Robust Voting under Uncertainty*

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Abstract

This paper proposes a normative criterion for voting rules under Knightian uncertainty about individuals' preferences to characterize a weighted majority rule (WMR). This criterion, which is referred to as robustness, stresses the significance of responsiveness: the probability that the social outcome coincides with the realized individual preferences. A voting rule is said to be robust if, for any probability distribution of preferences, the responsiveness of at least one voter is greater than one-half. The main result of this paper establishes that a voting rule is robust if and only if it is a WMR without any ties. Robustness is a stronger requirement than weak efficiency because a voting rule is weakly efficient if and only if it is a WMR in which ties are allowed with an arbitrary tie-breaking rule.

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1 Introduction

Consider a choice of a voting rule on a succession of two alternatives (such as "yes" or "no") by a group of individuals who are uncertain about their future preferences. Each individual presumes that the gain from the passage of a favorable issue equals the loss from the passage of an unfavorable issue. Imagine that someone proposes a voting rule such that the expected loss of every individual is greater than the expected gain. Then, the group will not agree to adopt it. In fact, such a voting rule is problematic because the probability that the outcome agrees with an individual's preference, i.e., responsiveness, is less than one-half for all individuals. This means that a group decision reflects minority preferences on average and that the decision can be eventually not only unfair ex post facto but also more likely incorrect.

To evaluate the expected net gain, individuals must know the true probability distribution of their preferences. However, in reality, they face Knightian uncertainty and have little confidence regarding the true probabilities.[1] This makes it difficult for them to figure out whether the expected net gain is positive or negative, which raises the following questions. Does there exist a voting rule such that the expected net gain of every individual is never negative whatever the underlying probability distribution is? If the answer is yes, what is it?

This paper proposes normative criteria for voting rules under Knightian uncertainty and provides answers to the above questions. A voting rule is said to be *robust*[2] if, for any probability distribution of preferences, responsiveness of at least one voter is strictly greater than one-half. Clearly, a voting rule is robust if and only if, for any probability distribution of preferences, the expected net gain of at least one voter is strictly positive. Even if a voting rule is robust, a collective decision can reflect minority preferences on average because the arithmetic mean of responsiveness of all individuals can be less than one-half. Thus, we also consider the following stronger concept. A voting rule is said to be *strongly robust* if the minimum arithmetic mean of responsiveness of all individuals, where the minimum is taken over all probability distributions of preferences, is greater than one-half. It should be noted that we are interested in the minimum responsiveness over all probability distributions rather than the maximum responsiveness over all voting rules (Rae, 1969). That is, we take into account the

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[1]Knight (1921) distinguishes risky situations, where a decision maker knows the probabilities of all events, and uncertain situations, where a decision maker does not know them.

[2]We borrow the term "robustness" from robust statistics, statistics with good performance for data drawn from a wide range of probability distributions (Huber, 1981).

worst case scenario under Knightian uncertainty, which is analogous to the maxmin expected utility model (Gilboa and Schmeidler, 1989).

Considering, first, anonymous rules, we show that a voting rule is robust and anonymous if and only if it is strongly robust. We also find that a simple majority rule is a unique robust anonymous rule when the number of individuals is odd, whereas no voting rule is robust and anonymous when the number of individuals is even. These results imply that an anonymous rule is not robust unless it is a majority rule or the number of individuals is odd. For example, a supermajority rule is not robust. To illustrate it by a numerical example, consider a two-thirds rule with a very large number of individuals. Each individual votes "yes" with probability p and "no" with probability $1 - p$ independently and identically. If $1/2 < p < 2/3$, then the collective decision is "no" with probability close to one, so responsiveness of each individual is close to $1 - p < 1/2$, which implies that a two-thirds rule is not robust.

Considering, next, nonanonymous rules, we show that a voting rule is robust if and only if it is a weighted majority rule without any ties. The proof, which is based upon the theorem of alternatives due to von Neumann and Morgenstern (1944), can be understood in terms of the following variant of the fundamental theorem of asset pricing (cf. Dybvig and Ross, 2003, 2008). Given n individuals with random preferences and a voting rule, consider the following imaginary asset for each individual i: one unit of asset i yields $+1$ if individual i's preference agrees with the collective decision and -1 otherwise. Using the fundamental theorem of asset pricing, we can show that there exists a portfolio with nonnegative weights in all assets (i.e. no short selling) yielding a strictly positive payoff in each state if and only if, for any arbitrage-free price vector, the price of at least one asset is strictly positive. The former condition is true if and only if the voting rule is a weighted majority rule and the latter condition is true if and only if responsiveness of at least one individual is greater than one-half, thus implying the equivalence of a robust rule and a weighted majority rule.

Our results and the Rae-Taylor-Fleurbaey (RTF) theorem (Rae, 1969; Taylor, 1969; Fleurbaey, 2008) have in common that responsiveness characterizes a weighted majority rule. The RTF theorem states that a voting rule is a weighted majority rule if and only if it maximizes the corresponding weighted sum of responsiveness over all individuals. By Harsanyi's characterization of utilitarianism (Harsanyi, 1955), the RTF theorem can also be understood as a characterization of weighted majority rules in terms of weak efficiency: a voting rule is a

weighted majority rule if and only if it is weakly efficient with respect to responsiveness.[3]

However, there are differences between our results and the RTF theorem in the use of responsiveness and in the characterized class of weighted majority rules. First, the RTF theorem uses the maximum responsiveness over all voting rules in order to achieve the optimal outcomes, whereas our results use the minimum responsiveness over all probability distributions in order to avoid the worst outcomes. In this respect, this paper together with the RTF theorem gives dual characterization of WMRs in terms of responsiveness.

Next, and importantly, the set of weighted majority rules characterized by our results is a proper subset of that characterized by the RTF theorem. This is because the former does not allow any ties, whereas the latter allows ties with any tie-breaking rules. The immediate implication is that a robust rule is weakly efficient, but a weakly efficient rule is not necessarily robust; that is, robustness is a stronger requirement than weak efficiency.

