Social Welfare for Independent Workers*

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Work In Progress

Abstract

This paper studies an insurance system for independent workers, characterized by multiple sources of incomes and ability to avoid taxation. Optimal dynamic contracts engineered by a public agency must satisfy incentive-compatibility constraints for workers to participate and to declare their entire income. A risk-averse worker has incentives to pay taxes today to be eligible for benefits in the future. The principal can thus tax workers and improve their lifetime utility simultaneously. The optimal contract takes the form of an individual portable account. This paper is intended to policy makers, as it proposes foundations for an implementable mechanism device.

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Introduction

A new form of employment is questioning the social welfare system of our modern societies. Internet triggered the boom of the "gig economy", driven by recent fast-growing companies.[1] New online platforms strongly reduced the frictions on many markets for services. Nowadays anyone can find a private teacher, a house cleaner or a gardener with a click on a website. The website provides you all the information needed to the consumer: public ratings and prices. This technological progress drastically increases the gain from creating one's own business relative to the gain from passing by an employer. This technology is extending to a variety of sectors in the economy, including transactions between professionals. Some occupations in the art industry for instance already benefits from accommodating labor regulations. Should we foster high flexibility with a risk of increasing precarity on labor markets? To answer this question, one needs at least an idea of a welfare system for independent workers. This is the aim of this article.

The present work develops a social welfare system for independent workers characterized by multiple contracts or employers. The current system is based on payroll taxes on the wage bill when a worker is employed and subsidies when she gets unemployed or sick. This system cannot be extended to independent workers for two reasons. First, the multiple sources of labor incomes, from different employers or customers, generate unstable revenue at a high frequency level (from one month to another). Whereas a unique employer could stabilize its employees' wage from part of the economic fluctuations, independent workers face a new risk which they cannot get insured against. Second, independent workers may work through the informal sector and avoid part of their fiscal duties. A firm is monitored by public authorities to prevent irregular practices. Independent workers can have incentives to lie to the public authority to pay less taxes or to receive more subsidies.

I study an insurance contract proposed by a public agency to insure a worker against idiosyncratic shocks on her income over time. The worker chooses to report her income and is either taxed or subsidized. The insurance device takes the form of an *individual portable account* giving rights to benefits. The theoretical mechanism design problem translates to a standard principal-agent model with one particular feature: the principal (the public insurance agency) does not observe the production (or revenue) of the agent (the worker), but the production reported by the agent. The independent worker thus can lie and divert part of her income. The optimal contract provides incentives to report the entire income. The worker accepts to pay taxes in a good states today in order to receive benefits in bad states in the future.

This work is related to a literature on multiple-period principal-agent models. Rogerson (1985), Lambert (1983), Spear and Srivastava (1987) study the repeated game of Holmstrom (1979): an employer designs a payment contract with a worker who produces a random outcome correlated to the unobserved worker's effort. Asymmetric informa-

[1] For a short list of the most famous ones: Uber, Lyft, TaskRabbit, Upwork, freelancer.com, Thumbtack, Spare5.

tion is different in my paper. The random outcome is unobserved by the principal and worker's efforts are ignored. My model extensively borrows from the lender-borrower model of Thomas and Worrall (1990). In their model, the lender is a principal that engineers a contract for the borrower. The optimal insurance contract in my paper has the same properties as their lending contract. The practicality of the contract proposed by Thomas and Worrall (1990) is questioned because of the so-called *immiserization result*: the debt of the borrower tends to infinity for any path of the process under the optimal contract. Williams (2011) extends their model to consider autocorrelated outcome in a continuous-time model, the immiserization result then does not hold anymore. Instead of adopting the same specification, I investigate the feasible contracts by considering lower bounds for taxes and utility, precisely the constraints for positive consumption and voluntary participation.

The optimal insurance contract provides insurance of unemployment risks as any source of income fluctuations. I depart from the literature on optimal unemployment insurance (Hopenhayn and Nicolini 1997 among others) by assuming away employment as an absorbing state. Pavoni (2007) also points the immiserization result of Hopenhayn and Nicolini 1997 and introduces a lower bound for the value of unemployment. I reproduce his approach with respect to Thomas and Worrall (1990).

The paper develops a model in the main section and introduces step-by-step the constraint from the first-best setting to the second-best setting with participation constraints.

