Bidimensional Risk Aversion: The Cardinal Sin

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Abstract. The preferences of a decision-maker who faces risk on two different dimensions depend both on the curvature of her indifference map under certainty, which is a distinctly ordinal property, and on the degree of concavity of the utility function representing preferences under risk, which is a cardinal property. The weight of these two components is shown to depend on the direction of the change in risk. Bidimensional stochastic orders accounting for risks on any possible direction are first defined by generalizing the concept of submodularity. Three different factors are shown to shape individual preferences under risk in the bidimensional case: the degree of substitutability among goods, the intensity of risk aversion on each single dimension and the degree of correlation aversion of the decision-maker. Their contributions to the amount of risk premium is assessed and a new notion of "compensated risk aversion" is introduced.

Keywords: Multivariate Risk Aversion, Correlation Aversion, Submodularity, Risk Premium, Compensated Risk Aversion.

JEL Classification: D31, D63, D81.

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1 Introduction

When choices under risk involve two dimensions, the Von Neumann-Morgenstern (VNM) utility functions used to represent individual preferences are defined on bundles of two attributes varying across two or more states of the world. As in the unidimensional case the effects of risk depend on the initial quantity of wealth, in the bidimensional the effects of risk on one attribute on individual preferences depend on the initial quantity of such an attribute, but also on the available quantity of the second good, which can also vary over states of the world. The nature of substitute/complement of the two goods and the association between good and bad outcomes in the different dimensions over states of the world then become crucial to understand the impact of risk on individual utility. The sign of the cross derivatives of the utility function is a key property, and has been used to describe risk aversion (Richard 1975), but it has also been used to identify complement and substitute goods under certainty (the so called ALEP substituability) [1] as well as for characterizing correlation aversion between good and bad outcomes in two dimensions over different states of the world (Eeckhoudt et al. 2007). It is however unclear how all these ingredients combine each other in order to describe individual preferences.

Kihlstrom and Mirman (1974) remarked that changes in risk aversion, which depend on cardinal properties of the utility function, can be mixed up with changes in the degree of substituability of goods, which is an ordinal property of individual preferences. To skip this difficulty, they limited their analysis of risk aversion to the case of agents with identical indifference curves under certainty. However, also under this simplification, it is difficult to separate the consequences of the ordinal properties of preferences under certainty from those depending on the cardinal properties of the utility function. The objective of this paper is to disentangle these different components, in the case where changes in risks are possible along any possible direction.

In Section 2 risks along general directions are considered and their effects on individual utility are shown to depend on the sign of cross derivatives. More precisely, stochastic

[1] Two goods are Auspitz and Lieben - Edgeworth - Pareto (ALEP) substitute (complement) at x if and only if $u_{12}(x) < 0$ ($u_{12}(x) > 0$). See Samuelson (1974).

orders are investigated to determine the broadest families of utility functions able to guarantee an increase in the level of the EU under well-defined types of changes in risk over two dimensions. The effects of changes in risk on the utility function are shown to depend on submodularity of the utility function, up to a change of variables depending on the direction of the risk. Section 3 develops new results on comparative risk aversion. The sensitivity of the risk premium to the cardinal and the ordinal properties of the utility function used to represent individual preferences is investigated. The risk premium is split into two components: the first one depends on ordinal properties of the utility function used to represent individual preferences over the two attributes in a certain environment. This component reflects the degree of substituability of the two attributes. Since the effects of an hypothetical risk moving an initial allocation to another one lying on the same indifference curve are not sensitive to the degree of concavity of the utility function, the more divergent is the risk direction from the slope of the indifference curve, the higher the impact of the degree of concavity of the utility function on individual preferences. This effect is embedded in the cardinal component of the risk premium. These results are discussed in Section 4, which concludes the paper with some suggestions for further developments.

2 Changes in risk and submodularity

Let $u : R^2 \to R$ be a non decreasing and concave VNM utility function differentiable as times as necessary, defined on two attributes x_1, x_2. Let $u_i(.)$ denote the marginal utility $\partial u(x)/\partial x_i$ and $u_{ij}(.)$ the second derivatives $\partial^2 u(x)/\partial x_i \partial x_j$, for $i, j = 1, 2$. Given an initial allocation, we study the effects of different zero-mean risks on the EU, depending on the direction and the intensity of gains and losses in each dimension. As in Eeckhoudt et al. (2007) we consider simple lotteries where two outcomes have an equally likely chance of occurrence.

