Contests with an uncertain number of prizes

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Abstract

We study multiple-prize contests where the set of prizes to be awarded is a random variable. We determine the unique symmetric Nash equilibrium of the contest game. We analyze the equilibrium outcome from the perspective of a contest designer aiming at maximizing the aggregate contest expenditure. We show that the aggregate contest expenditure is decreasing in the risk on the number of prizes (in the sense of second-order stochastic dominance) and is increasing in the number of contestants. Accordingly, a contest designer aiming at maximizing the aggregate contest expenditure should always reveal the number of prizes to be awarded and open the contest game to all potential contestants.

1 Introduction.

Sisak (2009) surveyed the recent literature on multiple-prize contests.[1] She outlines that this setting can be relevant in many situations, taking examples from rent-seeking activities, patents and R&D races, licences, labour markets, sports and so on. Sisak (2009) classifies the literature along two main dimensions, based on the choice of the contest success function (Tullock versus fully discriminating contest success function) and on the adoption of single versus multiple effort (the contestants exert an overall effort for all prizes or can allocate it more specifically to a sub-group of prizes). The central finding is that with risk-neutral and symmetric contestants, a contest designer aiming at maximizing the aggregate effort should always prefer to allocate a single prize rather than splitting it in several smaller prizes. However, dividing the prize can be optimal in situations with risk aversion and asymmetric players.

Surprizingly, the case of a contest with an uncertain number of prizes has never been investigated, although this is a natural and immediate extension of

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[1]The very first contributions on multiple-prize contests are Glazer and Hassin (1988) and Berry (1993).

the literature just surveyed. The pupose of the present paper is to provide a first attempt to fill the gap. We consider a Tullock contest success function with risk-neutral and symmetric players, assuming that the number of (identical) prizes to be awarded is a random variable, with common knowledge probability distribution. We characterize the Nash equilibrium of the corresponding contest game and identify some of its properties. In particular, we are interested in verifying whether the normative prescription from the literature, that a single prize contest maximizes the aggregate effort of the contestants, remains valid in our setting. We actually show that a multiple-prize contest with an uncertain number of prizes is dominated by the (certainty-equivalent) multiple-prize contest allocating the same expected number of prizes for sure. By transitity, this confirms that with risk-neutral and symmetric players, a contest designer aiming at maximizing the aggregate effort should always prefer to allocate a single prize for sure.

This paper can also be related to Münster (2006), Lim and Matros (2009), Myerson and Wärneryd (2006), and Kahana and Klunover (2015), which extend the contest literature to situations where the number of contestants is uncertain. They show that the (*ex-ante*) aggregate effort in a contest with population uncertainty is smaller than its counterpart in a contest with population certainty and the same expected number of contestants. Clearly, our paper gives the analog finding for contests with prize uncertainty.

The rest of the paper is organized as follows. Section 2 sets out the model. Section 3 analyses the Nash equilibrium under certainty. Section 4 characterizes Nash equilibrium under uncertainty. Section 5 compares the two situations. Section 6 deals with some relevant comparative statics results. Most proofs are given in the appendix.

2 The model.

We consider n (risk neutral) players competing in a nested contest awarding k prizes, with $1 \leq k < n$. The value of each prize is denoted $V(k)$.[2] Each contestant i simultaneously expends effort x_i to win one prize and no more. The vector of all efforts is denoted $\overline{x}$. The prizes are awarded in k rounds, according to the following iterative process (Clark and Riis, 1996). Let $N(r)$ denote the set of players still remaining in the contest at round r[3]. The conditional probability that a player i in $N(r)$ wins the prize of the r-th round is equal to

$$p_i^r(\overline{x}) = \frac{f(x_i)}{\sum_{j\in N(r)} f(x_j)},$$

where $f(0) = 0$ and $f(.)$ is a strictly increasing and concave function. *Ex post*, if player i really wins it, the set of the remaining players then evolves according

[2]For technical reason, it will be convenient to assume that $V(k)$ is defined and twice differentiable for all $k \in [1, n]$.

[3]Clearly, $N(1) = \{1, ..., n\}$.

to

$$N(r+1) = N(r) - \{i\}.$$

The process is repeated until all prizes are allocated. Let $P_i(\overline{x}; k)$ denote the (*ex ante*) probability that player i wins one prize at the end of the process.

