Sharing the revenues from broadcasting sport events*

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March 13, 2017

Abstract

We study the problem of sharing the revenue from broadcasting sport events, among participating players. We provide direct, axiomatic and game-theoretical foundations for two focal (and somewhat polar) rules: the *Shapley* rule and the *OLS* rule. The former allocates the revenues from each game equally among the participating players. The latter assigns to each player the revenue from the differential audience with respect to the average audience per game that the rest of the players yield (in the remaining games they play).

JEL numbers: *D63, C71.*

Keywords: *resource allocation, broadcasting, sport events, axioms, regression.*

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*Very preliminary and incomplete. Please do not quote. Proofs are subject to revision. Acknowledgment will be added later.

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1 Introduction

For most sports organizations, the sale of broadcasting and media rights is now the biggest source of revenue. A study of how much money various professional sports leagues generates shows that the NFL made \$13 billion in revenue last season.[1] The Major League Baseball, came second with \$9.5 billion and the Premier League third with \$5.3 billion.[2] Sharing these sizable revenues among participating teams is, by no means, a straightforward problem. Rules vary across the world. For instance, FC Barcelona and Real Madrid CF, the two Spanish giant football clubs, used to earn each more than 20% of the revenues generated by the Spanish football league. In England, however, the top two teams combined only make 13% of the revenues generated by the Premier league.[3]

The aim of this paper is to provide a formal model to study the problem of sharing the revenues from broadcasting sport events. Our model could be applied to different forms of competitions, but our running example will be a round robin tournament in which each competitor (usually, a team) plays in turn against every other (home and away). Thus, the input of our model will be a (square) matrix in which each entry will be indicating the revenues associated to broadcasting the game between the two corresponding competitors. For ease of exposition, we shall assume an equal *pay per view* fee to each game. Thus, broadcasting revenues can be simplified to audiences.

We shall take several approaches to analyze this problem. In each case, we shall derive focal rules to share the revenues from broadcasting sport events. Two salient rules will be what we shall call the *Shapley* rule and the *OLS* rule, each conveying somewhat polar forms of estimating the *fan effect*.

More precisely, we first take a direct approach, partly based on a regression analysis, which will lead us towards what we name the *OLS* rule. This rule assigns to each player the revenue from the differential audience with respect to the average audience per game that the rest of

[1]The study “Which Professional Sports Leagues Make the Most Money” is published by Howmuch.net, a cost information website. It can be accessed at https://howmuch.net/articles/sports-leagues-by-revenue.

[2]Four of the top five leagues in revenue are in North America. However, 14 of the 20 biggest earners are football leagues that are mostly based in Europe.

[3]This might explain why in the last 12 editions of the Spanish football league only 1 time the champion was different from FC Barcelona and Real Madrid CF, whereas the Premier League witnessed 4 different champions in its last 4 editions.

the players yield (in the remaining games they play).

Second, a strategic approach in which we deal with a natural cooperative game associated to the problem. This approach will lead us towards the *Shapley* rule. This rule allocates the revenues from each game equally among the two playing teams.

Third, we take an axiomatic approach formalizing axioms that reflect ethical or operational principles with normative appeal. It turns out the two rules mentioned above are characterized by three properties. Two properties are common in both characterizations. Namely, *equal treatment of equals*, which states that if two competitors have the same audiences, then they should receive the same amount, and *additivity*, which states that revenues should be additive on the audience matrix.[4] The third property in each characterization result comes from a pair of somewhat polar properties modeling the effect of *null* or *nullifying* players, respectively. More precisely, the *null player* property says that if nobody watches a single game of a given team (i.e., the team has a null audience), then such a team gets no revenue. On the other hand, the *nullifying player* property says that if a team nullifies the audience of all the games it plays (for instance, due to some kind of boycott), then the allocation of such a team should decrease exactly by the total audience of such a team.

Fourth, we take an indirect approach in which we focus on an associated problem of adjudicating conflicting claims.

The rest of the paper is organized as follows. In Section 2, we present the model. In Section 3, we perform a regression analysis that will lead to a first (direct) justification of one of the two rules mentioned above. In Section 4, we take the game-theoretical approach associating a suitable cooperative game to each problem. In Section 5, we deal with the axiomatic analysis. In Section 6, we associate our problems to claims problems and appeal to focal rules in the sizable literature dealing with these later problems to solve the former. In Section 7, we provide an empirical application. Finally, we conclude in Section 8.

[4] An interpretation is that the aggregation of the revenue sharing in two seasons (involving the same competitors) is equivalent to the revenue sharing in the hypothetical combined season aggregating the audiences of the corresponding games (involving the same teams) in both seasons.

2 The model

Let $\mathbb{N}$ represent the set of all potential competitors (**teams**) and let $\mathcal{N}$ be the family of all finite (non-empty) subsets of $\mathbb{N}$. An element $N \in \mathcal{N}$ describes a finite set of teams. Its cardinality is denoted by n. In what follows, we assume $n \geq 3$. Given $N \in \mathcal{N}$, let Π_N denote the set of all orders in N. Given $\pi \in \Pi_N$, let $Pre(i, \pi)$ denote the set of elements of N which come before i in the order given by π, *i.e.*

$$Pre(i, \pi) = \{j \in N \mid \pi(j) < \pi(i)\}.$$

For notational simplicity, given $\pi \in \Pi_N$, we denote the agent $i \in N$ with $\pi(i) = s$ as π_s.

For each pair of teams $i, j \in N$, we denote by a_{ij} the broadcasting audience (number of viewers) for the game played by i and j at i's stadium. We use the notational convention that $a_{ii} = 0$, for each $i \in N$. Let $A = (a_{ij})_{(i,j) \in N \times N}$ denote the resulting matrix with the broadcasting audiences generated in the whole tournament involving the teams within N.[5] Let $\mathcal{A}_{n \times n}$ denote the set of all possible such matrices (with zero entries in the diagonal), and $\mathcal{A} = \bigcup_n \mathcal{A}_{n \times n}$. For each $A \in \mathcal{A}$, let $||A|| = \sum_{i,j \in N} a_{ij}$.

A (broadcasting sports) **problem** is a duplet (N, A), where $N \in \mathcal{N}$ is the set of **teams** and $A = (a_{ij})_{(i,j) \in N \times N} \in \mathcal{A}_{n \times n}$ is the **audience matrix**. The family of all the problems described as such is denoted by $\mathcal{P}$. For each $(N, A) \in \mathcal{P}$, and each $i \in N$, let $\alpha_i(A)$ denote the total audience achieved by team i, i.e., $\alpha_i(A) = \sum_{j \in N}(a_{ij} + a_{ji})$. When no confusion arises we write α_i instead of $\alpha_i(A)$.

Consider the following example, which will be used often in the paper.[6]

Example 1 *Let (N, A) be such that $N = \{1, 2, 3\}$ and*

$$A = \begin{pmatrix} 0 & 1200 & 1030 \\ 1200 & 0 & 230 \\ 1030 & 230 & 0 \end{pmatrix}$$

Then $||A|| = 4920$ and $\alpha(A) = (\alpha_1(A), \alpha_2(A), \alpha_3(A)) = (4460, 2860, 2520)$.

[5] We assume a standard round robin tournament, i.e., a league in which each team plays each other team twice: once home, another away.

[6] The rationale underlying this example is the existence of three teams: a power house with 1000 fans, an average team with 200 fans and a small one with just 30 fans.