The link between the shape of the utility function and the directions of the change in risk that guarantee an EU improvement is well-known in the stochastic orders literature:

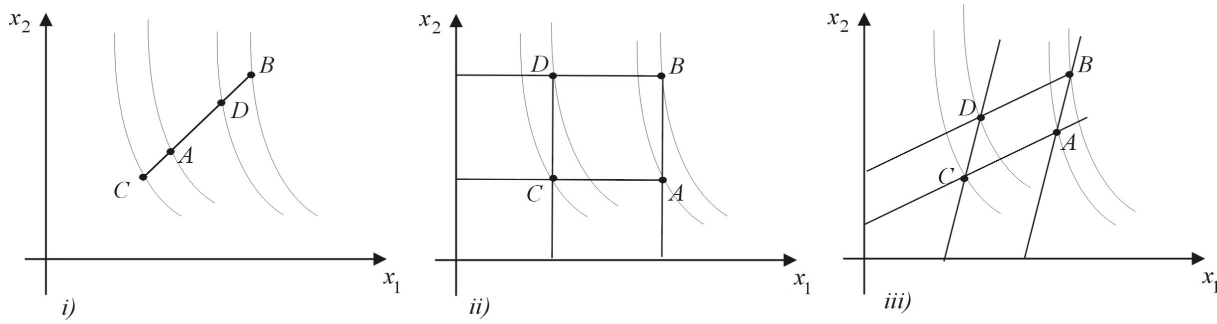


Figure 1: Changes in risk

Definition 2.1. *(Müller and Scarsini 2012, Müller 2013) Given a function $u : R^2 \to R$ and two lotteries with equally probable outcomes B and C, or A and D, then:*

$$u(C) + u(B) \leq u(A) + u(D)$$

i) *for any A, B, C, D in R^2 such that $A = \alpha B + (1-\alpha)C$, $D = (1-\alpha)B + (\alpha)C$, with $\alpha \in [0, 1]$ iff the function u is concave;*

ii) *for any A, B, C, D in R^2 such that $B = A \vee D$ and $C = A \wedge D$ (submodular changes in risk)* [2] *iff the function is submodular, that is $u_{ij} \leq 0$, for any $i \neq j$;*

iii) *for any A, B, C, D in R^2 such that $C \leq D \leq B$, $C \leq A \leq B$ and $A + D = C + B$ (directionally concave changes in risk) iff the function u is directionally concave*[3]*, that is $u_{ij} \leq 0$, for any $i, j = 1, 2$.*

The changes in risk considered in the definition above are illustrated in the three panels of Figure 1. In the first panel the alternative outcomes in equally-probable states of the world lie along the same line. An equally-probable lottery on the "internal" outcomes A and D is preferred to an equally-probable lottery on the "external" outcomes B and C if and only if u is concave.[4]

[2]We denote with $\vee$ and $\wedge$ the componentwise supremum and infimum, respectively.

[3]This function is called "inframodular" by Marinacci and Montrucchio (2005).

[4]This implies that the correspondence between concavity and risk-aversion can be extended from the unidimensional case to the bidimensional one only for a very restricted type of changes in risk. (See Section 3).