The originality of our paper is that we assume that the players ignore the exact number of prizes to be awarded and only know that it is distributed between 1 and K, according to a probability distribution $\pi(k)$. In this setting, each player i expects to win one prize with the probability

$$E[P_i(\overline{x}; k)] = \sum_{k=1}^{K} \pi(k) P_i(\overline{x}; k).$$

3 Equilibrium outcomes.

We characterize here the Nash equilibrium of the contest game. We consider first the case where the players observe the number of prizes k to be allocated before they choose their level of effort. We then deal with the case where the players only know that the number of prizes k is distributed between 1 and K, according the probability distribution $\pi(k)$.

The case where the number of prizes is known with certainty is solved in Clark and Riis (1996). Observing k, each player i chooses x_i to maximize

$$P_i(\overline{x}; k) V(k) - x_i.$$

An equilibrium of the contest game satisfies the following first-order condition

$$\frac{\partial}{\partial x_i} P_i(\overline{x}; k) V(k) - 1 = 0, \text{ for all } i.$$

Clark and Riis (1996) have shown that the contest game admits a unique symmetric equilibrium. Thus, letting $x_i = x$ for all i, we can calculate that[4]

$$\frac{\partial}{\partial x_i} P_i(\overline{x}; k) = \frac{n-k}{n} \left(\sum_{r=1}^{n} \frac{1}{r} - \sum_{r=1}^{n-k} \frac{1}{r} \right) \frac{f'(x)}{f(x)}. \tag{1}$$

It follows that the game admits a unique Nash equilibrium such that all players exert an effort satisfying

$$\frac{f(x)}{f'(x)} = A(k, n), \tag{2}$$

where[5]

$$A(k, n) = V(k) \frac{n-k}{n} \left(\sum_{r=1}^{n} \frac{1}{r} - \sum_{r=1}^{n-k} \frac{1}{r} \right). \tag{3}$$

[4]For reasons that will become clear below, the expression here slighly differs from that in Clark and Riis (1996). The proof is given in appendix.

[5]Clark and Riis (1996) use $A(k, n) = V(k) \left(k - \sum_{r=1}^{k-1} (k-r)/(n-r) \right)/n$, which is equivalent.

Consider now the case where the players choose their effort under uncertainty. Knowing that k is is distributed between 1 and K, according to a probability distribution $\pi(k)$, each player maximizes his expected utility according to his own effort

$$\sum_{k=1}^{K} \pi(k) P_i(\overline{x}; k) V(k) - x_i.$$

An equilibrium of the contest game satisfies the following first-order condition

$$\sum_{k=1}^{K} \pi(k) \frac{\partial}{\partial x_i} P_i(\overline{x}; k) V(k) - 1 = 0, \text{ for all } i.$$

Using (1), this game admits a symmetric Nash equilibrium where all players exert an effort satisfying

$$\frac{f(x)}{f'(x)} = \sum_{k=1}^{K} \pi(k) A(k, n). \tag{4}$$

4 Comparative statics.

We derive here some comparative statics of the equilibrium outcome under uncertainty. We implicitly adopt the point of view of a contest designer aiming at maximizing the aggregate contest expenditure. By assumption, the parameters that the contest designer may be able to manipulate are the value of the individual prizes, the probability distribution of the number of prizes and the number of participants.

4.1 Value of the prizes.

We consider here the choice of each individual prize's value. Unambiguously, for all k, a larger value $V(k)$ increases $A(k, n)$ and the right-hand side of (4). Since $f(x)/f'(x)$ is strictly increasing for all x (by concavity of $f(x)$), it is immediate that a larger value of the individual prizes induces a larger individual effort x and aggregate effort $X = nx$.

4.2 Number of prizes.

We consider here the choice of the distribution of the number of prizes. In particular, we analyse the effect of a greater uncertainty on the number of prizes. Let $\underline{\pi}(k)$ and $\overline{\pi}(k)$ be any pair of probability distributions, such that $\overline{\pi}(k)$ is obtained from $\underline{\pi}(k)$ by adding some uncorrelated noise (i.e., mean preserving spread). Denote by $\underline{x}$ and $\overline{x}$ the equilibrium individual efforts corresponding to $\underline{\pi}(k)$ and $\overline{\pi}(k)$ respectively. Below, we study the conditions such that for $\underline{x} > \overline{x}$.