1 Repeated principal-agent problem with income reporting

The framework is similar to Thomas and Worrall (1990) except that the distribution of incomes is continuous. The presentation slightly differs from theirs as I formulate the dual optimization problem to solve the principal-agent model.

1.1 Framework

Time is discrete and the horizon is infinite. A public agency wants to insure a risk-averse worker against income shocks. Both value future at a discount rate $\beta \leq 1$. The agency has access to perfect financial markets, whereas the worker has not. She cannot neither borrow nor save. At each period t, the worker produces a gross output x_t without using capital. The model is silent on how the worker receives this income. It can come from either professional or private, multiple or single, employers or trading partners. This income is random, x_t is drawn from a publicly-known distribution with probability density function $f(.)$ and support $(0, x_{max})$. The realized income, however, is only observable to the worker. The public agency observes $\tilde{x}_t$, the income reported by the worker in the interval $(0, x_t)$. To insure the worker, the agency taxes an amount τ_t given the reported income. τ_t can be negative, in which case the worker receives a subsidy. The worker enjoys current-period utility $u(x_t - \tau_t)$, with $u(.)$ a positive, strictly increasing and concave function. The inter-temporal worker's utility is the discounted sum of after-tax incomes, $\sum_{t=0}^{\infty} \beta^t u(x_t - \tau_t)$. The principal fulfills an inter-temporal budget constraint. g_t denotes the (possibly negative) amount of public funds available at the beginning of period t. Given the reported income, the government chooses a next-period stock of public funds g_{t+1} to clear its budget in expectation. g_t is public and it can be interpreted as a level of generosity or a negative mesure of the worker's indebtedness.[2] The timing is the following. At the beginning of period t, the public agency has access to public funds g_t. Then, the worker observes x_t and reports $\tilde{x}_t$ to the agency. The worker is then taxed τ_t, and the agency chooses the next-period funds g_{t+1}.

At time 0, the worker and the agency agree on a contract. It stipulates an initial exogenous level of generosity g_0, a level of tax at any time t, a level of generosity at any time t. Both taxes and levels of generosity depend on the history of reported incomes from 0 up to t. The writing of the principal-agent problem with explicit history dependence is cumbersome. It is now established that the problem is equivalent to find a "recursive contract".[3] The optimal contract consists in defining a tax function $\tau(.,.)$ for the current period and the next-period level of generosity $G(.,.)$, so that $\tau_t = \tau(\tilde{x}_t, g_t)$ and $g_{t+1} = G(\tilde{x}_t, g_t)$. In the recursive contract, generosity at period t, g, is a state variable that summarizes the information the agency needs to know for taxing the individual. The contract also specify a reporting strategy of the worker $\tilde{x}(.,.)$ such that $\tilde{x}_t = \tilde{x}(x_t, g_t)$. The insurance contract makes sure the worker gets the present-discounted value $U_t = \mathcal{U}(g_t)$, with

$$\mathcal{U}(g) = \int \left[u(x - \tau(\tilde{x}(x,g), g)) + \beta \mathcal{U}(G(\tilde{x}(x,g), g)) \right] f(x) dx. \tag{O}$$

Here, we depart from the literature which consider $U = \mathcal{U}(g)$ as the state variable instead of g. Both approaches are equivalent. $\tau(\tilde{x}(x,g), g))$ and $G(\tilde{x}(x,g), g)$ are the tax level and the generosity level given the realized income and the initial level of public founds. The budget constraint for the principal is

$$g + \int \left[\tau(\tilde{x}(x,g), g) - \beta G(\tilde{x}(x,g), g) \right] f(x) dx \geq 0. \tag{C1}$$

The budget constraint make explicit the inter-temporal trade-off of the principal. By increasing the tax today τ_t, the principal can be more generous tomorrow by increasing g_{t+1} without unbalancing its budget. Note the principal is able to redistribute across the different states of the economy. Thus, G is not necessarily such that $g + \tilde{x}(x_t, g_t) + \beta G(\tilde{x}(x,g), g) = 0$. The agent and the principal share the same objective, maximizing the expected worker's utility. They do not, however, face the same timing of information and the principal also wants to satisfy a budget constraint. The problem of self-reporting income is obvious in a one-period model. Imagine, the principal fully insures the worker,

[2] The term "indebtedness" is borrowed from Thomas and Worrall (1990).