To illustrate the latter difference, assume that the number of individuals is even, in which case a majority rule can result in a tie. According to the RTF theorem, a majority rule with any tie-breaking rule maximizes the sum of responsiveness and thus it is weakly efficient, whereas a majority rule with some tie-breaking rule is robust if and only if it is represented as a weighted majority rule without ties. For example, a majority rule with a casting (tie-breaking) vote is robust. Therefore, the requirement of robustness can justify the wide use of a casting vote even though it does not conform to some rules of parliamentary procedure such as Robert's Rules of Order (Roberts, 1971). In legislatures such as the New Zealand House of Representatives, the British House of Commons, and the Australian Senate, the presiding officers are required to vote in favor of the status quo or a tie is considered to be a defeat. This majority rule is not robust when the number of individuals is even because it is anonymous and no anonymous rule is robust in this case. On the other hand, in legislatures such as the United States Senate, the Australian House of Representatives, and the National Diet of Japan, the presiding officers hold casting votes to break ties. As mentioned above, this majority rule is robust.

This paper not only contributes to the literature on the axiomatic foundations of simple or weighted majority rules (May, 1952; Fishburn and Gehrlein, 1977) but also joins a recently growing literature on economic design with worst-case objectives. Most studies in the latter literature, however, have focused on mechanism design. For example, Chung and Ely (2007) consider a revenue maximization problem in a private value auction where the auctioneer does

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[3]See also Schmitz and Tröger (2012) and Azrieli and Kim (2014).

not know agents' belief structures exactly[4] and show that the optimal auction rule is a dominant-strategy mechanism when the auctioneer evaluates rules by their worst-case performance. On the other hand, Carroll (2015) considers a moral hazard problem where the principal does not know the agent's set of possible actions exactly and shows that the optimal contract is linear when the principal evaluates contracts by their worst-case performance.[5] In contrast to these papers, we consider a choice of voting rules with the worst-case objective (i.e., the minimum responsiveness) to characterize WMRs, where the constitution-maker does not know the probability distribution of preferences, thus demonstrating that this approach is also useful in the study of voting and social choice.

The rest of the paper is organized as follows. In Section 2, we introduce the concepts of robustness. Section 3 studies robustness of anonymous rules and Section 4 studies robustness of nonanonymous rules. In Section 5, we compare our results and the RTF theorem. We conclude the paper in Section 6.

2 Voting under Knightian uncertainty

Consider a group of individuals $N = \{1, \cdots, n\}$ that faces a choice between two alternatives (such as "yes" or "no"). The choice of individual $i \in N$ is represented by a decision variable $x_i \in \{-1, 1\}$. The choices of the group members are summarized by a decision profile $x = (x_i)_{i \in N}$. Let $\mathcal{X} = \{-1, 1\}^N$ denote the set of all possible profiles.

A voting rule is a mapping $\phi : \mathcal{X} \to \{-1, 1\}$. Let Φ denote the set of all voting rules. A voting rule $\phi \in \Phi$ is a weighted majority rule (WMR)[6] if there exists a weight vector $w = (w_i)_{i \in N} \in \mathbb{R}^N$ satisfying

$$\phi(x) = \begin{cases} 1 & \text{if } \sum_{i \in N} w_i x_i > 0, \\ -1 & \text{if } \sum_{i \in N} w_i x_i < 0. \end{cases}$$

Note that we allow negative weights in this definition. A simple majority rule (SMR) is a special case with positive equal weights, i.e., $w_i = w_j > 0$ for all $i, j \in N$. When there is a tie, i.e. $\sum_{i \in N} w_i x_i = 0$, a tie-breaking rule is used. For example, a SMR with an even number of

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individuals requires a tie-breaking rule.

The following characterization of WMRs, which is immediate from the definition, plays an important role in our analysis.

Lemma 1. *A voting rule $\phi \in \Phi$ is a WMR with a weight vector $w \in \mathbb{R}^N$ if and only if*

$$\phi(x) \sum_{i \in N} w_i x_i \geq 0 \text{ for all } x \in X.$$

A voting rule $\phi \in \Phi$ is a WMR with a weight vector $w \in \mathbb{R}^N$ allowing no ties if and only if

$$\phi(x) \sum_{i \in N} w_i x_i > 0 \text{ for all } x \in X.$$

Assume that $x \in X$ is randomly drawn according to a probability distribution $p \in \Delta(X) \equiv \{p \in \mathbb{R}_+^X : \sum_{x \in X} p(x) = 1\}$. Let

$$p(\phi(x) = x_i) \equiv p(\{x \in X : \phi(x) = x_i\})$$

be the probability that i's choice agrees with the collective decision, which is referred to as responsiveness or the Rae index (Rae, 1969). It is calculated as

$$p(\phi(x) = x_i) = (E_p[\phi(x)x_i] + 1)/2, \tag{1}$$

where $E_p[\phi(x)x_i] \equiv \sum_{x \in X} p(x)\phi(x)x_i$ is the expected value of $\phi(x)x_i$. This is because $\phi(x)x_i$ equals $+1$ if i's choice agrees with the collective decision and -1 otherwise, which implies that

$$E_p[\phi(x)x_i] = \sum_{x:\phi(x)=x_i} p(x) - \sum_{x:\phi(x)\neq x_i} p(x) = 2p(\phi(x) = x_i) - 1. \tag{2}$$

We can regard $E_p[\phi(x)x_i]$ as the expected net gain of individual i by assuming that the gain from the passage of a favorable issue and the loss from the passage of an unfavorable issue are equal and normalized to one. Note that $p(\phi(x) = x_i) \gtrless 1/2$ if and only if $E_p[\phi(x)x_i] \gtrless 0$.