From our previous analysis, we know that $\underline{x}$ and $\overline{x}$ satisfy

$$\frac{f(\underline{x})}{f'(\underline{x})} = \sum_{k=1}^{K} \underline{\pi}(k) A(k,n) \text{ and } \frac{f(\overline{x})}{f'(\overline{x})} = \sum_{k=1}^{K} \overline{\pi}(k) A(k,n).$$

As $f(x)/f'(x)$ is strictly increasing for all x, the condition for $\underline{x} > \overline{x}$ writes

$$\sum_{k=1}^{K} \underline{\pi}(k) A(k,n) > \sum_{k=1}^{K} \overline{\pi}(k) A(k,n). \tag{5}$$

According to Rothschild and Stiglitz (1970), given that $\underline{\pi}(k)$ is less risky than $\overline{\pi}(k)$, this condition will hold true if $A(k,n)$ is concave in k.

Our strategy for studying the "curvature" of $A(k,n)$ is the following[6]. For all $\chi \in [1,n]$, we first construct a twice differentiable function $F(\chi)$, such that $F(k) = A(k,n)$ for all (integers) $k < n$. We then deduce sufficient conditions for the concavity of $F(\chi)$ from its second-order derivative.

For all $\chi \in [1,n]$, let us define

$$F(\chi) = V(\chi) \frac{n-\chi}{n} (\psi(n+1) - \psi(n-\chi+1)),$$

where ψ is the psi (or digamma) function (Abramowitz and Stegun, 1964)

$$\psi(\chi) = -\gamma + \sum_{j=0}^{\infty} \frac{\chi - 1}{(j+1)(j+\chi)}, \tag{6}$$

with γ the Euler constant. Knowing that

$$\psi(\eta+1) = -\gamma + \sum_{j=1}^{\eta} \frac{1}{j},$$

where η denotes any positive integer (Abramowitz and Stegun, 1964), we can show that

$$F(k) = V(k) \frac{n-k}{n} \left(\sum_{r=1}^{n} \frac{1}{r} - \sum_{r=1}^{n-k} \frac{1}{r} \right) = A(k,n)$$

for all (integers) $k < n$. Moreover, knowing that the first- and second-order derivatives of ψ are (Abramowitz and Stegun, 1964)

$$\psi'(\chi) = \sum_{j=0}^{\infty} \frac{1}{(j+\chi)^2} \text{ and } \psi''(z) = -\sum_{j=0}^{\infty} \frac{2}{(j+\chi)^3}, \tag{7}$$

[6] Remark that (3) is a discrete (non differentiable) function of k because of the term $\sum_{r=1}^{n-k} 1/r$.

we know that $F(\chi)$ is twice differentiable and we can calculate that

$$\begin{aligned}F''(\chi) =& \frac{V''(\chi)}{n}\sum_{j=0}^{\infty}\frac{(n-\chi)\chi}{(j+n+1)(j+n-\chi+1)} \\ &-2\frac{V'(\chi)}{n}\sum_{j=0}^{\infty}\frac{(j+n-\chi+1)\chi-(j+n+1)(n-\chi)}{(j+n+1)(j+n-\chi+1)^2} \\ &-2\frac{V(\chi)}{n}\sum_{j=0}^{\infty}\frac{j+1}{(j+n-\chi+1)^3}.\end{aligned}$$

Recall that if $F''(\chi)<0$, then $F(\chi)$ is concave, condition (5) is true and $\underline{x} > \overline{x}$. Hence, the expression of $F(\chi)$ implicitly defines a class of contest games, as represented by the schedule $V(k)$ of the prizes values, such that the contestants will exert less effort when there is greater uncertainty on the number of prizes.

We will not try to fully characterize the set of functions $V(k)$ such that $F''(\chi)<0$. Instead, we first note that it is stable to positive combinations of its elements. Indeed, it should be clear that if $V_1(\chi)$ and $V_2(\chi)$ are such that $F''(\chi)<0$, then $V(\chi)=\lambda V_1(\chi)+\mu V_2(\chi)$, with $\lambda>0$ and $\mu>0$, also implies that $F''(\chi)<0$.Moreover, we show that all functions $V(\chi)$ of the form V/χ^σ, where $0\le\sigma\le 1$, belong to this set.[7] Indeed, we then have

$$F''(\chi) = -\frac{V}{n\chi^{1+\sigma}}\left[\begin{array}{c}\sigma(1-\sigma)\sum_{j=0}^{\infty}\frac{n-\chi}{(j+n+1)(j+n-\chi+1)} \\ +\sum_{j=0}^{\infty}\frac{(j+1)\chi^2}{(j+n+1)(j+n-\chi+1)^3} \\ +2(1-\sigma)\chi\sum_{j=0}^{\infty}\frac{j+1}{(j+n-\chi+1)^3}\end{array}\right]$$

which is negative under the condition that $0\le\sigma\le 1$.