[3] See Spear and Srivastava (1987); Thomas and Worrall (1990); Williams (2011) among others.

then the taxation function is increasing with production, negative for low incomes and positive for high incomes. Then the worker has an incentive to declare less than her actual income, to receive subsidies. In other words, the worker wants the principal to believe she is unlucky. The problem of income diversion translates to an adverse selection problem. The worker of type x may prefer to behave like a type $\tilde{x}(x, g)$. In the one-period model, one can show that the principal cannot insure the worker, precisely the optimal contract specifies a constant tax or subsidy depending on generosity g. In the repeated game, the principal can provide incentives to declare income truthfully because it can reward the worker in the future through G.

Once the individual receives her gross income x, her inter-temporal utility from reporting $\tilde{x} \leq x$ is the integrand in expression (O). The optimal reporting strategy is such that the utility is maximized,

$$\tilde{x}(x, g) \in \underset{0 \leq \tilde{x} \leq x}{\operatorname{argmax}} \{u(x - \tau(\tilde{x}, g)) + \beta \mathcal{U}(G(\tilde{x}, g))\}. \tag{C2}$$

Definition 1 *Given an initial stock of public funds g_0, an insurance contract is a 4-tuple $\langle \tilde{x}(.,.), \tau(.,.), G(.,.), \mathcal{U}(.) \rangle$, which specifies*

- *a reporting strategy $\tilde{x}(x, g)$ as a function of the realized production x and generosity g,*
- *a taxation function $\tau(\tilde{x}, g)$ as a function of the reported income $\tilde{x}$ and generosity g,*
- *a next-period level of generosity $G(\tilde{x}, g)$ as a function of the reported income $\tilde{x}$ and generosity g,*
- *an expected present-discounted worker's utility $\mathcal{U}(g)$ as a function of generosity g.*

In the following, I will study first the optimal contracts when the agency observes the realized income (first-best setting) and then I will account for the contracts when income is reported (second-best setting). In addition, I will distinguish in each case the *ideal* solution to the *practical* solution. For the ideal contracts, the consumption is unbouded below so that the agency is able to tax an infinite amount, and the worker is forced to participate to the contract. The practical contracts incorporate the constraints $x - \tau(x, g) \geq 0$ and $\mathcal{U}(g) \geq \bar{U} = \int \frac{u(x)}{1-\beta} f(x) dx$. The first constraint imposes positive consumption, the second constraint imposes an expected utility under the contract higher than the expected utility in autarky (i.e. a participation constraint).

For the rest of the analysis, I focus on differentiable solutions so that a first-order approch is valid.

1.2 Observable income, no asymmetric information

In this subsection, the principal observes the realized income so that it can make sure that $\tilde{x}(x, g) = x$ at any time, without self-enforcing equation (C2).

1.2.1 Ideal contract

The problem of the principal is to find a taxation function τ and the next-period spending G that maximizes the worker's utility under the budget constraint. Both functions are defined on the entire space $\mathbb{R}^2$. The principal solves the following problem for any level of generosity g:

$$\mathcal{U}(g) = \max_{\tau, G} \int \left[u(x - \tau(x,g)) + \beta \mathcal{U}(G(x,g))\right] f(x)dx \tag{1}$$
$$\text{s.t. } g + \int \left[\tau(x,g) - \beta G(x,g)\right] f(x)dx \geq 0.$$

Before solving the problem with a first-order approach, we establish three properties of any solution: i) the budget constraint always binds; ii) $\mathcal{U}$ is increasing and strictly concave, iii) the tax τ and the level of generosity G are increasing with income x. If the constraint does not bind, the agency could build a new insurance contract and tax less, which is a better contract. With higher public funds g, the principal can offer at least the same contract and so $\mathcal{U}$ is increasing. The concavity proof of $\mathcal{U}$ is more technical and given in appendix. It is a consequence of the concavity of u and the convexity of the constraint set. If τ was not increasing in x, then the principal could improve the worker's current-period utility by taxing a fix amount $\int \tau(x,g)f(x)dx$ for any state x instead. Such a modification would not change the budget constraint and would make the worker better-off because of risk-aversion. As $\mathcal{U}$ is concave, an analogous proof works for G. Although the solution is simple to derive analytically, these properties turn out to hold for optimal contracts in a more complex setting.