Imagine that the individuals agree not to adopt a voting rule such that the responsiveness of every individual is less than one-half; that is, the expected net gain of every individual is nonpositive. However, if they have no information about the true probability distribution of their preferences facing Knightian uncertainty, they cannot evaluate the responsiveness or the expected net gain. Then, under these circumstances, the individuals require that, for each $p \in \Delta(X)$, the responsiveness of every individual should not be less than one-half. We call this requirement robustness.

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Definition 1. A voting rule $\phi \in \Phi$ is *robust* if, for each $p \in \Delta(\mathcal{X})$, responsiveness of at least one individual is strictly greater than one-half:

$$\max_{i \in N} p(\phi(x) = x_i) > 1/2 \text{ for all } p \in \Delta(\mathcal{X}). \tag{3}$$

For example, a WMR with nonnegative weights is robust if there are no ties. To see this, note that $\sum_{i \in N} w_i E_p[\phi(x)x_i] > 0$ for all $p \in \Delta(\mathcal{X})$ by Lemma 1. This implies that, for each $p \in \Delta(\mathcal{X})$, there exists $i \in N$ such that $w_i E_p[\phi(x)x_i] > 0$, i.e., $p(\phi(x) = x_i) > 1/2$ by (1).

Even if a voting rule is robust and responsiveness of at least one individual is strictly greater than one-half, the arithmetic mean of responsiveness of all individuals can be less than one-half, in which case a collective decision reflects minority preferences on average. To avoid this problem under Knightian uncertainty, a voting rule must satisfy the following stronger requirement.

Definition 2. A voting rule $\phi \in \Phi$ is *strongly robust* if, for each $p \in \Delta(\mathcal{X})$, the minimum arithmetic mean of responsiveness, where the minimum is taken over all $p \in \Delta(\mathcal{X})$, is strictly greater than one-half:

$$\sum_{i \in N} p(\phi(x) = x_i)/n > 1/2 \text{ for all } p \in \Delta(\mathcal{X}). \tag{4}$$

For example, a SMR with an odd number of individuals is strongly robust. In fact, Lemma 1 implies that, for each $p \in \Delta(\mathcal{X})$, it holds that $\sum_{i \in N} E_p[\phi(x)x_i] > 0$, i.e., $\sum_{i \in N} p(\phi(x) = x_i)/n > 1/2$ by (1).

We also consider the following weaker requirement

Definition 3. A voting rule $\phi \in \Phi$ is *weakly robust* if, for each $p \in \Delta(\mathcal{X})$, responsiveness of at least one individual is not equal to one-half.

To understand the implication of weak robustness, note that some individuals are more likely to have correct choices and other individuals are more likely to have wrong choices. However, if the responsiveness of every individual is equal to one-half, then it is difficult to extract information from individuals in order to arrive at a correct group decision. A weakly robust rule does not face this problem for any probability distributions.

For example, a WMR is weakly robust if there are no ties. To see this, note that $\sum_{i \in N} w_i E_p[\phi(x)x_i] > 0$ for all $p \in \Delta(\mathcal{X})$ by Lemma 1. This implies that, for each $p \in \Delta(\mathcal{X})$, there exists $i \in N$ such that $w_i E_p[\phi(x)x_i] > 0$, i.e., $p(\phi(x) = x_i) \neq 1/2$ by (1) because w_i can be negative.

3 Robustness of anonymous rules

In this section, we study robustness of anonymous rules. A voting rule ϕ is anonymous if it is symmetric in its n variables; that is, $\phi(x) = \phi(x^{\pi})$ for each $x \in X$ and each permutation $\pi : N \to N$, where $x^{\pi} = (x_i^{\pi})_{i \in N}$ with $x_i^{\pi} = x_{\pi(i)}$. We first show that if a voting rule is anonymous then robustness is equivalent to strong robustness.

Lemma 2. *An anonymous rule is robust if and only if it is strongly robust.*

Proof. Because strong robustness implies robustness (i.e., (4) implies (3)), it is enough to show that a robust anonymous rule ϕ is strongly robust. For $p \in \Delta(X)$ and a permutation $\pi : N \to N$, consider $p^{\pi} \in \Delta(X)$ given by $p^{\pi}(x) = p(x^{\pi})$. Because $\phi(x) = \phi(x^{\pi})$ for each $x \in X$ by anonymity,

$$p^{\pi}(\phi(x) = x_i) = p(\phi(x^{\pi}) = x_i^{\pi}) = p(\phi(x) = x_{\pi(i)}).$$

Consider $q \in \Delta(X)$ given by $q = \sum_{\pi} p^{\pi}/n!$, where the summation is taken over all permutations. Then,

$$q(\phi(x) = x_i) = \frac{\sum_{\pi} p^{\pi}(\phi(x) = x_i)}{n!} = \frac{\sum_{\pi} p(\phi(x) = x_{\pi(i)})}{n!} = \frac{\sum_{j \in N} p(\phi(x) = x_j)}{n} \quad (5)$$

for all $i \in N$, which implies that $q(\phi(x) = x_i)$ is the same for all i and thus $q(\phi(x) = x_i) > 1/2$ because ϕ is robust. Therefore, we must have $\sum_{i \in N} p(\phi(x) = x_i)/n > 1/2$ by (5), so ϕ is strongly robust. $\qquad\square$

On the basis of Lemma 2, we study strong robustness as well as robustness of anonymous rules. First, suppose that the number of individuals is odd. Then, a SMR is anonymous and strongly robust as demonstrated in the previous section. Moreover, we can show that any strongly robust rule must be a SMR, which implies that a SMR is a unique robust anonymous rule. Thus, the following proposition holds.