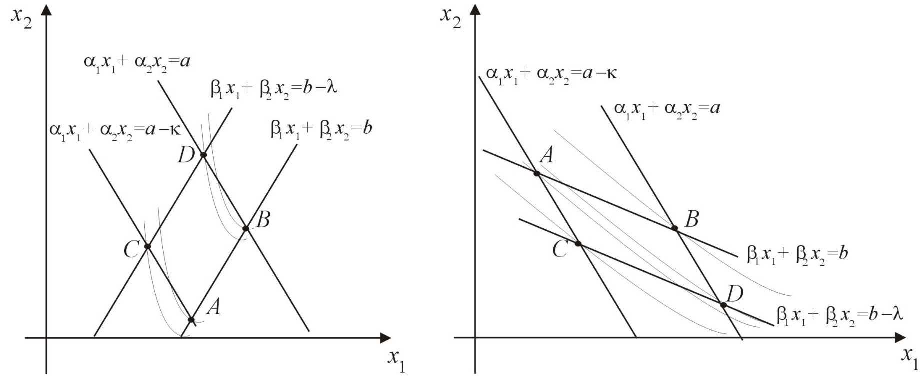


Figure 2: Generalized directional changes

The second panel illustrates case ii), where a negative sign of the cross derivative u_{ij} describes an individual who prefers a lottery where the shortage of one good is compensated for by an increase of the other good in each state of the world, rather than a lottery where affluence in both attributes in one state of the world is opposed to shortage of both attributes in the other state of the world (see Richard 1975 and Crainich et al. 2013). [5] The third panel illustrates case iii), where more general directions are allowed for losses and gains in the two dimensions, such that the rectangle of case ii) becomes a parallelogram. In case iii) the lottery over the best outcome B and the worst outcome C is compared to a lottery over A and D where gains and losses on both dimensions are allowed. Both in cases ii) and iii) the order structure of the vector space, joined with the monotonicity of u, guarantees that points B and C can be ranked in terms of all non-decreasing utility functions, while A and D guarantee a level of utility intermediate between $u(B)$ and $u(C)$ but can be ranked only specifying the functional form of u.

In the next paragraph the previous definition ii) is extended by allowing risk in any direction, such that the following condition holds:

$$u(B) \geq u(D) \geq u(C) \text{ and } u(B) \geq u(A) \geq u(C). \tag{2.1}$$

Looking at the lotteries' oucomes in Figure 2, neither C, A nor D, B can be ranked by using the orthant order, but (2.1) holds.

[5] To save space we skip the opposite case where "joining bad with bad and good with good" increases the EU.

To analyze this case, it is convenient to introduce a more general order structure in R^2. Let $B = (x_1, x_2)$ and $C = (y_1, y_2) = (x_1 - k, x_2 - \ell)$, with $k, \ell > 0$, two points that can be ranked as before by the elementary order structure of R^2 and the monotonicity of u. The outcomes A and D are constructed such that the rectangle of case ii) becomes a parallelogram.

Drawing the lines joining A, B, C and D, as illustrated in Figure 2, it is possible to rank C, A and D, B in the space Ω composed of all the couples (a, b), such that $a = \alpha_1 x_1 + \alpha_2 x_2$ and $b = \beta_1 x_1 + \beta_2 x_2$. More precisely, in the space Ω, we get:[6]

$$\begin{aligned}
B &= (a, b),\\
C &= (a - \kappa, b - \lambda),\\
A &= (a - \kappa, b),\\
D &= (a, b - \lambda).
\end{aligned}$$

Note that in the first panel of Figure 2, we have $-\beta_1/\beta_2 > \ell/k$. This condition guarantees that the relative positions of the worst off and the best off individuals are preserved, when $\beta_1/\beta_2 < 0$.[7]

The notion of submodularity is then extended by defining a function $\Psi : \Omega \to R$, such that $\Psi(a, b) \equiv u(x_1, x_2)$.

According to Definition 2.1, a function $\Psi : \Omega \to R$ is submodular if for any quadruple A, B, C, D in Ω such that $B = A \vee D$ and $C = A \wedge D$, the following inequality holds:

$$\Psi(C) + \Psi(B) \leq \Psi(A) + \Psi(D). \tag{2.2}$$

Then a lottery on A, D is preferred to another on C, B if and only if $\dfrac{\partial^2 \Psi}{\partial a \partial b}(a, b) \leq 0$.

Observe that just as the partial derivatives are taken with respect to a change in one variable, the directional derivatives evaluate the rate at which the utility changes when both inputs move, with positive or negative dependence.

The conditions on the cross derivative of Ψ can be also expressed in terms of the second derivatives of u:

[6]It is also necessary that $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$ for the invertibility of the transformation

[7]Note that, in the opposite case, i.e. $-\beta_1/\beta_2 < \ell/k$, we have $B = (a, b - \lambda)$, $C = (a - \kappa, b)$, $A = (a - \kappa, b - \lambda)$, $D = (a, b)$.

