint, G_t defines productive public spending and X_t stands for unproductive spending (assimilated to lump-sum transfers to households).[8]

For years, works on the public debt have shown that the stability of the government budget constraint is based on the possibility that the primary surplus can adjust to the

[7]See, e.g., Minea and Villieu (2010, 2012), Nishimura et al. (2015a), Nishimura et al. (2015b), and Menuet et al. (2015) for an extension to money financing.

[8]These expenditures could improve household's welfare without qualitative change in our model. More generally, G_t represents public spending that positively impacts economic growth, while X_t describes all variables in the primary budget balance that have no effect on long-run growth, like, e.g., "consumption" or wasteful expenditures, lump-sum transfers, and so one.

debt burden (see, e.g. Blinder and Solow, 1973, Sargent and Wallace, 1981, or, more recently, Ostry et al., 2010, 2015). Recent contributions on so-called "fiscal space" have emphasized that high public debt ratios can impair the feasibility of such an adjustment, because there is a debt limit beyond which the dynamics of the public debt-to-GDP ratio is explosive (Ostry et al., 2010). This limit comes from the confrontation between two features: the capacity to levy taxes or to cut current expenditures (namely the fiscal space, reflecting the necessary fiscal adjustment to achieve debt sustainability); and the effective debt burden. Specifically, the size of fiscal space negatively depends on the debt-to-GDP ratio, because the government is subject to more difficulties to raise additional revenues as the debt increases. Many empirical papers (see, e.g. Abiad and Ostry, 2005; Mendoza and Ostry, 2008) have shown that the marginal response of the primary balance to debt is significantly weaker at high levels of debt (in particular, in the presence of political pressures).

Additionally, the feasibility of fiscal adjustments also depends on the type of public expenditures that are subject to budget cuts. Indeed, rising public debt ratios lead more likely to reductions in public investment programs (that roughly corresponds to our definition of productive expenditures) rather than in transfers or wages expenditures, as well documented in the empirical literature.[9] Alesina and Perotti (1997) explain this feature by "political realities" suggesting a greater facility to cut back investment spending than current expenditures. As a result, productive expenditures become the adjustment variable of fiscal programmes, with possible detrimental consequences on economic growth and debt sustainability. Therefore, the fiscal space can be narrowed by the propensity of households to accept cuts in unproductive spending.

To account for this mechanism in our model, we suppose that a fraction $x \in (0, 1)$ of the debt burden can be accommodated by unproductive spending, namely $X_t = x r_t B_t$, and that there is some resistance to fiscal adjustment, such that this fraction gets lower as the public debt ratio increases, namely $x := x(B_t/Y_t)$, where $x' < 0$. Hence, as the debt ratio rises, the adjustment to the growing debt burden will rely increasingly on new debt (which inflates future debt burden) or on productive public spending (with a detrimental effect on economic growth). Consequently, the government budget constraint becomes, with $\eta(\cdot) := 1 - x(\cdot)$ the residual debt burden

$$\dot{B}_t = \eta\left(B_t/Y_t\right) r_t B_t + G_t - \tau Y_t. \tag{9}$$

Function $\eta(\cdot)$ defines the "tenseness" on the adjustment of unproductive primary expenditures and reflects social resistance to such adjustment, in accordance with "fiscal space" mechanism. The tenseness function $\eta(\cdot)$ is differentiable, strictly increasing, non-negative, with elasticity $\varepsilon(s) := s\eta'(s)/\eta(s) > 0$, for any $s \geq 0$, and $\eta(0) =: \eta_0 \in [0, 1]$.

Finally, to close the model, we must introduce a hypothesis on the public debt path. Clearly, to obtain an endogenous growth path in the long-run, the trajectory of productive public spending must be endogenous in (9). To determine this trajectory, a restriction on the public debt path is needed. Barro (1990) introduces such a restriction by considering only the balanced-budget rule (BBR) with zero public debt (namely $B_t = \dot{B}_t = 0$ in (9)). In this paper, we extend this analysis by considering general deficit rules $\dot{B}_t = \theta Y_t$, with

[9] See, for example, de Haan et al. (1996), Balassone and Franco (2000), or, more recently, IMF (2014).

a constant deficit-to-output ratio $\theta \geqslant 0$.[10] The case $\theta = 0$ characterizes the BBR, which corresponds to a constant, but non necessarily zero public debt. Contrary to the BBR, this deficit rule authorizes permanent deficits in the long run, thus reproducing stylized facts since, from 1970 to 2005 (namely, before the Great Recession of 2007) the average public deficit in the OECD countries is about 2.5% of GDP (namely $\theta = 0.025$).

2.4. Equilibrium

To solve the model, we express variables as ratios that are stationary in the long run. In this way, we deflate all growing variables by the stock of capital (and we henceforth remove time indexes for notational convenience), namely: $b_k = B/K$, $y_k = Y/K$, $c_k = C/K$ and $g_k = G/K$. Thus, the path of the capital stock is obtained from the goods market equilibrium

$$\frac{\dot{K}}{K} = y_k - c_k - g_k - \delta, \tag{10}$$

with technology defined as $y_k = Ag_k^{1-\alpha}$, and the government budget constraint becomes

$$\theta y_k = (1 - x(\cdot))\, r b_k + g_k - \tau y_k, \tag{11}$$

where the real interest rate is $r_t = (1-\tau)\, q_t - \delta = \alpha(1-\tau)\, y_k - \delta$.

We first solve the steady-state, before studying local and global dynamics.

3. The steady-state

In the steady-state, the rate of interest is constant, together with all capital-deflated variables. Therefore, all variables in level grow at the same balanced rate (with a star denoting steady-state values)

$$\gamma^* = \dot{C}/C = \dot{K}/K = \dot{B}/B = \dot{G}/G = \dot{Y}/Y.$$

3.1. The multiplicity of BGPs

To compute the steady-state, we proceed by induction. First, the long-run value of the public debt ratio is: $b_k^* = (B/\dot{B})(\dot{B}/K) = \theta y_k^*/\gamma^*$. Knowing this value, we derive the productive public spending ratio in (11), namely: $g_k^* = \tau y_k^* + \theta y_k^* \left[1 - (1 - x(\theta/\gamma^*))\, r^*/\gamma^*\right]$. Simplifying by $g_k^{*1-\alpha}$ and introducing the real interest rate $r^* = \alpha(1-\tau)\, g_k^{*1-\alpha} - \delta$, we get

$$\frac{\gamma}{\eta(\theta/\gamma)} = r\left\{1 + \frac{1}{\theta}\left(\tau - \frac{1}{A}\left[\frac{r+\delta}{\alpha(1-\tau)}\right]^{\alpha/(1-\alpha)}\right)\right\}^{-1}. \tag{12}$$

Eq. (12) provides a first relation between the long-run rate of economic growth (γ^*) and the real interest rate (r^*). The second relation comes from the Keynes-Ramsey rule (4) in steady-state

$$\gamma^* = S(r^* - \rho). \tag{13}$$

We obtain the BGPs at the intersection of these two relations. The following Proposition establishes the existence and multiplicity of BGPs.

[10] One could also specify a deficit rule with a gradual adjustment path of the deficit-to-output ratio to a long-run target (say θ^*), as in Menuet et al. (2015). This would only add one autonomous equation, associated with a stable eigenvalue (equal to the speed of adjustment of the deficit ratio), in the reduced form used in Section 4, without change in the model.

Proposition 1. *(Multiplicity of BGPs) There is a non-empty set of parameters $\mathscr{C}$, such that, for small $\theta \geqslant 0$ and $\eta(\cdot) \in [0,1]$, two and only two BGPs characterize the long-run solution of the model: a high BGP (γ^H) and a low BGP (γ^L), where $0 < \gamma^L < \gamma^H$.*

Proof: We adopt a two-step proof. The first step shows that there are two and only two BGPs for $\theta = 0$, and the second step extends the proof for small $\theta > 0$.

Step 1. A BGP is defined by a couple (r^*, γ^*) that simultaneously satisfies Eqs. (12) and (13). For $\theta = 0$ and $\gamma > 0$, Eq. (12) coincides with

$$r^* = \alpha(1-\tau)(A\tau)^{\frac{1-\alpha}{\alpha}} - \delta =: r^B, \tag{14}$$

and the associated rate of economic growth is, by (13): $\gamma^B = S\left(r^B - \rho\right) > 0$. The couple (r^B, γ^B) characterizes the "Barro solution", which is obtained at point B in Figure 1. For $\theta = 0$ and $\gamma = 0$, equation (12) is not defined, but the interest rate that corresponds to the no-growth solution $\gamma^S = 0 < \gamma^B$ is simply $r^S = \rho < r^B$ in Eq. (13). The couple (r^S, γ^S), depicted by point S in Figure 1, characterizes the "Solow solution". The associated values of the productive expenditure and output ratios are, respectively,

$$g_k^B = (\tau A)^{1/\alpha}, \; y_k^B = A\left(g_k^B\right)^{1-\alpha}; \tag{15}$$

$$g_k^S = \left[\frac{\rho+\delta}{\alpha(1-\tau)A}\right]^{\frac{1}{1-\alpha}}, \; y_k^S = \frac{\rho+\delta}{\alpha(1-\tau)}; \tag{16}$$

and the public debt ratio is $b_k^B = 0$ along the Barro BGP, while along the Solow BGP, it is defined by $b_k^S = b_y^S / y_k^S > 0$, where the debt to output ratio b_y^S is implicitly determined in the following relation

$$\eta\left(b_y^S\right) b_y^S = \frac{1}{A\rho}\left\{A\tau - \left(g_k^S\right)^\alpha\right\}.$$

Step 2. For $\theta > 0$, Eqs (12)-(13) intersect twice, as described in Figure 1, where point H characterizes the high BGP, while point L denotes the low BGP. On the one hand, since θ is small, Eq. (12) characterizes an increasing convex relation between γ and r.[11] On the other hand, Eq. (13) defines an increasing line between γ and r. Then, there is a non-empty set of parameters, denoted by $\mathscr{C}$, such that Eqs. (12) and (13) intersect twice in the plane $\mathbb{R}^+ \times \mathbb{R}^+$.

For θ small enough, we can compute a approximate value of both solutions by using a linear approximation of Eq. (13) in the neighborhood of S and B respectively. By rewriting Eq. (13) as

$$f(r^*) := \frac{1}{A}\left[\frac{r^*+\delta}{\alpha(1-\tau)}\right]^{\frac{\alpha}{1-\alpha}} - \tau = (\theta/\gamma^*)\left[\gamma^* - \eta(\theta/\gamma^*) r^*\right], \tag{17}$$

the two approximate solutions are (see Appendix A)

$$\gamma^H \approx \gamma^B + \frac{S\theta}{\bar{\eta} f'(r^B)}(\bar{\eta} - \eta_0), \tag{18}$$

[11] Indeed, when θ is small and $\eta(0) < +\infty$, we have $\partial\{\gamma/\eta(\theta/\gamma)\}/\partial r = (\partial\gamma/\partial r)/\eta_0$.

and

$$\gamma^L \approx \frac{\theta}{b_y^S}, \tag{19}$$

where $\bar{\eta} := \gamma^B / r^B \in (0,1)$ and $f'(r^B) > 0$.

$\square$

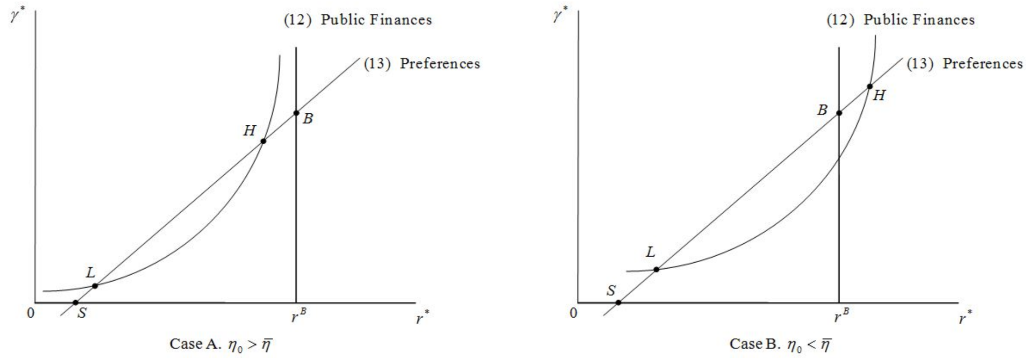


Figure 1: Multiplicity of BGPs

The intuitive explanation of this multiplicity is the following. A high real interest rate is synonymous with high profitability of capital, which, in turn, implies a high rate of economic growth in the Keynes-Ramsey relationship (13). Simultaneously, a high rate of economic growth means that the debt burden will be relatively low, with little crowding-out effect on productive public expenditures in the government budget constraint (12), which ensures a high profitability of private capital and a high real interest rate. Consequently, there can be two equilibria, one associated with high expected real interest rate and economic growth, and the other associated with expectations of low growth and capital profitability. High growth, by reducing the debt burden, ensures high profitability of private capital, which further enhances growth, while low growth magnifies the crowding-out effect of debt on productive public spending, which, in turn, decreases growth.