Denote $\lambda(g)$ the Lagrange parameter associated with condition in problem (1), or equivalently the shadow price of increasing the principal's revenue by a marginal unit. The Lagrangian can be written for a given g:

$$\mathcal{L}(\tau, G, \lambda; g) = \int \left[u(x - \tau(x,g)) + \beta \mathcal{U}(G(x,g)) + \lambda(g)(\tau(x,g) - \beta G(x,g) + g)\right] f(x)dx. \tag{2}$$

We use a variational argument to derive the first-order conditions. Once predefined x, we consider a small tax increase $\Delta\tau$ on a small interval $(x, x + \Delta x)$. Such an infinitesimal increase should let unchanged the value of the Lagrangian. Repeating the analysis for G, we obtain the conditions:

$$u'(x - \tau(x,g)) = \lambda(g) \tag{3}$$
$$\mathcal{U}'(G(x,g)) = \lambda(g) \tag{4}$$

The optimal contract is such that i) the worker is fully insured as the current-period net income $x - \tau(x,g)$ and next-period generosity $G(x,g)$ do not depend on income x; ii) the current-period marginal utility equals the next-period discounted marginal utility,

$u'(x-\tau(x,g)) = \mathcal{U}'(G(x,g))$. Denote the mean of the distribution f as

$$\bar{x} = \int x f(x) dx.$$

Proposition 1 *A first-best ideal optimal contract is such that*

$$\begin{aligned} x - \tau(x,g) &= \bar{x} + (1-\beta)g, \\ G(x,g) &= g, \\ \mathcal{U}(g) &= \frac{u(\bar{x} + (1-\beta)g)}{1-\beta}. \end{aligned}$$

There is a unique optimal contract for any initial level g_0.

Proof. Using the envelope condition, we obtain that $\mathcal{U}'(g) = \lambda(g)$. If $\mathcal{U}$ is concave, then $\mathcal{U}'$ is monotonous and so $G(x,g) = g$. Then, the first equality of the Proposition derives from the binding budget constraint and the fact that $x - \tau(x,g)$ is constant. Write the expected utility as the infinite sum to obtain the last equality. ■

1.2.2 Feasible contract

In practice, the principal is limited in how it can tax the worker. In particular, it may not be able to raise any amount of tax $g_0 < 0$. The principal solves the following problem for any level of generosity g:

$$\mathcal{U}(g) = \max_{\tau,G} \int \left[u(x-\tau(x,g)) + \beta \mathcal{U}(G(x,g))\right] f(x) dx \tag{5}$$
$$\text{s.t.} \begin{cases} g + \int \left[\tau(x,g) - \beta G(x,g)\right] f(x) dx \geq 0 \\ x - \tau(x,g) \geq 0 \\ \int \left[u(x-\tau(x,g)) + \beta \mathcal{U}(G(x,g))\right] f(x) dx \geq \bar{U} \end{cases}.$$

The principal cannot tax more than x at each period. So if $g_0 < \frac{x}{1-\beta}$, the principal cannot clears its budget constraint.

Proposition 2 *The public agency and the worker agree on an optimal feasible contract if and only if $g_0 \geq g_{min}$. The threshold g_{min} is implicitly defined such that*

$$u(\bar{x} + (1-\beta)g_{min}) = \int u(x) f(x) dx.$$

The unique contract they implement for a given g_0 is the ideal first-best contract.

Proof. The proof relies on how the public agency can raise positive taxes, $g_0 < 0$. The only way to tax her is to extract part of the worker's risk premium. Suppose the agency does not fully insured the worker, then we could find a contract that insures more the worker and that increases the agency's revenue. The lower bound for g_0 is such that the first-best feasible contract provides the same utility of the worker as in autarky. ■

1.3 Unobservable income, income reporting

We turn now to the key assumption, the inability to observe directly the worker's gross income. One can check that condition (C2) is not satisfied in the first-best setting. As the worker receives a constant net income, she could lie and report having a zero income to obtain more subsidies. Define a truth-revealing contract as an optimal contract where the reporting strategy is the identity function, $\tilde{x}(x, g) = x$. Because the principal and the agent share the same objective function, the principal can always modify an optimal contract such that the agent truthfully reports its income. This is the Revelation Principle.