This result implies that under standard and reasonable assumptions, a contest designer aiming at maximizing the aggregate contest expenditure should always reveal the number of prizes to be awarded before the contest takes place.

4.3 Number of contestants

Consider finally the effect of a larger population of contestants. To ease the presentation, assume that $f(x)=ax^r$, with $a>0$ and $0<r\le 1$.[8] Condition (4) then writes as

$$\frac{x}{r}=\sum_{k=1}^{K}\pi(k)A(k,n),$$

[7] This includes as particular cases $V(k)=V$ and $V(k)=V/k$, which are standard in the literature.

[8] This specification is standard in the literature. Moreover, we generalize our result in the appendix to the class of all functions $f(x)$ such that the elasticity of $g(x)=f(x)/f'(x)$ is larger than 1.

implying that the aggregate contest expenditure is

$$X = nr\sum_{k=1}^{K} \pi\left(k\right) A\left(k,n\right).$$

Using

$$A\left(k,n\right) = \frac{V\left(k\right)}{n}\left(k - \sum_{j=0}^{k-1}\frac{k-j}{n-j}\right),$$

we can calculate

$$X = r\sum_{k=1}^{K} \pi\left(k\right) V\left(k\right)\left(k - \sum_{j=0}^{k-1}\frac{k-j}{n-j}\right).$$

From this, it is clear that the aggregate contest expenditure is an increasing and concave function of n. Moreover, it converges to $r\sum_{k=1}^{K}\pi\left(k\right)kV\left(k\right)$ when n tends to infinity. In the special case where $r = 1$, we obtain that the (expected) rent is fully dissipated when there is free entry to the contest. This is a standard result in the literature.

5 Numerical illustrations

THIS PART IS IN PROGRESS

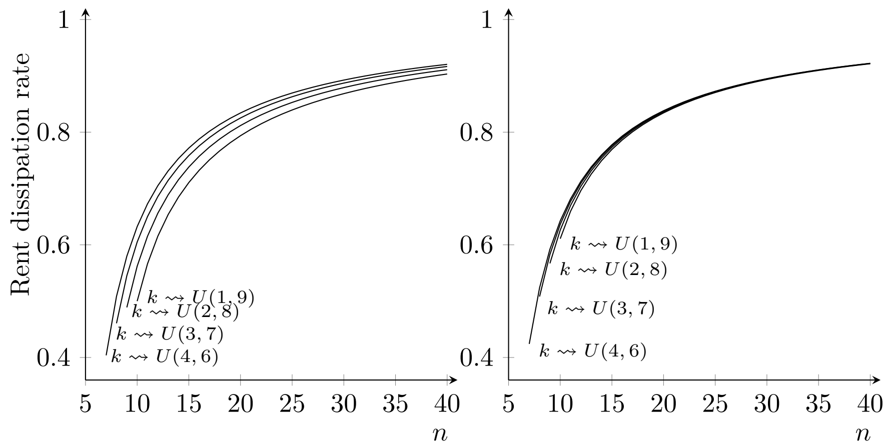
In this section, we propose some examples to highlight our results. We consider that $f(x) = x$ and we are particularly interested in looking at the rate of rent dissipation, defined as total effort divided by the total value distributed in the contest, *i.e.* total spending per unit of prize.

In the next subsection, we compare rent dissipation rate for different uniform distributions by fixing the expected number of prizes and varying the risk.

5.1 The uniform distribution

Figures 1(a) and (b) represent rent dissipation rate as a function of the number of players when the number of prizes follows a uniform distribution. Several plots are obtained using distributions with the same expectancy, but differently risky[9]. We verify that given the number of players and the expected number of prizes, total effort is higher for contests where the risk on the number of prizes is lower.

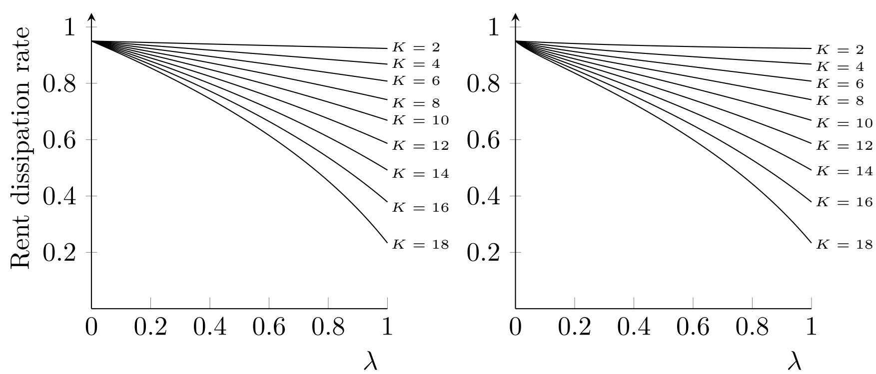
In passing, we also verify that the rent dissipation rate converges to 1 when the number n of players increases.