Fundamentally, the multiplicity comes from the interaction of the government budget constraint and the Keynes-Ramsey relationship governing households' saving behaviour, as explained in Minea and Villieu (2012). What is new here is that, with flexible values of $\eta(\cdot)$, the high-growth BGP can exceed the Barro solution (as in Case B of Figure 1), contrary to the solution considered in Minea and Villieu (2012) with $\eta = 1$ (as in Case A of Figure 1). This change in the model has very strong implications, both on short-run and on long-run dynamics, because, as we will show, the topological characteristics of the low BGP are very sensitive to the tenseness function $\eta(\cdot)$.

3.2. The effect of public deficit and debt in the long-run

This Subsection examines comparative statics of BGPs with respect to changes in the deficit ratio.

Proposition 2. *For a given long-run economic growth* (γ^*)*:*

(i) *any increase in the deficit ratio rises economic growth along the low BGP;*

(ii) *any increase in the deficit ratio reduces economic growth along the high BGP if* $\eta_0 > \bar{\eta}$*, but rises it if* $\eta_0 < \bar{\eta}$*, where:* $\bar{\eta} := \gamma^B/r^B \in (0,1)$*.*

Proof: By (18) and (19) we obtain immediately

$$\left.\frac{d\gamma^H}{d\theta}\right|_{\theta\to 0} = \frac{S(\bar{\eta}-\eta_0)}{\bar{\eta}f'(r^B)} \text{ and, } \left.\frac{d\gamma^L}{d\theta}\right|_{\theta\to 0} = 1/b_y^S > 0.$$

□

The intuition of Proposition 2 is the following. The effect of deficit on public finance is (i) to increase resources for productive expenditure, and (ii) to rise the debt burden. The permanent flux of new resources provided by the deficit is simply $\dot{B}$, while the debt burden which is not sterilized by unproductive spending is $\eta r B$. Along the low BGP, the increase in deficit always promotes economic growth, because it reduces the debt burden (as a share of output) in steady-state. Indeed, productive spending and economic growth are so low that the new resources provided by the deficit allow reducing the debt ratio in the long-run. However this pleasant effect of deficit only holds if the return of productive public spending is high enough, namely in the neighborhood of the no-growth trap. In contrast, along the high BGP, an increase in the deficit ratio will promote economic growth if and only if $\dot{B} > \eta r B$, namely, for small deficit ratio, iff $\gamma^B/r^B > \eta_0$, hence the results in Proposition 2. Thus, along the high BGP, an increase in the deficit ratio will impede long-run economic growth if $\eta_0 > \bar{\eta}$ and, consequently, the high-growth solution associated with a positive deficit ratio cannot exceed the Barro solution with zero deficit (Case A in Figure 1). In contrast, if $\eta_0 < \bar{\eta}$, any increase in the deficit ratio promotes long-run economic growth, and the high-growth BGP can exceed the Barro solution (Case B in Figure 1).

More generally, our model produces a threshold effect of the debt-to-output ratio along the high BGP ($b_y^H := \theta/\gamma^H$), which is established in the following Proposition.

Proposition 3. *(Nonlinear effect of public debt) Along the high BGP, there are two bounds* $\underline{\alpha}$ *and* $\bar{\eta}_0$*, such that, if* $\varepsilon'(\cdot) \geq 0$*,* $\alpha > \underline{\alpha}$*, and* $\eta_0 < \bar{\eta}_0$*, there is a critical level of the public debt ratio* $\bar{b}_y^H > 0$*, such that, for* b_y^H *in the neighborhood of* $\bar{b}_y^H$*, we have*

$$\frac{d\gamma^H}{db_y^H} \geq 0 \Leftrightarrow \; b_y^H \leq \bar{b}_y^H.$$

Proof: See Appendix A.

This threshold effect is explained as follows. As previously noticed, public deficits bring additional resources for productive expenditures but also increase the debt burden. As long as the public debt remains sufficiently low, the debt burden can be mainly absorbed by a downward adjustment of unproductive expenditures, and the positive effect of debt on productive public spending and economic growth prevails. As government debt increases, however, so does the resistance of unproductive spending to downward pressures; and, from a certain threshold, the negative effect of the residual debt burden to be financed by a decrease in productive spending overcomes the positive effect of deficits. This threshold is defined by $\bar{b}_y^H$, notwithstanding conditions of Proposition 3.

This is an important finding because, to our knowledge, there is no other theoretical model capable of explaining the threshold effects of public debt on growth replicated by many empirical results.[12] Indeed, previous endogenous growth models with wasteful (Saint-Paul, 1992; Futagami and Shibata, 1998) or productive (Minea and Villieu, 2010, 2012) public expenditures have show that higher debts and deficits always impede economic growth. These findings also prevail in our setting, but, only in a high debt context. If public debt is sufficiently low, however, the debt burden can be covered by cuts in unproductive public spending, while the new resources provided by the deficit can be devoted to productive expenditures. Hence, the emergence of a threshold in the relationship between economic growth and public debt.

Let us now turn our attention to local dynamics of the model.

4. Local Dynamics

Outside the steady-state, the dynamics of the model can be summarized by a two-variable reduced-form describing the laws of motion of the consumption and the debt ratios. Effectively, from (4), (9) and (10), we obtain the following system

$$\frac{\dot{b}_k}{b_k} = \frac{\dot{B}}{B} - \frac{\dot{K}}{K} = \frac{\theta A g_k^{1-\alpha}}{b_k} + \delta + g_k + c_k - A g_k^{1-\alpha}, \tag{20}$$

$$\frac{\dot{c}_k}{c_k} = \frac{\dot{C}}{C} - \frac{\dot{K}}{K} = S(r-\rho) + \delta + g_k + c_k - A g_k^{1-\alpha}, \tag{21}$$

where the real interest rate is determined by the marginal return of private capital

$$r = \alpha(1-\tau)A g_k^{1-\alpha} - \delta =: r\left(g_k\right). \tag{22}$$

The relationship between the public spending ratio and the public debt ratio comes from the government budget constraint (11)

$$g_k = (\theta+\tau)A g_k^{1-\alpha} - \eta(b_k/A g_k^{1-\alpha}) r\left(g_k\right) b_k. \tag{23}$$

Equation (23) provides an implicit function (ψ) such that: $b_k = \psi(g_k)$, whose properties are established in the following Lemma.

Lemma 1. *There is a differentiable function $\psi : \mathbb{R}^+ \to [0, \bar{b}_k]$, such that*

(i) $b_k = \psi(g_k)$;

(ii) $\psi(0) = 0$;

(iii) *If* $b_k = 0$, $g_k = 0$ *or* $g_k = \bar{\bar{g}}_k := (\theta+\tau)^{1/\alpha} > g_k^B$.

[12] See, e.g.,Reinhart and Rogoff (2010); Checherita and Rother (2012); Baum et al. (2013); Pescatori et al. (2014); Eberhardt and Presbitero (2015); Egert (2015).

Proof: See Appendix B.

From Lemma 1, function $\psi(\cdot)$ is maximized on $[0, \bar{\bar{g}}_k]$ at the unique point $\bar{g}_k$ such that $\psi'(\bar{g}_k) = 0$, as described in Figure 2.

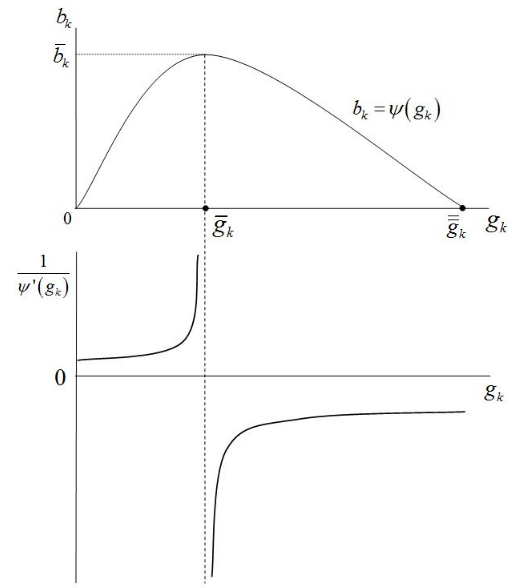


Figure 2: The relation between g_k and b_k

As shows Figure 2, there is a nonlinear relation between the public debt ratio and the productive public spending ratio. The decreasing part of $\psi(\cdot)$ comes, as usual, from the inverse relationship between the debt burden and government spending, for unchanged tax rate and deficit ratio. The increasing part of $\psi(\cdot)$ is more novel and can be explained as follows. As productive spending rises, so does output, which reduces the debt-to-output ratio. Social acceptance to use unproductive spending to finance the debt burden ($\eta(\cdot)$) then increases, which reduces the effective debt burden on productive spending an allows financing more public debt. Hence, for the same debt ratio, there are two possible values of productive public spending, depending on the tenseness on non-distorsive ways of finance.

As a corollary, the implicit function $\psi(\cdot)$ defines a maximum sustainable public debt ratio $\bar{b}_k$ beyond which there is no more possibility to finance productive public spending, according to the fiscal Laffer curve. In this way, our model provides a quite natural definition of the sustainable public debt: given the downward resistance of non-productive expenditures, the public debt ratio cannot exceed a certain threshold, failing which no distorting tax rate would allow to finance the productive expenditure and the residual share of the debt burden. The sustainable public debt ratio depends in particular on the social acceptance to adjust "consumption" public spending, through the tenseness function $\eta(\cdot)$. In this regard, any increase in $\eta(\cdot)$ lowers the maximum sustainable ratio of government debt, because weaker acceptance to resort to non-distorsive ways of finance will lead to a greater weight of the debt burden on productive expenditure, which will impede long-run economic growth and make harder the repayment of public debt.

Noteworthy, in our model, the threshold defining the sustainable public debt ratio is *endogenous* and does not resort on some *ad hoc* definition of public debt regimes.

Lemma 2. *For $\psi'(g_k) \neq 0$, we define the following coefficient*

$$\left.\frac{dg_k/g_k}{db_k/b_k}\right|_i = \frac{\psi(g_k)}{g_k\psi'(g_k)} =: -\mathcal{B}(g_k),$$

where

$$\mathcal{B}(g_k) = \frac{r(g_k)\psi(g_k)\left[\eta(\cdot) + \eta'(\cdot)\psi(g_k)/Ag_k^{1-\alpha}\right]}{\alpha g_k - (1-\alpha)\psi(g_k)\left[-\eta(\cdot)\delta + \eta'(\cdot)r(g_k)\psi(g_k)/Ag_k^{1-\alpha}\right]}.$$

Proof: We directly apply the Implicit Function Theorem in Eq. (17).

Interestingly, since function $\psi(g_k)$ goes through a maximum at $\bar{g}_k$, we can observe in Figure 2b, that coefficient $\mathcal{B}$, which is inversely related to $\psi'(g_k)$, can take virtually any value as high or as low as desired, if equilibrium is located near the asymptote $\bar{g}_k$.

In what follows, to examine wether changes in the stability of steady-states can occur for some critical values of parameters (namely, bifurcations), we will use coefficient $\mathcal{B}^i := \mathcal{B}(g_k^i)$ as our *bifurcation coefficient* for any steady state $i \in \{H, L\}$. Specifically, through g_k^i, coefficient $\mathcal{B}^i$ depends on primer parameters of the model $(\alpha, \rho, S, \delta, ...)$. Therefore, if a bifurcation occurs for a specific value of $\mathcal{B}^i$ (say, $\hat{\mathcal{B}}_i$), we can easily retrieve, by using Lemma 2, the value of a specific primer parameter (or combination of parameters), say $\hat{\alpha}, \hat{\rho}, \hat{S}$, or $\hat{\delta}$,..., that generates the bifurcation (see Subsection 4.3 for an application to the elasticity of the tenseness function $\eta(\cdot)$).