Lemma 1 ***(Revelation Principle)*** *Any optimal insurance contract is equivalent to an optimal truth-revealing contract.*

Proof. Consider an optimal contract $\langle \tilde{x}_0(.), \tau_0(.,.), G_0(.,.), \mathcal{U}_0(.) \rangle$. Denote Id as the identity function. We define the truth-revealing contract $\langle \mathrm{Id}(.), \tau_1(.,.), G_1(.,.), \mathcal{U}_0(.) \rangle$ with $\tau_1(x, g) = \tau_0(\tilde{x}_0(x), g)$ and $G_1(x, g) = G_0(\tilde{x}_0(x), g)$. Given that the first contract is optimal, the second is optimal too. ■

In the following, I will focus on the optimal truth-revealing contracts. Denote $H(x, \tilde{x}, g)$ the inter-temporal utility of declaring $\tilde{x} \leq x$ once the worker knows its production x,

$$H(x, \tilde{x}, g) = u(x - \tau(\tilde{x}, g)) + \beta \mathcal{U}(G(\tilde{x}, g)).$$

The constraint of truthful income-reporting (C2) is such that $x \in \underset{0 \leq \tilde{x} \leq x}{\operatorname{argmax}}\, H(x, \tilde{x}, g)$. This condition straightforwardly writes as a first-order condition, $\frac{\partial H}{\partial \tilde{x}}(x, \tilde{x}, g) \geq 0$ for any g, for any x and for any $\tilde{x}$ such that $\tilde{x} \leq x$. It consists in a constraint on three dimensions. The next Lemma shows that the number of dimensions can be reduced to two. It is equivalent to Lemma 4 of Thomas and Worrall (1990) with a discrete distribution of incomes.

Lemma 2 *Under the assumption that the taxation increases with reported income,*

$$\forall g, \forall x, \qquad \frac{\partial \tau}{\partial x}(x, g) \geq 0,$$

the condition for workers to report their entire income is equivalent to the condition

$$\forall g, \forall x, \qquad \frac{\partial H}{\partial \tilde{x}}(x, x, g) \geq 0. \tag{6}$$

Proof. Straightforwardly, $\frac{\partial H}{\partial \tilde{x}}(x, \tilde{x}, g) \geq 0$ implies $\frac{\partial H}{\partial \tilde{x}}(x, x, g) \geq 0$ as a particular case. Now suppose $\frac{\partial H}{\partial \tilde{x}}(x, x, g) \geq 0$ for any g and x and show $\frac{\partial H}{\partial \tilde{x}}(x, \tilde{x}, g) \geq 0$. Set g as fixed.

The assumption implies

$$u'(\tilde{x}-\tau(\tilde{x},g)).\frac{\partial\tau}{\partial x}(\tilde{x},g) \leq \beta\mathcal{U}'(G(\tilde{x},g)).\frac{\partial G}{\partial x}(\tilde{x},g).$$

Because the tax is increasing with income and the current-period utility function is concave, for any $x \geq \tilde{x}$,

$$u'(x-\tau(R,\tilde{x})).\frac{\partial\tau}{\partial x}(\tilde{x},g) \leq u'(\tilde{x}-\tau(\tilde{x},g)).\frac{\partial\tau}{\partial x}(\tilde{x},g).$$

By combining these two inequalities, we obtain $\frac{\partial H}{\partial \tilde{x}}(x,\tilde{x},g) \geq 0$. ■

In the optimal contract, the tax is increasing with income because the principal can always improve worker's welfare by a fix tax if it implements a decreasing taxation.

1.3.1 Ideal contract

The problem of the principal is to find a taxation function τ and the next-period spending G that maximizes the worker's utility under the budget constraint. Both functions are defined on the entire space $\mathbb{R}^2$. In addition to the problem define previously we add the constraint $\frac{\partial H}{\partial \tilde{x}}(x,x,g) \geq 0$. The principal solves the following problem for any level of generosity g:

$$\mathcal{U}(g) = \max_{\tau,G} \int \left[u(x-\tau(x,g)) + \beta\mathcal{U}(G(x,g))\right] f(x)dx \tag{7}$$

$$\text{s.t.} \quad \begin{cases} g + \int \left[\tau(x,g) - \beta G(x,g)\right] f(x)dx \geq 0 \\ -u'(x-\tau(x,g)).\frac{\partial\tau}{\partial x}(x,g) + \beta\mathcal{U}'(G(\tilde{x},g)).\frac{\partial G}{\partial x}(x,g) \geq 0 \end{cases}.$$

We formulate three properties of any solution under the additional assumption that the utility function as the non-increasing absolute risk aversion property (NIARA): i) the budget constraint always binds; ii) $\mathcal{U}$ is increasing and strictly concave, iii) the tax τ and the level of generosity G are increasing with income x. The proof are identical to before, the concavity of $\mathcal{U}$ is shown in appendix. Note that the second constraint will bind but it will be shown in the analytic solution.

