A Two Country Model of Public Infrastructure Capital: Trade Patterns and Trade Gains in the Long Run *

Akihiko Yanase †
Graduate School of Economics, Nagoya University
Makoto Tawada ‡
Faculty of Economics, Aichi Gakuin University

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Abstract

This paper develops a two-country dynamic trade model with a public intermediate good whose stock has a positive effect on private sectors' productivity. Under the assumptions of one primary factor (labor), the national government that determines the level of the public good so as to maximize the discounted sum of a representative household's utility, and the stock of the public intermediate good as a kind of "unpaid factors," this paper examines the economy's trade pattern and the long-run effects of trade. It is shown that in the case of Lindahl pricing for the provision of the public intermediate good, in which each country takes the world prices as given, the country with a smaller labor endowment, a lower depreciation rate of the public-good stock, and/or a lower rate of time preference will become an exporter of a good which is more dependent on the stock of the public intermediate good, and this country will unambiguously gains from trade. We also discuss the case of Nash provision of the public intermediate good, in which each country noncooperatively determines the public good recognizing their effect on the terms of trade.

Key Words: Public infrastructure Capital; Two-country Trade; Trade pattern; Gains/losses from trade
JEL classification: F11; H41

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†Graduate School of Economics, Nagoya University. Furo-cho, Chikusa-ku, Nagoya 464-8601, JAPAN. E-mail: yanase@soec.nagoya-u.ac.jp
‡mtawada2@dpc.agu.ac.jp

1 Introduction

In economic activities, including international trade, the role of public intermediate goods becomes more important as seen in food security, national defense, promotion of innovation activities, transportation and communication systems, and so on. Although there have been a number of studies examining trade models with public intermediate goods (Manning and McMillan, 1979; Tawada and Okamoto, 1983; Tawada and Abe, 1984; Ishizawa, 1988; Abe, 1990; Altenburg, 1992; Suga and Tawada, 2007), all of their studies are unfortunately confined in a static framework. In reality, many public intermediate goods have a characteristic of durable or capital goods; research and development activities, national defense, and transportation and communication infrastructures are typical examples.

An exception is the study of McMillan (1978), which considers stock effect of a public intermediate good in an open economy. He considers a three-sector (two private goods and a public intermediate good), one-factor (labor) small open economy with optimal supply of the public intermediate good. It is shown that the stock of public intermediate good determines the slope of the production possibility frontier and thus determines the pattern of international trade. McMillan's model is recently re-examined by Yanase and Tawada (2012), who show the possibility of multiple steady states and history-dependent dynamic paths. Moreover, they discuss whether trade is gainful or not in McMillan's model.

In both McMillan (1978) and Yanase and Tawada (2012), the public intermediate good is assumed to have an impact similar to the "creation of atmosphere" type externality classified by Meade (1952). That is, in their models, the technology of each private sector exhibits constant returns to scale in primary factors of production only. In one-factor model, this assumption implies that for a given stock of the public intermediate good, the production possibility frontier becomes linear, as in the standard Ricardian model, and thus the economy hardly diversifies production.

There is another class of public intermediate goods, which can be interpreted as "unpaid factors of production," according to Meade's terminology (1952). If the public good is of this type, the production function of each private sector is characterized by constant returns to scale in all inputs, including the public intermediate good. Following Tawada(1980) and Tawada and Okamoto(1983) we adopt the alternative term "semi-public intermediate good" to describe this kind of goods.[1] In the case of unpaid factor type, congestion arises within an industry though not among industries. For example, public transportation system like highways may be included in this type. The agriculture industry uses the highway system in a certain season or region, while the manufacturing industry uses it in a different time or region. This highway system can be considered as a public infrastructure of unpaid factor type. Similarly, the communication system may be supposed to be of this type. This paper focuses on this kind of public intermediate goods and presents a dynamic trade model in which the stock of a public good has a positive effect on private production. Following McMillan (1978) and Yanase and Tawada (2012), we consider an open economy in which two tradable private consumption goods and one non-tradable public intermediate good are produced by one primary input, say labor. However, different from McMillan (1978) and Yanase and Tawada (2012), we suppose that the private goods are produced under constant returns to scale with respect to the stock of the public intermediate good as well as labor. Thus, the production possibility frontier becomes strictly concave to the origin even in the case of one primary factor. Therefore, incomplete specialization would carry over in general even after trade.

[1] The public intermediate good that has an impact similar to the "creation of atmosphere" type externality is also referred to as "pure" public good.

We begin with a dynamic two-country model in which the national government, taking the prices of tradables as given, determines the optimal levels of the public intermediate good by using the Lindahl pricing rule. Then, the dynamic general equilibrium of the world economy and the steady state equilibrium are characterized. It is shown that there exists a unique and saddle-point stable steady state if the economic fundamentals of the two countries are slightly different. We then investigate the trade pattern of the economy. It is shown that if the economy is initially at the autarkic steady state, after opening trade, a country with a smaller (larger) labor endowment tends to be an exporter of a good that is more (less) intensive to the stock of the public intermediate good. We also discuss whether each country gains or loses from trade by comparing the steady state welfare level under free trade with the one at autarky. It is shown that, in comparison with the autarkic steady state, free trade raises (reduces) the steady state stock of the public intermediate good and thereby expands (curtails) the long-run production possibility in a country exporting a good that is more (less) dependent on the public intermediate good. An interesting result is that a smaller country unambiguously gains from trade in the long-run whereas a larger country may lose from trade.

We also consider an alternative situation where the governments strategically determine their provision of public intermediate goods, taking the effect of each country's choice on the world market into consideration. If each country recognizes the effect on the world price, i.e., the terms of trade effect, the level of the public intermediate good in each country can be controlled in pursuit of national interests. That is, in comparison with the non-strategic case, the country exporting (importing) the the good that is more dependent on the public-good stock would under-accumulate (over-accumulate) the public intermediate good.

2 Model

We consider a world economy consisting of two countries, home and foreign, in which two private and one public production sectors and one primary factor exist. The primary factor is supposed to be labor. The public sector produces a public intermediate good with non-increasing returns to scale technology with respect to labor. The public intermediate good can be accumulated and its accumulated stock serves in production in the private sectors as a positive external effect without congestion between sectors. The two private sectors are supposed to be sectors 1 and 2 where goods 1 and 2 are produced, respectively, under constant returns to scale technology with respect to labor and the stock of the public intermediate good. Total labor endowment is assumed to be given and constant over time.