4.1. Local Dynamics and Bifurcations

This Subsection analyzes the deterministic dynamics defined by the reduced-form (20)-(21) in the neighborhood of the low and the high BGP. We first compute the Jacobian matrix by linearizing the model in the neighborhood of both steady-states, before studying the associated characteristic polynomial to assess local determinacy or indeterminacy of the BGPs.

By linearization, in the neighborhood of steady state i, the system (20)-(21) behaves according to $(\dot{b}_k, \dot{c}_k) = \mathbf{J}^i(b_k - b_k^i, c_k - c_k^i)$, where $\mathbf{J}^i$ is the Jacobian matrix. The reduced-form includes one jump variable (the consumption ratio c_k) and one predetermined variable (the public debt ratio b_k, since the initial stocks of public debt B_0 and private capital K_0 are predetermined). For BGP i to be well determined, $\mathbf{J}^i$ must possesses two opposite-sign eigenvalues. The following Proposition computes the characteristic polynomial of the Jacobian matrix for "small" deficit ratio (formally, $\theta \to 0$).[13]

[13]Consequently, our results are establish for BGPs $i \in \{B, S\}$ strictly speaking. By continuity, they remain valid for BGPs $i \in \{H, B\}$ if the deficit ratio is "small" (see Lemma 4 below), which is generally the case in developed countries. A numeric Appendix, available on demand, shows that the stability properties of both steady-states do not change, even with quite high values (exceeding 10% of the GDP) of the deficit ratio.

Proposition 4. *The characteristic polynomial associated to the Jacobian matrix in the neighborhood of BGP i, $i \in \{B, S\}$, is defined by $Q^i(s) := s^2 - T^i s + D^i = 0$, where $T^i := Tr(\mathbf{J}^i)$ is the trace, and $D^i := \det(\mathbf{J}^i)$ the determinant.*
(i) In the neighborhood of the Barro BGP, we have

$$D^B = -\gamma^B c_k^B < 0.$$

(ii) In the neighborhood of the Solow BGP, we have

$$T^S = c_k^S - \mathcal{B}^S (g_k^S)^{1-\alpha}[(g_k^S)^\alpha - (\tilde{g}_k)^\alpha], \tag{24}$$

$$D^S = \Omega^{-1}\mathcal{B}^S, \tag{25}$$

where $\tilde{g}_k := [(1-\alpha)\,A]^{1/\alpha}$ and $\Omega^{-1} := S(1-\alpha)(\rho+\delta)c_k^S > 0$.
Proof: See Appendix B.

Regarding the topological behavior of the high BGP, Proposition 4 directly gives rise to the following result.

Proposition 5. *(Determinacy of the Barro BGP) The Barro BGP is saddle-path stable. Thus, it is well determined for all parameters belonging to the set $\mathscr{C}$.*

Proof: From Proposition 4, as $D^B = -\gamma^B c_k^B < 0$, we have $Q^B(0) = D^B < 0$. Therefore, the Jacobian matrix has one real root with real parts less than 0, and one real root with real parts greater than 0, which ensures the determinacy of the Barro BGP. □

Regarding the Solow BGP, stability depends on parameters, since bifurcations can occur. To assess whether the two eigenvalues of the Jacobian matrix, namely the roots of the quadratic polynomial $Q^S(\cdot)$, have a real part less or greater than 0, we adopt a simple graphical analysis[14] (see Figure 3) that delimitates three areas. First, if $D^S < 0$, the two eigenvalues are of opposite sign independently of T^S, and the Solow BGP is *locally determined (saddle-path-stable)*. Second, if $D^S > 0$ and $T^S > 0$, the two eigenvalues have positive real parts and the Solow BGP is *over-determined* (*unstable*). Third, if $D^S > 0$ and $T^S < 0$, the two eigenvalues have positive real parts and the Solow BGP is *locally undetermined* (*stable*). In addition, the eigenvalues are complex iff $(T^S)^2 < 4D^S$, and real otherwise. Figure 3 resumes these properties, with the Λ curve representing the separatrix between complex and real eigenvalues, namely $D^S = (T^S)^2/4$.

The same diagram allows studying local bifurcations, i.e., changes of stability of the Solow BGP resulting from small variations in the coefficient $\mathcal{B}^S$. In Figure 3, the line Δ is the parametric curve (T^S, D^S) with respect to $\mathcal{B}^S$, and a simple geometrical way to look at stability and bifurcations is to locate this line for a given value of $\mathcal{B}^S$. Especially, if Δ crosses the x-axis, one of the eigenvalues goes through 0, which defines a *saddle-none bifurcation.* If Δ crosses the y-axis in its positive part, there is a change of stability such that the two eigenvalues are complex conjugate with real-part crossing zero, i.e., a *Hopf bifurcation.*

[14]See e.g., Azariadis (1993), p.93; Grandmont and Laroque (1996); Grandmont et al. (1998).

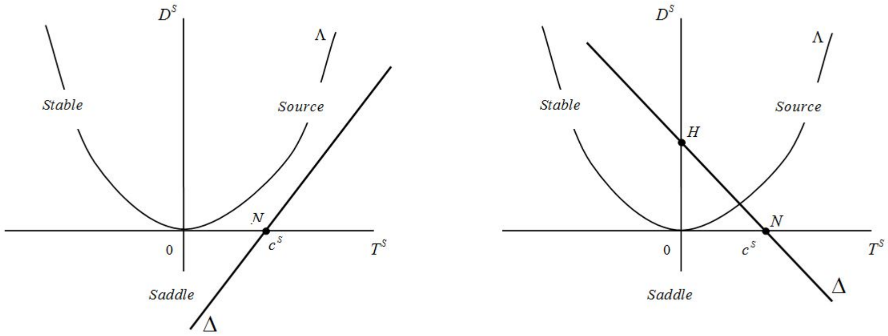


Figure 3: Determination of the Solow BGP ($\mathcal{B}^S > 0$)

Using Eqs. (24) and (25), the line Δ is defined by the parametric equation

$$T^S = c_k^S - (g_k^S)^{1-\alpha}[(g_k^S)^\alpha - (\tilde{g}_k)^\alpha]\Omega D^S. \tag{26}$$

The line Δ crosses the x-axis at the point $N = (c_k^S, 0)$ for the critical value $\Omega D^S = \mathcal{B}^S = 0$. Even if, in strict logic, such a *saddle-node bifurcation* cannot occur (since, by Lemma 2, $\mathcal{B}^S \neq 0$), the situation is analogous to a saddle-node bifurcation, because, on both sides of point N, the sign of one eigenvalue changes as $\mathcal{B}^S > 0$ or $\mathcal{B}^S < 0$. In addition, from (26), the slope of the line Δ changes sign as $g_k^S < (>) \tilde{g}_k$. In this way, the topological behavior of the Solow BGP fundamentally depends on the value of coefficient $\mathcal{B}^S$ and on the gap between g_k^S and $\tilde{g}_k$, as shows the following Proposition.

Proposition 6. *(Determinacy of the Solow BGP) The topological behavior of the Solow BGP can be described by three cases.*

Case A: If $\mathcal{B}^S < 0$, the Solow BGP is *locally determined* (*saddle-path stable*),
Case B: If $\mathcal{B}^S > 0$ *and* $g_k^S \leq \tilde{g}_k$, the Solow BGP is *locally over-determined* (*unstable*),
Case C: If $\mathcal{B}^S > 0$ *and* $g_k^S > \tilde{g}_k$*, a Hopf bifurcation occurs.*

Proof:

- *Case A* – $\mathcal{B}^S < 0$, then, $D^S < 0$. Thus, $\mathbf{J}^S$ possesses two opposite sign eigenvalues. The Solow BGP is a saddle-path, independently of the value of $\mathcal{B}^S$, and we do not need to study the location of the Δ line.

- *Case B* – $\mathcal{B}^S > 0$ and $\mathrm{g}_k^S \leq \tilde{g}_k$. The line Δ is positively sloped (see Figure 3a) and we have $T^S > 0$ and $D^S > 0$. Thus, the Solow BGP is *locally over-determined* (*unstable*).[15]

- *Case C* – $\mathcal{B}^S > 0$ and $g_k^S > \tilde{g}_k$. The line Δ is negatively sloped (see Figure 3b) and $D^S > 0$. By (26), T^S negatively depends on $\mathcal{B}^S$ and there is a critical level $\mathcal{B}_h^S$

[15] Eventually the Δ line may intersect with the Λ curve, without qualitative change in the stability of the Solow BGP, which becomes an unstable focus above Λ and an unstable node below Λ.

such that: $T^S > 0$ if $\mathcal{B}^S < \mathcal{B}^S_h$ and $T^S < 0$ if $\mathcal{B}^S > \mathcal{B}^S_h$. In this respect, a *Hopf bifurcation* arises at the point H corresponding to the unique level $\mathcal{B}^S = \mathcal{B}^S_h$ such that Δ crosses the y-axis. At this point, a periodic orbit through a local change in the stability properties of the Solow BGP appears. For $\mathcal{B}^S > \mathcal{B}^S_h$, the steady-state is *locally undetermined* (*stable*) in the complex half-plane, while for $0 < \mathcal{B}^S < \mathcal{B}^S_h$, the steady-state is (asymptotically) *unstable*. By inspection of relations (24)-(26), we can establish the following Lemma.

Lemma 3. *The Hopf bifurcation occurs at the unique value $\mathcal{B}^S_h$, where*

$$\mathcal{B}^S_h = \frac{c^S_k}{(g^S_k)^{1-\alpha}[(g^S_k)^\alpha - (\tilde{g}_k)^\alpha]}.$$

□

In order to illustrate these findings, the following Subsection numerically characterizes the topological behavior of the Solow in the constant elasticity case.

4.2. The constant elasticity case

An explicit reduced-form of the model can be found by assuming an iso-elastic tenseness function such that[16]

$$\eta(b^S_y) = \eta_1(b^S_y)^\varepsilon, \text{with } \eta_1 > 0. \tag{27}$$

In the following, we will use $\varepsilon > 0$ as the bifurcation parameter that generates the bifurcation coefficient $\mathcal{B}$. In particular, having computed the analytic value of $\mathcal{B}^S_h$ in Lemma 3, we can easily obtain the corresponding value of ε, according to Appendix F [17]

$$\varepsilon_h = \frac{(1 + \alpha\mathcal{B}^S_h)(g^S_k)^\alpha - \tau A}{[1 + (1-\alpha)\mathcal{B}^S_h][\tau A - (g^S_k)^\alpha]} > 0, \text{with } d\varepsilon_h/d\mathcal{B}^S_h > 0.$$

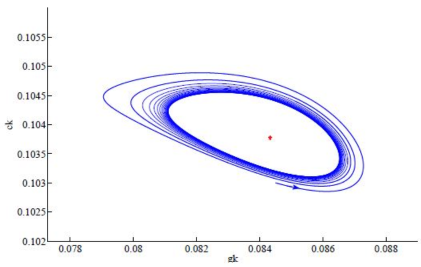
The elasticity of the tenseness function is inversely related to the propensity to accept non-distorsive-based fiscal adjustments. This elasticity can be very large for strong resistance to fiscal adjustment and, in the limiting case where the agents completely reject the adjustment, ε becomes infinite, notwithstanding the upper bound $\eta(\cdot) = 1$. To clearly illustrate this point, let us compute the effective burden that the debt exerts on public finance, namely $\eta(b_y)rb_y$. Assuming a constant interest rate, the elasticity of this burden to the debt ratio is one if η is constant (possibility one). But, with $\eta(\cdot)$ defined in (27), this elasticity becomes $1 + \varepsilon$. Thus, the response of the effective debt burden to an increase in public debt can be considerably higher, even if, in absolute value, this effective burden is lower (i.e. $\eta(\cdot) < 1$).[18]

[16] For coherence, in this numerical Subsection, we restrict our analysis to a parameter space such that $b_y > 1$ in equilibrium, so that $d\eta(\cdot)/d\varepsilon > 0$, and we focus on not-bounded solutions for $\eta(\cdot)$.

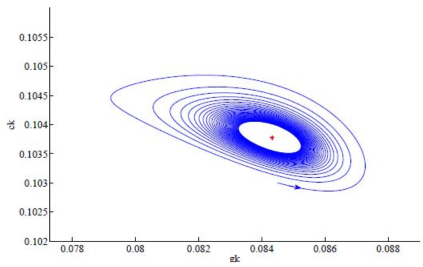
[17] The same reasoning applies for the low BGP with positive values of the deficit ratio.