Proposition 2.1. *Given $\Psi : \Omega \to R$, such that $\Psi(a, b) = u(x_1, x_2)$, with $a = \alpha_1 x_1 + \alpha_2 x_2$ and $b = \beta_1 x_1 + \beta_2 x_2$.*

$$\frac{\partial^2 \Psi}{\partial a \partial b}(a, b) < 0 \Leftrightarrow \alpha_2\beta_2 u_{11}(x_1, x_2) + \alpha_1\beta_1 u_{22}(x_1, x_2) - (\alpha_1 b_2 + \alpha_2\beta_1)u_{12}(x_1, x_2) > 0. \quad (2.3)$$

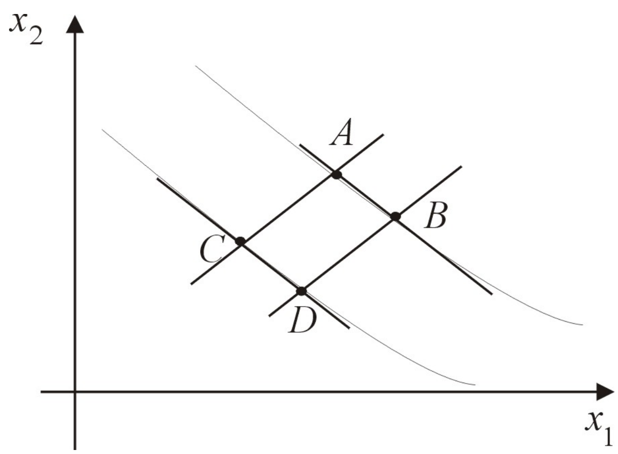
Proof. From conditions $a = \alpha_1 x_1 + \alpha_2 x_2$ and $b = \beta_1 x_1 + \beta_2 x_2$ and applying the substitution $x_1 = (a\beta_2 - b\alpha_2)/(\alpha_1\beta_2 - \alpha_2\beta_1)$ and $x_2 = (b\alpha_1 - a\beta_1)/(\alpha_1\beta_2 - \alpha_2\beta_1)$, we calculate the cross derivative of $\Psi(a, b) = u(x_1, x_2)$. Then $\frac{\partial \Psi(a,b)}{\partial a} = (\beta_2 u_1(x_1, x_2) - \beta_1 u_2(x_1, x_2))/(\alpha_1\beta_2 - \alpha_2\beta_1)$, and $\frac{\partial^2 \Psi}{\partial a \partial b} = -(\alpha_2\beta_2 u_{11}(x_1, x_2) + \alpha_1\beta_1 u_{22}(x_1, x_2) - (\alpha_1\beta_2 + \alpha_2\beta_1)u_{12}(x_1, x_2))/(\alpha_1\beta_2 - \alpha_2\beta_1)^2$. □

This result shows that all the second derivatives of u, combined with the directions of the risk, determine the effects of the risk on the EU.

Note that, by the introduction of the function Ψ, all the standard cases, presented in Definition 2.1 and represented in Figure 1, can be covered. In fact, we can obtain case i) in a degenerate case, where Ψ is a function of only one variable and computing the second order derivative of Ψ, for instance, with respect to the first component a,[8] and it is negative if and only if the utility function u is concave by the definition of negative semidefinite Hessian matrix. Case ii) corresponds to independent changes in risk, that is $a = \alpha_1 x_1$ and $b = \beta_2 x_2$, see panel ii) of Figure 1. In this case, $\frac{\partial^2 \Psi}{\partial a \partial b} = \alpha_1\beta_2 u_{12}(x_1, x_2)$ and it is negative if and only if u_{12} is negative, with $\alpha_1, \beta_2 > 0$. Finally, to recover case iii) we need that the slopes of $a = \alpha_1 x_1 + \alpha_2 x_2$ and $b = \beta_1 x_1 + \beta_2 x_2$ are both positive. The only feasible cases are given by α_i, α_j and β_i, β_j discordant with also α_i, β_i discordant $\forall i = 1, 2$. In this case, if all the second order derivatives of u are negative, then Ψ_{ab} is negative.