Let us denote $\lambda(g)$ and $\mu(x,g)$ the two Lagrangian parameters associated to these conditions. The corresponding Lagrangian can be expressed:

$$\mathcal{L}(\tau,G,h,\lambda,\mu;g) = \int \left[u(x-\tau(x,g)) + \beta\mathcal{U}(G(x,g)) + \lambda(g)(\tau(x,g) - \beta G(x,g) + g)\right] f(x)dx$$
$$+ \int \mu(x,g)\left[-u'(x-\tau(x,g)).\frac{\partial\tau}{\partial x}(x,g) + \beta\mathcal{U}'(G(\tilde{x},g)).\frac{\partial G}{\partial x}(x,g)\right] dx.$$

I use a variational argument on $\tau(0,g)$, $\frac{\partial\tau}{\partial x}(x,g)$, $G(0,g)$ and $\frac{\partial G}{\partial x}(x,g)$ instead of $\tau(x,g)$

and $G(x,g)$. The technical details are in the appendix.[4] We can show that:

$$u'(x-\tau(x,g)) = \frac{\lambda(g)f(x)}{f(x)-\frac{\partial\mu}{\partial x}(x,g)} + \frac{\mu(x,g)}{f(x)-\frac{\partial\mu}{\partial x}(x,g)}u''(x-\tau(x,g)), \tag{8}$$

$$\mathcal{U}'(G(x,g)) = \frac{\lambda(g)f(x)}{f(x)-\frac{\partial\mu}{\partial x}(x,g)}. \tag{9}$$

The mathematical reason for the difference between the first and the second equality is due to the presence of x in the current-period utility $u(x-\tau(x,g))$, whereas x is not an argument of $\mathcal{U}$ except through G. We have established the equivalent first-order conditions of Thomas and Worrall (1990). We also have $\mu(0,g) = \mu(x_{max},g)$. As $\mu(x,g) \geq 0$, it is possible to show that the self-enforcing constraint is always binding. If there exists x_0 such that the constraint do not bind, then $\mu(x_0,g) = 0$ and so there exist x_1 and x_2 such that $\frac{\partial\mu}{\partial x}(x_1,g) = \frac{\partial\mu}{\partial x}(x_2,g) = 0$. Because G is increasing this condition cannot be true.

The envelope theorem implies

$$\frac{1}{\mathcal{U}'(g)} = \int \frac{1}{\mathcal{U}'(G(x,g))}f(x)dx. \tag{10}$$

If we introduce the inverse function, $V(.)$ such that $\mathcal{U}(V(g)) = g$, then $V(U)$ is the discounted value of the public budget. This equation means that the marginal present-discounted budget at time t is then equal to the expected present-discounted budget at $t+1$, as $V'(U) = \frac{1}{\mathcal{U}(V(U))}$. It is not possible to provide a general explicit definition of the optimal insurance contract. However two important properties can be shown. The first one is informative of the type of insurance provided to the worker, the second one is the immiserazation result.

Proposition 3 *The second-best contract is such that*

- *the worker is partly insured across states within a period, there is a x such that $g + \int[\tau(x,g) - \beta G(x,g)]f(x)dx \neq 0$;*
- *any path of public funds g_t diverges to minus infinity.*

Proof. Suppose $g + \int[\tau(x,g) - \beta G(x,g)]f(x)dx = 0$ for any x. Differentiating this equation and $\mathcal{U}(g)$ would imply that $\int \frac{\mu(x,g)}{f(x)-\frac{\partial\mu}{\partial x}(x,g)}u''(x-\tau(x,g))f(x)dx = 0$, which cannot be true. For the second property, notice $\frac{1}{\mathcal{U}'(g_t)}$ is a positive martingale which is upper-bounded. It thus converges to a random variable almost surely. We can show that this random variable is 0.[5] ■

The first property proves that the agency can do more than providing access to a riskless asset, meaning only borrowings and savings. The second property illustrates the non-feasibility of such a contract in practice.

[4]Note the problem looks like a standard optimal control program solvable through a Hamitonian. This is more complicated here because of the inequality constraint.

[5]See the similar proof of Thomas and Worrall (1990).