A (sharing) **rule** is a mapping that associates with each problem an allocation indicating the amount each team gets from the total revenue generated by broadcasting games. Without loss of generality, we normalize the revenue generated by each game to 1 (to be interpreted as the "pay per view" fee). Thus, formally, $R : \mathcal{P} \to \mathbb{R}^n$ is such that, for each $(N, A) \in \mathcal{P}$,

$$\sum_{i \in N} R_i(N, A) = ||A||.$$

Two rules will be central to our analysis. First, the rule that allocates the revenues from each game equally among the two playing teams. Equivalently, given our normalization convention, each team is awarded half of its total audience.[7] Formally,

Shapley, **S**: For each $(N, A) \in \mathcal{P}$, and each $i \in N$,

$$S_i(N, A) = \frac{\alpha_i}{2}.$$

Second, a somewhat less intuitive rule, resulting from a specific linear combination of the Shapley rule just described, and the rule splitting the total revenue equally among all teams.[8] Formally,

OLS, **O**: For each $(N, A) \in \mathcal{P}$, and each $i \in N$,

$$O_i(N, A) = \frac{(n-1)\alpha_i - ||A||}{n-2}. \tag{1}$$

As $||A|| = \alpha_i + \sum_{j,k \in N \setminus \{i\}} (a_{jk} + a_{kj})$, we can express the rule in the following alternative way:

$$O_i(N, A) = \alpha_i - \frac{\sum_{j,k \in N \setminus \{i\}} (a_{jk} + a_{kj})}{n-2}. \tag{2}$$

Notice that $\sum_{j,k \in N \setminus \{i\}} (a_{jk} + a_{kj})$ is the total audience in the $(n-1)(n-2)$ games played by the rest of the teams. Thus, $\frac{\sum_{j,k \in N \setminus \{i\}} (a_{jk} + a_{kj})}{n-2}$ is the average audience per game in the games played by each of the rest of the teams. Thus, the rule is assigning to each team the differential audience with respect to the average audience per game that the rest of the teams yield (in the remaining games they play).

[7] The reader is referred to Section 4 for a plausible reason to name this rule after Shapley (1953). A similar rule was introduced by Ginsburgh and Zang (2003), and characterized by Bergantiños and Moreno-Ternero (2015), for the so-called museum pass problem.

[8] The reader is referred to the next section for a convincing reason to name this rule as OLS.

In **Example 1** we have that

Rule/Team	1	2	3
Equal Awards	1640	1640	1640
Shapley	2230	1430	1260
OLS	4000	800	120

We can safely argue that, in general, one might become a viewer of a game involving teams i and j for several reasons:

1. Because of being a fan of this sport per se (in which case one would be eager to watch all the games, independently of the teams playing).
2. Because of being a fan of team i (in which case one would be eager to watch all the games involving team i).
3. Because of being a fan of team j (in which case one would be eager to watch all the games involving team j).
4. Because of some other reason, different than the ones stated above.

Let a, a_i, a_j and u_{ij} denote, respectively, the number of viewers in each of the above categories. Then, it seems reasonable to allocate $a_{ij} + a_{ji}$ (the overall audience of the games involving teams i and j) among team i and team j as follows:

$$\left(\frac{a}{2} + a_i + \frac{u_{ij}}{2}, \frac{a}{2} + a_j + \frac{u_{ij}}{2}\right)$$

In words, the audience generated by each team is assigned to such a team. The rest of the audience is divided equally among both teams.[9]

As mentioned in footnote 5, **Example 1** was defined assuming that $a_1 = 1000$, $a_2 = 200$, $a_3 = 30$ and the remaining parameters as 0. Thus, the allocation proposed by the above principle would be $(4000, 800, 120)$, which is precisely the OLS allocation depicted at the table above.

[9] This is the same logic underlying the so-called *concede-and-divide* mechanism (e.g., Thomson, 2003), which can be traced back to the Babylonian Talmud.

Now, in practice, we do not know the parameters a, a_i, a_j and u_{ij}. One might interpret the Shapley rule as a naive attempt to deal with this issue, assuming $a_i = a_j$. But, obviously, this is a very strong assumption. It is more natural to assume that teams are heterogeneous when it comes to their numbers of fans. Some teams have more fans than others and, consequently, they drive larger audiences. This aspect seems to be indeed captured by the actual revenue sharing process used in professional sports, where the amount assigned to each team depends on some parameters that try to capture such heterogeneity. As mentioned in the introduction, the way in which this idea is implemented varies across countries and sports.

A somewhat polar option to the attempt conveyed by the Shapley rule can also be considered. More precisely, it seems plausible to assume that the "worst" scenario for measuring the fan effect is what the Shapley rule conveys, as it could be interpreted as saying that no team has fans. Similarly, the "best" scenario for measuring the fan effect is what the previous example showed, where it was assumed that all individuals are fans of some team. We shall develop this scenario further with the help of the regression analysis elaborated in the next section. As we shall see, it will drive us towards the OLS rule introduced above.

3 Regression Analysis

We take an econometric approach to our problem in this section. More precisely, we consider the following linear regression model:

$$Y = b_0 + \sum_{i \in N} b_i X_i + \varepsilon,$$

where Y is the audience of a game, X_i is the team dummy variable (i.e., $X_i = 1$ if team i plays the game and 0 otherwise) and ε is the error term. Thus, with our notation,

$$a_{ij} = b_0 + b_i X_i + b_j X_j + \varepsilon_{ij},$$

for each pair $i, j \in N$, with $i \neq j$.

Let the estimation of the parameters be denoted by $\hat{b}_0$, $\left\{\hat{b}_i\right\}_{i \in N}$ and $\{\hat{\varepsilon}_{ij}\}_{i,j \in N, i \neq j}$, respectively. We then assume the following:

$(C1)$ $\hat{b}_0$ is divided equally among teams.

$(C2)$ $\hat{b}_i$ is assigned to team i.

$(C3)$ $\hat{\varepsilon}_{ij}$ is divided equally between teams i and j.

Applying those principles we can define a rule where, for each problem $(N, A) \in \mathcal{P}$ and each $i \in N$, the audience assigned to agent i is[10]

$$(n-1)\,\widehat{b_0} + 2\,(n-1)\,\widehat{b_i} + \sum_{j \in N \setminus \{i\}} \frac{\widehat{\varepsilon_{ij}} + \widehat{\varepsilon_{ji}}}{2}.$$

We now estimate the parameters using the ordinary least squares (OLS) estimator. That is,

$$\begin{aligned} \left(\widehat{b_i}\right)_{i \in N} &= Cov\,(X, X)^{-1}\,Cov\,(X, Y) \text{ and} \\ \widehat{b_0} &= \overline{Y} - \sum_{i=1}^{n} \widehat{b_i}\overline{X_i} \end{aligned} \tag{3}$$

where

$$\begin{aligned} Cov\,(X, X) &= \left(Cov\,(X_i, X_j)\right)_{i,j \in N} \text{ and} \\ Cov\,(X, Y) &= \left(Cov\,(X_i, Y)\right)_{i \in N} \end{aligned}$$

In general, given two variables U, V taking the values $\{(u_k, v_k)\}_{k=1}^{m}$we have that

$$Cov\,(U, V) = \frac{\sum\limits_{k=1}^{m} u_k v_k}{m} - \left(\frac{\sum\limits_{k=1}^{m} u_k}{m}\right)\left(\frac{\sum\limits_{k=1}^{m} v_k}{m}\right).$$

We now apply the previous formula to some cases.