(a) $V(k) = 1$, $\mathbb{E}[k] = 5$

(b) $V(k) = 1/k$, $\mathbb{E}[k] = 5$

Figure 1: Rent dissipation rate when $n = 40$, $\mathbb{E}[k] = 5$



(a) $V(k) = 1/k$

(b) $V(k) = 1$

Figure 2: Rent dissipation rate when $n = 20$

5.2 The binomial distribution

In this subsection we examine the situation where the number of prizes k is distributed according to a binomial distribution. As we consider in our model that there is at least one prize to win, we suppose that the number of prizes awarded is equal to one plus a variable following a binomial distribution of parameters $(K-1, \lambda)$. We focus on the impact of λ on the rent dissipation rate.

In the case $V(k) = V/k$, the result is immediate : indeed, Clark and Riis

[9] For instance, the uniform distribution on the interval $[\![3,7]\!]$ stochastically dominates at the second order the uniform distribution on the interval $[\![1,9]\!]$, but the expected number of prizes is identical and equal to 5.

(1996) have shown that in a certain contest, total effort is decreasing with the number of prizes. Therefore, as $k = 1$ maximizes equilibrium spending in a contest under certainty, and as effort is strictly decreasing with the number of prizes, then $\lambda = 0$ maximizes total effort. Figure 2(a) illustrates our purpose : total effort is strictly decreasing with λ and the maximum effort is therefore obtained for $\lambda = 0$.

In the case $V(k) = V$, we observe a similar result : the maximal rent dissipation rate is obtained for $\lambda = 0$, meaning $K = 1$ (figure 2(b)).

5.3 First order Stochastic Dominance ($V(k) = V/k$)

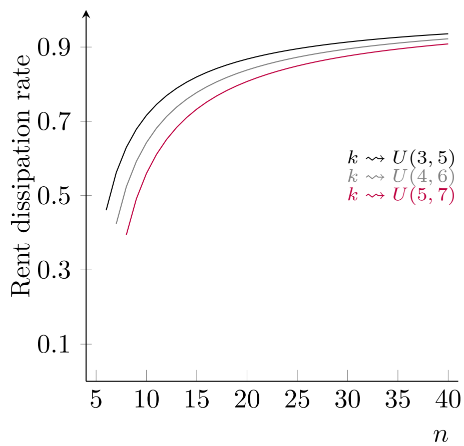


Figure 3: Rent dissipation rate when $n = 40$

Figure 3 represents effort produced by all the players when the number of prizes follows uniform distributions with a similar structure, but differ from their expected number of prizes.

For instance, the uniform distribution on the interval $[\![5, 7]\!]$ stochastically dominates at the first order the uniform distribution on the interval $[\![4, 6]\!]$. We obtain that in the case V/k, total effort is higher for contests where the expected number of prizes is lower : this comes from the fact that total effort is decreasing with the number of prizes (Clark and Riis, 1996). Note that this relation is true in the situation $V(k) = V/k$ but is in general not checked for other value functions.

6 Conclusion

THIS PART IS IN PROGRESS.

7 Appendix

Proof of equation (1)

Clark and Riis (1996) have shown that the contest game admits a unique symmetric equilibrium. From this, considering player i's point of view, we assume that $x_j = x$, for all $j \neq i$. Then, for all r, we can write

$$p_i^r\left(\overline{x}\right) = \frac{f(x_i)}{f(x_i) + (n-r)\, f(x)}.$$

If $k = 1$, player i chooses x_i to maximize

$$\frac{f(x_i)}{f(x_i) + (n-1)\, f(x)} V\left(1\right) - x_i.$$

Therefore, the first-order condition for x_i to maximize player i's payoff is

$$\frac{(n-1)\, f(x) f'(x_i)}{\left(f(x_i) + (n-1)\, f(x)\right)^2} V\left(1\right) - 1 = 0.$$