1.3.2 Feasible contract

The principal solves the following problem for any level of generosity g:

$$\mathcal{U}(g) = \max_{\tau,G} \int \left[u(x - \tau(x,g)) + \beta \mathcal{U}(G(x,g)) \right] f(x)dx \tag{11}$$

$$\text{s.t.} \begin{cases} g + \int \left[\tau(x,g) - \beta G(x,g) \right] f(x)dx \geq 0 \\ -u'(x - \tau(x,g)).\frac{\partial \tau}{\partial x}(x,g) + \beta \mathcal{U}'(G(\tilde{x},g)).\frac{\partial G}{\partial x}(x,g) \geq 0 \\ x - \tau(x,g) \geq 0 \\ \int \left[u(x - \tau(x,g)) + \beta \mathcal{U}(G(x,g)) \right] f(x)dx \geq \bar{U} \end{cases}.$$

As the constraint set a subset of the constraint set in the first-best feasible case, we also have that the public agency does not propose any contract when $g_0 \leq g_{min}$. Denote $\nu(x,g)$ and $\gamma(g)$ the new Lagrange parameters of the feasibility constraints. The marginal utilities satisfy

$$u'(x - \tau(x,g)) = \frac{\lambda(g) f(x)}{(1+\gamma(g))f(x) - \frac{\partial \mu}{\partial x}(x,g)} + \frac{\mu(x,g) u''(x - \tau(x,g))}{(1+\gamma(g))f(x) - \frac{\partial \mu}{\partial x}(x,g)} - \frac{\nu(x,g)}{(1+\gamma(g))f(x) - \frac{\partial \mu}{\partial x}(x,g)}, \tag{12}$$

$$\mathcal{U}'(G(x,g)) = \frac{\lambda(g) f(x)}{(1+\gamma(g))f(x) - \frac{\partial \mu}{\partial x}(x,g)}. \tag{13}$$

The martingale property disappears as

$$\frac{1}{\mathcal{U}'(g)} = \frac{1}{1+\gamma(g)} \int \frac{1}{\mathcal{U}'(G(x,g))} f(x)dx. \tag{14}$$

Obviously the two feasibility constraints do not always bind.

2 Numerical Exercise

(Coming Soon)

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A Appendix

A.1 Concavity of $\mathcal{U}$

The infinite-horizon model can be characterized as the limit of the finite-horizon model. This characterization is necessary for a concavity proof of $\mathcal{U}$ following Thomas and Worrall (1990). Fix the horizon at T, the problem at period $T-k$ is defined recursively given $\mathcal{U}_{T-k+1}$,

$$\mathcal{U}_{T-k}(g) = \max_{\tau_{T-k}, G_{T-k}} \int \left[u(x-\tau_{T-k}(x,g)) + \beta \mathcal{U}_{T-k+1}(G_{T-k}(x,g))\right] f(x)dx \qquad (15)$$
$$\text{s.t. } g + \int \left[\tau_{T-k}(x,g) - \beta G_{T-k}(x,g)\right] f(x)dx \geq 0.$$

The terminal condition, solution of

$$\mathcal{U}_T(g) = \max_{\tau_T} \int u(x-\tau_T(x,g)) f(x)dx \qquad (16)$$
$$\text{s.t. } g + \int \tau_T(x,g) f(x)dx \geq 0,$$

is $\mathcal{U}_T(g) = u(\bar{x}+g)$.

$\mathcal{U}_T$ is concave. Suppose $\mathcal{U}_{T-k+1}$ concave, we show that $\mathcal{U}_{T-k}$ is concave too. Take $g^0 < g^1$ and $0 \leq \alpha \leq 1$, and show that $\mathcal{U}_{T-k}(\alpha g^0 + (1-\alpha)g^1) \geq \alpha \mathcal{U}_{T-k}(g^0) + (1-\alpha)\mathcal{U}_{T-k}(g^1)$. Define $\tilde{\tau}$ and $\tilde{G}$ such that

$$\tilde{\tau}(x) = \alpha \tau_{T-k+1}(x, g^0) + (1-\alpha)\tau_{T-k+1}(x, g^1), \qquad (17)$$
$$\tilde{G}(x) = \alpha G_{T-k+1}(x, g^0) + (1-\alpha)G_{T-k+1}(x, g^1). \qquad (18)$$

The pair $(\tilde{\tau}, \tilde{G})$ belongs to the constraint set for $\mathcal{U}_{T-k}(\alpha g^0 + (1-\alpha)g^1)$ and provide a level of utility equivalent to $\alpha \mathcal{U}_{T-k}(g^0) + (1-\alpha)\mathcal{U}_{T-k}(g^1)$. Hence, it is a lower bound for

$\mathcal{U}_{T-k}(\alpha g^0 + (1-\alpha)g^1)$ as a maximum. This property holds as a limit when T goes to infinity.