[18] To fix ideas, consider the hypothetical case with $r = 0.05$, and the debt ratio jumps from 110% to 121% of GDP. If $\eta = 1$, the primary budget surplus must jumps from 5% to 5.5% of GDP to finance the increased debt burden (the elasticity is one). Using (27), with, say, $\eta_1 = 0.01$ and $\varepsilon = 15$, η jumps from 0.042 to 0.17, and the effective debt burden jumps from $0.042 \times 110\% \times 5\% = 0.23\%$ to $0.17 \times 120\% \times 5\% = 1.06\%$ of GDP. The effective debt burden remains less than under $\eta = 1$ but it has been multiplied by more than 4.

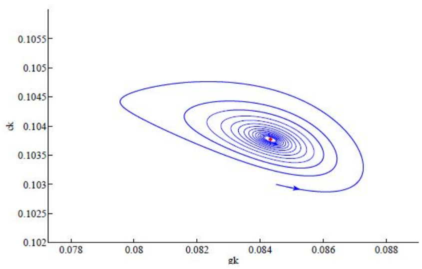
Our numerical results are based on reasonable values for parameters. We choose $\rho = \delta = 0.05$, and the consumption elasticity of substitution is, as a rule, fixed at $S = 1$. Regarding the technology, we set $A \in (0.5, 0.6)$ to obtain realistic rates of economic growth, and the capital share in the production function is $\alpha = 0.7$, as in Gomes et al. (2013), close to the value (0.715) used by Gomme et al. (2011). Such a capital share allows us to reproduce the empirical results of Munnell (1990) on the elasticity of output to productive public spending ($1 - \alpha = 0.3$). Regarding the Government's behavior, the income tax rate is $\tau \in (0.36, 0.44)$, according to, e.g., Trabandt and Uhlig (2011) and Gomes et al. (2013), and we take the deficit ratio to $\theta \in (0, 0.03)$, consistent with long-run average values in the US or OECD data from 1950 to 2015. In our baseline calibration, we take $\tau = 0.4$, which corresponds to the value that allows the highest amplitude of the deficit ratio consistent with the existence of an equilibrium exhibiting Hopf bifurcations. We fix $\eta_1 = 0.01$, to obtain realistic share of the debt burden that is financed by new debt $\eta(\cdot) \in (0, 1)$, and the elasticity ε will be scanned over a large range of values to verify the presence (or not) a Hopf bifurcation.



4a – $\mathcal{B}^S < \mathcal{B}_h^S$



4b – $\mathcal{B}^S = \mathcal{B}_h^S$



4c – $\mathcal{B}^S > \mathcal{B}_h^S$

Figure 4 – The supercritical Hopf bifurcation

Figure 4 depicts a typical Hopf bifurcation arising at the Solow BGP. For our benchmark calibration,[19] the bifurcation occurs for $\mathcal{B}^S = \mathcal{B}_h^S \simeq 8.65$ (Figure 4b), corresponding to $\varepsilon \simeq 13.16$, with the Solow BGP being stable for $\mathcal{B}^S > \mathcal{B}_h^S$ (Figure 4a) or unstable for $\mathcal{B}^S < \mathcal{B}_h^S$ (Figure 4c). However, the presence of a Hopf bifurcation does not necessarily

[19]Figure 4 is built for $A = 0.5$, $\tau = 0.4$, $\eta = 0.01$, $\rho = \delta = 0.05$ and $\alpha = 0.7$.

generate a limit-cycle. As it is well known, such a limit-cycle only occurs if the so-called *first Lyapunov coefficient* is negative. To verify the presence of a limit-cycle we run some simulations with ©matcont and compute the associated Lyapunov coefficient. For all configurations of parameters in Table 1, simulations show that the associated first Lyapunov coefficient is negative, revealing the presence of a limit-cycle, as in Figure 4, revealing a *supercritical* Hopf bifurcation. Thus, for values of $\mathcal{B}^S$ slightly lower than $\mathcal{B}^S_h$, there is a stable limit-cycle around the Solow fixed-point, which gets larger as $\mathcal{B}^S$ decreases.

The limit-cycle does not only occur for the Solow BGP but also along the low BGP with positive deficit ratio, as shown the following Lemma.

Lemma 4. *For small deficit values, the behavior of the low BGP is topologically equivalent to the Solow BGP. Hence,* $\lim_{\theta\to 0}\mathcal{B}^L(\cdot) = \mathcal{B}^S$.

Proof: See Appendix D.

Lemma 4 establishes, in particular, that $\mathcal{B}^L_h \to \mathcal{B}^S_h$ when $\theta \to 0$. Table 1 computes the bifurcation coefficient and the associated Lyapunov coefficient for different values of parameters, and specifically different deficit ratios, showing that limit-cycles arise in a large constellation of values. Moreover, such limit-cycles are associated with reasonable public debt ratios, between 106% and 198% of GDP, consistent with observations in highly indebted countries. In addition, the share of the debt burden that can be financed by non-distorsionary resources $(1-\eta(\cdot))$ is realistic, since it is between 20% and 89%. Table 1 also shows that the BGP is robust to changes in parameters, since the consumption ratio and the productive spending ratio remain fairly unchanged.

θ	0			0.005	0.01	0.02	0.03
τ	0.36	0.4	0.44	0.4	0.4	0.4	0.4
A	0.5	0.5	0.5	0.5	0.52	0.54	0.56
$\mathcal{B}^L_h$	101.93	8.65	3.44	4.11	5.18	4.86	8.21
ε_h	13.37	13.16	20.26	43.96	11.57	8.23	5.81
c^L_k	0.105	0.104	0.99	0.096	0.105	0.109	0.117
g^L_k	0.068	0.084	0.106	0.098	0.093	0.096	0.090
γ^L	0	0	0	0.0046	0.0071	0.0123	0.0142
b^L_y	1.39	1.29	1.2	1.06	1.39	1.57	1.98
$\eta(b^L_y)$	0.80	0.35	0.41	0.11	0.45	0.42	0.54
ℓ	-3.19e+004	-1.32e+003	-9.85e+002	-1.51e+004	-4.01e+002	-1.18e+002	-3.46e+001

Table 1: Bifurcation-points and Lyapunov coefficients for a constellation of parameters

In the following Section, we extend these results by fully characterizing the dynamic system through global analysis and we show the existence of global bifurcations.

5. Global Dynamics

In order to study global dynamics, we rely on a graphical analysis. To describe the phase portrait of the system in a simple way, it is convenient to reformulate the reduced-form (20)-(21) in the (g_k, c_k)-plane owing to Lemma 1. By so doing, the reduced-form (20)-(21) becomes

$$\psi'(g_k)\dot{g}_k = \theta A g_k^{1-\alpha} + \psi(g_k)[\delta + g_k + c_k - A g_k^{1-\alpha}], \tag{28}$$

$$\frac{\dot{c}_k}{c_k} = S\left[\alpha\,(1-\tau)\,A g_k^{1-\alpha} - \rho - \delta\right] + \delta + g_k + c_k - A g_k^{1-\alpha}. \tag{29}$$

Figures 6.1-2-3 describe the adjustment of c_k, g_k and b_k using a two-area graph. The upper area depicts the phase portrait of (c_k, g_k) *stricto-sensu*, while the bottom area simply represents the nonlinear relation between b_k and g_k through function $\psi(g_k)$. Clearly, the asymptote $\overline{g}_k$ in the upper area corresponds to the public spending ratio that maximizes $\psi(g_k)$ in the bottom area. The stationarity points of c_k and g_k, are represented by two hump-shaped curves $\Phi_c(\cdot)$ and $\Phi_g(\cdot)$, which take a maximum at $\hat{g}_k$ and (for small deficit ratios) $\tilde{g}_k$, respectively, with $\max\Phi_c(\cdot) = \hat{g}_k < \tilde{g}_k = \max\Phi_g(\cdot)$, as discussed in Appendix E.

The global dynamics of the model crucially depends on the location of the public spending ratio at the low steady-state g_k^L with respect to the critical public spending ratio $\tilde{g}_k$ and to the asymptote $\overline{g}_k$. Especially, according to Section 4, there are three possible cases: (i) $g_k^L > \overline{g}_k$ and $g_k^L \leq \tilde{g}_k$, (ii) $g_k^L < \overline{g}_k$, (iii) $g_k^L > \overline{g}_k$ and $g_k^L > \tilde{g}_k$, which are addressed in the three following subsections, respectively.

5.1. *Global determinacy:* $\overline{g}_k < g_k^L \leq \tilde{g}_k$

The first configuration corresponds to the *Case B* of the preceding Section. In this configuration, the low BGP is unstable, because $\mathcal{B}^L > 0$ (thus $\overline{g}_k < g_k^L$) and $g_k^L \leq \tilde{g}_k$, while, in accordance with Proposition 2, the high BGP is saddle-path stable. Figure 6.1 provides a typical phase portrait in this case. The high BPG (point H) is reached by the stable manifold $\mathcal{M}_H$, while all paths run away the low BGP (point L). Therefore, provided that the initial public debt ratio is no too large (i.e. $b_{k0} \leq b_k^m$), the system is globally determined.[20] Indeed, for any predetermined initial public debt ratio b_{k0}, there are unique initial values of the jumpable variables g_{k0} (through function ψ) and c_{k0} putting the economy on the stable manifold $\mathcal{M}_H$ that converges to the high BGP, which defines the only long-run equilibrium.[21]

[20] If the economy start from a high public debt ratio, i.e. $b_{k0} > b_k^m = \psi\left(g_k^m\right)$, there is no jump of the control variables that can place the equilibrium path on the stable manifold. g_k^m is defined by the projection onto the g_k-axis of the leftmost point of $\mathcal{M}_H$.

[21] Depending on the position of the asymptote $\overline{g}_k$, there can be (as in Figure 5.1), or not, a heteroclinic connection between the low and the high BGPs. See the discussion in Subsection 6.3 below

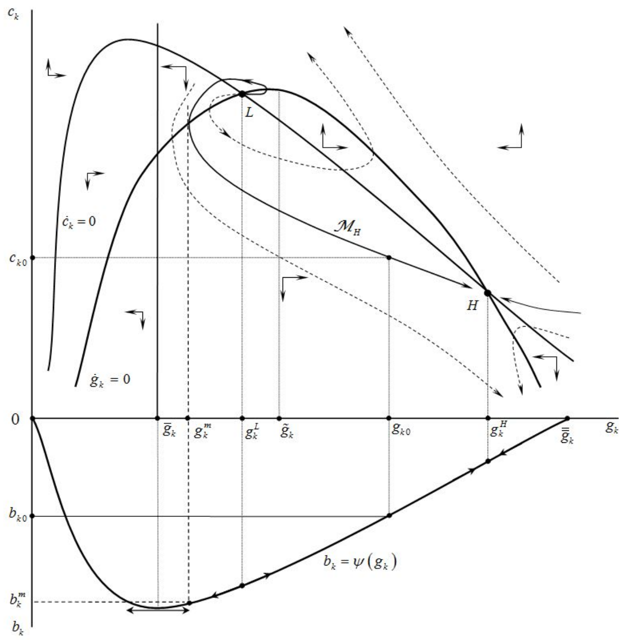


Figure 5.1: Phase Portrait for $\bar{g}_k < g_k^L < \tilde{g}_k$

5.2. Global indeterminacy: $g_k^L < \bar{g}_k$

In this second configuration, $\mathcal{B}^L < 0$, which corresponds to the *Case A* of Section 4. The low BGP is now located in the increasing part of function $\psi(\cdot)$, and becomes saddle-path stable. Therefore, if (c_k, g_k) belongs to the half-plane $\mathbb{R}^+ \times (0, \bar{g}_k)$, there is a unique stable manifold $\mathcal{M}_L$ that converges towards the low BGP. Hence, both BGPs are locally determined and can be reached by one stable manifold, respectively $\mathcal{M}_L$ or $\mathcal{M}_H$, thus generating global indeterminacy.