Proposition 1. *Suppose that n is odd. The following four statements are equivalent.*

  (i) *A voting rule is anonymous and robust.*

 (ii) *A voting rule is anonymous and strongly robust.*

(iii) *A voting rule is strongly robust.*

 (iv) *A voting rule is a SMR.*

Proof. We already know that (iv) implies (i), (ii), and (iii). By Lemma 2, (i) is equivalent to (ii). Clearly, (ii) implies (iii). Thus, it is enough to show that (iii) implies (iv). If ϕ is not a SMR, there exist $y \in \mathcal{X}$ and $S \subsetneq N$ such that $|S| \leq (n-1)/2$ and $\phi(y) = y_i$ if and only if $i \in S$. Let $p \in \Delta(\mathcal{X})$ be such that $p(y) = 1$. Then,

$$\frac{2}{n} \sum_{i \in N} p(\phi(x) = x_i) - 1 = \frac{1}{n} E_p \left[\sum_{i \in N} \phi(x) x_i \right] = \frac{1}{n}(|S| - |N \setminus S|) < 0.$$

This implies that $\sum_{i \in N} p(\phi(x) = x_i)/n < 1/2$. Therefore, ϕ is not strongly robust. $\square$

Next, suppose that the number of individuals is even. In this case, neither a robust anonymous rule nor a strongly robust rule exists as shown by the following proposition.

Proposition 2. *Suppose that n is even. Then, no voting rule is both anonymous and robust. Moreover, no voting rule is strongly robust.*

Proof. It is enough to show that no voting rule is strongly robust by Lemma 2. Let $x \in \mathcal{X}$ be such that $x_i = 1$ for $i \leq n/2$ and $x_i = -1$ for $i \geq n/2 + 1$. For $p \in \Delta(\mathcal{X})$ with $p(x) = 1$ and any $\phi \in \Phi$, it holds that

$$\frac{2}{n} \sum_{i \in N} p(\phi(x) = x_i) - 1 = \frac{1}{n} E_p \left[\sum_{i \in N} \phi(x) x_i \right] = 0.$$

This implies that $\sum_{i \in N} p(\phi(x) = x_i)/n = 1/2$. Therefore, ϕ is not strongly robust. $\square$

The above results imply the following corollary, which is helpful in understanding what voting rules are not robust.

Corollary 3. *An anonymous rule is not robust if the number of individuals is even or if it is not a SMR.*

For example, a supermajority rule is not robust because it is anonymous. To illustrate it by an numerical example, consider a two-thirds rule with a very large number of individuals. Suppose that $x_i = 1$ with probability $p \in (1/2, 2/3)$ independently and identically for each $i \in N$. By the law of large numbers, the group decision is -1 with probability close to one, so responsiveness of each individual is close to $1 - p < 1/2$, which implies that a two-thirds rule is not robust when n is very large. Of course, even if n is not so large, we can find $p \in \Delta(\mathcal{X})$ such that the responsiveness of every individual is less than one-half.

4 Robustness of nonanonymous rules

In this section, we study robustness of nonanonymous rules. As discussed in Section 2, a WMR with nonnegative weights is robust if there are no ties. The following main result of this paper establishes the equivalence of robust rules and WMRs with nonnegative weights allowing no ties.

Proposition 4. *A voting rule is robust if and only if it is a WMR with nonnegative weights such that there are no ties.*

In the proof, we use the following inequality symbols. For vectors ξ and η, we write $\xi \geq \eta$ if $\xi_i \geq \eta_i$ for each i, $\xi > \eta$ if $\xi_i \geq \eta_i$ for each i and $\xi \neq \eta$, and $\xi \gg \eta$ if $\xi_i > \eta_i$ for each i.

We enumerate elements in X as $\{x^j\}_{j \in M}$ with an index set $M \equiv \{1, \ldots, m\}$ with $m = 2^n$. Consider an $n \times m$ matrix

$$L = [l_{ij}]_{n \times m} = \left[\phi(x^j)x_i^j\right]_{n \times m}.$$

Note that l_{ij} equals $+1$ if i's choice agrees with the collective decision and -1 otherwise. Using this matrix, we can restate the conditions in Proposition 4 as follows.

(a) By Lemma 1, a voting rule ϕ is a WMR with nonnegative weights allowing no ties if and only if there exists $w = (w_i)_{i \in N} \geq 0$ such that

$$\sum_{i \in N} w_i l_{ij} = \sum_{i \in N} w_i (\phi(x^j)x_i^j) > 0$$

for each $j \in M$, or equivalently, $w^\top L \gg 0$.

(b) By definition, a voting rule is *not* robust if and only if there exists $p = (p_j)_{j \in M} > 0$ such that

$$\sum_{j \in N} l_{ij} p_j = \sum_{j: \phi(x^j) = x_i^j} p_j - \sum_{j: \phi(x^j) \neq x_i^j} p_j \leq 0$$

for each $i \in N$, or equivalently, $Lp \leq 0$.

To prove Proposition 4, it is enough to show that exactly one of (a) and (b) holds. The following theorem of alternatives due to von Neumann and Morgenstern (1944)[7] guarantees that exactly one of them is true. This result also appears in Gale (1960, Theorem 2.10) as a corollary of Farkas' lemma.

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[7]von Neumann and Morgenstern (1944) use this result to prove the minimax theorem.

Lemma 3. *Let A be an $m \times n$ matrix. Exactly one of the following alternatives holds.*

- *There exists $\xi \in \mathbb{R}^n$ satisfying*

$$\xi^\top A \gg 0, \ \xi \geq 0.$$

- *There exists $\eta \in \mathbb{R}^m$ satisfying*

$$A\eta \leq 0, \ \eta > 0.$$

We can interpret Lemma 3 as a corollary of the fundamental theorem of asset pricing,[8] which is equivalent to Farkas' lemma. Thus, we can explain why Proposition 4 is true in terms of arbitrage-free pricing in an imaginary asset market.

Let M and N be the set of states and the set of assets, respectively. One unit of asset $i \in N$ yields a payoff l_{ij} when state $j \in M$ is realized. Recall that l_{ij} equals $+1$ if i's choice agrees with the collective decision and -1 otherwise. The matrix L is referred to as the payoff matrix. We denote by $q = (q_i)_{i \in N}$ the vector of prices of the n assets.