A.2 First-order conditions

Write $\tau(x,g) = \tau(0,g) + \int_0^x \frac{\partial \tau}{\partial x}(y,g)dy$ and $G(x,g) = G(0,g) + \int_0^x \frac{\partial G}{\partial x}(y,g)dy$. We maximize the Lagrangian with respect to $\tau(0,g)$, $G(0,g)$, $\frac{\partial \tau}{\partial x}(x,g)$ and $\frac{\partial G}{\partial x}(x,g)$:

$$\begin{aligned}\mathcal{L}(\tau, G, h, \lambda, \mu; g) = &\int \left[u(x-\tau(x,g)) + \beta\mathcal{U}(G(x,g)) + \lambda(g)(\tau(x,g) - \beta G(x,g) + g)\right] f(x)dx \\ &+ \int \mu(x,g)\left[-u'(x-\tau(x,g)).\frac{\partial \tau}{\partial x}(x,g) + \beta\mathcal{U}'(G(\tilde{x},g)).\frac{\partial G}{\partial x}(x,g)\right]dx.\end{aligned}$$

A standard derivation method works for $\tau(0,g)$ and $G(0,g)$,

$$\int \left[-u'(x-\tau(x,g))f(x) + \lambda(g)f(x) + \mu(x,g)u''(x-\tau(x,g)).\frac{\partial \tau}{\partial x}(x,g)\right]dx = 0, \quad (19)$$

$$\int \left[\beta\mathcal{U}'(G(x,g))f(x) - \lambda(g)\beta f(x) + \mu(x,g)\beta\mathcal{U}''(G(\tilde{x},g)).\frac{\partial G}{\partial x}(x,g)\right]dx = 0. \quad (20)$$

Fix x in $(0, x_{max})$. Consider a small deviation $\Delta\tau$ of $\frac{\partial \tau}{\partial x}$ on the small interval $(x, x+\Delta x)$. This infinitesimal shift should let the Lagrangian unchanged. Therefore,

$$\begin{aligned}&\int_x^{x_{max}} \left[-u'(y-\tau(y,g))f(y) + \lambda(g)f(y) + \mu(y,g)u''(y-\tau(y,g)).\frac{\partial \tau}{\partial x}(y,g)\right]dy \\ &- \mu(x,g)u'(x-\tau(x,g)) = 0. \end{aligned} \quad (21)$$

The analogous derivation works for G,

$$\begin{aligned}&\int_x^{x_{max}} \left[\mathcal{U}'(G(y,g))f(x) - \lambda(g)f(y) + \mu(y,g)\mathcal{U}''(G(y,g)).\frac{\partial G}{\partial x}(y,g)\right]dx \\ &+ \mu(x,g)\mathcal{U}'(G(x,g)) = 0. \end{aligned} \quad (22)$$

Note we can show that $\mu(0,g) = \mu(x_{max},g) = 0$. Now we differentiate the last two equations over x:

$$\begin{aligned}&u'(x-\tau(x,g))f(x) - \lambda(g)f(x) - \mu(x,g)u''(x-\tau(x,g)).\frac{\partial \tau}{\partial x}(x,g) \\ &= \mu(x,g)u''(x-\tau(x,g)).\left(1 - \frac{\partial \tau}{\partial x}(x,g)\right) + \frac{\partial \mu}{\partial x}(x,g)u'(x-\tau(x,g)), \end{aligned} \quad (23)$$

$$\begin{aligned}&-\mathcal{U}'(G(x,g))f(x) + \lambda(g)f(x) - \mu(x,g)\mathcal{U}''(x-\tau(x,g)).\frac{\partial G}{\partial x}(x,g) \\ &= -\mu(x,g)\mathcal{U}''(G(x,g)).\frac{\partial G}{\partial x}(x,g) - \frac{\partial \mu}{\partial x}(x,g)\mathcal{U}'(G(x,g)). \end{aligned} \quad (24)$$

By simplifying these two equations, we obtain the two first-order conditions.