1. $i, j \in N$ with $i \neq j$.

$$\begin{aligned} Cov\,(X_i, X_j) &= \frac{2}{n\,(n-1)} - \left(\frac{2\,(n-1)}{n\,(n-1)}\right)\left(\frac{2\,(n-1)}{n\,(n-1)}\right) \\ &= \frac{2}{n\,(n-1)} - \frac{4}{n^2} = \frac{2\,(2-n)}{n^2\,(n-1)}. \end{aligned}$$

2. $i \in N$.

$$\begin{aligned} Cov\,(X_i, X_i) &= \frac{2\,(n-1)}{n\,(n-1)} - \left(\frac{2\,(n-1)}{n\,(n-1)}\right)\left(\frac{2\,(n-1)}{n\,(n-1)}\right) \\ &= \frac{2}{n} - \frac{4}{n^2} = \frac{2\,(n-2)}{n^2}. \end{aligned}$$

[10] Note that each team plays $2\,(n-1)$ games.

3. $i \in N$.

$$\begin{aligned} Cov\left(X_i, Y\right) &= \frac{\alpha_i}{n\left(n-1\right)} - \left(\frac{2\left(n-1\right)}{n\left(n-1\right)}\right)\left(\frac{\left|\left|A\right|\right|}{n\left(n-1\right)}\right) \\ &= \frac{\alpha_i}{n\left(n-1\right)} - \frac{2\left|\left|A\right|\right|}{n^2\left(n-1\right)} \\ &= \frac{n\alpha_i - 2\left|\left|A\right|\right|}{n^2\left(n-1\right)} = \left(\alpha_i - \frac{2\left|\left|A\right|\right|}{n}\right)\frac{1}{n\left(n-1\right)} \end{aligned}$$

Now,

$$Cov\left(X, Y\right) = \frac{1}{n^2\left(n-1\right)}\begin{pmatrix} n\alpha_1 - 2\left|\left|A\right|\right| \\ ... \\ n\alpha_n - 2\left|\left|A\right|\right| \end{pmatrix} \tag{4}$$

$$Cov\left(X, X\right) = \frac{2\left(2-n\right)}{n^2\left(n-1\right)}\begin{pmatrix} 1-n & 1 & ... & 1 \\ 1 & 1-n & ... & 1 \\ 1 & ... & ... & 1 \\ 1 & 1 & 1 & 1-n \end{pmatrix} \tag{5}$$

Unfortunately, $Cov\left(X, X\right)$ has a zero determinant (and, thus, cannot be inverted). Thus, we have a problem of colinearity in the regression model. It is easy to see that, for each $k = 1, ..., n$, we have that

$$X_k = 2_A - \sum_{i \in N \setminus \{k\}} X_i$$

where 2_A is the vector with all coordinates equal to 2.

We now remove one of the independent variables in order to avoid the colinearity issue. Thus, given $k \in N$ we consider the regression where the set of independent variables is $\{X_i\}_{i \in N \setminus \{k\}}$.

In this new regression the expressions for $Cov\left(X, Y\right)$ and $Cov\left(X, X\right)$ are the same as in formulas (4) and (5). But now $Cov\left(X, X\right)$ is a matrix of $\left(n-1\right) \times \left(n-1\right)$ dimension (instead of $n \times n$ as in (5)). It is not difficult to show that

$$Cov\left(X, X\right)^{-1} = \frac{n\left(n-1\right)}{2\left(n-2\right)}\begin{pmatrix} 2 & 1 & ... & 1 \\ 1 & 2 & ... & 1 \\ 1 & ... & ... & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \tag{6}$$

Because of (3), we have that, for each $j \in N\backslash\{k\}$,

$$\begin{aligned}
\widehat{b_j} &= \frac{n(n-1)}{2(n-2)}\frac{1}{n^2(n-1)}\left[2\left(n\alpha_j - 2\,||A||\right) + \sum_{i\in N\backslash\{j,k\}}\left(n\alpha_i - 2\,||A||\right)\right] \\
&= \frac{1}{2(n-2)\,n}\left[2n\alpha_j - 4\,||A|| + n\sum_{i\in N\backslash\{j,k\}}\alpha_i - 2(n-2)\,||A||\right] \\
&= \frac{1}{2(n-2)\,n}\left[2n\alpha_j + n\sum_{i\in N\backslash\{j,k\}}\alpha_i - 2n\,||A||\right].
\end{aligned}$$

As $\sum_{i\in N}\alpha_i = 2\,||A||$, we have that

$$\begin{aligned}
\widehat{b_j} &= \frac{1}{2(n-2)\,n}\left[2n\alpha_j + n\left(2\,||A|| - (\alpha_j+\alpha_k)\right) - 2n\,||A||\right] \\
&= \frac{1}{2(n-2)\,n}\left[2n\alpha_j + 2n\,||A|| - n(\alpha_j+\alpha_k) - 2n\,||A||\right] \\
&= \frac{1}{2(n-2)\,n}\left[n(\alpha_j-\alpha_k)\right] = \frac{\alpha_j-\alpha_k}{2(n-2)}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\widehat{b_0} &= \overline{Y} - \sum_{j\in N\backslash\{k\}}\widehat{b_j}\overline{X_j} = \frac{||A||}{n(n-1)} - \sum_{j\in N\backslash\{k\}}\frac{\alpha_j-\alpha_k}{2(n-2)}\frac{2(n-1)}{n(n-1)} \\
&= \frac{||A||}{n(n-1)} - \sum_{j\in N\backslash\{k\}}\frac{\alpha_j-\alpha_k}{n(n-2)} \\
&= \frac{||A||}{n(n-1)} - \frac{1}{n(n-2)}\left[\sum_{j\in N\backslash\{k\}}\alpha_j - (n-1)\,\alpha_k\right] \\
&= \frac{||A||}{n(n-1)} - \frac{1}{n(n-2)}\left[2\,||A|| - \alpha_k - (n-1)\,\alpha_k\right] \\
&= \frac{||A||}{n(n-1)} - \frac{2\,||A||}{n(n-2)} + \frac{\alpha_k}{n-2} = -\frac{||A||}{(n-1)(n-2)} + \frac{\alpha_k}{n-2}.
\end{aligned}$$

Once we have estimated the parameters we have that

$$\begin{aligned}
a_{ij} &= \widehat{b_0} + \widehat{b_i} + \widehat{b_j} + \widehat{\varepsilon_{ij}} \quad \text{if } i,j \in N\backslash\{k\} \\
a_{ik} &= \widehat{b_0} + \widehat{b_i} + \widehat{\varepsilon_{ik}} \qquad \text{if } i \in N\backslash\{k\} \\
a_{ki} &= \widehat{b_0} + \widehat{b_i} + \widehat{\varepsilon_{ki}} \qquad \text{if } i \in N\backslash\{k\}.
\end{aligned}$$

Given $i,j\in N\backslash\{k\}$,

$$\begin{aligned}\widehat{\varepsilon_{ij}} &= a_{ij}-\widehat{b_0}-\widehat{b_i}-\widehat{b_j}=\\ &= a_{ij}+\frac{||A||}{(n-1)(n-2)}-\frac{\alpha_k}{n-2}-\frac{\alpha_i-\alpha_k}{2(n-2)}-\frac{\alpha_j-\alpha_k}{2(n-2)}\\ &= a_{ij}+\frac{||A||}{(n-1)(n-2)}-\frac{\alpha_i+\alpha_j}{2(n-2)}.\end{aligned}$$

Given $i\in N\backslash\{k\}$,

$$\begin{aligned}\widehat{\varepsilon_{ik}} &= a_{ik}-\widehat{b_0}-\widehat{b_i}=\\ &= a_{ik}+\frac{||A||}{(n-1)(n-2)}-\frac{\alpha_k}{n-2}-\frac{\alpha_i-\alpha_k}{2(n-2)}\\ &= a_{ik}+\frac{||A||}{(n-1)(n-2)}-\frac{\alpha_i+\alpha_k}{2(n-2)}.\end{aligned}$$

Analogously, we have that

$$\widehat{\varepsilon_{ki}}=a_{ki}+\frac{||A||}{(n-1)(n-2)}-\frac{\alpha_i+\alpha_k}{2(n-2)}.$$

Notice that, for each pair $i,j\in N$,

$$\widehat{\varepsilon_{ij}}=a_{ij}+\frac{||A||}{(n-1)(n-2)}-\frac{\alpha_i+\alpha_j}{2(n-2)}. \tag{7}$$

We now compute a rule by applying principles $(C1)$, $(C2)$ and $(C3)$ in this regression. We consider two cases.