A portfolio defined by a vector $w = (w_i)_{i \in N}$ consists of w_i units of asset i for each $i \in N$. It yields a payoff $\sum_{i \in N} w_i l_{ij}$ when state $j \in M$ is realized, which is summarized in $w^\top L = (\sum_{i \in N} w_i l_{ij})_{j \in M}$. The price of the portfolio is $w^\top q = \sum_{i \in N} q_i w_i$.

A price vector q is arbitrage-free if $w^\top L \geq 0$ implies $w^\top q \geq 0$. That is, the price of any portfolio yielding a nonnegative payoff in each state is nonnegative.

A price vector q is determined by a nonnegative linear pricing rule if there exists a nonnegative vector $p = (p_j)_{j \in M} > 0$, which is referred to as a state price, such that $q = Lp$.

The fundamental theorem of asset pricing establishes the equivalence of an arbitrage-free price and the existence of a nonnegative linear pricing rule, which is immediate from Farkas' lemma.

Claim 1. *A price vector q is arbitrage-free if and only if it is determined by a nonnegative linear pricing rule. That is, the set of all arbitrage-free price vectors is $\{q : q = Lp, \ p > 0\}$.*

Its corollary, which is immediate from Lemma 3 as a corollary of Farkas' lemma, states the following.

Claim 2. *There exists a portfolio with nonnegative weights in all assets (i.e. no short selling) yielding a strictly positive payoff in each state if and only if, for any arbitrage-free price vector, the price of at least one asset is strictly positive.*

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[8]For details on the fundamental theorem of asset pricing, see Dybvig and Ross (2003, 2008) and references therein.

The former condition is restated as $w^\top L \gg 0$ for some $w \geq 0$ and the latter condition is restated as $Lp \nleq 0$ for all $p > 0$. Therefore, Claim 2 implies the equivalence of a robust rule and a WMR with nonnegative weights.

We can also characterize a weakly robust rule using another theorem of alternatives called Gordan's theorem. The following proposition establishes the equivalence of a weakly robust rule and a WMR (with possibly negative weights) allowing no ties.

Proposition 5. *A voting rule is weakly robust if and only if it is a WMR such that there are no ties.*

Proof. See Appendix A. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5 Robustness vs. weak efficiency

Rae (1969) and Taylor (1969) were the first to use responsiveness to characterize voting rules, followed by Straffin (1977) and Fleurbaey (2008). In this section, we discuss their results in comparison to our results.

Note that, by Lemma 1, $\phi \in \Phi$ is a WMR with a weight vector $w \in \mathbb{R}^N$ if and only if

$$\phi(x) \sum_{i \in N} w_i x_i \geq \phi'(x) \sum_{i \in N} w_i x_i \tag{6}$$

for all $\phi' \in \Phi$ and $x \in \mathcal{X}$.[9] This is true if and only if, for all $p \in \Delta(\mathcal{X})$, it holds that

$$\sum_{i \in N} w_i E_p[\phi(x)x_i] = \max_{\phi' \in \Phi} \sum_{i \in N} w_i E_p[\phi'(x)x_i], \tag{7}$$

or equivalently,

$$\sum_{i \in N} w_i p(\phi(x) = x_i) = \max_{\phi' \in \Phi} \sum_{i \in N} w_i p(\phi'(x) = x_i). \tag{8}$$

That is, a necessary and sufficient condition for a voting rule to be a WMR is that it maximizes the corresponding weighted sum of responsiveness over all voting rules for each $p \in \Delta(\mathcal{X})$. This result is summarized in the following proposition due to Fleurbaey (2008),[10] where the sufficient condition is weaker. We call it the Rae-Taylor-Fleurbaey (RTF) theorem because it generalizes the Rae-Taylor theorem which focuses on a SMR.

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[9]If ϕ is a WMR then $\phi(x) \sum_{i \in N} w_i x_i = |\phi(x)|| \sum_{i \in N} w_i x_i| = |\phi'(x)|| \sum_{i \in N} w_i x_i| \geq \phi'(x) \sum_{i \in N} w_i x_i$ for all $\phi' \in \Phi$. If ϕ satisfies this condition then we must have $\phi(x) \sum_{i \in N} w_i x_i \geq 0$.

[10]See also Brighouse and Fleurbaey (2010), who discuss the implication of this result for democracy.

Proposition 6. *If a voting rule is a WMR with a weight vector w, then* (8) *holds for each* $p \in \Delta(X)$. *For fixed* $p \in \Delta(X)^\circ \equiv \{p \in \Delta(X) : p(x) > 0$ for each $x \in X\}$, *where every x is possible, if* (8) *holds, then the voting rule is a WMR with a weight vector w.*

To see why the latter part is true, suppose that ϕ is not a WMR. Then, (6) does not hold for some $\phi' \in \Phi$ and $x \in X$, which contradicts (7) and (8) for each $p \in \Delta(X)^\circ$. In this way, the property of a WMR in Lemma 1 is essential in both the RTF theorem and our results, whereas the latter is based upon the theorem of alternatives as well.

Propositions 5 and 6 consider WMRs (with possibly negative weights), but there are two differences. First, the former uses weak robustness in order to avoid the worst outcomes, but the latter uses the maximization of the weighted sum of responsiveness in order to achieve the optimal outcomes. In this respect, Propositions 5 and 6 give dual characterizations of WMRs. Next, the former does not allow any ties, but the latter allows them. We will elaborate this difference later in this section.