In order to separate the aspects of preferences under certainty from those depending on the cardinal properties of the utility function, it is useful to start from Kannai (1980), who noticed that, for any utility function u, ALEP complements can become substitutes by applying a sufficiently concave transformation f to u. In fact, given $v = f(u(x))$, its second cross derivative is $v_{ij} = f_{uu} u_i u_j + f_u u_{ij}$ and its sign can be changed by selecting an appropriate concave function f. More precisely, if the goods are ALEP complement

[8] Recall that $\Psi_{aa} = (\beta_2^2 u_{11} + \beta_1^2 u_{22} - 2\beta_1\beta_2 u_{12})/(\alpha_1\beta_2 - \alpha_2\beta_1)^2$.



$(u_{ij} > 0)$, they can become substitute $(v_{ij} > 0)$. However, such a transformation alters only the cardinality of the utility function. Therefore, the cross derivative of u used in the ALEP definition cannot capture ordinal aspects, as substitutability or complementarity among goods. Submodularity is not preserved in general and this is also true for generalized submodular functions. The sensitivity of submodularity to concave transformations, however, depends on the direction of the risk. For instance, if the risk changes the allocations along the direction of the MRS, then generalized submodularity is not affected by the cardinalization of the utility function. If $\alpha_1/\alpha_2 = -\beta_1/\beta_2 = u_1/u_2$, with $\Psi(a,b) = u(x_1,x_2)$ and $\tilde{\Psi}(a,b) = v(x_1,x_2) = f(u(x_1,x_2))$, where f is concave, it is easy to check that $\Psi_{ab}(a,b)$ and $\tilde{\Psi}_{ab}(a,b)$ have the same sign. In a neighbourhood of a starting position C, moving along the MRS, without changing the level curve, going to D and doing the opposite from B to A, does not change the preferences of the agent towards the goods, even after a concave transformation of the given utility function, as showed in Figure 2

The links between the direction of the risk, the risk premium and the cardinalization of the VNM utility is analyzed in the next section, which studies comparative risk aversion in the bidimensional case.

3 Comparative risk aversion with two attributes

3.1 Basic concepts

We recall the definitions of risk aversion in the one-dimensional case. Let $\tilde{\varepsilon}$ be a zero-mean risk:

Definition 3.1. *An individual is risk averse if he dislikes all zero-mean risks at all wealth levels:*

$$Eu(x_0 + \tilde{\varepsilon}) \leq u(x_0), \ \forall \tilde{\varepsilon} \ s.t. E(\tilde{\varepsilon}) = 0. \tag{3.1}$$

An individual (with utility) v is more risk averse than an individual (with utility) u if v dislikes all lotteries that u dislikes, for any common level of initial wealth. Pratt (1964) characterized comparative risk aversion by introducing absolute and relative measures based on the curvature of the utility function. He demonstrated that any increasing and concave transformation $v = f(u(x))$ raises the degree of risk aversion and the amount of the risk premium (see Gollier 2001 for details).

Richard (1975) studied multivariate risk aversion in terms of the preferences of an agent for mixing bad and good outcomes, and provided a characterization based on the cross derivative of the utility function (see Richard 1975, p.13). Unfortunately, this definition leads to confound two different aspects: risk aversion and correlation aversion. Duncan (1977) and Karni (1979) proposed to complement the Arrow-Pratt absolute risk aversion coefficient on single dimensions i, defined as $RA_i(x) = -\frac{u_{ii}}{u_i}(x)$, by an index of absolute correlation aversion:

Definition 3.2. *The absolute correlation aversion index between attributes i and j (computed with respect to the ith attribute) is*