- Team $i\in N\backslash\{k\}$.

  The audience assigned to team i is made of three components:

  By $(C1)$, team i receives

  $$(n-1)\widehat{b_0}=-\frac{||A||}{n-2}+\frac{(n-1)\alpha_k}{n-2}.$$

  By $(C2)$, team i receives

  $$2(n-1)\widehat{b_i}=\frac{(n-1)(\alpha_i-\alpha_k)}{n-2}.$$

  By $(C3)$, team i receives

  $$\begin{aligned}\sum_{j\in N\backslash\{i\}}\frac{\widehat{\varepsilon_{ij}}+\widehat{\varepsilon_{ji}}}{2} &= \frac{1}{2}\sum_{j\in N\backslash\{i\}}(a_{ij}+a_{ji})+\frac{||A||}{(n-2)}-\frac{(n-1)\alpha_i+\sum_{j\in N\backslash\{i\}}\alpha_j}{2(n-2)}\\ &= \frac{\alpha_i}{2}+\frac{||A||}{n-2}-\frac{(n-1)\alpha_i+2||A||-\alpha_i}{2(n-2)}\\ &= \frac{\alpha_i}{2}+\frac{||A||}{(n-2)}-\frac{\alpha_i}{2}-\frac{||A||}{n-2}=0.\end{aligned}$$

Thus, team i receives

$$-\frac{||A||}{n-2}+\frac{(n-1)\,\alpha_k}{n-2}+\frac{(n-1)\,(\alpha_i-\alpha_k)}{n-2}=\frac{(n-1)\,\alpha_i-||A||}{n-2}$$

- Team k.

  The audience assigned to team k is also made of three components:

  By $(C1)$, team k receives

$$(n-1)\,\widehat{b_0}=-\frac{||A||}{n-2}+\frac{(n-1)\,\alpha_k}{n-2}.$$

  By $(C2)$, team k receives nothing.

  Analogously to the previous case, by $(C3)$, team k receives nothing.

  Thus, team k receives

$$\frac{(n-1)\,\alpha_k-||A||}{n-2}.$$

Notice that the audience assigned to any team i is independent of the variable X_k removed from the initial list of independent variables. Thus, the regression analysis drives precisely towards the OLS rule introduced above. This seems to be a strong argument to endorse the OLS rule.

4 The (cooperative) game-theoretical approach

We now take a game-theoretical approach and model our problem as a cooperative game. A **cooperative game with transferable utility**, briefly a **TU game**, is a pair (N, v), where N denotes a set of agents and $v : 2^N \to \mathbb{R}$ satisfies that $v(\varnothing) = 0$. We say that (N, v) is **convex** if, for each pair $S, T \subset N$ and $i \in N$ such that $S \subset T$ and $i \notin T$,

$$v(T\cup\{i\})-v(T)\geq v(S\cup\{i\})-v(S).$$

Given $S \subset N$, the **unanimity game** associated with S is defined as the TU game (N, u_S) where $u_S(T) = 1$ if $S \subset T$, and $u_S(T) = 0$ otherwise. Given a TU game (N, v), there exists a unique family of numbers $\{\delta_S\}_{S\subset N}$ such that $v = \sum\limits_{S\subset N} \delta_S u_S$.

We present some well-known solutions for TU games. First, the **core**, defined as the set of feasible payoff vectors, for which no coalition can improve upon. Formally,

$$C(N,v)=\left\{x\in\mathbb{R}^N \text{ such that } \sum_{i\in N}x_i=v(N) \text{ and } \sum_{i\in S}x_i\geq v(S)\text{, for each } S\subset N\right\}.$$

The **Shapley value** (Shapley, 1953) is the linear function that, for each unanimity game, splits each unit equally among the members of the coalition (and only among them). Formally, for each $i\in N$, $Sh_i(N,v)=\sum\limits_{S\subset N}\delta_S Sh_i(N,u_S)$, where

$$Sh_i(N,u_S)=\begin{cases}\frac{1}{|S|} & \text{if } i\in S\\ 0 & \text{otherwise.}\end{cases}$$

Alternatively, we can define it as follows:

$$Sh_i(N,v)=\frac{1}{n!}\sum_{\pi\in\Pi_N}\left[v\left(Pre(i,\pi)\cup\{i\}\right)-v\left(Pre(i,\pi)\right)\right],$$

for each $i\in N$.

We associate with each (broadcasting sports) problem $(N,A)\in\mathcal{P}$ a TU game (N,v_A) where, for each $S\subset N$, $v_A(S)$ denotes the total audience of the games played by the teams in S. Namely,

$$v_A(S)=\sum_{\substack{i,j\in S\\ i\neq j}}a_{ij}=\sum_{\substack{i,j\in S\\ i<j}}(a_{ij}+a_{ji}).$$

Notice that, for each problem $(N,A)\in\mathcal{P}$ and each $i\in N$, $v_A(\{i\})=0$.

In **Example 1** we have that

S	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v_A(S)$	2400	2060	460	4920

and

$$Sh(N,v_A)=(2230,1430,1260)=S(N,A).$$

The next result summarizes our main findings regarding the game v_A. First, we show that the game is convex. Consequently, the core is easily characterized. Namely, for each pair of teams, we divide the audience of the games played by both teams in any way among them. Each team receives the aggregation of these amounts, across the rest of the teams.

Proposition 1 *Let $(N, A) \in \mathcal{P}$ and (N, v_A) be its associated TU game. The following statements hold:*

(a) (N, v_A) is convex.

(b) $x = (x_i)_{i \in N} \in C(N, v_A)$ if and only if, for each $i \in N$, there exist $\left(x_i^j\right)_{j \in N \setminus \{i\}}$ satisfying three conditions:

(i) $x_i^j \geq 0$, for each $j \in N \setminus \{i\}$;

(ii) $\sum\limits_{j \in N \setminus \{i\}} x_i^j = x_i$, for each $i \in N$;

(iii) $x_i^j + x_j^i = a_{ij} + a_{ji}$, for each pair $i, j \in N$, with $i < j$.

Proof. Let $(N, A) \in \mathcal{P}$ and (N, v_A) be its associated TU game.

(a) Let $S, T \subset N$ and $i \in N$ such that $S \subset T$ and $i \notin T$. Then,

$$\begin{aligned} v_A(T \cup \{i\}) - v_A(T) &= \sum_{j,k \in T \cup \{i\}} a_{jk} - \sum_{j,k \in T} a_{jk} \\ &= \sum_{j \in T} (a_{ij} + a_{ji}) \geq \sum_{j \in S} (a_{ij} + a_{ji}) \\ &= \sum_{j,k \in S \cup \{i\}} a_{jk} - \sum_{j,k \in S} a_{jk} \\ &= v_A(S \cup \{i\}) - v_A(S). \end{aligned}$$

(b) We first prove that if $x = (x_i)_{i \in N}$ is such that for each $i \in N$, there exists $\left(x_i^j\right)_{j \in N \setminus \{i\}}$ satisfying the three conditions, then $x \in C(N, v_A)$.