Effectively, according to Lemma 1, two initial public-spending ratios, located on both sides of the asymptote $\bar{g}_k$, are consistent with any predetermined public debt ratio $b_{k0} \in \left[0, \bar{b}_k\right]$. Thus, at time 0, the public-spending ratio can jump either at $g_{k0}^A < \bar{g}_k$ (point A in Figure 5.2), or at $g_{k0}^B > \bar{g}_k$ (point B). Accordingly, the initial consumption ratio (c_{k0}) can jump respectively to the value c_{k0}^A putting the economy on the stable manifold $\mathcal{M}_L$ that converges towards the low BGP or to the value c_{k0}^B corresponding to the stable manifold $\mathcal{M}_H$ that reaches the high BGP. In the first case, the public debt ratio increases until point A', while it decreases towards point B' in the second case. Consequently, starting from b_{k0}, two self-fulfilling perfect-foresight paths are feasible.

In this configuration, our model gives rise to global indeterminacy without local indeterminacy: for a given initial public debt ratio (b_{k0}), the equilibrium path that leads to both steady-states is well determined, but the economy can converge towards the low or the high BGP, depending on agents' expectations. The coexistence of two feasible equilibrium paths illustrates the possibility of self-fulfilling prophecies: if everyone thinks that the economy will end up at the high BGP, then it will, whereas if the low BGP is expected, then it will too. In such a case, the transition path and the long-run solution of the model are subject to optimistic or pessimistic views on the future. Do the agents expect strong economic growth, and the economy will reach the high BGP; do they anticipate a low-growth trap, and the economy will be condemned to join the low BGP.

The intuition of such an indeterminacy is the following. If high future economic growth is expected, the return of private investment is expected to be high, and, at the initial time, the agents increase their savings, such that the initial consumption ratio c_{k0}^B is low. For a given initial stock of private capital (k_0), this means that, in equilibrium, initial private investment and productive public spending ratio g_{k0}^B will be high, generating a (self-fulfilling) high growth path in the future. On the contrary, if agents expect low future economic growth, they choose a high initial consumption ratio c_{k0}^A because the (perfectly expected) return on their savings will be low. In equilibrium, such a consumption ratio crowds out private investment and productive public spending (g_{k0}^A), thus the economy goes towards the low-growth trap.

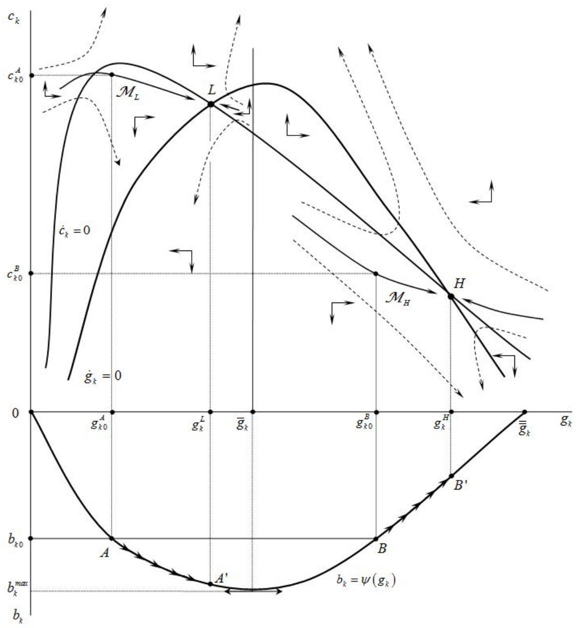


Figure 5.2: Phase Portrait for $g_k^L < \overline{g}_k$.

5.3. Possibility of global and local indeterminacy: $\tilde{g}_k < g_k^L$ *and* $\overline{g}_k < g_k^L$*.*

The third configuration corresponds to the *Case C* of Section 4. As in the first configuration, both BGPs are located in the decreasing side of $\psi(\cdot)$, namely $\mathcal{B}^L > 0$, but the low BGP is now located on the left side of the maximum of $\Phi_g(\cdot)$, namely $g_k^L > \tilde{g}_k$. The high BGP is still saddle-point stable, but a Hopf bifurcation occurs in the neighborhood of the low BGP (at $\mathcal{B}^L = \mathcal{B}_h^L$). Thus, this steady-state can be characterized either by (i) stable or (ii) unstable oscillations, depending on the value of $\mathcal{B}^L$, in accordance with the local stability analysis in the preceding Section.

In case (i), $\mathcal{B}^L > \mathcal{B}_h^L$ and the economy is characterized by both local and global indeterminacy. Effectively, around the low BGP, there are an infinity of trajectories that converge towards the low BGP (local indeterminacy),[22] and for an initial public debt ratio b_{k0}, the economy can reach either the high or the low BGP (global indeterminacy).

In case (ii), $\mathcal{B}^L < \mathcal{B}_h^L$ and the low BGP is locally unstable. For values slightly less than $\mathcal{B}_h^L$, there is a stable limit-cycle and the economy is still characterized by local and global indeterminacy. Figure 5.3. depicts a typical phase portrait in this configuration. There are an infinity of trajectories that converge towards the limit-cycle (local indeterminacy) and the long-run equilibrium can be the high BGP, if the initial consumption ratio jumps to c_{k0}^A, or a growth cycle around the low BGP, if it jumps to (e.g.) c_{k0}^B. Global indeterminacy then comes from the fact that, for an initial (predetermined) public debt ratio b_{k0}, the initial public expenditure ratio g_{k0} is also predetermined, but the initial consumption ratio can take different values, depending on "optimistic" or "pessimistic" views of Households. In contrast, if $\mathcal{B}^L$ gets lower, the limit-cycle vanishes and only the saddle-path leading to the high growth BGP can be reached, therefore, removing indeterminacy. In this case, the economy is characterized by an unique equilibrium (the high BGP), as in Subsection 5.1.

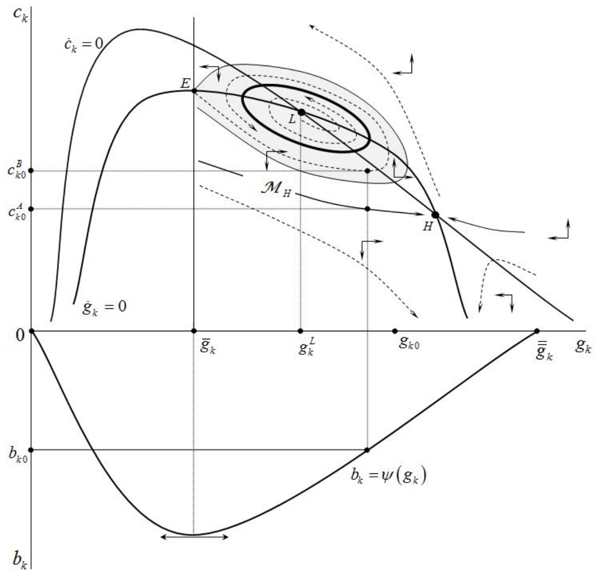


Figure 5.3: Phase Portrait for $\overline{g}_k < g_k^L$ and $\tilde{g}_k < g_k^L$

As Figure 5.3 emphasizes, the existence of limit-cycles is linked to the presence of a non-empty compact *trapping region* containing the fixed-point L. As we will show below, such condition is, in particular, ensured by the presence of a *homoclinic orbit* enclosing point L. This kind of orbit describes a trajectory that connects a saddle point to itself, and occurs at the intersection of the stable and the unstable manifolds. In our model, the existence of such an orbit is due to the presence of the singular point

[22] In deterministic perfect-foresight models, local indeterminacy can be associated to the existence of *sunspot equilibria* (see, e.g., Woodford, 1986a,b; Matsuyama, 1991; Benhabib and Farmer, 1999).

E, which corresponds to the crossing point between the stationary locus of g_k and the asymptote $\bar{g}_k$. This point has interesting implications (that we numerically illustrate in the following subsection).

First, E defines the unique point that can be reached at the asymptote ($\bar{g}_k$). Effectively, if $g_k = \bar{g}_k$ (or, identically, if $b_k = \bar{b}_k$), then $\psi'(\bar{g}_k) = 0$, and system (27)-(28) has no sense unless $\dot{g}_k = 0$, that is, at point E. This means that, if the economy starts with a public debt ratio $b_{k0} = \bar{b}_k$, the initial consumption ratio c_{k0} must immediately jumps at the unique level $c_k^E =: \Phi_g(\bar{g}_k)$, which is the only one that satisfies relation (27). In the same way, if the asymptote $\bar{g}_k$ was to be reached during the dynamics, this can only be at point E. Outstandingly, equilibrium trajectories cannot (asymptotically) end on the asymptote $\bar{g}_k$. Indeed, this could only be the case above point E (since the trajectories deviate from the asymptote below this point). Yet, in the right side of point E, we have

$$\lim_{g_k \to \bar{g}_k^+} \left(\frac{\dot{g}_k}{g_k} \right) = \frac{\psi(\bar{g}_k)}{\bar{g}_k} \left(c_k - c_k^E \right) \lim_{g_k \to \bar{g}_k^+} \left\{ \frac{1}{\psi'(g_k)} \right\} = \begin{cases} +\infty \text{ if } c_k < c_k^E, \\ -\infty \text{ if } c_k > c_k^E, \end{cases}$$

hence, above point E, we have $\lim_{g_k \to \bar{g}_k^+} \dot{b}_k / b_k = +\infty$, which is not consistent with the standard transversality condition.

Second, the singular point E behaves as a saddle, because the inward manifold (denoted by $\mathcal{M}_E^s$) converges towards E and the outward manifold (denoted by $\mathcal{M}_E^u$) diverges from it with infinite speed. In other words, the stable orbit "rebounds" at very high (infinite) speed on the asymptote, while it remains smooth for turning points slightly higher than $\bar{g}_k$. This collision creates an angle in the trajectory that passes through E, such that two$\mathcal{M}_E^s$ and $\mathcal{M}_E^u$ are locally invariant, like in a saddle point. Figure 6 illustrates the "reflection symmetry" of E (since there is no fixed-point on the left-hand side of the asymptote $\bar{g}_k$, we will focus on the right-hand side).

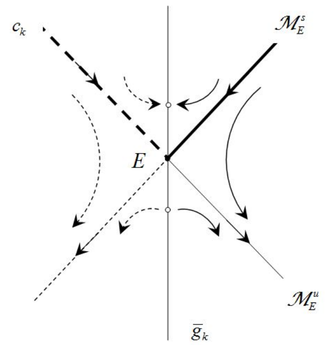


Figure 6: Dynamics around point E

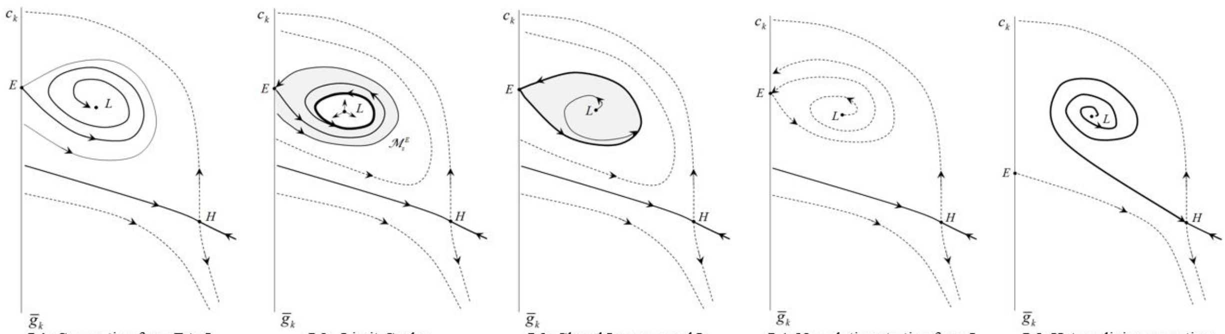
Based on this discussion, we are able to exhaustively characterize the global dynamics of the model, simply by analyzing the behavior of incoming and outgoing trajectories of point E. To this end, the typology of possible orbits starting at the asymptote is presented in Figure 7.

Let us consider the hypothetical case where the economy starts at the maximal feasible initial debt ratio $b_{k0} = \bar{b}_k$. In this case, the initial point is necessarily at point $E =$

$(c_k^E, \bar{g}_k)$, which is reached by an immediate jump of c_{k0}, and the trajectory then can follow different paths, as in Figure 7. Since, in accordance with local dynamics, the system is characterized by two fixed-points: the saddle H and the (stable or unstable) node L, the equilibrium path must converge to these points (or towards a limit-cycle around point L). Global dynamics is then characterized by five typical cases that can be categorized according to their implications on (in)determinacy.