2.1 Production side

Let us focus on the home country. The foreign country, whose variables are denoted with an asterisk (*), has a similar economic structure.

The production function of each private sector is assumed to take the following form:

$$Y_i = R^{\alpha_i} L_i^{1-\alpha_i}, \quad 0 \leq \alpha_i < 1, \quad i = 1, 2, \tag{1}$$

where Y_i is the output of good i, R is the stock of the public intermediate good, and L_i is the labor input in sector i. It is clear that the labor productivity in each private sector is exhibited by $\partial Y_i / \partial L_i = (1 - \alpha_i) R^{\alpha_i} L_i^{-\alpha_i}$ and is dependent on the stock of the public intermediate good R.

In the following analysis, we make the following assumption regarding the impact of the public intermediate good to industries:

Assumption 1 $\alpha_1 > \alpha_2$.

Because α_i is the production elasticity of the public intermediate good stock in sector i, i.e. $\alpha_i = (\partial Y_i/\partial R) \cdot (R/Y_i)$, Assumption 1 can be interpreted that *sector 1 is more dependent on the stock of the public intermediate good than sector 2.* At the same time, Assumption 1 also implies that sector 2 is more labor intensive than sector 1, as usual in the standard Heckscher-Ohlin-Samuelson model.

The production function of the public intermediate good is expressed as $r = f(L_R)$, where r is the output and L_R is a labor input in the public sector. Concerning $f(L_R)$, we assume that this function is increasing and strictly concave ($f' > 0 > f''$), and $f(0) = 0$. Given initial stock $R_0 > 0$, the public intermediate good is assumed to accumulate over time according to[2]

$$\dot{R} = f(L_R) - \beta R, \tag{2}$$

where $\beta > 0$ is the depreciation rate of the stock of the public intermediate good.

At each moment of time, the economy must face the following full employment constraint on labor:

$$L_1 + L_2 + L_R = L, \tag{3}$$

where $L > 0$ is labor endowment and is assumed to be given and constant over time.

2.2 GDP function

Let $l \equiv L - L_R$ is the sum of labor inputs in the private sectors. Then, the production side of the economy is characterized by the following GDP function:

$$G(p, R, l) = \max_{L_1, L_2} \left\{ pR^{\alpha_1} L_1^{1-\alpha_1} + R^{\alpha_2} L_2^{1-\alpha_2} \quad \text{s.t.} \quad L_1 + L_2 = l \right\}, \tag{4}$$

where p is a world price of good 1 in terms of good 2. Let us define the Lagrangian:

$$\mathcal{L}(L_1, L_2, w, p, R, l) = pR^{\alpha_1} L_1^{1-\alpha_1} + R^{\alpha_2} L_2^{1-\alpha_2} + w(l - L_1 - L_2).$$

The first-order conditions for maximizing $\mathcal{L}$ are

$$(1 - \alpha_1) p \frac{Y_1}{L_1} = w, \tag{5}$$

$$(1 - \alpha_2) \frac{Y_2}{L_2} = w, \tag{6}$$

$$L_1 + L_2 = l. \tag{7}$$

Applying the envelope theorem to the GDP function, we obtain the following derivatives:[3]

$$G_p = Y_1, \qquad G_R = \frac{\alpha_1 p Y_1 + \alpha_2 Y_2}{R}, \qquad G_l = w. \tag{8}$$

[2] A dot over a variable denotes time derivative. To reduce the complexity of notation, we may omit time arguments when no confusion is caused by doing so.

[3] The subscripts denote partial derivatives: $G_p = \partial G/\partial p$, and so on.

Moreover, the second-order derivatives of the GDP function can be derived as follows:[4]

$$G_{pp} = \frac{(1-\alpha_1)(1-\alpha_2)Y_1Y_2}{\{(1-\alpha_1)\alpha_2 pY_1 + (1-\alpha_2)\alpha_1 Y_2\}p} > 0, \tag{9a}$$

$$G_{RR} = -\frac{\alpha_1\alpha_2(wl)^2}{\{(1-\alpha_1)\alpha_2 pY_1 + (1-\alpha_2)\alpha_1 Y_2\}R^2} < 0, \tag{9b}$$

$$G_{ll} = -\frac{\alpha_1\alpha_2 w^2}{(1-\alpha_1)\alpha_2 pY_1 + (1-\alpha_2)\alpha_1 Y_2} < 0, \tag{9c}$$

$$G_{pR} = \frac{\{(1-\alpha_1)\alpha_1\alpha_2 pY_1 + (1-\alpha_2)[\alpha_1 - (1-\alpha_1)\alpha_2]Y_2\}Y_1}{\{(1-\alpha_1)\alpha_2 pY_1 + (1-\alpha_2)\alpha_1 Y_2\}R}, \tag{9d}$$

$$G_{pl} = \frac{w(1-\alpha_1)\alpha_2 Y_1}{(1-\alpha_1)\alpha_2 pY_1 + (1-\alpha_2)\alpha_1 Y_2} > 0, \tag{9e}$$

$$G_{Rl} = \frac{\alpha_1\alpha_2 w^2 l}{\{(1-\alpha_1)\alpha_2 pY_1 + (1-\alpha_2)\alpha_1 Y_2\}R} > 0. \tag{9f}$$

2.3 Consumption Side

The consumption side of the economy is described by a representative household, whose lifetime utility is given by:

$$U = \int_0^\infty e^{-\rho t}\left[\gamma \log C_1 + (1-\gamma)\log C_2\right] dt, \tag{10}$$

where C_i is consumption of good i ($i = 1, 2$), ρ is the rate of time preference, and $\gamma \in (0, 1)$ is a parameter. Let us denote the household's total expenditure at each moment of time by E, and assume that no borrowing or lending is permitted. Then, the household's optimal consumption must satisfy $C_1 = \gamma E/p$ and $C_2 = (1-\gamma)E$.

The private goods are traded between countries, while the public intermediate good is assumed to be nontradable. In addition, we assume away the international borrowing and lending. Therefore, national income must equal total expenditure at all points in time: $E = G(p, R, l)$. Substituting the household's optimal consumption into the lifetime utility (10), the its indirect lifetime utility is derived as

$$V = \int_0^\infty e^{-\rho t}\left\{\log[G(p, R, L - L_R)] - \gamma \log p + \Gamma\right\} dt, \tag{11}$$

where $\Gamma \equiv \gamma \log \gamma + (1-\gamma)\log(1-\gamma)$.