By (ii),

$$\sum_{i \in N} x_i = \sum_{i \in N} \sum_{j \in N \setminus \{i\}} x_i^j = \sum_{\substack{i,j \in N \\ i<j}} \left(x_i^j + x_j^i\right).$$

By (iii),

$$\sum_{\substack{i,j \in N \\ i<j}} \left(x_i^j + x_j^i\right) = \sum_{\substack{i,j \in N \\ i<j}} (a_{ij} + a_{ji}) = v_A(N).$$

Analogously, for each $S \subset N$,

$$\sum_{i \in S} x_i = \sum_{i \in S} \sum_{j \in N \setminus \{i\}} x_i^j \geq \sum_{i \in S} \sum_{j \in S \setminus \{i\}} x_i^j = \sum_{\substack{i,j \in S \\ i<j}} \left(x_i^j + x_j^i\right) = \sum_{\substack{i,j \in S \\ i<j}} (a_{ij} + a_{ji}) = v_A(S).$$

Then, $x \in C(N, v_A)$.

Conversely, let $x = (x_i)_{i \in N} \in C(N, v_A)$. As (N, v_A) is convex, the core is the convex hull of the vector of marginal contributions. Thus, there exists $(y_\pi)_{\pi \in \Pi_N}$ with $y_\pi \geq 0$ for each $\pi \in \Pi_N$ and $\sum_{\pi \in \Pi_N} y_\pi = 1$ such that, for each $i \in N$,

$$x_i = \sum_{\pi \in \Pi_N} y_\pi \left[v_A \left(Pre(i, \pi) \cup \{i\} \right) - v_A \left(Pre(i, \pi) \right) \right].$$

Because of the definition of v_A, we have that

$$x_i = \sum_{\pi \in \Pi_N} y_\pi \left[\sum_{j \in Pre(i,\pi)} (a_{ij} + a_{ji}) \right] = \sum_{j \in N \setminus \{i\}} (a_{ij} + a_{ji}) \sum_{\pi \in \Pi_N, j \in Pre(i,\pi)} y_\pi.$$

For each pair $i, j \in N$, with $i \neq j$, we define

$$x_i^j = (a_{ij} + a_{ji}) \sum_{\pi \in \Pi_N, j \in Pre(i,\pi)} y_\pi.$$

Thus, $x_i^j \geq 0$, for each $j \in N \setminus \{i\}$, and for each $i \in N$, i.e., (i) holds.

Furthermore, $\sum_{j \in N \setminus \{i\}} x_i^j = x_i$, i.e., (ii) holds.

Let $i, j \in N$ with $i \neq j$. Then,

$$\begin{aligned} x_i^j + x_j^i &= \left((a_{ij} + a_{ji}) \sum_{\pi \in \Pi_N, j \in Pre(i,\pi)} y_\pi \right) + \left((a_{ij} + a_{ji}) \sum_{\pi \in \Pi_N, i \in Pre(j,\pi)} y_\pi \right) \\ &= (a_{ij} + a_{ji}) \sum_{\pi \in \Pi_N} y_\pi = a_{ij} + a_{ji}, \end{aligned}$$

i.e., (iii) holds. ■

Statement (b) of the above proposition states that, in order to satisfy the core constraints, we should divide the revenue generated by the audience of a game between the two teams playing the game. There is complete freedom within those bounds. For instance, assigning all the revenue to one of the teams would be admissible. The Shapley rule states that the revenue generated by the audience of a game be divided equally between both teams. Thus, the allocations that the Shapley rule yields satisfy the core constraints. This is, however, not the case with the OLS rule, which might allocate less than the total revenue generated by the audience of a game to the two teams involved.[11]

[11] Take, for instance, the problem $(N, A) \in \mathcal{P}$, where $N = \{1, 2, 3, 4\}$ and

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 150 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 200 & 0 \end{pmatrix}$$

The next result establishes a correspondence between the Shapley rule and the Shapley value for TU-games described above, which justifies the name given to the rule.

Theorem 1 *For each $(N, A) \in \mathcal{P}$, $Sh(N, v_A) = S(N, A)$.*

Proof. Let $(N, A) \in \mathcal{P}$ and (N, v_A) be its associated TU game. For each pair $i, j \in N$ with $i \neq j$ we define the characteristic function v_A^{ij} as follows. For each $S \subset N$,

$$v_A^{ij}(S) = \begin{cases} a_{ij} + a_{ji} & \text{if } \{i, j\} \subset S \\ 0 & \text{otherwise.} \end{cases}$$

Consider the resulting TU-game $\left(N, v_A^{ij}\right)$. It is straightforward to see that, for such a game, agents i and j are symmetric, whereas the remaining agents in $N \backslash \{i, j\}$ are null teams. Thus,

$$Sh_k\left(N, v_A^{ij}\right) = \begin{cases} \frac{a_{ij}+a_{ji}}{2} & \text{if } k \in \{i, j\} \\ 0 & \text{otherwise.} \end{cases}$$

For each $S \subset N$,

$$v_A(S) = \sum_{\substack{i,j \in S \\ i<j}} (a_{ij} + a_{ji}) = \sum_{\substack{i,j \in N \\ i<j}} v_A^{ij}(S).$$

As the Shapley value is additive on v, we have that

$$Sh(N, v_A) = \sum_{\substack{i,j \in N \\ i<j}} Sh\left(N, v_A^{ij}\right).$$

Thus, for each $k \in N$,

$$Sh_k(N, v_A) = \sum_{\substack{i,j \in N \\ i<j}} Sh_k\left(N, v_A^{ij}\right) = \sum_{j \in N} Sh_k\left(N, v_A^{kj}\right) = \sum_{j \in N} \frac{a_{kj} + a_{jk}}{2} = \frac{\alpha_k}{2}.$$

■

It is well known that when the cooperative game is convex the Shapley value belongs to the core. Thus, it follows from Proposition 1(a) and Theorem 1 that the Shapley rule always yields stable allocations, in the sense formalized by the core. Formally, $S(N, A) \in C(N, v_A)$, for each problem (N, A).[12]

Then $||A|| = 350$, $\alpha = (150, 150, 200, 200)$, $O(N, A) = (50, 50, 125, 125)$, and $v_A(1, 2) = 150$.

[12]This had been implicitly mentioned above, right after the proof of Proposition 1.

This is a strong argument to endorse the Shapley rule. Teams are corporations and, as such, any subgroup of teams could potentially secede and form another (smaller) competition. Thus, if the rule selects allocations within the core, it provides stable outcomes, in the sense of dismissing incentives for team secessions. As shown above, in this case, the core is non-empty and very large.[13] Thus, it seems reasonable to select one allocation within the core.

5 The axiomatic approach

The previous two sections provided arguments to endorse, respectively, the two focal rules of this work. First, the OLS rule was the outcome of a linear regression analysis of our problem. Second, the Shapley rule was shown to coincide with the Shapley value of a natural convex TU-game, thus guaranteeing stable outcomes (as formalized by the core of such a game). In this section, we provide normative foundations for both rules. As we shall see, each rule is characterized by a combination of three axioms (among which, two are common for both results).

The first axiom we consider says that if two teams have the same audiences, then they should receive the same amount.

Equal treatment of equals: For each $(N, A) \in \mathcal{P}$, and each pair $i, j \in N$ such that $a_{ik} = a_{jk}$, and $a_{ki} = a_{kj}$, for each $k \in N \setminus \{i, j\}$,

$$R_i(N, A) = R_j(N, A).$$

The second axiom says that revenues should be additive on A. Formally,

Additivity: For each pair (N, A) and $(N, A') \in \mathcal{P}$

$$R\left(N, A + A'\right) = R\left(N, A\right) + R\left(N, A'\right).$$

The third axiom says that if nobody watches a single game of a given team (i.e., the team has a null audience), then such a team gets no revenue.