The first configuration appears when low BGP is stable. Then, there is a direct connection between E and L (Figure 7.1). In contrast, when the low BGP is unstable, namely slightly after the Hopf bifurcation, there is a limit-cycle that attracts the manifold departing from E (Figure 7.2). This configuration arises when the asymptote $\bar{g}_k$ moves toward the left. The third case describes the largest feasible limit-cycle, when the outward manifold from E collides with the asymptote at E, generating a closed loop (Figure 7.3). Since E behaves as a saddle but is not a fixed-point of the system, we qualify this situation as a *quasi-homoclinic* orbit. In the preceding three cases, there is global indeterminacy if the economy starts with a high public debt ratio. Indeed, if $\hat{b}_k \leq b_{k0} < \bar{b}_k$ (where $\hat{b}_k$ is the debt ratio that corresponds to the rightmost point of the stable manifold $\mathcal{M}_E^s$), the initial consumption ratio can jumps either on the saddle path that converges to H or on the stable spiral that converges to L or to the limit-cycle.

If the asymptote $\bar{g}_k$ moves again to the left, by contrast, the limit-cycle vanishes and the unstable manifolds that depart from L crash against the asymptote and cannot be equilibrium paths (Figure 7.4). In this case, there is no indeterminacy. Only the saddle path that converges to H can be reached by an initial jump of the consumption ratio (as soon as $b_{k0} < \bar{b}_k$). Finally, if the asymptote $\bar{g}_k$ moves again to the left, a heteroclinic orbit that directly connects the low with the high BGPs can appear (Figure 7.5).[23]



7.1: Connection from E to L 7.2: Limit-Cycles 7.3: Closed Loop around L 7.4: No solution starting from L 7.5: Heteroclinic connection

Figure 7: A typology of global dynamics

From a global analysis perspective, the existence of a homoclinic orbit explains the birth of a limit-cycle. Indeed, the set of points (c_k, g_k) that are enclosed by the homoclinic

[23] In this case, as in Figure 5.1, there is no equilibrium solution if the economy begins with an initial debt ratio such that $b_{k0} > \tilde{b}_k$, where $\tilde{b}_k$ is the debt ratio that corresponds to the leftmost point of the heteroclinic manifold from L to H.

manifold defines a non-empty compact space (denoted by V) in $\mathbb{R}^2_+$. This set (V) is a *trapping region*, because, for any $(c_k, g_k) \in V$, all trajectories remain in V. Furthermore, this set includes only one fixed-point (L). According to Poincaré-Bendixson theorem, if L is unstable, there is a unique limit-cycle in V, as illustrated in Figure 7.3.

Following the same logic, the existence of limit-cycles can be extended to configuration 7.2, because the manifold $\mathcal{M}^s_E$ converging to E forms an envelope curve defining a trapping region around point L. Let us denote by W the set of points enclosed by this orbit (including only one fixed-point L). Since all trajectories starting in W remain in the same set, it follows that the associated ω-limit of any point in W remains in W (where the ω-limit of point x is the limit of the trajectory starting at x). Then, according to the Poincaré-Bendixson theorem, if L is unstable, there is a unique limit-cycle in W.

A noteworthy feature of Figures 7 is the presence of an non-generic global bifurcation of the system, since, in the case 7.3, the limit-cycle collides with the quasi-homoclinic orbit that passes through E. The following Subsection numerically characterizes this global bifurcation.

5.4. A quantitative exploration of global bifurcations

In this Subsection, we provide a numerical illustration of global dynamics in the Case 3. Very interestingly, in the constant elasticity case, the global bifurcation is codim 1, and can be generated, simply by modifying the elasticity of the tenseness function (ε). To establish numerical results, we use the benchmark calibration of Subsection 4.3. Case 3 emerges if $\mathcal{B}^L > 0$, namely if $\varepsilon < \hat{\varepsilon}$, where $\hat{\varepsilon}$ corresponds to the critical value of the elasticity such beyond which the economy switches into Case 2.[24]

As we have seen, there is a (local) Hopf bifurcation around the low BGP at $\varepsilon = \varepsilon^h = 13.16$, and, for slightly inferior values of ε, there is a stable limit-cycle. As ε decreases, the limit-cycle becomes larger, as in Figure 8. Effectively, the value of ε determines the location of the asymptote $\bar{g}_k$. The smaller the elasticity of the tenseness function, the more the asymptote moves to the left. However, at some point, the limit-cycle collides with the asymptote $\bar{g}_k$ and can no longer grow bigger. This defines point E in Figure 6. Consequently, the largest limit-cycle consistent with the existence of a periodic equilibrium defines the *quasi-homoclinic* orbit with respect to E. This orbit corresponds to $\varepsilon = \bar{\varepsilon} = 12.7$.

[24] For small values of the deficit ratio, we have $\hat{\varepsilon} := \alpha(g^S_k)^\alpha/[(1-\alpha)(\tau A - (g^S_k)^\alpha)] \simeq 18$.

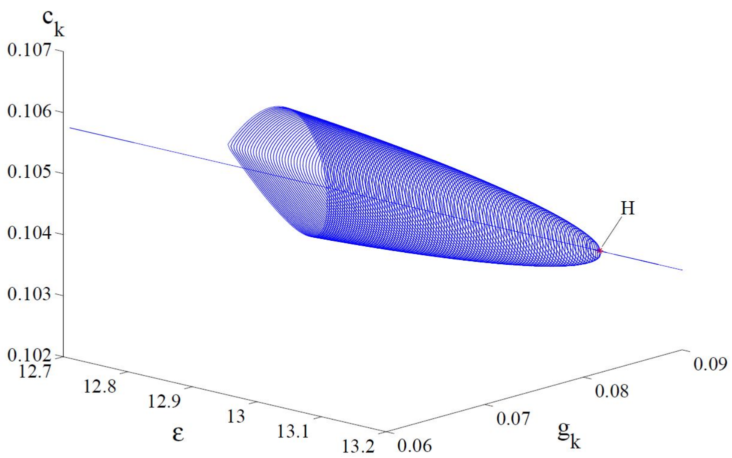
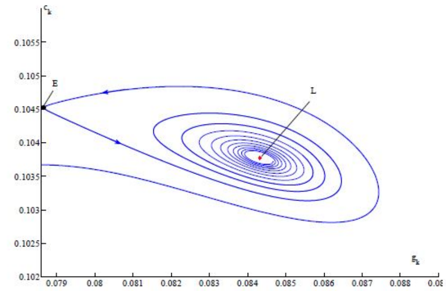
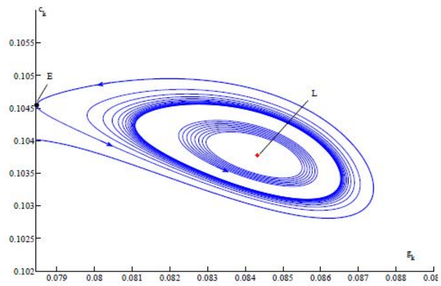


Figure 8: Limit-cycles as a function of ε

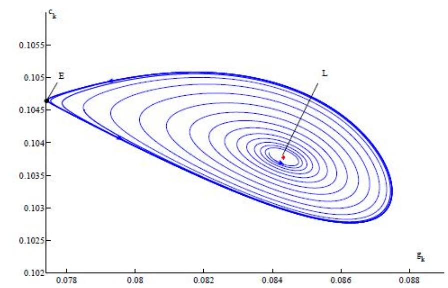
Consequently, for $\hat{\varepsilon} > \varepsilon > \varepsilon^h = 13.16$, the low equilibrium is stable, and the model is characterized by local and global indeterminacy, as in Figure 7.1. At $\varepsilon = \varepsilon^h \simeq 13.16$, the Hopf bifurcation occurs and, for $\varepsilon \in (\varepsilon^h, \bar{\varepsilon})$, there is still local and global indeterminacy, since the economy can reach any path that converges towards the limit-cycle (local indeterminacy) or the saddle path towards the high BGP (global indeterminacy), as in Figure 7.2. The corresponding numerical findings are respectively depicted in Figures 9.1 and 9.2. At $\varepsilon = \bar{\varepsilon} \simeq 12.7$, there is a closed-loop connection which passes "through" E, thus delimiting a separatrix curve between the (out of equilibrium) trajectories which are absorbed by the asymptote and those which continue along a cycle (see Figure 9.3). This connection therefore constitutes a global bifurcation. Effectively, when the elasticity decreases again, the limit-cycle disappears and becomes an unstable spiral, that collides with the asymptote $\bar{g}_k$, as in Figure 7.4. Thus, the point $\varepsilon = \bar{\varepsilon}$ defines the global bifurcation of the model.



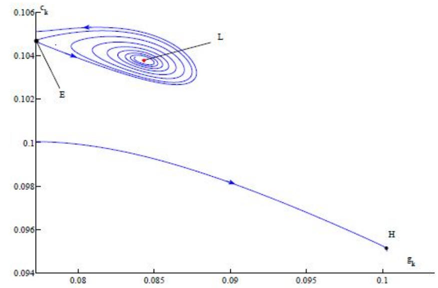
9.1: $\varepsilon = 13.5$ – L stable



9.2: $\varepsilon = 13$ – Limit-cycle



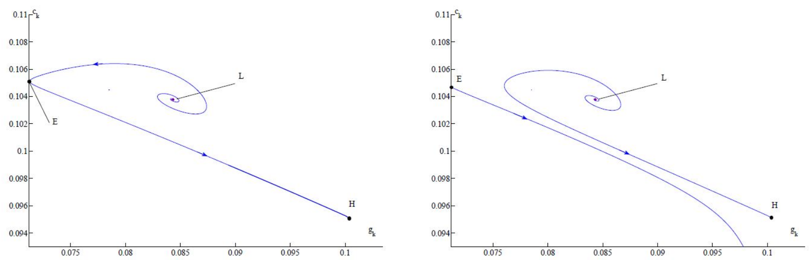
9.3: $\varepsilon = 12.7$ – Quasi-homoclinic orbit



9.4: $\varepsilon = 12.5$ – Unstable spiral

Figure 9a: The global bifurcation

For lower values of the elasticity ($\varepsilon < \bar{\varepsilon}$), the economy is well determined, since only the high equilibrium can be reached by a unique saddle-path stable manifold. In this case, there are two configurations, depending on whether the unstable trajectory which leaves the low steady-state collides or not with the asymptote. In the first case, there is no long-run solution for trajectories that depart from the neighborhood of the low BGP, and the equilibrium path immediately jumps on the saddle path that converges towards the high BGP, as in Figure 7.4. In the second case, the saddle-path that converges towards the high BGP is the one emerging from the low steady-state, namely we are in the presence of a heteroclinic connection, as in Figure 7.5. Numerically, the value of the elasticity giving rise to the largest feasible heteroclinic connection between L and H is found to be $\tilde{\varepsilon} \simeq 9.4$, such that, if $\tilde{\varepsilon} < \varepsilon < \bar{\varepsilon}$, there is no equilibrium manifold departing from L, while, if $\varepsilon < \tilde{\varepsilon}$, there is a heteroclinic connection between L and E. These findings are illustrated in Figures 9.4-5-6.



9.5: $\varepsilon = 9.4$ – Heteroclinic connection through E 9.6: $\varepsilon = 9$ – Direct heteroclinic connection

Figure 9b: Heteroclinic connections

5.5. Discussion

Based on the analysis of local and global dynamics, we can resume the properties of the economy in Figure 10, which highlights three typical configurations: local and global determinacy; local determinacy and global indeterminacy; local and global indeterminacy. These configurations crucially depend on three factors: the initial level of public debt (b_{k0}), the elasticity of the tenseness function (ε) that determines the location of the asymptote $\bar{g}_k$, and the gap $\tilde{g}_k - g_k^L$ that reflects the response of economic growth to an increase in productive expenditures along the low BGP. [25]

In accordance with the discussion of Lemma 1, public debt is positively (negatively) linked to productive expenditures if $g_k^L < (>) \bar{g}_k$, namely if $\varepsilon > (<) \hat{\varepsilon}$.

To provide a simple intuition of the (in)stability of low BGP, let us first consider the first configuration ($\varepsilon < \hat{\varepsilon}$), by distinguishing two cases. On the one hand, if $g_k^L < \tilde{g}_k$, an increase in public debt reduces public expenditures and economic growth along the low BGP. Slower growth makes it harder to stabilize the public debt-to-output ratio, and the low BGP is unstable, for any initial public debt ratio such that $b_{k0} \in (0, b_k^m]$, as developed in Subsection 5.1. On the other hand, if $g_k^L > \tilde{g}_k$, an increase in public debt reduces public expenditures but rises economic growth. This alleviates the debt burden and makes it easier to stabilize public debt. This configuration gives rise to complex scenarios, because the low BGP can be locally stable or unstable, as detailed in Subsection 5.3.