3 Trading Equilibrium under Non-strategic Behavior of Governments

In this and the next sections, we assume that governments in both countries take the world price p as given when they determine the time paths of the stock of public intermediate good stock.

3.1 The Optimal Resource Allocation

We characterize the economy's resource allocation as a dynamic optimization problem faced by a social planner. The same solution can be obtained by competitive equilibrium with appropriate Lindahl pricing for the public intermediate good.

[4] See Appendix for derivation.

Consider a social planner who seeks to maximize the representative household's indirect lifetime utility (11) by choosing the time path of the allocation of labor to the public sector subject to the constraint (2), taking p as given. Let us define the current-value Hamiltonian as

$$\mathcal{H}(L_R, R, \theta) = \log[G(p, R, L - L_R)] - \gamma \log p + \Gamma + \theta \{f(L_R) - \beta R\}.$$

Then the optimal control must satisfy

$$\frac{\partial \mathcal{H}}{\partial L_R} = 0 \quad \Rightarrow \quad \theta f'(L_R) = \frac{G_l(p, R, L - L_R)}{G(p, R, L - L_R)}. \tag{12}$$

Moreover, the adjoint equation and the transversality condition are, respectively, expressed as

$$\dot{\theta} = \rho\theta - \frac{\partial \mathcal{H}}{\partial R} = (\rho + \beta)\theta - \frac{G_R(p, R, L - L_R)}{G(p, R, L - L_R)}, \tag{13}$$

$$\lim_{t \to \infty} e^{-\rho t}\theta(t)R(t) = 0. \tag{14}$$

3.2 Dynamic equilibrium path

At each moment in time the world market for each private good has to hold along an equilibrium path. We assume that the two countries share the identical production technologies, meaning that the GDP function in the foreign country is given by $G(p, R^*, L^* - L_R^*)$, and have the same felicity function, meaning that $\gamma = \gamma^*$. Then, the world market-clearing condition for good 1, $C_1 + C_1^* = Y_1 + Y_1^*$, can be rewritten as

$$\frac{\gamma}{p}\{G(p, R, L - L_R) + G(p, R^*, L^* - L_R^*)\} = G_p(p, R, L - L_R) + G_p(p, R^*, L^* - L_R^*). \tag{15}$$

The dynamic equilibrium of the world economy is characterized by the home country's dynamic equations and first-order condition (2), (12), (13), their foreign counterparts, and the world market-clearing condition (15).

3.3 Steady state

At the steady state, it holds that $\dot{R} = \dot{R}^* = \dot{\theta} = \dot{\theta}^* = 0$. In light of (2), (12), and (13), conditions for the free-trade, steady-state equilibrium are given by

$$f(L_R) = \beta R, \tag{16}$$

$$\rho + \beta = \frac{G_R(p, R, L - L_R)}{G_l(p, R, L - L_R)} f'(L_R), \tag{17}$$

$$f(L_R^*) = \beta^* R^*, \tag{18}$$

$$\rho^* + \beta^* = \frac{G_R(p, R^*, L^* - L_R^*)}{G_l(p, R^*, L^* - L_R^*)} f'(L_R^*), \tag{19}$$

and (15).

From (16) and (17), the steady-state level of L_R can be represented as a function of p (and the parameters β, ρ, and L): $L_R = \psi(p)$. Under Assumption 1, the following lemma is obtained.

Lemma 1 *(i)* $\psi'(p) > 0$. *(ii) There exist* $\overline{L}_R$ *and* $\underline{L}_R$ *such that* $0 < \underline{L}_R < \overline{L}_R < L$, $\lim_{p \to 0} \psi(p) = \underline{L}_R$, *and* $\lim_{p \to \infty} \psi(p) = \overline{L}_R$.

Proof. See Appendix.

From (16), the steady-state stock of the public intermediate good is denoted as $R = f(\psi(p))/\beta$. Analogously for the foreign country, from (18) and (19), we have $L_R^* = \psi^*(p)$, which has the similar properties to Lemma 1, and $R^* = f(\psi^*(p))/\beta^*$. Substituting these expressions into (15), the steady-state, market-clearing condition is given by

$$\frac{\gamma}{p}\left\{G\left(p, \frac{f(\psi(p))}{\beta}, L-\psi(p)\right) + G\left(p, \frac{f(\psi^*(p))}{\beta^*}, L^* - \psi^*(p)\right)\right\}$$
$$= G_p\left(p, \frac{f(\psi(p))}{\beta}, L-\psi(p)\right) + G_p\left(p, \frac{f(\psi^*(p))}{\beta^*}, L^* - \psi^*(p)\right). \tag{20}$$

Let us denote the solution of (20), i.e., the steady-state equilibrium solution for the relative price of good 1 in the world market by p_{ss}. The steady-state stocks of public intermediate good in the home and foreign country are then derived as $R_{ss} = f(\psi(p_{ss}))/\beta$ and $R_{ss}^* = f(\psi^*(p_{ss}))/\beta^*$, respectively. The existence, uniqueness, and stability of the steady state is characterized by the following theorem.

Theorem 1 *There exist at least one steady-state, free-trade equilibrium. If the differences in discount rates, depreciation rates, and labor endowments between the two countries are not very large, there exists a unique and saddlepoint-stable steady state, in which production is diversified in both countries.*

Proof. See Appendix.

4 Trade Patterns and Trade Gains

4.1 Trade patterns

Let us focus on the home country's excess demand for good 1 in the steady state:

$$ED(p) \equiv \frac{\gamma}{p}G\left(p, \frac{f(\psi(p))}{\beta}, L-\psi(p)\right) - G_p\left(p, \frac{f(\psi(p))}{\beta}, L-\psi(p)\right). \tag{21}$$

The foreign country's excess demand in the steady state, $ED^*(p)$, can be analogously defined. Let us denote the autarkic steady-state prices in the home country and foreign country by p_a and p_a^*, respectively. Then, in the autarkic steady-state equilibrium, it holds that $ED(p_a) = 0 = ED^*(p_a^*)$. It is verified that $ED(p)$ and $ED^*(p)$ are downward sloping in the neighborhood of the steady-state, autarkic-equilibrium price.