$$CA_i(x) = -\frac{u_{ij}}{u_i}(x).$$

The attitude towards risk is not fully captured by the cross derivative of the utility function and correlation aversion. Risk aversion implies that one prefers with certainty $(5, 5)$, instead of the lottery $\{(0, 0), (10, 10)\}$ with equal probability of the outcomes, regardless of the sign of the cross derivative. In fact, risk aversion is related with the degree of concavity of the utility function also in the multivariate case, as pointed out by Kihlstrom and Mirman (1974). These authors associate "more risk aversion" to a "more concave" utility function (p. 366), extending the Pratt approach to the bidimensional case. Under a specific definition of risk premium (directional and multiplicative, see Proposition 1, p. 368), they also extend the Pratt result that a higher risk premium is paid by a more risk averse agent. Nevertheless, their definition of risk premium does not allow to fully generalize equation (3.1) in the multidimensional context, due to their definition of risk, a share of the vector of the attributes, that scales all the components in the same way. In the next paragraphs, we investigate three different factors that play a role when an agent faces risk over two dimensions along more general directions: 1) differences in preferences on the goods under certainty (the goods can be complement or substitute), 2) differences in risk attitudes (preferences for sure outcomes instead of risky ones, in terms of utility values) and 3) differences in risk characteristics: the risks can be negatively or positively correlated.

3.2 The risk-premium for a risk along two dimensions

Let us consider the random vector $(\tilde{\varepsilon}, \tilde{\delta})$, where $\tilde{\varepsilon}$ is a zero-mean risk on the first variable x_1, and $\tilde{\delta}$ is a zero-mean risk on x_2. We denote with $V = [\sigma_{ij}]$ the positive semi-definite variance-covariance matrix. If the two risks are simultaneous, we get:

$$Eu(x_1 + \tilde{\varepsilon}, x_2 + \tilde{\delta}) \leq u(x_1, x_2), \ \forall \tilde{\varepsilon}, \tilde{\delta} \text{ s.t. } E(\tilde{\varepsilon}) = 0, E(\tilde{\delta}) = 0. \tag{3.2}$$

The second order approximation of this expression around (x_1, x_2) leads to:

$$u_{11}(x_1, x_2)\sigma_1^2 + u_{22}(x_1, x_2)\sigma_2^2 + 2u_{12}(x_1, x_2)\sigma_{12} \leq 0. \tag{3.3}$$

Notice that with negative correlation between the two risks and a positive cross derivative of u, (3.3) is always satisfied and the agent is risk averse for any (x_1, x_2).

The corresponding definition of risk premium π_1 to pay in one dimension, for instance x_1, to get rid of risk on both dimensions is given by:[9]

$$Eu(x_1 + \tilde{\varepsilon}, x_2 + \tilde{\delta}) = u(x_1 - \pi_1, x_2), \ \forall \tilde{\varepsilon}, \tilde{\delta}, \text{ s.t. } E(\tilde{\varepsilon}) = 0, E(\tilde{\delta}) = 0. \tag{3.4}$$

[9] We follow Karni (1979) and Courbage (2001).

$$\pi_1 \approx -1/2 \left[\frac{\sigma_1^2 u_{11} + \sigma_2^2 u_{22} + 2\sigma_{12} u_{12}}{u_1} \right]. \tag{3.5}$$

Notice that the premium is positive if and only if the utility function is concave (see Theorem 1, Karni 1979).

From expression (3.3) we will explore the combined role of risk and correlation aversion coefficients in determining individual preferences under risk. Following Kihlstrom and Mirman (1974), we consider the effects of a concave transformation $f \circ u$. It is well-known that it raises the risk-aversion coefficient on each single dimension while it also raises the absolute correlation aversion coefficient, as shown in the previous section.

In what follows, we study the impact of these effects on the intensity of risk aversion, measured in terms of the amount of the risk premium. For instance, with negatively correlated risks, it might be the case that a concave transformation, by raising the risk aversion coefficient in each dimension, increases the risk premium, but its effect on the correlation aversion decreases it. This "compensative effect" is the main object of the next paragraph.

3.3 Compensated risk aversion and risk premium

The effect of the cross derivative of the utility function on the risk premium depends on the correlation between the outcomes of the risks. Considering the simple case of equiprobable lotteries as in Richard (1975), for negatively correlated outcomes the risk premium is given by:

$$1/2[u(x_1 + \varepsilon_1, x_2 - \delta_1) + u(x_1 - \varepsilon_1, x_2 + \delta_1)] = u(x_1 - \pi_1^N, x_2), \tag{3.6}$$

$$\pi_1^N \approx -1/2 \left[\frac{\varepsilon_1^2 u_{11} + \delta_1^2 u_{22} - 2\delta_1 \varepsilon_1 u_{12}}{u_1} \right], \text{ or equivalently} \tag{3.7}$$

$$\pi_1^N \approx \varepsilon_1^2/2 \left[RA_1 + \frac{\delta_1^2 u_2}{\varepsilon_1^2 u_1} RA_2 - 2\frac{\delta_1}{\varepsilon_1} CA_1 \right]. \tag{3.8}$$