The special case of Proposition 6 for a SMR is established by Rae (1969) and Taylor (1969) and elaborated by Straffin (1977), which is referred to as the Rae-Taylor theorem in the literature.[11]

Corollary 7. *If a voting rule is a SMR, then the arithmetic mean of responsiveness is greater than that of any voting rule for each* $p \in \Delta(X)$. *For fixed* $p \in \Delta(X)^\circ$, *if the arithmetic mean of responsiveness is greater than that of any voting rule, then the voting rule is a SMR.*

The normative implication of Proposition 6 is weak efficiency or optimality of WMRs in terms of Paretian social preferences, which is immediate from Harsanyi's utilitarianism theorem (Harsanyi, 1955).[12]

To give the formal definitions of weak efficiency and optimality in this context, we consider preferences over the set of random voting rules. For $\phi, \phi' \in \Phi$ and $\lambda \in [0, 1]$, the convex combination $\lambda\phi + (1-\lambda)\phi' : X \to [-1, 1]$ is given by $(\lambda\phi + (1-\lambda)\phi')(x) = \lambda\phi(x) + (1-\lambda)\phi'(x)$ for each $x \in X$. The convex hull of Φ is denoted by co(Φ). We regard $\phi \in$ co(Φ) as the following random voting rule: the collective decision is $+1$ with probability $(1 + \phi(x))/2$ and -1 with probability $(1 - \phi(x))/2$. For each $\phi \in$ co(Φ), the responsiveness of individual i is calculated

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[11]Assuming that individuals' choices are independently and identically distributed, Rae (1969) and Taylor (1969) establish the first part of Corollary 7 and Straffin (1977) establishes the second part.

[12]Fleurbaey (2008) assumes that w_i is proportional to i's utility and the weighted sum of responsiveness is the total sum of utilities, in which sense a WMR is optimal. Thus, he does not formally discuss this issue.

as $(E_p[\phi(x)x_i] + 1)/2$.[13] Taking responsiveness (or equivalently, $E_p[\phi(x)x_i]$) as an individual's utility, we can define weak efficiency and optimality as follows.

Definition 4. Fix $p \in \Delta(\mathcal{X})$. A voting rule $\phi \in \Phi$ is weakly efficient if there is no $\phi' \in \mathrm{co}(\Phi)$ such that $E_p[\phi'(x)x_i] > E_p[\phi(x)x_i]$ for all $i \in N$.

Definition 5. Fix $p \in \Delta(\mathcal{X})$. A voting rule $\phi \in \Phi$ is optimal with respect to a Paretian von Neumann-Morgenstern (vNM) welfare function if there exists a linear welfare function $v : \mathrm{co}(\Phi) \to \mathbb{R}$ such that (i) $v(\phi) \geq v(\phi')$ for all $\phi' \in \mathrm{co}(\Phi)$, and (ii) for $\phi', \phi'' \in \mathrm{co}(\Phi)$, $E_p[\phi'(x)x_i] \geq E_p[\phi''(x)x_i]$ for each $i \in N$ implies $v(\phi') \geq v(\phi'')$.

Harsanyi's utilitarianism theorem states that if a linear welfare function $v : \mathrm{co}(\Phi) \to \mathbb{R}$ satisfies the condition (ii) then there exists a nonnegative vector $w \in \mathbb{R}_+^N$ such that $v(\phi) = \sum_{i \in N} w_i E[\phi(x)x_i]$ for each $\phi \in \mathrm{co}(\Phi)$.[14] Thus, Proposition 6 can be understood as the following normative characterization of weighted majority rules.[15]

Corollary 8. *Fix $p \in \Delta(\mathcal{X})^\circ$. The following three statements are equivalent.*

 (i) *A voting rule is a WMR with nonnegative weights.*

 (ii) *A voting rule is weakly efficient.*

 (iii) *A voting rule is optimal with respect to a Paretian vNM welfare function.*

We compare robustness and weak efficiency (i.e. optimality) on the basis of Proposition 4 and Corollary 8, which immediately imply the following corollary.

Corollary 9. *A robust rule is weakly efficient. A weakly efficient rule is not necessarily robust.*

Both robust rules and weakly efficient rules are WMRs with nonnegative weights, but a weakly efficient WMR allows ties with any tie-breaking rule, while there must be no ties in a robust WMR. Therefore, robustness is a stronger requirement than weak efficiency.[16]

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[13]This is because, given $x \in \mathcal{X}$, the conditional probability that i's decision agrees with the collective decision is $(1 + x_i)/2 \cdot (1 + \phi(x))/2 + (1 - x_i)/2 \cdot (1 - \phi(x))/2 = (\phi(x)x_i + 1)/2$.

[14]See Domotor (1979), Weymark (1993), Mandler (2005), and references therein.

[15]The theorem of Wald (1950) on admissible decision functions or that of Pearce (1984) on strictly dominated strategies implies that a weakly efficient rule is a WMR. For completeness, we give a proof in Appendix B.

[16]Even if a voting rule is efficient rather than weakly efficient, i.e., it is a WMR with strictly positive weights, there can be ties. An example is a SMR with an even number of individuals.

To illustrate the above difference between robustness and weak efficiency, suppose that n is even. All SMRs with all tie-breaking rules are weakly efficient. For example, consider a SMR with a tie-breaking rule in which the collective decision is "no" whenever there is a tie. This rule is weakly efficient, but it is not robust because it is anonymous and no anonymous rule is robust when n is even by Corollary 3.

On the other hand, a SMR with some tie-breaking rule is robust if and only if it is represented as a WMR without ties. For example, a SMR with a casting (tie-breaking) vote is robust. To see this, we consider two cases. First, assume that the presiding officer with a casting vote is a member of the group of n individuals. This rule is equivalent to a WMR such that the presiding officer's weight is slightly greater than the others' weights. Next, assume that the presiding officer is not a member of the group of n individuals and that he or she votes only when there is a tie. This rule is equivalent to a WMR with $n + 1$ individuals including the presiding officer such that the presiding officer's weight is very close to zero. Each WMR does not have ties and it is robust.