Labor endowments Evaluating at the autarkic steady state, we obtain the shift in the excess demand in response to a change in the labor endowment L as follows:

$$\left.\frac{\partial ED(p)}{\partial L}\right|_{p=p_a} = \left(\frac{G_p G_R}{G} - G_{pR}\right)\frac{f'}{\beta}\frac{\partial\psi(p)}{\partial L} + \left(\frac{G_p G_l}{G} - G_{pl}\right)\left(1 - \frac{\partial\psi(p)}{\partial L}\right), \tag{22}$$

where

$$\frac{\partial\psi}{\partial L} = \frac{\left(G_{Rl} - \frac{G_R G_{ll}}{G_l}\right)f'}{\left(\frac{G_R G_{lR}}{G_l} - G_{RR}\right)\frac{(f')^2}{\beta} + \left(G_{lR} - \frac{G_R G_{ll}}{G_l}\right)f' - G_R f''} > 0$$

from (16) and (17).[5] After some calculations given in Appendix, it can be verified that the sign of $\partial ED(p)/\partial L$ is equal to that of $(\alpha_2 - \alpha_1)f''$. Therefore, $\partial ED(p)/\partial L$ is positive under Assumption 1. Suppose that the home has a smaller labor endowment than the foreign country: $L < L^*$. Then, it holds that $ED(p) < ED^*(p)$ in the neighborhood of autarkic equilibrium prices, as illustrated in Figure 1, and thus $p_a < p_a^*$. Hence, the home country exports good 1 in the steady state.

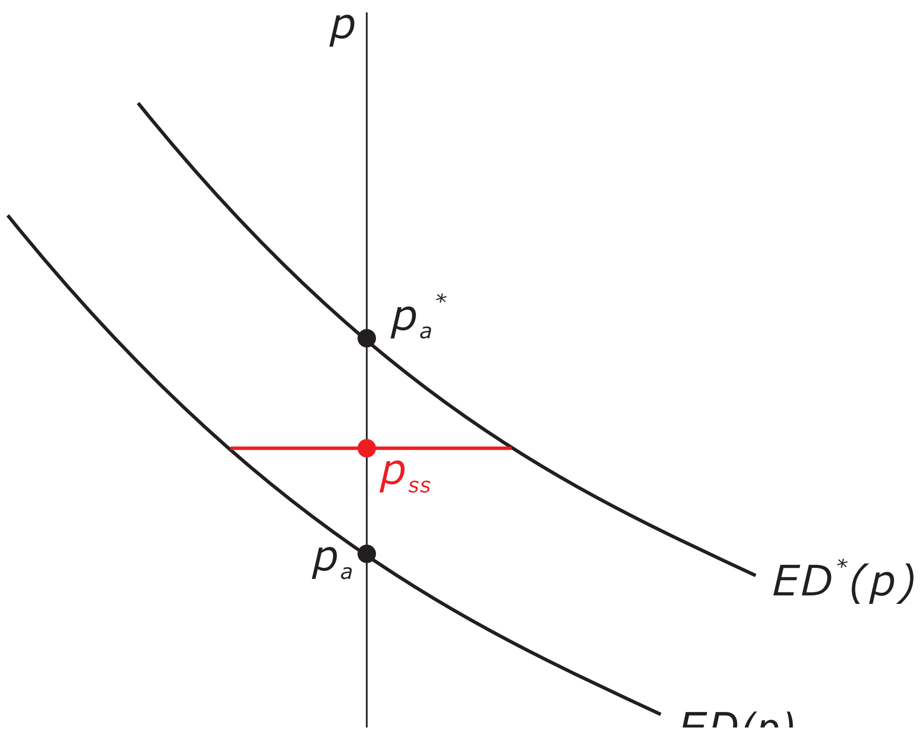

Figure 1: Steady-state equilibrium prices ($L < L^*$)

McMillan (1978) and Yanase and Tawada (2012) consider a dynamic model of a small open economy with a stock of public intermediate good in which the production technology of each private good exhibits a Ricardian property, i.e., constant returns to scale with respect to labor. They show that if the labor endowment is sufficiently large (small), a small open country completely specializes in a good whose productivity is more (less) sensitive to the public intermediate good. This implies that after opening of trade, a country with a higher labor endowment becomes an exporter of a good whose productivity is more sensitive to the public intermediate good. However, in the present model with a constant-returns technology with respect to labor and the public-good stock, the result is reversed.

Intuitively, the difference between the present model and McMillan's model can be interpreted as follows. In McMillan (1978) and Yanase and Tawada (2012), the private sectors' production function is given by $Y_i = g_i(R)L_i$, where $g_i(R)$ is increasing and concave in R, and thus each country's comparative advantage depends on the current stock of

[5]See (A.16) in the Appendix for derivation.

the public intermediate good. The labor endowment affects comparative advantage only indirectly, via the accumulation of the public good, and a country with a higher labor endowment will have a higher R. Hence, a larger country tends to specialize in a good which is more dependent on R. In the present model, by contrast, each country's comparative advantage depends on both R and L in a manner such that higher R (resp. L) increases Y_1 to a larger (resp. lesser) extent to Y_2 Although a higher labor endowment implies a higher steady-state stock of R, the steady-state relative factor endowment becomes decreasing in L. This implies that a country with a higher L has comparative advantage in the labor-intensive good, which is, in turn, less dependent on R.

Depreciation rates The shift in the excess demand in response to a change in the depreciation rate of the public-good stock β is derived as

$$\left.\frac{\partial ED(p)}{\partial \beta}\right|_{p=p_a} = \left(\frac{G_p G_R}{G} - G_{pR}\right)\left(\frac{f'}{\beta}\frac{\partial \psi(p)}{\partial \beta} - \frac{f}{\beta^2}\right) - \left(\frac{G_p G_l}{G} - G_{pl}\right)\frac{\partial \psi(p)}{\partial \beta}, \tag{23}$$

where

$$\frac{\partial \psi}{\partial \beta} = \frac{\left(\frac{G_R G_{lR}}{G_l} - G_{RR}\right)\frac{ff'}{\beta^2} - G_l}{\left(\frac{G_R G_{lR}}{G_l} - G_{RR}\right)\frac{(f')^2}{\beta} + \left(G_{lR} - \frac{G_R G_{ll}}{G_l}\right)f' - G_R f''}.$$

After some calculations given in Appendix, it can be verified that $\partial ED(p)/\partial \beta > 0$ under Assumption 1. Suppose that the home country has a lower depreciation rate of the public-good stock than the foreign country: $\beta < \beta^*$. In this case, it holds that $ED(p) < ED^*(p)$ in the neighborhood of autarkic equilibrium prices, and thus $p_a < p_a^*$. The home country exports good 1 in the steady state. The intuition behind this result is straightforward. Other things being equal, a smaller β implies a higher steady-state stock of the public intermediate good, which leads to higher relative supply of good 1 under Assumption 1 and thus lower relative price of good 1 under autarky.