In the case of a narrow fiscal space (namely if the elasticity of the tenseness function is high, $\varepsilon > \bar{\varepsilon}$), as public debt increases, society hardly accepts a public consumption-based fiscal adjustment, and the debt burden will be covered by cutting productive expenditures, or by issuing new public debt. As a result, for high initial public debt ratios, the economy is trapped in a region with high public debt and low growth, which can lead either to a stagnation equilibrium (L) (if $\varepsilon > \varepsilon_h$), or a low-growth cycle with large oscillating public debt (if $\varepsilon < \varepsilon_h$). In the case of a large fiscal space ($\varepsilon < \bar{\varepsilon}$), in contrast, the fiscal adjustment will be based more on cuts in public consumption, preserving productive expenditures. Thus, the economy can escape from the low-growth (unstable) equilibrium and converge towards the high BGP.

[25] Indeed, from goods market equilibrium (10), we directly obtain: $\text{sign}(\partial \gamma_k^L / \partial g_k) = \text{sign}(\tilde{g}_k^\alpha - (g_k^L)^\alpha)$.

Let us now turn our attention to the second configuration ($\varepsilon > \hat{\varepsilon}$). In this case, an increase in public debt does not reduce but *rise* productive expenditures, leading to an uncertain effect on debt stabilisation. Indeed, the debt burden and economic growth increase together and can offset each other along the unique stable saddle path to the low BGP, as described in subsection 5.2. Since the high BGP is still saddle-path stable, this configuration leads to a global indeterminacy. In this respect, both the long-run equilibrium and the transition path will be subject to "animal spirits" in the form of pessimistic or optimistic views.

The multitude of configuration emerging from our model highlights the crucial role of the tenseness function ($\eta(\cdot)$). To avoid local and global indetermination and escape from a low-growth trap, possibly marked by large, undesirable, periodic fluctuations in economic growth, it is important that the social acceptance to cut-back unproductive expenditures in response to increasing public debt is large enough.

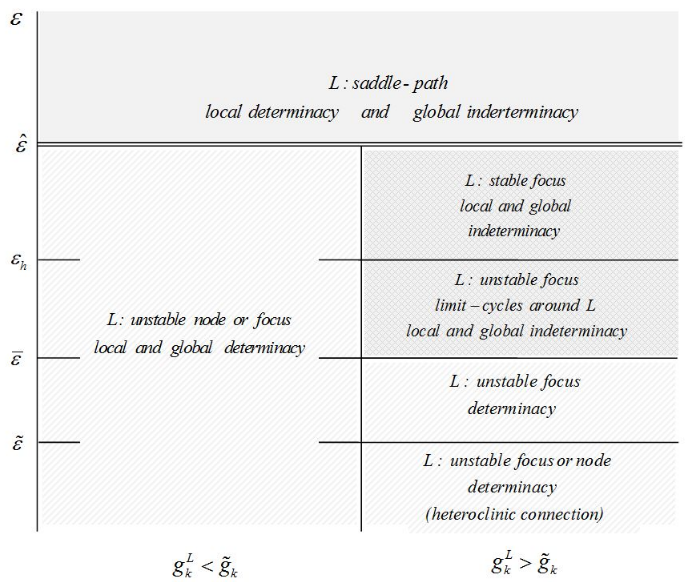


Figure 10: Characterization of equilibria

6. Conclusion

In this paper we have developed a new theoretical analysis for studying interactions between economic growth and public debt. The economy is fully characterized by a very simple endogenous-growth setup based on a continuous-time two-dimensional dynamic system. The model provides many substantial features. First, it exhibits an original explanation of the threshold effect of public debt on economic growth in steady-state. Second, it can generate multiple equilibria, giving rise to limit-cycles together with local and global bifurcations, depending on the adjustment of public finances to the debt burden. Specifically, the way the non-distorsive components of public budget adjust to the debt burden is shown to be a key factor determining the local and global dynamics of the economy. Narrow fiscal space in heavily indebted economies then can lead to undesirable endogenous growth fluctuations with oscillating public debt in steady-state.

From a methodological perspective, our setup shows that a small dimensional macroeconomics model can generate quite complex dynamics, allowing to capture different historical experiences of indebted countries. In some sense, our model provides an illustration of the "History versus Expectations" scenario developed by Krugman (1991) and Matsuyama (1991). Depending on the elasticity of the tenseness function, "history", in the form of the initial public debt ratio, or "expectations", in the form of self-fulfilling prophecies, will determine the long-run growth path of the economy. As in other endogenous growth models, the presence of a productive externality is an essential ingredient in generating multiplicity of equilibrium paths, but this not a sufficient condition, since, in our model, indeterminacy arises or not, depending on the composition of fiscal adjustment to the debt burden. This feature offers a fresh approach for studying the nonlinearities between public debt and economic growth, since it shows that, beyond the current debt ratios there is room for various sociopolitical factors in determining the future growth path of indebted countries.

Since indeterminacy and endogenous growth cycles arise for plausible parameters values, our results also have first-order policy implications. First, our model may illustrate the erratic debt dynamics observed since the beginning of the Great Recession, in particular in some south-European countries (Greece being the most quoted case). Second, our setup shows the importance of the social agreement to cover the debt burden, identifying the "fiscal space", in the determination of the BGP. Building fiscal space is a matter of political and societal choice, and, as shown by our analysis, can be achieved through a proper management of the composition of public spending. Therefore, a natural extension of our analysis would consist in providing a political economy mechanism to endogenize the tenseness function. The social acceptance to cover public debt through public consumption could be viewed as the outcome of a coordination scheme (possibly through a voting process) or a conflict between different groups about the agreement of increasing taxes. In this way, our tenseness function could be related to the political polarization of the society, viewed as a bifurcation parameter, such that, if the debt burden requires drastic cuts in primary expenditures, or tax increases, social cohesion would be difficult to achieve, leading to a collapse of the political support in favor of deficit rules.

Without dimensional change of the dynamic system, a second extension would regard the introduction of a stochastic setting, in the spirit of Grandmont et al. (1998). By including random shocks (affecting the technology factor, for example), our bifurcation analysis would emphasize crucial changes in the propagation mechanism. From a rational expectation perspective, dynamics would then give rise to stochastic endogenous fluctuations through the emergence of stochastic sunspot equilibria. Finally, an interesting way of future research would determine how monetary policies (in particular, by the monetization of public debt, as in Menuet et al., 2015) would alter the dynamics and could or not improve the configuration associated to endogenous fluctuations.

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Appendix A.

Approximate values of the BGPs Using Eqs. (13) and (17), we obtain

$$\mathcal{F}\left(r^{*},\theta\right):=\theta\left[1-\eta\left(\frac{\theta}{S\left(r^{*}-\rho\right)}\right)\frac{r^{*}}{S\left(r^{*}-\rho\right)}\right]-f\left(r^{*}\right)=0.$$

Along the high BGP, a linear approximation of this relation around the couple $\left(r^{B},0\right)$, provides $\mathcal{F}\left(r^{H},\theta\right)\approx\mathcal{F}_{r}(r^{H}-r^{B})+\mathcal{F}_{\theta}(\theta-0)=0$, where $\mathcal{F}_{\theta}(r^{B},0)=1-\eta_{0}r^{B}/\gamma^{B}$ and $\mathcal{F}_{r}(r^{B},0)=-f'(r^{B})$. Hence, $r^{H}-r^{B}\approx\theta r^{B}\left(\bar{\eta}-\eta_{0}\right)/\gamma^{B}f'\left(r^{B}\right)$ in the neighborhood of the Barro BGP. In the neighborhood of the Solow BGP, since $\theta/\gamma^{S}=b_{k}^{S}/y_{k}^{S}$, we find immediately $r^{L}-\rho\approx\theta/Sb_{y}^{S}$. Using Eq. (13), we obtain the approximate values of the rate of economic growth along the high and the low BGP in the main text (18)-(19).

Proof of Proposition 3. Let $\mathcal{V}(a)$ be the set of neighborhoods of point a. From Eqs. (13) and (17), we can write

$$f\left(r^{H}\right)=b_{y}^{H}\left[S(r^{H}-\rho)-\eta\left(b_{y}^{H}\right)r^{H}\right]. \tag{A.1}$$

By the Implicit Function Theorem, we obtain

$$\frac{dr^{H}}{db_{y}^{H}}=\frac{\xi_{1}(r^{H},b_{y}^{H})}{\xi_{2}(r^{H},b_{y}^{H})}, \tag{A.2}$$

where

$$\xi_{1}(r^{H},b_{y}^{H}):=S(r^{H}-\rho)-r^{H}\eta(b_{y}^{H})[1+\varepsilon(b_{y}^{H})], \tag{A.3}$$
$$\xi_{2}(r^{H},b_{y}^{H}):=f'(r^{H})-b_{y}^{H}\left[S-\eta(b_{y}^{H})\right], \tag{A.4}$$

First, for a given r^{H}, since $\varepsilon(\cdot)$ is assumed to be increasing, continuous and positive, the function defined by $b_{y}^{H}\mapsto\xi_{1}(r^{H},b_{y}^{H})$ is continuous and decreasing, with $\xi_{1}(r^{H},b_{y}^{H})\to-\infty$ when $b_{y}^{H}\to+\infty$. In addition, we have $\xi_{1}(r^{H},0)>0\Leftrightarrow\eta(0)<\bar{\eta}_{0}:=S(r^{H}-\rho)/r^{H}>0$. In this way, if $\eta_{0}<\bar{\eta}_{0}$, according to the Intermediate Value Theorem, there is a unique level $\overline{b}_{y}^{H}>0$ such that: $\xi_{1}(\gamma^{H},b_{y}^{H})\geq0$ if and only if $b_{y}^{H}\geq\overline{b}_{y}^{H}$. Specifically, by (A.3), $\overline{b}_{y}^{H}$ is given by

$$\eta\left(\overline{b}_{y}^{H}\right)=\frac{\gamma^{H}}{r^{H}[1+\varepsilon(\overline{b}_{y}^{H})]}.$$

Second, we show that there is a level $\bar{\alpha}\in(0,1)$, such that $\xi_{2}(r^{H},b_{y}^{H})>0$ for $b_{y}^{H}\in\mathcal{V}(\overline{b}_{y}^{H})$, under the sufficient (unnecessary) condition $\alpha\geq\bar{\alpha}$. On the one hand, by (17), we compute

$$f'(r)=\frac{1}{A(1-\alpha)(r+\delta)}[\tau+Af(r)],$$

hence, using (17),

$$\xi_{2}(r^{H},b_{y}^{H})=\frac{\tau}{A(1-\alpha)(r+\delta)}+b_{y}^{H}\tilde{\xi}(r^{H},b_{y}^{H}),$$

where

$$\tilde{\xi}(r^H, b_y^H) := S - \eta\left(b_y^H\right) - \frac{\alpha r^H}{(1-\alpha)(r+\delta)}\left(\frac{\gamma^H}{r^H} - \eta\left(b_y^H\right)\right).$$

For $b_y^H \in \mathcal{V}(\bar{b}_y^H)$, we have $\tilde{\xi}(r^H, b_y^H) \geq 0 \Leftrightarrow \alpha \geq \underline{\alpha}$, where

$$\bar{\alpha} := \frac{(r^H\varepsilon(\cdot)+\rho)(r^H+\delta)}{(r^H\varepsilon(\cdot)+\rho)(r^H+\delta)+r^H(r^H-\rho)\varepsilon(\cdot)} \in (0,1).$$

Consequently, if $\alpha \geq \bar{\alpha}$, $\xi_2(\gamma^H, b_y^H) > 0$ for $b_y^H \in \mathcal{V}(\bar{b}_y^H)$, and there is a unique threshold $\bar{b}_y^H$, such that $d\gamma^H/db_y^H \geq 0 \Leftrightarrow\ b_y^H \leq \bar{b}_y^H$. Notice that, for small levels of ρ and δ, this threshold appears for the reasonable condition that $\alpha \geq \underline{\alpha} = 1/2$. □

Appendix B. Proof of Lemma 1

We can rewrite Eq. (24) as an implicit relation $\zeta(g_k, b_k) = 0$, where

$$\zeta(g_k, b_k) := g_k - (\theta+\tau)Ag_k^{1-\alpha} + \eta(b_k/Ag_k^{1-\alpha})b_k r(g_k), \tag{B.1}$$

with $r(g_k) := \alpha(1-\tau)Ag_k^{1-\alpha} - \delta$. As η is assumed to be differentiable, ζ is also differentiable on $\mathbb{R}_+ \times \mathbb{R}_+^*$.