The risk premium locally depends on the sum of the risk aversion coefficient of the reference good, that of the other good weighted by the marginal rate of substitution, and negatively on the correlation aversion. A positive CA means that a lottery allowing to substitute to some extent one good with another, reduces the total riskiness. [10]

In the unidimensional case the Pratt absolute coefficient measures the relative change in the shape of the utility function, when income changes. When we deal with two variables, the relative variation of the total differential of u_i, due to a change dx_i, can be computed:

$$-\frac{du_i(x_i,x_j)/dx_i}{u_i(x_i,x_j)} \approx -\frac{u_{ii}}{u_i} - \frac{u_{ij}}{u_i}\frac{dx_j}{dx_i}. \tag{3.9}$$

If this approximation is computed at a fixed level of utility, then $\frac{dx_j}{dx_i} = -\frac{u_i}{u_j}$ and we get:

$$\frac{du_i(x_i,x_j)/dx_i}{u_i(x_i,x_j)} \approx -\frac{u_{ii}}{u_i} + \frac{u_{ij}}{u_j}. \tag{3.10}$$

Hence, this means that the change in the marginal utility measured by the usual Pratt coefficient is compensated by taking by the other variable's indirect effect.

We then define the Compensated Risk Aversion (CRA) coefficients:

$$CRA_1 = -\frac{u_{11}}{u_1} + \frac{u_{12}}{u_2} = RA_1 + MRS_{21} \times CA_1 \tag{3.11}$$

$$CRA_2 = -\frac{u_{22}}{u_2} + \frac{u_{12}}{u_1} = RA_2 + MRS_{12} \times CA_2 \tag{3.12}$$

The previous results prove the following

Proposition 3.1. *Given the risks $(\tilde{\varepsilon}, \tilde{\delta})$, the risk premium can be decomposed as follows:*

$$\pi_1^N \approx \frac{1}{2}\varepsilon_1^2\left(CRA_1 + \frac{\delta_1^2 u_2}{\varepsilon_1^2 u_1}CRA_2\right) - \frac{1}{2}\varepsilon_1^2\frac{u_{12}}{u_2}\left(1 - \frac{\delta_1 u_2}{\varepsilon_1 u_1}\right)^2. \tag{3.13}$$

The interest in this decomposition of the risk premium is supported by the following property of CRA.

Proposition 3.2. *The measure of risk aversion CRA_i is invariant to any monotonic transformation of the utility function.*

[10] Another way to restate equation (3.7) can be used to show that the impact of the risk depends both on the absolute correlation aversion and on the Arrow-Pratt indices:

$$\pi_1^N \approx -\frac{\varepsilon_1^2}{2}\left(\frac{u_{11}}{u_1} - \frac{\delta_1}{\varepsilon_1}\frac{u_{12}}{u_1}\right) - \frac{\delta_1^2}{2}\frac{u_2}{u_1}\left(\frac{u_{22}}{u_2} - \frac{\varepsilon_1}{\delta_1}\frac{u_{12}}{u_2}\right).$$

Proof. By considering the transformation $v = f \circ u$, from the definition of absolute risk and correlation aversion we get:

$$-\frac{v_{ij}}{v_j} = -\frac{f_{uu}}{f_u} u_i - \frac{u_{ij}}{u_j} \text{ and } -\frac{v_{ii}}{v_i} = -\frac{f_{uu}}{f_u} u_i - \frac{u_{ii}}{u_i}$$

and, hence,

$$\frac{v_{ij}}{v_j} - \frac{v_{ii}}{v_i} = \frac{u_{ij}}{u_j} - \frac{u_{ii}}{u_i}. \tag{3.14}$$

□

The CRA is invariant to concave transformations of the utility function, which implies that the increase of the absolute risk aversion coefficient on a single variable is perfectly balanced by an increase of correlation aversion $\frac{u_{ij}}{u_i}$.