Discount rates The shift in the excess demand in response to a change in the discount rate ρ is derived as

$$\left.\frac{\partial ED(p)}{\partial \rho}\right|_{p=p_a} = \left(\frac{G_p G_R}{G} - G_{pR}\right)\frac{f'}{\beta}\frac{\partial \psi(p)}{\partial \rho} - \left(\frac{G_p G_l}{G} - G_{pl}\right)\frac{\partial \psi(p)}{\partial \rho}, \tag{24}$$

where

$$\frac{\partial \psi}{\partial \rho} = -\frac{G_l}{\left(\frac{G_R G_{lR}}{G_l} - G_{RR}\right)\frac{(f')^2}{\beta} + \left(G_{lR} - \frac{G_R G_{ll}}{G_l}\right)f' - G_R f''} < 0.$$

From (A.9) and (A.10) it holds that $G_p G_R/G - G_{pR} < 0$ and $G_p G_l/G - G_{pl} > 0$ under Assumption 1. Therefore, it follows that $\partial ED(p)/\partial \rho > 0$, i.e., an increase in ρ shifts $ED(p)$ upwards. Suppose that the home consumer is more patient than the foreign consumer: $\rho < \rho^*$. In this case, it holds that $ED(p) < ED^*(p)$ in the neighborhood of autarkic equilibrium prices, and thus $p_a < p_a^*$. The home country exports good 1 in the steady state.

The intuition behind this result is as follows. Other things being equal, a smaller ρ implies that people are more patient and put weight on future consumption than the current consumption. The future consumption can be enhanced by the accumulation of public intermediate good, which augments outputs of the final goods in the future.

To sum up, we establish the following theorem regarding each country's trade pattern.

Theorem 2 *Other things being equal, (i) the country with a smaller (larger) labor endowment, (ii) the country with a lower (higher) depreciation rate of the public-good stock, and/or (iii) the patient (impatient) country exports (imports) the good that is more dependent on the public-good stock in the steady state.*

4.2 Gains from trade

Let us denote the autarkic steady-state stock of the public intermediate good in the home and foreign countries by R_a and R_a^*, respectively. Suppose that the home country exports good 1 in the free-trade, steady-state equilibrium, i.e., $p_a^* > p_{ss} > p_a$. Then, from Lemma 1 it holds that $R_{ss} > R_a$ and $R_{ss}^* < R_a^*$. That is, in comparison with the autarkic steady state, trade liberalization increases the steady-state stock of the public intermediate good in the country exporting a good that is more dependent on the public-good stock, while it reduces the public-good stock in the other country.

Theorem 3 *The country exporting a good that is more dependent on the stock of the public intermediate good unambiguously gains from trade in the long run, in the sense that the country enjoys the higher steady-state welfare under free trade than under autarky.*

Proof. Let us define the expenditure function as

$$E(p,u) = \min_{C_1,C_2} \{pC_1 + C_2 \quad \text{s.t.} \quad \gamma \log C_1 + (1-\gamma)\log C_2 \geq u\}.$$

It is easily verified that $E_u > 0$. Let us also denote the steady-state utility level under autarky and free trade by u_a and u_{ss}, respectively. In light of (**??**) and (**??**), we have the following expression:

$$\begin{aligned} &E(p_{ss},u_{ss}) - E(p_{ss},u_a) \\ &= G(p_{ss},R_{ss},l_{ss}) - (p_{ss}Y_{1a} + Y_{2a}) + (p_{ss}C_{1a} + C_{2a}) - E(p_{ss},u_a), \end{aligned} \tag{25}$$

where l_{ss} is the free-trade, steady-state level of l, and Y_{ia} and C_{ia} are autarkic steady-state level of output and consumption, respectively, of good $i = 1,2$. From the definition of the expenditure function, it holds that $p_{ss}C_{1a} + C_{2a} \geq E(p_{ss},u_a)$. Using the GDP function, the second term in the right-hand side of (25) can be rewritten as $G(p_{ss},R_a,l_a)$, where l_a is the autarkic steady-state level of l. Because the stead-state labor input in the public sector is rewritten as $L_R = f^{-1}(\beta R)$, the effect of an increase in R on the maximized GDP for a given price level can be derived as

$$\begin{aligned} &G_R(p,R,L-f^{-1}(\beta R)) - \frac{\beta}{f'}G_l(p,R,L-f^{-1}(\beta R)) \\ &= G_R(p,R,L-f^{-1}(\beta R)) - \frac{\beta}{\rho+\beta}G_R(p,R,L-f^{-1}(\beta R)) \\ &= \frac{\rho}{\rho+\beta}G_R(p,R,L-f^{-1}(\beta R)) > 0, \end{aligned} \tag{26}$$

where we used (17). Eq.(26) implies that $G(p_{ss},R_{ss},l_{ss}) > p_{ss}Y_{1a} + Y_{2a}$. To sum up, it follows that the sign of (25) is unambiguously positive, and thus $u_{ss} > u_a$. □

For the country importing a good that is more dependent on the stock of the public intermediate good, international trade reduces the steady-state stock of the public intermediate good and thus the steady-state national income evaluated at the post-trade prices.

If this reduction in the national income outweighs the efficiency gains from specialization and exchange, the country will suffer steady-state losses from trade.

Notice that the above discussion on gains/losses from trade is from a long-run viewpoint; we have focused on the comparison of steady-state welfare levels. Let us now discuss the welfare effects of trade along the transition path. Suppose that home country exports good 1 and that both countries are initially under autarky and open international trade. Then, in the short run, both countries enjoy welfare improvement because their consumption possibilities expand. Along the transition path, however, the stock of public intermediate good increases over time in the home country and decreases in the foreign country. This implies that the home country continues to improve welfare, but in the foreign country the instantaneous welfare decreases over time and may be lower than autarkic welfare after a certain point in time.