[13] As a matter of fact, the core is made of all the allocations induced by the rules satisfying non negativity (i.e., no team gets a negative amount) and two axioms considered in the next section (namely, additivity and null team).

Null team: For each $(N, A) \in \mathcal{P}$, and each $i \in N$, such that $a_{ij} = 0 = a_{ji}$, for each $j \in N$,

$$R_i(N, A) = 0.$$

Alternatively, the next axiom says that if a team nullifies the audience of all the games it plays (for instance, due to some kind of boycott), then the allocation of such a team should decrease exactly by the total audience of such a team.[14] Formally,

Nullifying team: For each $(N, A), (N, A') \in \mathcal{P}$ such that there exists $k \in N$ (the nullifying team) satisfying $a'_{ij} = a_{ij}$ when $k \notin \{i, j\}$ and $a'_{ij} = 0$ when $k \in \{i, j\}$ we have that

$$R_k(N, A') = R_k(N, A) - \alpha_k(A).$$

The next result provides the characterizations of the two rules.

Theorem 2 *The following statements hold:*

(a) *A rule satisfies equal treatment of equals, additivity and null team if and only if it is the Shapley rule.*

(b) *A rule satisfies equal treatment of equals, additivity and nullifying team if and only if it is the OLS rule.*

Proof.

(a) It is obvious that the Shapley rule satisfies the three axioms. Conversely, let $(N, A) \in \mathcal{P}$. For each pair $i, j \in N$, with $i \neq j$, let A^{ij} denote the matrix with the following entries:

$$a^{ij}_{kl} = \begin{cases} a_{ij} & \text{if } (k, l) = (i, j) \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $a^{ij}_{ji} = 0$.

Let $k \in N$. By *additivity*,

$$R_k(N, A) = \sum_{i,j \in N : i \neq j} R_k\left(N, A^{ij}\right).$$

By *null team*, for each pair $i, j \in N$ with $i \neq j$, and for each $l \in N \backslash \{i, j\}$, we have $R_l\left(N, A^{ij}\right) = 0$. Thus,

$$R_k(N, A) = \sum_{l \in N \backslash \{k\}} \left[R_k\left(N, A^{lk}\right) + R_k\left(N, A^{kl}\right)\right].$$

[14] A similar axiom was introduced in cooperative transferable utility games by van den Brink (2007).

By *equal treatment of equals*, $R_k\left(N, A^{lk}\right) = R_l\left(N, A^{lk}\right)$. As $\left|\left|A^{lk}\right|\right| = a_{lk}$ we have that $R_k\left(N, A^{lk}\right) = \frac{a_{lk}}{2}$. Similarly, $R_k\left(N, A^{kl}\right) = \frac{a_{kl}}{2}$. Thus,

$$R_k(N, A) = \sum_{i \in N \setminus \{k\}} \left[\frac{a_{lk}}{2} + \frac{a_{kl}}{2}\right] = \frac{\alpha_k}{2} = S_k(N, A).$$

(b) It is obvious that the OLS rule satisfies equal treatment of equals and additivity. Let $(N, A), (N, A') \in \mathcal{P}$ and $k \in N$ be as in the definition of nullifying team. By (2),

$$\begin{aligned}
O_k(N, A') &= \alpha_k(N, A') - \frac{\sum\limits_{j,k \in N \setminus \{i\}} \left(a'_{jk} + a'_{kj}\right)}{n-2} \\
&= -\frac{\sum\limits_{j,k \in N \setminus \{i\}} \left(a_{jk} + a_{kj}\right)}{n-2} \\
&= \alpha_k(N, A) - \frac{\sum\limits_{j,k \in N \setminus \{i\}} \left(a_{jk} + a_{kj}\right)}{n-2} - \alpha_k(N, A) \\
&= O_k(N, A) - \alpha_k(N, A)
\end{aligned}$$

Then, O satisfies nullifying team.

Conversely, let R be a rule satisfying the three axioms in the statement. Let $(N, A) \in \mathcal{P}$. We define the problems A^{ij} as in the proof of the previous item. As R satisfies additivity and $A = \sum\limits_{i,j \in N : i \neq j} A^{ij}$, it is enough to prove that R is uniquely determined in each problem (N, A^{ij}).

Let $0_{N,N}$ be the matrix with all entries equal to 0. As R satisfies additivity, for each $k \in N$ and each $m \in \mathbb{N}$, we have that

$$R_k(N, 0_{N,N}) = R_k\left(N, \sum_{l=1}^{m} 0_{N,N}\right) = \sum_{l=1}^{m} R_k(N, 0_{N,N}) = mR_k(N, 0_{N,N}).$$

Thus, $R_k(N, 0_{N,N}) = 0$.

As (N, A^{ij}), $(N, 0_{N,N})$, and $k = i$ are under the hypothesis of nullifying team,

$$0 = R_i(N, 0_{N,N}) = R_i\left(N, A^{ij}\right) - a_{ij}.$$

Thus, $R_i(N, A^{ij}) = a_{ij}$. Analogously, we can prove that $R_j(N, A^{ij}) = a_{ij}$.

By equal treatment of equals, for each $k, l \in N \setminus \{i, j\}$ we have that $R_k(N, A^{ij}) = R_l(N, A^{ij})$. Let x denote such an amount. Thus,

$$a_{ij} = \left|\left|A^{ij}\right|\right| = \sum_{k \in N} R_k\left(N, A^{ij}\right) = a_{ij} + a_{ij} + (n-2)x,$$

from where it follows that $x = \frac{-a_{ij}}{n-2}$.

Hence, R is uniquely determined in (N, A^{ij}). ■

Remark 1 *The axioms of Theorem 2 are independent.*

Let R^1 be the rule in which, for each game $(i,j) \in N \times N$ the revenue goes to the team with the lowest number of the two. Namely, for each problem $(N, A) \in \mathcal{P}$, and each $i \in N$,

$$R_i^1(N, A) = \sum_{j \in N : j > i} (a_{ij} + a_{ji}).$$

R^1 satisfies null team and additivity, but not equal treatment of equals.

The equal awards rule satisfies equal treatment of equals and additivity, but not null team.

Let R^2 be the rule that divides the total revenue among all teams proportionally to their total audiences. Namely, for each problem $(N, A) \in \mathcal{P}$, and $i \in N$,

$$R_i^2(N, A) = \frac{\alpha_i(A)}{\sum_{j \in N} \alpha_j(A)} ||A||.$$

R^2 satisfies equal treatment of equals and null team, but not additivity.

The Shapley rule satisfies additivity and equal treatment of equals but fails nullifying team.

The rule imposing that only the team with the smallest index in $N \setminus \{i,j\}$ gets a non-null award from A^{ij} (and extended to general problems by additivity) satisfies additivity and nullifying team but fails equal treatment of equals.

The rule imposing the same solution as the OLS rule for each A^{ij}, but not extended to general problems by additivity, satisfies equal treatment of equals and nullifying team, but fails additivity.

Theorem 2 not only provides a characterization of our two focal rules, but also a common ground for them. More precisely, it states that both rules are characterized by the combination of equal treatment of equals, additivity, and a third axiom. This third axiom (null player in one case; nullifying player in the other case) formalizes the behavior of the rule with respect to somewhat peculiar teams (those with no audience in one case; those killing audiences in the other case). It turns out, nevertheless, that this only difference, reflected in the mentioned pair of axioms, is substantial as the axioms are incompatible. Namely, there is no rule satisfying both the null team axiom and the nullifying team axiom. Consider the problem (N, A^{12}) defined as in the proof of Theorem 2 where $N = \{1, 2, 3\}$ and $a_{12} > 0$. If R satisfies null team we have that $R_3(N, A^{12}) = 0$ and $R_i(N, 0_{N,N}) = 0$ for each $i \in N$. Suppose that R also satisfies nullifying team. Using arguments similar to the ones used in the proof of Theorem 2 we can deduce that $R_1(N, A^{12}) = R_2(N, A^{12}) = a_{12}$. Thus, $R_3(N, A^{12}) = -a_{12}$ which is a contradiction.