If $x_i = x$, this simplifies to

$$V\left(1\right) \frac{n-1}{n^2} \frac{f'(x)}{f(x)} - 1 = 0.$$

If $k > 1$, player i chooses x_i to maximize

$$P_i\left(\overline{x}; k\right) V\left(k\right) - x_i,$$

where

$$P_i\left(\overline{x}; k\right) = \frac{f(x_i)}{f(x_i) + (n-1)\, f(x)} + \sum_{r=2}^{k} \left[\prod_{s=1}^{r-1} \left(1 - \frac{f(x_i)}{f(x_i) + (n-s)\, f(x)}\right) \right] \frac{f(x_i)}{f(x_i) + (n-r)\, f(x)}$$

An equivalent expression is

$$P_i\left(\overline{x}; k\right) = \sum_{r=1}^{k} \frac{f(x_i)}{(n-r)\, f(x)} \left(\prod_{s=1}^{r} \frac{(n-s)\, f(x)}{f(x_i) + (n-s)\, f(x)} \right).$$

By differentiation, we can get

$$\frac{\partial}{\partial x_i} P_i\left(\overline{x}; k\right) = \sum_{r=1}^{k} \left[\begin{array}{c} \frac{f'(x_i)}{(n-r) f(x)} \left(\prod_{s=1}^{r} \frac{(n-s) f(x)}{f(x_i) + (n-s) f(x)} \right) \\ + \frac{f(x_i)}{(n-r) f(x)} \left(- \sum_{s=1}^{r} \frac{(n-s) f(x) f'(x_i)}{(f(x_i) + (n-s) f(x))^2} \prod_{t \neq s} \frac{(n-t) f(x)}{f(x_i) + (n-t) f(x)} \right) \end{array} \right].$$

If $x_i = x$, this simplifies to

$$\frac{\partial}{\partial x_i} P_i\left(\overline{x};k\right) = \frac{1}{n}\sum_{r=1}^{k}\left(1-\sum_{s=1}^{r}\frac{1}{n+1-s}\right)\frac{f'(x)}{f(x)}.$$

We can show by induction that

$$\sum_{r=1}^{k}\left(1-\sum_{s=1}^{r}\frac{1}{n-s+1}\right) = (n-k)\left(\sum_{r=1}^{n}\frac{1}{r}-\sum_{r=1}^{n-k}\frac{1}{r}\right),$$

which implies that

$$\frac{\partial}{\partial x_i} P_i\left(\overline{x};k\right) = \frac{n-k}{n}\left(\sum_{r=1}^{n}\frac{1}{r}-\sum_{r=1}^{n-k}\frac{1}{r}\right)\frac{f'(x)}{f(x)}.$$

Comparative statics with respect to n

To simplify the notations in this proof, let us denote

$$g\left(x\right) = f\left(x\right)/f'\left(x\right)$$

and

$$B\left(n\right) = \sum_{k=1}^{K}\pi\left(k\right)A\left(k,n\right).$$

In equilibrium, from condition (4), we have

$$g\left(x\right) = B\left(n\right).$$

Using the implicit function theorem, this implies that

$$\frac{\mathrm{d}x}{\mathrm{d}n} = \frac{B'\left(n\right)}{g'\left(x\right)}$$

and

$$\frac{\mathrm{d}X}{\mathrm{d}n} = x + n\frac{B'\left(n\right)}{g'\left(x\right)}.$$

From this, we can show that

$$\frac{\mathrm{d}X}{\mathrm{d}n} > 0 \Leftrightarrow \frac{g'\left(x\right)x}{g\left(x\right)} > -\frac{nB'\left(n\right)}{B\left(n\right)}.$$

Now, using

$$nB\left(n\right) = n\sum_{k=1}^{K}\pi\left(k\right)A\left(k,n\right)$$

and

$$A\left(k,n\right)=\frac{V\left(k\right)}{n}\left(k-\sum_{j=1}^{k-1}\frac{k-j}{n-j}\right),$$

we calculate

$$nB\left(n\right)=\sum_{k=1}^{K}\pi\left(k\right)V\left(k\right)\left(k-\sum_{j=1}^{k-1}\frac{k-j}{n-j}\right).$$

Since $nB\left(n\right)$ is clearly increasing in n, we can write

$$B\left(n\right)+nB'\left(n\right)>0$$

and

$$-\frac{nB'\left(n\right)}{B\left(n\right)}<1.$$

This finally implies that a sufficient condition for $dX/dn>0$ is

$$\frac{g'\left(x\right)x}{g\left(x\right)}\geq 1>-\frac{nB'\left(n\right)}{B\left(n\right)}.$$

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