4.3 Possibility of loss from trade: An example

As indicated in the previous subsection, the country importing a good that is more dependent on the stock of the public intermediate good may suffer a lower steady-state welfare than under autarky. In this subsection, we demonstrate this possibility by considering a simplified version of the model.

Let us consider a more simplified version of the model, where

- $f(L_R) = L_R - L_R^2/2$,
- $\alpha_1 = \alpha > 0 = \alpha_2$,
- $L = L^* = 1$, and
- $\beta^* = \lambda\beta$ and $\rho^* = \lambda\rho$, where $\lambda > 1$.

That is, we assume that only good 1 is dependent on the stock of the public intermediate good and the two countries are identical except for the rates of depreciation and time preference. From Theorem 2 (ii) and (iii), the home (foreign) country becomes an exporter of good 1 (good 2).[6] This assertion is verified in this example. Let us define $q \equiv (1-\alpha)^{\frac{1-\alpha}{\alpha}}\alpha p^{\frac{1}{\alpha}}$. Then, as shown in the Appendix, we obtain[7]

$$q_a = \sqrt{\frac{(\rho+\beta)\{(1+\alpha\gamma)\beta + (1-\alpha\gamma)\rho\}}{1-\alpha\gamma}}, \quad q_a^* = \lambda q_a, \quad q_{ss} = \sqrt{\lambda} q_a, \tag{27}$$

and thus $q_a < q_{ss} < q_a^*$.

Remark Substituting $q = q_{ss}$ into each country's steady-state equilibrium level of labor input in the public sector, we can conclude that both $L_R > 0$ and $L_R^* > 0$ hold only if

$$\frac{(1+\alpha\gamma)\beta + (1-\alpha\gamma)\rho}{(1-\alpha\gamma)(\rho+\beta)} > \lambda > 1. \tag{28}$$

In the following analysis, we assume that this condition is satisfied.

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[6] Alternatively, we can consider a situation in which the labor endowments differ between countries (e.g., $L = 1$ and $L^* = 1+\epsilon$, where $\epsilon > 0$. Theorem 2 (i) and 3 can be verified under this alternative specification, although the calculation becomes more complicated.

[7] Because both α and γ are between 0 and 1, $1-\alpha\gamma > 0$.

Denoting the steady-state welfare level by $u \equiv \gamma \log C_1 + (1-\gamma)\log C_2$, we obtain the welfare comparison result in the home country:

$$u_{ss} - u_a = \log\left\{1 + \frac{(\lambda-1)[(1+\alpha\gamma)\beta + (1-\alpha\gamma)\rho]}{2\beta}\right\} - \frac{1+\alpha\gamma}{2}\log\lambda. \tag{29}$$

Theorem 3 purports that the home country obtains unambiguously higher welfare under free trade than under autarky. This assertion is also verified because $u_{ss} > u_a$ holds.[8] However, for the foreign country, the following expression holds:

$$u^*_{ss} - u^*_a = \log\left\{1 + \frac{(\lambda-1)(1-\alpha\gamma)(\beta-\rho)}{2\beta}\right\} - \frac{1-\alpha\gamma}{2}\log\lambda, \tag{30}$$

the sign of which ambiguous. In particular, if ρ becomes higher (i.e., people are more impatient), the sign of (30) is more likely to be negative.

Yanase and Tawada (2012) add the gains-from-trade analysis to McMillan's (1978) model with a pure public intermediate good and show that a country unambiguously gains from trade in the long run only if the country has a comparative advantage in a good with productivity more sensitive to the public intermediate good; if the country has a comparative advantage in a good with productivity less sensitive to the public intermediate good, the economy may lose from trade in the long run. In the present model, we obtain the similar result. However, in their model, the country that gains (resp. may lose) from trade is the larger (resp. smaller) country measured in labor endowments. By contrast, as implied by Theorem 2, the country that gains (resp. may lose) from trade is the smaller (resp. larger) country in the present model. In this sense, we can conclude that the results on gains/losses from trade obtained in Yanase and Tawada (2012) are not robust and are dependent on the property of the public intermediate good.

5 Trading Equilibrium under Dynamic Nash Provision of Public Goods

We have so far assumed that the national government in each country is a price taker in the world commodity markets and the public intermediate good is provided by the government using the Lindahl pricing rule without strategic motives In this section, we consider a situation in which the governments act strategically in providing the public good; i.e., the governments noncooperatively determine the scale of the public intermediate good supply, recognizing that it can affect the international prices.[9]

The market-clearing condition for good 1, i.e., (15), derives the equilibrium price of good 1 as a function of the public intermediate-good stock in each country and the total labor allocated to private sectors in each country; let us denote $p = P(R, R^*, l, l^*)$. Under Assumption 1, the function P has the following properties:

$$P_{R^{(*)}} = \frac{\gamma G_R^{(*)} - pG_{pR}^{(*)}}{(1-\gamma)(G_p + G_p^*) + p(G_{pp} + G_{pp}^*)}, \quad P_{l^{(*)}} = \frac{\gamma G_l^{(*)} - pG_{pl}^{(*)}}{(1-\gamma)(G_p + G_p^*) + p(G_{pp} + G_{pp}^*)}.$$

The governments no longer take the world price p as given; rather, they take into account the effects of the change in the resource allocation at each moment in time and

[8]The right-hand side of (29) is continuous in λ, becomes zero if $\lambda = 1$, and strictly increasing in λ.

[9]In a static model, Shimomura (2007) proves that if governments determine the levels of public goods noncooperatively, free trade is beneficial to all countries.

the stock of public intermediate good on the price. The home government's problem is to maximize welfare (11) subject to the dynamics of the domestic public intermediate good (2) and $p = P(R, R^*, L - L_R, L^* - L_R^*)$. Note also that each government recognizes that the rival country's choice of consumption and production of public intermediate good can affect the world price. In this sense, there exists dynamic and strategic interaction, and thus the trading equilibrium is characterized as a differential game between national governments that determine the appropriate levels of their domestic public intermediate goods.