6 The conflicting claims approach

O'Neill (1982) is credited for introducing one of the simplest (and yet useful) models to study distributive justice. The so-called problem of *adjudicating conflicting claims* refers to a situation in which an insufficient amount of a perfectly divisible good (endowment) has to be allocated among a group of agents who hold claims against it. This basic framework is flexible enough to accommodate a variety of related situations that trace back to ancient sources such as Aristotle's essays and the Talmud.[15] It turns out that, as we show in this section, our problems of sharing the revenue from broadcasting sport events could also be seen as a specific instance of the problem of adjudicating conflicting claims.

A *problem of adjudicating conflicting claims* (or, simply, a claims problem) is a triple consisting of a population $N \in \mathcal{N}$, a claims profile $c \in \mathbb{R}^n_+$, and an *endowment* $E \in \mathbb{R}_+$ such that $\sum_{i \in N} c_i \geq E$. Let $C \equiv \sum_{i \in N} c_i$. To avoid unnecessary complication, we assume $C > 0$. Let $\mathcal{D}^N$ be the domain of bankruptcy problems with population N and $\mathcal{D} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{D}^N$.

Given a problem $(N, c, E) \in \mathcal{D}^N$, an *allocation* is a vector $x \in \mathbb{R}^n$ satisfying the following two conditions: (i) for each $i \in N$, $0 \leq x_i \leq c_i$ and (ii) $\sum_{i \in N} x_i = E$. We refer to (i) as *boundedness* and (ii) as *balance*. A *rule* on $\mathcal{D}$, $R \colon \mathcal{D} \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^n$, associates with each problem $(N, c, E) \in \mathcal{D}$ an allocation $R(N, c, E)$ for the problem.

The so-called *constrained equal-awards* rule, CEA, selects, for each $(N, c, E) \in \mathcal{D}$, the vector $(\min\{c_i, \lambda\})_{i \in N}$, where $\lambda > 0$ is chosen so that $\sum_{i \in N} \min\{c_i, \lambda\} = E$. The so-called *constrained equal-losses* rule, CEL, selects, for each $(N, c, E) \in \mathcal{D}$, the vector $(\max\{0, c_i - \lambda\})_{i \in N}$, where $\lambda > 0$ is chosen so that $\sum_{i \in N} \max\{0, c_i - \lambda\} = E$. The so-called *Talmud* rule is a hybrid between the above two. More precisely, for each $(N, c, E) \in \mathcal{D}$, it selects

$$T(N, c, E) = \begin{cases} CEA(N, \frac{1}{2}c, E) & \text{if } E \leq \frac{1}{2}C \\ \frac{1}{2}c + CEL(N, \frac{1}{2}c, E - \frac{1}{2}C) & \text{if } E \geq \frac{1}{2}C \end{cases}$$

Finally, the so-called *proportional* rule, P, yields awards proportionally to claims, i.e., for each $(N, c, E) \in \mathcal{D}$, $P(N, c, E) = \frac{E}{C} \cdot c$.

In a (broadcasting sports) problem (N, A), as formalized above, the issue is to allocate the aggregate audience in the tournament $(||A||)$ among the participating teams (N). If one

[15] The reader is referred to Thomson (2003, 2015, 2017) for excellent surveys of the sizable literature dealing with this model.

considers each team claims the overall audience of the games it was involved ($\alpha_i(A)$), then we obviously have a problem of adjudicating conflicting claims. More precisely, we associate with each (broadcasting sports) problem (N, A) a claims problem $\left(N, c^A, E^A\right)$ where $c_i^A = \alpha_i(A)$, for each $i \in N$, and $E^A = ||A||$. We sometimes write (N, c, E) instead of $\left(N, c^A, E^A\right)$, if no confusion arises.

Notice that $E = \frac{C}{2}$. Thus, $P(N, c, E) = T(N, c, E) = c/2$ for each problem. In words, the Talmud rule and the proportional rule will always yield the same awards; namely half of its claim for each team. In what follows, we shall then just refer to the proportional rule, instead of the Talmud rule.

Now, by definition, $S_i(N, A) = \frac{\alpha_i}{2}$, for each $i \in N$. Thus, $P\left(N, c^A, E^A\right) = T\left(N, c^A, E^A\right) = S(N, A)$, for each $(N, A) \in \mathcal{P}$.

In **Example 1** we have that $E = 4920$ and

i	1	2	3
c_i	4460	2860	2520
$P_i(E, c)$	2230	1430	1260
$CEA_i(E, c)$	1640	1640	1640
$CEL_i(E, c)$	2820	1220	880
$T_i(E, c)$	2230	1430	1260

The next result summarizes the stability properties of the above rules. As stated therein, only the proportional rule guarantees allocations within the core. This is due to the fact that, as mentioned above, the proportional rule yields the same outcomes as the Shapley rule.

Proposition 2 *The following statements hold:*

(a) $P\left(N, c^A, E^A\right) \in C(N, v_A)$, *for each* $(N, A) \in \mathcal{P}$.

(b) $CEA\left(N, c^A, E^A\right) \notin C(N, v_A)$ *for some* $(N, A) \in \mathcal{P}$.

(c) $CEL\left(N, c^A, E^A\right) \notin C(N, v_A)$ *for some* $(N, A) \in \mathcal{P}$.

Proof. *(a)*. As mentioned above, $P\left(N, c^A, E^A\right) = S(N, A)$, for each $(N, A) \in \mathcal{P}$. By Theorem 1 *(c)*, $Sh(N, v_A) = S(N, A)$, for each $(N, A) \in \mathcal{P}$. By Theorem 1 *(a)*, (N, v_A) is convex and, therefore, $Sh(N, v_A) \in C(N, v_A)$. Altogether, we have $P\left(N, c^A, E^A\right) \in C(N, v_A)$, for each $(N, A) \in \mathcal{P}$.

(b) Let $(N, A) \in \mathcal{P}$ be such that $N = \{1, 2, 3, 4\}$ and

$$A = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Then, $E^A = 8$, $c^A = (6, 6, 2, 2)$. Thus, $CEA\left(N, c^A, E^A\right) = (2, 2, 2, 2)$. As

$$CEA_1\left(N, c^A, E^A\right) + CEA_2\left(N, c^A, E^A\right) = 4 < 6 = a_{12} + a_{21},$$

it follows from Theorem 1 (b) that $CEA\left(N, c^A, E^A\right) \notin C\left(N, v_A\right)$.

(c) Let $(N, A) \in \mathcal{P}$ be such that $N = \{1, 2, 3, 4\}$ and

$$A = \begin{pmatrix} 0 & 9 & 0 & 0 \\ 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Then, $E^A = 20$, $c^A = (18, 18, 2, 2)$ and $CEL\left(N, c^A, E^A\right) = (10, 10, 0, 0)$. As

$$CEL_3\left(N, c^A, E^A\right) + CEL_4\left(N, c^A, E^A\right) = 0 < 2 = a_{34} + a_{43},$$

it follows from Theorem 1 (b) that $CEL\left(N, c^A, E^A\right) \notin C\left(N, v_A\right)$ ■

7 An empirical application

In this section, we present an empirical application of our model resorting to La Liga, the Spanish Football League.[16]

La Liga is a standard round robin tournament involving 20 teams. Thus, each team plays 38 games, facing each time one of the other 19 teams (once home, another away). The available data we have (retrieved from one of the major sport newspapers in Spain and La Liga's website) refers to the average audience of each team during the last completed season (2015-2016).[17] From there, we can derive the necessary parameters of our model; namely, the total audience

[16] http://www.laliga.es/en

[17] It is important to note that, in general, all games were broadcasted nationally in different time windows (normally, during the weekend) that did not overlap. Now, for each day (weekend) of competition, one game was

achieved by each team ($\alpha_i(A)$), and the aggregate audience in the league ($||A||$).[18] We also have data on the actual sharing of the revenues obtained that season. They are all collected in Table 1 below.