Proof of (i). By (B.1), we compute

$$\frac{\partial\zeta(g_k, b_k)}{\partial b_k} = r(g_k)\eta(b_k/Ag_k^{1-\alpha}) + b_k\eta'(b_k/Ag_k^{1-\alpha})\frac{r(g_k)}{Ag_k^{1-\alpha}}.$$

As η is assumed to be positive ($\eta(s) \geq 0$) and strictly increasing ($\eta'(s) > 0$, for any $s \geq 0$), we obtain that $\partial\zeta(g_k, b_k)/\partial b_k > 0$, for any $(g_k, b_k) \in \mathbb{R}_+^2$. Consequently, according to the Implicit Function Theorem, there is a unique differentiable function $\psi : \mathbb{R}_+^* \to \mathbb{R}_+$, such that, if the couple (g_k, b_k) ensures condition (B.1), we have $b_k = \psi(g_k)$.

Proof of (ii). Let introduce $\|\cdot\|$ the standard euclidian norm, and the vector $v_k := (g_k, b_k) \in \mathbb{R}_+$.[26] By assuming (g_k, b_k) close to the origin (hence, $g_k < 1$), we establish that $\|v_k\|^{1-\alpha} \geq g_k^{1-\alpha} > g_k$; hence,

$$0 \leq \frac{b_k}{Ag_k^{1-\alpha}} \leq \frac{\|v_k\|}{A\,\|v_k\|^{1-\alpha}} = \frac{\|v_k\|^{\alpha}}{A}.$$

According to the Squeeze Theorem, we have: $b_k/Ag_k^{1-\alpha} \to 0$ when $(g_k, b_k) \to (0,0)$. In this way, since $\eta(0) < +\infty$, we obtain, using (B.1): $\zeta(g_k, b_k) \to 0$ when $(g_k, b_k) \to (0,0)$, and thus, $\psi(0) = 0$. Thanks to this result, ψ can be extended by continuity at $g_k = 0$.

Proof of (iii). Let fix $b_k = 0$. As $\eta(0) < +\infty$, by (B.1), we have: $\zeta(g_k, 0) = g_k - (\theta+\tau)Ag_k^{1-\alpha}$. Therefore, there are two (and only two) solutions solving $\zeta(g_k, 0) = 0$, namely, $g_k = 0$ and $g_k = \bar{\bar{g}}_k := (\theta+\tau)^{1/\alpha}$. □

[26] Therefore, $(g_k, b_k) \to (0,0) \Leftrightarrow\ \|v_k\| \to 0$, with $0 \leq g_k \leq \|v_k\|$, and $0 \leq b_k \leq \|v_k\|$.

Appendix C. Stability of the BGPs

Let us define by c_k^i, g_k^i, r^i and γ^i steady-state levels of variables at BGP i, $i \in \{H, L\}$. The Jacobian Matrix of system (20)-(21) is, in the neighborhood of steady-state i

$$J^i = \begin{bmatrix} -\left[1+(1-\alpha)\mathcal{B}^i\right]\gamma^i - \mathcal{B}^i\left(g_k^i\right)^{1-\alpha}\left[\left(g_k^i\right)^\alpha - (\tilde{g}_k)^\alpha\right] & b_k^i \\ -\frac{\mathcal{B}^i c_k^i \left(g_k^i\right)^{1-\alpha}}{b_k^i}\left[\alpha S(1-\tau)(1-\alpha)A + \left(g_k^i\right)^\alpha - (\tilde{g}_k)^\alpha\right] & c_k^i \end{bmatrix}$$

where $\tilde{g}_k := [(1-\alpha)A]^{1/\alpha}$ and coefficient $\mathcal{B}^i$ is defined in Lemma 2.

The determinant and the trace of this matrix are, respectively

$$D^i = -\left[1+(1-\alpha)\mathcal{B}^i\right]\gamma^i c_k^i + S(1-\alpha)(\rho+\delta)\mathcal{B}^i c_k^i, \tag{C.1}$$

$$T^i = c_k^i - \left[1+(1-\alpha)\mathcal{B}^i\right]\gamma^i - \mathcal{B}^i\left(g_k^i\right)^{1-\alpha}\left[\left(g_k^i\right)^\alpha - (\tilde{g}_k)^\alpha\right]. \tag{C.2}$$

From Lemma 2, in the neighborhood of the Barro steady-state, we have $\mathcal{B}^B = 0$, and $D^B = -\gamma^B c_k^B$. In the neighborhood of the Solow steady-state, we have $\gamma^S = 0$, $D^S = \Omega^{-1}\mathcal{B}^S$, and $T^S = c_k^S - \mathcal{B}^S\left(g_k^S\right)^{1-\alpha}\left[\left(g_k^S\right)^\alpha - (\tilde{g}_k)^\alpha\right]$.

Appendix D. Proof of Lemma 4

For θ small enough, this Appendix proves the topological equivalence between the low and the Solow BGPs, and between the high and the Barro BGPs.

First, we show that the properties of system (24)-(25) are continuous when $\theta \to 0$. Regarding the steady-states, Proposition 1 argues that there are two steady-states: a high BGP (c_k^H, g_k^H) that converges to the Barro BGP (c_k^B, g_k^B), and a low BGP (c_k^L, g_k^L) that converges to the Solow BGP (c_k^S, g_k^S). Specifically, using (18) and (19), $\gamma^H \to \gamma^B > 0$ and $\gamma^L \to \gamma^S = 0$ when $\theta \to 0$.

Second, as the (local) topological behaviour of steady-states are fully characterized by the trace (T^i) and the determinant (D^i) of the Jacobian matrix (computed by an approximation in the neighborhood of steady states i), we must ensure that $D^L \to D^S, D^H \to D^B$ and $T^H \to T^B$, $T^L \to T^S$ for $\theta \to 0$.

From Proposition 4, since $(c_k^H, g_k^H) \to (c_k^B, g_k^B)$ and $(\mathrm{c}_k^L, g_k^L) \to (c_k^S, g_k^S)$, we have, using Eqs. (C.2) (**??**), $D^H \to D^B$ and $T^H \to T^B$ if and only if $\mathcal{B}^H \to \mathcal{B}^B$ and; $D^L \to D^S$ and $T^L \to T^S$ if and only if $\mathcal{B}^L \to \mathcal{B}^S$. The function ζ defined in (B.1), is continuous in θ, while $\partial\zeta/\partial b_k$ is clearly independent of θ. Consequently, the functions $\psi(\cdot)$ and $\psi'(\cdot)$ are continuous in θ, namely, $\psi(g_k)|_{\theta>0} \to \psi(g_k)|_{\theta=0}$, and $\psi'(g_k)|_{\theta>0} \to \psi'(g_k)|_{\theta=0}$ when $\theta \to 0$, for any g_k. In addition, $\psi(g_k)$ and $\psi'(g_k)$ are also continuous in g_k. Thanks to this dual continuity, we can establish that, from Lemma 2, $\mathcal{B}^H(g_k^H) \to \mathcal{B}^B(g_k^B)$, $\mathcal{B}^L(g_k^L) \to \mathcal{B}^S(g_k^S)$. Consequently, when $\theta \to 0$, $D^H \to D^B$; $T^H \to T^B$; $D^L \to D^S$ and $T^L \to T^S$. Thus, if θ is small enough, the high BGP (resp. the low BGP) is locally topological equivalent to the Barro BGP (resp. the Solow BGP). $\square$

Appendix E. Construction of the phase portrait

The stationarity locus of the public spending ratio ($\dot{g}_k = 0$) is, by (28),

$$c_k = Ag_k^{1-\alpha} - g_k - \frac{\theta A g_k^{1-\alpha}}{\psi(g_k)} - \delta =: \Phi_g(g_k).$$

We study the behavior of the function $\Phi_g(\cdot)$ by assuming a small deficit ratio (θ). In this respect, $\Phi_g \in C^{+\infty}(\mathbb{R}_+^*)$ with $\Phi_g(0) = -\delta$, $\lim_{s\to+\infty} \Phi_g(s) = -\infty$, and $\Phi_g'(g_k) \approx A(1-\alpha)g_k^{-\alpha} - 1 > 0 \Leftrightarrow g_k < \tilde{g}_k := [A(1-\alpha)]^{1/\alpha}$ if $\theta \approx 0$. Therefore, Φ_g describes a hump-shaped curve in the (c_k, g_k)-plane, with a maximum at $\tilde{g}_k$. Outside this curve, Eq. (28) highlights that the dynamics of the public spending ratio crucially depends on the sign of $\psi'(\cdot)$. If $\psi'(\cdot) > 0$, the public spending ratio increases (decreases) iff (c_k, g_k) lies above (below) the relation $\Phi_g(g_k)$. Conversely, if $\psi'(\cdot) < 0$, the public spending ratio increases (decreases) iff (c_k, g_k) lies below (above) the relation $\Phi_g(g_k)$. In this respect, there is a discontinuity of the law of motion of g_k at the *asymptote-point* $\overline{g}_k$ such that $\psi'(\overline{g}_k) = 0$. The stationarity locus of the consumption ratio ($\dot{c}_k = 0$) is, by (29),

$$c_k = Ag_k^{1-\alpha} - g_k - \delta - S\left[\alpha(1-\tau)Ag_k^{1-\alpha} - \rho - \delta\right] =: \Phi_c(g_k).$$

Regarding stability, if (c_k, g_k) lies above (below) the relation $\Phi_c(g_k)$, the consumption ratio increases (decreases). In addition, we observe that $\Phi_c \in C^{+\infty}(\mathbb{R}_+^*)$, with $\Phi_c(0) = S\rho - \delta$, $\lim_{s\to+\infty} \Phi_c(s) = -\infty$, and we compute $\Phi_c'(g_k) = A(1-\alpha)g_k^{-\alpha}[1 - S\alpha(1-\tau)] - 1$. Therefore, function Φ_c describes a hump-shaped curve in the (c_k, g_k)-plane, and the maximum is reached at $\hat{g}_k$, where $\hat{g}_k^\alpha = (1-\alpha)A[1 - S\alpha(1-\tau)]$. To ensure $\hat{g}_k > 0$, we assume that $S < 1/[\alpha(1-\tau)]$, and, since $1 - S\alpha(1-\tau) < 1$ we directly obtain: $\max \Phi_c(\cdot) = \hat{g}_k < \tilde{g}_k = \max \Phi_g(\cdot)$.

Appendix F. The reduced-form with constant elasticity of substitution

With Eqs. (27) and (23), we obtain an explicit relation between the public debt-to-GDP ratio (b_y) and the productive spending ratio

$$b_y = \left[\frac{(\theta+\tau) - g_k^\alpha/A}{\eta_1(\alpha(1-\tau)Ag_k^{1-\alpha} - \delta)}\right]^{\frac{1}{1+\varepsilon}} =: b_y(g_k).$$

From Lemma 2, we can compute

$$\mathcal{B}^L(g_k) := \frac{(1+\varepsilon)\left[(\theta+\tau)Ag_k^{1-\alpha} - g_k\right]}{\alpha g_k + (1-\alpha)\left[(\theta+\tau)Ag_k^{1-\alpha} - g_k\right][\delta/r - \varepsilon]},$$

with $r = \alpha(1-\tau)Ag_k^{1-\alpha} - \delta$. Consequently, the reduced-form (27)-(28) becomes

$$\frac{\dot{g}_k}{g_k} = -\mathcal{B}^L(g_k)\left[\frac{\theta}{b_y(g_k)} + \delta + g_k + c_k - Ag_k^{1-\alpha}\right],$$

$$\frac{\dot{c}_k}{c_k} = S\left[\alpha(1-\tau)Ag_k^{1-\alpha} - \rho - \delta\right] + \delta + g_k + c_k - Ag_k^{1-\alpha}.$$

This is the form we introduce in ©matcont to compute the Lyapunov coefficient in Section 4 and numerically illustrate global dynamics in Section 5. □