In the following analysis, we assume for simplicity that each government uses an open-loop strategy, in which the government chooses the whole time paths of the choice variables, taking the time paths of rival's choice variables as given, at the beginning of the game.

5.1 The open-loop Nash equilibrium

The current-value Hamiltonian for the home country's problem is now defined as

$$\begin{aligned}
&\mathcal{H}(L_R, R, \theta; L_R^*, R^*) \\
&= \log[G(P(R, R^*, L - L_R, L^* - L_R^*), R, L - L_R)] \\
&\quad - \gamma \log P(R, R^*, L - L_R, L^* - L_R^*) + \Gamma + \theta \left\{ f(L_R) - \beta R \right\}.
\end{aligned}$$

Then the optimal controls and adjoint equations must satisfy

$$\frac{\partial \mathcal{H}}{\partial L_R} = 0 \quad \Rightarrow \quad \theta f'(L_R) = \frac{G_l + (G_p - \gamma G) P_l}{G}, \tag{31}$$

$$\dot{\theta} = \rho\theta - \frac{\partial \mathcal{H}}{\partial R} = (\rho + \beta)\theta - \frac{G_R + (G_p - \gamma G) P_R}{G}. \tag{32}$$

The foreign government solves the similar problem to the home country's and determines the time paths of the foreign variables. The dynamic equilibrium of the world economy is characterized by a profile of each country's open-loop strategies, i.e., an open-loop Nash equilibrium.[10]

5.2 Steady state

The steady state of the open-loop Nash equilibrium must satisfy (16), (18),

$$\rho + \beta = \frac{G_R + (G_p - \gamma G) P_R}{G_l + (G_p - \gamma G) P_l} f'(L_R), \tag{33}$$

and

$$\rho^* + \beta^* = \frac{G_R^* + (G_p^* - \gamma G^*) P_{R^*}}{G_l^* + (G_p^* - \gamma G^*) P_{l^*}} f'(L_R^*). \tag{34}$$

Because the derivatives, e.g., G_R, G_p, P_R, and P_l, involve the price function P and the GDP function G, the formal analysis of the steady state becomes much more complicated than the analysis in the non-strategic case. Therefore, let us provide some rough but heuristic discussions on the properties of the steady state in the presence of strategic interactions.

[10]The open-loop Nash equilibrium is founded on the assumption that each player can make a credible precommitment (Long, 2010). An alternative equilibrium concept is a Markov-perfect Nash equilibrium, in which each player's current action is conditioned on the state vector (in the present model, R and R^*). The Markov-perfect Nash equilibrium satisfies subgame perfection.

Let us compare the steady-state conditions (33) and (34) with those in the non-strategic steady-state equilibrium, i.e., (17) and (19). First of all, notice that the left-hand side of these respective equations, the sum of the discount and depreciation rates, can be interpreted as the long-run marginal cost of providing the stock of public intermediate good, whereas the right-hand side, shadow value of the public-good stock measured by labor, can be interpreted as the marginal benefit of the public intermediate good. Other things being equal, the existence of the terms of trade effect changes the marginal benefit of providing the public intermediate good. Specifically, comparing (17) and (33) in light of the facts that $\gamma G = C_1$ and $G_p = Y_1$, we have

$$\left\{ \frac{G_R - (C_1 - Y_1)P_R}{G_l - (C_1 - Y_1)P_l} - \frac{G_R}{G_l} \right\} f'(L_R) = \frac{(C_1 - Y_1)(G_{pR}G_l - G_{pl}G_R)}{\{G_l - (C_1 - Y_1)P_l\}G_l(G_{pp} + G^*_{pp})} f'(L_R),$$

where $G_{pR}G_l - G_{pl}G_R$ can be rewritten as

$$G_{pR}G_l - G_{pl}G_R = \frac{wY_1Y_2(\alpha_1 - \alpha_2)}{\{(1-\alpha_1)\alpha_2 pY_1 + (1-\alpha_2)\alpha_1 Y_2\}R} > 0.$$

Therefore, if the home country exports good 1, which is more intensive to R, the marginal benefit of the public intermediate good becomes smaller. Because the marginal benefit is decreasing in L_R (see the proof of Lemma 1), this implies that in comparison with the non-strategic case, the home government has an incentive to under-accumulate the public-good stock. It is also verified in a similar manner that the foreign country over-accumulates the public-good stock compared with the non-strategic case if the country imports good 1. These under- and over-accumulation are due to the terms of trade effect; because an increase in the stock of public intermediate good augments the relative output of good 1, its relative price will fall. The exporter country will suffer a deterioration in its terms of trade, and thus the country has an incentive to reduce the public-good provision in order to reduce the welfare loss.

6 Concluding Remarks

In this paper, we have dealt with a dynamic two-country model of international trade accommodating a semi-public intermediate good. In the case where the government in each country acts as a price taker in the world commodity market, we revealed that if the economy is initially at the autarky steady state, a country with a smaller (larger) labor endowment, a lower (higher) depreciation rate of the public-good stock, and/or a lower (higher) rate of time preference exports (imports) a good that is more dependent on the stock of the public intermediate good. We also showed that the country exporting the good that is more dependent on the public-good stock unambiguously gains from trade in the long-run but the country importing that good may lose from trade in the long-run.

We also considered an alternative situation where the governments strategically determine their provision of public intermediate goods, taking the effect of each country's choice on the world commodity market into consideration. Because of such a terms of trade effect, in comparison with the non-strategic case, the country exporting (importing) the good that is more dependent on the public-good stock would under-accumulate (over-accumulate) the public intermediate good.

We should notice that most of the existing studies concentrated on national public intermediate goods so far. On the other hand, in reality, international public goods are increasing recently. For example, railway and highway constructions are carried out across national boundaries. Various communication systems become available also across boundaries. Even national defense becomes international in some regions by making a military alliance. Thus, of more importance is to accommodate international public intermediate goods into the present model and re-examine the trade theorems. There seems to be much work to do once our attention is directed to this sort of extension.