It is then possible to separate the risk premium in two terms: $\tilde{\pi}_1^N + \rho$. The first term $\tilde{\pi}_1^N = \frac{1}{2}\varepsilon_1^2 \left(CRA_1 + \frac{\delta_1^2 u_2}{\varepsilon_1^2 u_1} CRA_2 \right)$ is invariant to any transformation of the utility function, while the second term $\rho = -\frac{1}{2}\varepsilon_1^2 \frac{u_{12}}{u_2} \left(1 - \frac{\delta_1 u_2}{\varepsilon_1 u_1} \right)^2$ is sensitive to different cardinalizations of the utility function. Notice that when $\varepsilon_1/\delta_1 = u_2/u_1$, then $\pi_1^N = \tilde{\pi}_1^N$: when the risk follows exactly the direction of the marginal rate of substitution, this invariance property implies that the risk premium does not change, even after a concave transformation of the utility function.

As the risk direction moves away from the MRS, a more concave utility function implies a stronger impact on the risk premium. It is then important to disentangle the cardinal and the ordinal components for determining individual preferences under risk.

The cross derivative of u is not itself a marker of the presence of multivariate risk aversion, but has a role in measuring the intensity of risk aversion and to determine the sign of ρ, as shown in the following example.

Example 3.1. *Consider an agent with the Cobb-Douglas utility function* $u(x_1, x_2) = (x_1 x_2)^{1/2}$ *. Suppose now that she faces a lottery with negatively correlated risks along the directions* $\varepsilon_1 = 1$ *and* $\delta_1 = 0.3$*, starting from the initial position* $\bar{x}_1 = 100$ *and* $\bar{x}_2 = 5$*. The EU results* $\frac{1}{2}\left[u(100-1,\ 5+0.3) + u(100+1,\ 5-0.3)\right]$*. Using (3.7), the risk premium is* $\pi_1^N = 0.1225$*. From (3.13) we find that the "ordinal" component is* $\tilde{\pi}_1^N = 0.185$ *while the "cardinal" component is* $\rho = -0.0625$*.*

We can consider a more risk averse agent, with ρ equal to zero, because of the null cross derivative. We have, for instance, $v(x_1, x_2) = \ln(x_1x_2)^{1/2}$, and the risk premium becomes $\pi_1^N = 0.185$, that coincides with the ordinal component.

Otherwise, we can also have a more risk averse agent, that is also correlation averse. In this case, we consider $v(x_1, x_2) = -2(x_1x_2)^{-1/4}$, with risk premium $\pi_1^N = 0.21625$, decomposable in $\tilde{\pi}_1^N = 0.185$ and $\rho = 0.03125$.

Therefore, if the risks are negatively correlated, and the agent is correlation averse, she is more willing to pay for the risk elimination, then an agent indifferent or prone to correlation.

The following corollary links the previous results with gereralized submodularity, defined in Section 2 as the negative cross derivative of $\Psi(a, b)$.

Corollary 3.1. $\Psi_{ab} < 0$ *is equivalent to* $\pi_1^N < \pi_1^P$.

Proof. Equation (2.3) determines the sign of the cross derivatives of Ψ on the basis of the derivatives of u, for a risk along general directions. Then, the result immediately follows from equations (2.2), (3.6), (**??**) and Proposition 2.1. □

Therefore, we can conclude that, as the rate of substitution of the goods shows how much one individual is willing to exchange one good with another under certainty and the cross derivative of u, given orthogonal variations of the goods, indicates preferences for joining or spread gains and losses across states of the world, the cross derivative of Ψ describe preferences for changes in the quantity of the goods across states of the world which are not orthogonal, but arise along general directions.

4 Concluding Remarks

In this paper we have studied the properties of the directional derivatives of an utility function that associate to directional risks an increase of expected utility. We have investigated the consequences of directional changes in risk on the risk premium and we have proposed a decomposition of the risk premium in order to disentangle the degree of substituability among the attributes from the degree of risk and correlation aversion. Our results on multidimensional risk assessment can be of interest for inequality measurement, whose formal analogy with the risk setting is well known in the economic literature (see Atkinson 1970 for the unidimensional case or Gajdos and Weymark 2012 for a survey).

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