TEAMS	Average Audience	$\alpha_i(A)$	Revenues	%
RM	$4139,81$	$157312,78$	140	$14,467$
BCN	$2739,97$	$104118,86$	140	$14,467$
ATM	$1387,43$	$52722,34$	$69,08$	$7,138$
SVQ	$651,89$	$24771,82$	$48,52$	$5,014$
BET	$619,37$	$23536,06$	$33,94$	$3,507$
VAL	$582,95$	$22152,1$	$53,8$	$5,559$
CEL	$580,92$	$22074,96$	$33,03$	$3,413$
DPV	$524,63$	$19935,94$	$31,68$	$3,274$
ATH	$486,28$	$18478,64$	$47,88$	$4,948$
RVL	$473,97$	$18010,86$	$32,59$	$3,368$
RSC	$454,72$	$17279,36$	$38,56$	$3,985$
VIL	$451,07$	$17140,66$	$41,72$	$4,311$
LPA	$439,03$	$16683,14$	$27,65$	$2,857$
SPO	$417,73$	$15873,74$	$29,84$	$3,083$
MLG	$414,32$	$15744,16$	$38,95$	$4,025$
GRA	$409,77$	$15571,26$	$30,99$	$3,202$
EIB	$394,29$	$14983,02$	$28,18$	$2,912$
ESP	$384,45$	$14609,1$	$35,57$	$3,676$
LEV	$384,07$	$14594,66$	$33,81$	$3,494$
GET	$287,52$	$10925,76$	$31,96$	$3,303$

Table 1. Audiences and revenues for the Spanish Football League in 2005/2016.

Table 1 lists the 20 teams, their average audiences (in thousands), their global audiences (in

broadcasted in a non-subscription channel. We do not treat those latter games distinctively in our empirical analysis. The data refers only to national broadcasting (within Spain). Although large audiences are also obtained abroad, not all games are broadcasted abroad. In order to avoid making the empirical analysis biased in favor of the teams that are more frequently broadcasted, we decided to dismiss those data from our analysis.

[18]Recall that $||A|| = \sum_{i \in N} \alpha_i(A)/2$.

thousands) and the actual revenues they made (in millions of euros), as well as in percentage terms. As we can see, two teams dominated the sharing collecting a combined 30% of the pie.

Table 2 lists the allocations proposed by our two rules (Shapley and OLS). The numbers are normalized under the premise of our model; namely, each viewer pays a pay-per-view fee of 1 euro per game, and the overall amount is allocated. That is, $||A|| = 308259610$ euros. This is almost one third of the real revenues that the teams made combined. That is why we also provide the percentage levels obtained by each team under both rules.

TEAMS	$S_i(N,A)$	%	$O_i(N,A)$	%
RM	$78656,39$	$25,52$	$148926,845$	$48,31$
BCN	$52059,43$	$16,89$	$92777,707$	$30,10$
ATM	$26361,17$	$8,55$	$38525,825$	$12,50$
SVQ	$12385,91$	$4,02$	$9022,498$	$2,93$
BET	$11768,03$	$3,82$	$7718,085$	$2,50$
VAL	$11076,05$	$3,59$	$6257,238$	$2,03$
CEL	$11037,48$	$3,58$	$6175,813$	$2,00$
DPV	$9967,97$	$3,23$	$3917,958$	$1,27$
ATH	$9239,32$	$3,00$	$2379,697$	$0,77$
RVL	$9005,43$	$2,92$	$1885,929$	$0,61$
RSC	$8639,68$	$2,80$	$1113,791$	$0,36$
VIL	$8570,33$	$2,78$	$967,385$	$0,31$
LPA	$8341,57$	$2,71$	$484,447$	$0,16$
SPO	$7936,87$	$2,57$	$-369,919$	$-0,12$
MLG	$7872,08$	$2,55$	$-506,698$	$-0,16$
GRA	$7785,63$	$2,53$	$-689,204$	$-0,22$
EIB	$7491,51$	$2,43$	$-1310,124$	$-0,42$
ESP	$7304,55$	$2,37$	$-1704,817$	$-0,55$
LEV	$7297,33$	$2,37$	$-1720,059$	$-0,56$
GET	$5462,88$	$1,77$	$-5592,787$	$-1,81$

Table 2. The Shapley and OLS outcomes for the Spanish Football League in 2005/2016.

Several conclusions can be derived from our analysis. Maybe the most obvious one is that eight teams would be awarded negative values under the OLS rule. That is, they should be compensating the other teams (for an overall amount of almost 4% of the pie) because they are not bringing enough audiences on their own to the tournament, and they are somewhat benefitting from competing in this tournament.

As we can also see, and contrary to what some might argue, the actual revenue sharing seems to be biased against the two powerhouses. In particular, Real Madrid, should be obtaining a quarter of the pie with the Shapley rule and almost half of it with the OLS rule. Barcelona would also go up (from 14.5% to almost 17% and 30%, respectively). Atletico de Madrid would increase considerably too. All the other teams would decrease, with the exception of Celta de Vigo and Real Betis Balompié (the greatest team on earth) who would increase if the Shapley rule would be implemented (but not if the OLS rule would be implemented).

Finally, under the Shapley rule, the two powerhouses would be obtaining (combined) slightly above 40% of the pie. Under the OLS rule, they would be obtaining a staggering 78.4%. The latter distribution, which also exhibits the feature of making eight teams pay (rather than receive), thus seems difficult to be accepted in this case. Nevertheless, if the real outcome is the result of a bargaining process among the participating teams, we cannot deny the fact that the two powerhouses have a very strong bargaining power, which might largely influence the final outcome.

8 Discussion

We have presented a stylized model to deal with the problem of sharing the revenues from broadcasting sports events. We have provided normative, empirical and strategic foundations for rules sharing each game's revenues equally or proportionally among the participating teams. Both rules have distinguishing merits. One (the OLS rule) is supported by a linear regression analysis and thus, it reflects the (potentially different) fan base of each team. Another (the Shapley rule) is supported by a powerful (and normatively appealing) stability property preventing secessions from participating players.

We have also provided as a case study an empirical application deriving what both rules would suggest for the Spanish Football League (La Liga). Our results largely differ from the current existing schemes, which we find (somewhat surprisingly) biased against the three teams driving the largest audiences.

It is left for further research to enrich the model in plausible ways. For instance, some games are offered for free (in non-subscription channels), instead of pay per view. That might influence the audience numbers. In our case study (La Liga), not all teams are broadcasted under that option. And its broadcasting rights are negotiated independently. Thus, it might well make sense to talk about two different budgets: one coming from subscription channels (to which all team have access) and another coming from non-subscription channels (to which not all team have access, and which might be associated to different audience figures).

Similarly, several games might be broadcasted simultaneously, which might reduce the number of viewers for some games. And if all games are broadcasted in exclusive time windows (as it happens, for instance, in our case study), prime time is only awarded to some games. All these aspects might have an important impact on audience figures, which has been ignored in our model.

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