The above discussion presents a justification for the wide use of a casting vote in terms of robustness although it does not conform to some rules of parliamentary procedure such as Robert's Rules of Order (Roberts, 1971). In legislatures such as the New Zealand House of Representatives, the British House of Commons, and the Australian Senate, the presiding officers are required to vote in favor of the status quo or a tie is considered to be a defeat. This SMR is not robust. On the other hand, in legislatures such as the United States Senate, the Australian House of Representatives, and the National Diet of Japan, the presiding officers hold casting votes to break ties. As mentioned above, this SMR is robust.

Finally, we compare the number of robust SMRs and that of all SMRs. The following proposition demonstrates that the former is substantially smaller than the latter when there are a large number of individuals.

Proposition 10. *Suppose that n is even. The number of SMRs with some tie-breaking rules is $2^{\binom{n}{n/2}}$. The number of SMRs with some tie-breaking rules that can be represented as WMRs with no ties is at most $2^{\binom{n}{n/2}/2}$. Thus, the ratio of the latter to the former is at most $2^{-\binom{n}{n/2}/2}$.*

Proof. The number of decision profiles with ties is $\binom{n}{n/2}$. A tie-breaking rule assigns 1 or -1 to each of $\binom{n}{n/2}$ decision profiles. Thus, the number of tie-breaking rules is $2^{\binom{n}{n/2}}$. If a voting rule is a WMR, then the collective decision to x is the negative of that to $-x$. Thus, the number of tie-breaking rules represented as WMRs is at most $2^{\binom{n}{n/2}/2}$. $\qquad\square$

For example, when $n = 6$, the number of SMRs with some tie-breaking rules is 2^{20}, while the number of SMRs with some tie-breaking rules that can be represented as WMRs with no ties is at most 2^{10}. Thus, the ratio of the latter to the former is at most $2^{-10} = 1/1024$.

6 Conclusion

The justification of WMRs and, in particular, SMRs based on efficiency arguments or axiomatic characterizations has yielded some of the celebrated contributions to the social choice and voting literature. The two paramount examples rationalizing a SMR within a dichotomous setting are Condorcet's jury theorem and May's theorem,[17] where the rationalization of a voting rule is based on asymptotic (i.e., infinite-individual) probabilistic criteria or deterministic criteria. An alternative approach based on non-asymptotic (i.e., finite-individual) probabilistic criteria was pioneered by Rae (1969), who suggested the aggregate expected net gain or the aggregate responsiveness as a meaningful criterion for evaluating the performance of a voting rule in the constitutional stage, namely, where the veil of ignorance prevails.

This paper contributes to the latter literature. That is, we introduce a normative criterion for voting rules under Knightian uncertainty about individuals' preferences and establish that a voting rule is a WMR without any ties if and only if it is robust. Robustness of a voting rule requires that, for any probability distribution of preferences, responsiveness of at least one voter is strictly greater than one-half. In our setting, robustness is satisfied if and only if, for any probability distribution of preferences, the expected net gain of at least one voter is strictly positive. Our focus on the minimum responsiveness over all probability distributions rather than the maximum responsiveness over all voting rules implies that we take into account the worst case scenario under Knightian uncertainty. That is, this paper contributes to the literature on the axiomatic foundations of WMRs and SMRs by joining the recently growing literature on economic design with worst-case objectives discussed in the introduction.

Our results and that of Rae (1969), Taylor (1969), and Fleurbaey (2008) have in common that responsiveness characterizes a WMR. Their RTF theorem states that a voting rule is a WMR if and only if it maximizes the corresponding weighted sum of responsiveness of all individuals over all voting rules. By Harsanyi's characterization of utilitarianism (Harsanyi, 1955), the RTF theorem can also be understood as a characterization of WMRs in terms of weak efficiency:

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[17]See May (1952), Fishburn (1973), and Dasgupta and Maskin (2008).

a voting rule is a WMR if and only if it is weakly efficient with respect to responsiveness. However, there are significant differences between our results and the RTF theorem in the use of responsiveness and in the characterized class of weighted majority rules. Most importantly, as already noted, the RTF theorem uses the maximum responsiveness over all voting rules in order to achieve the optimal outcomes, whereas our results use the minimum responsiveness over all probability distributions in order to avoid the worst outcomes. Hence, our results complement the renowned RTF theorem by providing a dual characterization of WMRs and, in particular, of SMRs in terms of responsiveness under Knightian uncertainty.

Appendix

A   Proof of Proposition 5

We can restate the conditions in Proposition 5 as follows.

(a) By Lemma 1, a voting rule ϕ is a WMR with nonzero weights allowing no ties if and only if there exists $w = (w_i)_{i \in N} \neq 0$ such that

$$\sum_{i \in N} w_i l_{ij} = \sum_{i \in N} w_i (\phi(x^j) x_i^j) > 0$$

for each $j \in M$, or equivalently, $w^\top L \gg 0$.

(b) By the definition, a voting rule is *not* weakly robust if and only if there exists $p = (p_j)_{j \in M} > 0$ such that

$$\sum_{j \in N} l_{ij} p_j = \sum_{j : \phi(x^j) = x_i^j} p_j - \sum_{j : \phi(x^j) \neq x_i^j} p_j = 0$$

for each $i \in N$, or equivalently, $Lp = 0$.

To prove Proposition 5, it is enough to show that exactly one of (a) and (b) holds. The following theorem of alternatives, which is referred to as Gordan's theorem, guarantees that exactly one of them is true. This result also appears in Gale (1960, Theorems 2.9) as a corollary of Farkas' lemma.

Lemma A. *Let A be an $m \times n$ matrix. Exactly one of the following alternatives holds.*

- *There exists $\xi \in \mathbb{R}^n$ satisfying*

$$\xi^\top A \gg 0.$$

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- *There exists $\eta \in \mathbb{R}^m$ satisfying*

$$A\eta = 0, \ \eta > 0.$$

We can interpret Lemma A as the following corollary of the fundamental theorem of asset pricing. By using it, we can explain why Proposition 5 is true in terms of arbitrage-free pricing in an imaginary asset market, as we did so in the case of Proposition 4.

Claim A. *There exists a portfolio yielding a strictly positive payoff in each state if and only if, for any arbitrage-free price vector, the price of at least one asset is nonzero.*

B   Proof of Corollary 8

We give a proof of the "only if" part. Define

$$U_p = \{(E_p[\phi(x)x_i])_{i \in N} \in \mathbb{R}^n : \phi \in \mathrm{co}(\Phi)\},$$

which is convex because $\mathrm{co}(\Phi)$ is convex. Let $\phi^* \in \Phi$ be weakly efficient and write $u^* \equiv (E_p[\phi^*(x)x_i])_{i \in N} \in U_p$. For $V_p = \{v \in \mathbb{R}^n : v \gg u^*\}$, it holds that $U_p \cap V_p = \emptyset$. Because U_p and V_p are convex, there exists $w \in \mathbb{R}^n$ and $r \in \mathbb{R}$ such that

$$\langle w, u \rangle \leq r \leq \langle w, v \rangle \text{ for all } u \in U_p \text{ and } v \in V_p$$

by the separation theorem. Because $u^* \in U_p$ is on the boundary of V_p, $\langle w, u^* \rangle = r$ and thus $\sum_{i \in N} w_i E_p[\phi^*(x)x_i] \geq \sum_{i \in N} w_i E_p[\phi(x)x_i]$ for all $\phi \in \mathrm{co}(\Phi)$. By Proposition 6, ϕ^* must be a WMR and its weight vector w must be nonnegative because $\langle w, v - u^* \rangle \geq 0$ for all $v \in V_p$.

References

Azrieli, Y., Kim, S., 2014. Pareto efficiency and weighted majority rules. Int. Econ. Rev. 55, 1067–1088.

Bergemann, D., Morris, S., 2005. Robust mechanism design. Econometrica 73, 1771–1813.

Bergemann, D., Schlag, K., 2011. Robust monopoly pricing. J. Econ. Theory 146, 2527–2543.

Brighouse, H., Fleurbaey, M., 2010. Democracy and proportionality. J. Polit. Philos. 18,137–155.

Carroll, G., 2015. Robustness and linear contracts. Amer. Econ. Rev. 105, 536–563.

Carroll, G., 2016. Informationally robust trade and limits to contagion. J. Econ. Theory 166, 334–361.

Chung, K. S., Ely, J. C., 2007. Foundations of dominant-strategy mechanisms. Rev. Econ. Stud. 74, 447–476.

Dasgupta, P., Maskin, E., 2008. On the robustness of majority rule. J. Eur. Econ. Assoc. 6, 949–973.

Domotor, Z., 1979. Ordered sum and tensor product of linear utility structures. Theory Dec. 11, 375–399.

Dybvig, P., Ross, S., 2003. Arbitrage, state prices and portfolio theory. In: Constantinides, G., Harris, M., Stulz, R. M. (Eds.), Handbook of the Economics of Finance 1. Elsevier, pp. 605–637.

Dybvig, P., Ross, S., 2008. Arbitrage. In: Durlauf, S., Blume, L. (Eds.), The New Palgrave Dictionary of Economics. Palgrave Macmillan, pp. 188–197.

Fishburn, P. 1973. The Theory of Social Choice. Princeton University Press.

Fishburn, P. C., Gehrlein, W. V., 1977. Collective rationality versus distribution of power for binary social choice functions. J. Econ. Theory 15, .72–91.

Fleurbaey, M., 2008. One stake one vote. Working paper.

Gale, D., 1960. The Theory of Linear Economic Models. University of Chicago Press.

Gilboa, I., Schmeidler, D., 1989. Maxmin expected utility with a non-unique prior. J. Math. Econ. 18, 141–153.

Harsanyi, J., 1955. Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility. J. Polit. Econ. 63, 309–321.

Huber, P. J., 1981. Robust Statistics. John Wiley & Sons.

Knight, F. H., 1921. Risk, Uncertainty, and Profit. Houghton Mifflin.

Mandler, M. 2005. Harsanyi's utilitarianism via linear programming. Econ. Letters 88, 85–90.

May, K., 1952. A set of independent necessary and sufficient conditions for simple majority decisions. Econometrica 20, 680–684.

Neumann, J. V., Morgenstern, O., 1944. Theory of Games and Economic Behavior. Princeton University Press.

Nitzan, S., Paroush, J., 1981. The characterization of decisive weighted majority rules. Econ. Letters 7. 119–124.

Pearce, D. G., 1984. Rationalizable strategic behavior and the problem of perfection. Econometrica 52, 1029–1050.

Rae, D. W., 1969. Decision-rules and individual values in constitutional choice. Am. Pol. Sci. Rev. 63, 40–56.

Roberts, G. H. M., 1971. Robert's Rules of Order, Revised. William Morrow and Company.

Schmitz, P. W., Tröger, T., 2012. The (sub-)optimality of the majority rule. Games Econ. Behav. 74, 651–665.

Straffin, P. D., 1977. Majority rule and general decision rules. Theory Dec. 8, 351–360.

Taylor, A., Zwicker, W., 1992. A characterization of weighted voting. Proc. Amer. Math. Soc. 115, 1089–1094.

Taylor M., 1969. Proof of a theorem on majority rule. Behav. Sci. 14, 228–231.

Yamashita, T., 2015. Implementation in weakly undominated strategies: Optimality of second-price auction and posted-price mechanism. Rev. Econ. Stud. 82, 1223–1246.

Wald, A., 1950. Statistical decision functions. Chapman & Hall.

Weymark, J., 1993. Harsanyi's social aggregation theorem and the weak Pareto principle. Soc. Choice Welfare 10, 209–221.