Appendix

A.1 Properties of the GDP function

Substituting (5) and (6) into (1) and (7) yield

$$Y_1 = R^{\alpha_1}\left[\frac{(1-\alpha_1)pY_1}{w}\right]^{1-\alpha_1}, \tag{A.1}$$

$$Y_2 = R^{\alpha_2}\left[\frac{(1-\alpha_2)Y_2}{w}\right]^{1-\alpha_2}, \tag{A.2}$$

$$(1-\alpha_1)pY_1 + (1-\alpha_2)Y_2 = wl. \tag{A.3}$$

Totally differentiating eqs.(A.1), (A.1), and (A.3), we have

$$\begin{bmatrix} \frac{\alpha_1}{Y_1} & 0 & \frac{1-\alpha_1}{w} \\ 0 & \frac{\alpha_2}{Y_2} & \frac{1-\alpha_2}{w} \\ (1-\alpha_1)p & 1-\alpha_2 & -l \end{bmatrix}\begin{bmatrix} dY_1 \\ dY_2 \\ dw \end{bmatrix} = \begin{bmatrix} \frac{\alpha_1}{R}dR + \frac{1-\alpha_1}{p}dp \\ \frac{\alpha_2}{R}dR \\ -(1-\alpha_1)Y_1dp + wdl \end{bmatrix}. \tag{A.4}$$

Solving (A.4) for dY_1, we obtain

$$dY_1 = \frac{wY_1Y_2\left\{\frac{(1-\alpha_1)\alpha_2}{pw}dp + \frac{(1-\alpha_1)\alpha_1\alpha_2pY_1+(1-\alpha_2)[\alpha_1-(1-\alpha_1)\alpha_2]Y_2}{wRY_2}dR + \frac{(1-\alpha_1)\alpha_2}{Y_2}dl\right\}}{(1-\alpha_1)\alpha_2pY_1+(1-\alpha_2)\alpha_1Y_2}. \tag{A.5}$$

Because $G_{pp} = \partial Y_1/\partial p$, $G_{pR} = \partial Y_1/\partial R$, and $G_{pl} = \partial Y_1/\partial l$, (A.5) derives the first, fourth, and fifth expressions in (9).

In order to obtain G_{RR} and G_{Rl}, let us solve (A.4) for dY_2 with $dp = 0$:

$$dY_2 = \frac{wY_1Y_2\left\{\frac{(1-\alpha_1)[\alpha_2-(1-\alpha_2)\alpha_1]pY_1+(1-\alpha_2)\alpha_1\alpha_2Y_2}{wRY_1}dR + \frac{(1-\alpha_2)\alpha_1}{Y_1}dl\right\}}{(1-\alpha_1)\alpha_2pY_1+(1-\alpha_2)\alpha_1Y_2}. \tag{A.6}$$

Because $G_R = (\alpha pY_1 + \alpha_2Y_2)/R$, it follows that

$$G_{RR} = \frac{\left(\alpha p\frac{\partial Y_1}{\partial R} + \alpha_2\frac{\partial Y_2}{\partial R}\right)R - (\alpha pY_1+\alpha_2Y_2)}{R^2}, \tag{A.7a}$$

$$G_{Rl} = \frac{\alpha_1}{R}p\frac{\partial Y_1}{\partial l} + \frac{\alpha_2}{R}\frac{\partial Y_2}{\partial l}. \tag{A.7b}$$

Substituting (A.5) and (A.6) into (A.7), we obtain the second and last expressions in (9).

Solving (A.4) for dw with $dp = dR = 0$, we have

$$dw = \frac{\alpha_1\alpha_2w^2}{(1-\alpha_1)\alpha_2pY_1+(1-\alpha_2)\alpha_1Y_2}dl. \tag{A.8}$$

Because $G_l = w$, it follows that $G_{ll} = \partial w/\partial l$ and thus we obtain the third expression in (9).

Finally, let us provide some calculation results that are useful in the subsequent analysis:

$$G_{pR} - \frac{G_pG_R}{G} = \frac{(\alpha_1-\alpha_2)wlY_1Y_2}{\{(1-\alpha_1)\alpha_2pY_1+(1-\alpha_2)\alpha_1Y_2\}(pY_1+Y_2)R}, \tag{A.9}$$

$$G_{pl} - \frac{G_pG_l}{G} = -\frac{(\alpha_1-\alpha_2)wY_1Y_2}{\{(1-\alpha_1)\alpha_2pY_1+(1-\alpha_2)\alpha_1Y_2\}(pY_1+Y_2)}, \tag{A.10}$$

$$G_{Rl} - \frac{G_RG_l}{G} = -\frac{(\alpha_1-\alpha_2)^2wpY_1Y_2}{\{(1-\alpha_1)\alpha_2pY_1+(1-\alpha_2)\alpha_1Y_2\}(pY_1+Y_2)R} < 0, \tag{A.11}$$

$$G_{pR} - \frac{G_RG_{lp}}{G_l} = \frac{(\alpha_1-\alpha_2)Y_1Y_2}{\{(1-\alpha_1)\alpha_2pY_1+(1-\alpha_2)\alpha_1Y_2\}R}, \tag{A.12}$$

$$G_{RR} - \frac{G_RG_{Rl}}{G_l} = -\frac{\alpha_1\alpha_2wl(pY_1+Y_2)}{\{(1-\alpha_1)\alpha_2pY_1+(1-\alpha_2)\alpha_1Y_2\}R^2} < 0, \tag{A.13}$$

$$G_{Rl} - \frac{G_RG_{ll}}{G_l} = \frac{\alpha_1\alpha_2w(pY_1+Y_2)}{\{(1-\alpha_1)\alpha_2pY_1+(1-\alpha_2)\alpha_1Y_2\}R} > 0. \tag{A.14}$$

Under Assumption 1, the signs of (A.9) and (A.12) are positive, whereas the sign of (A.10) is negative.

A.2 Proof of Lemma 1

(i) Substituting (16) into (17) yields

$$\rho+\beta=\frac{G_R(p,f(L_R)/\beta,L-L_R)}{G_l(p,f(L_R)/\beta,L-L_R)}f'(L_R). \tag{A.15}$$

Totally differentiating (A.15) and rearranging, we have

$$\begin{aligned}&\left\{\left(\frac{G_RG_{lR}}{G_l}-G_{RR}\right)\frac{(f')^2}{\beta}+\left(G_{Rl}-\frac{G_RG_{ll}}{G_l}\right)f'-G_Rf''\right\}dL_R\\&=\left(G_{Rp}-\frac{G_RG_{lp}}{G_l}\right)f'dp+\left(G_{Rl}-\frac{G_RG_{ll}}{G_l}\right)f'dL\\&\quad+\left\{\left(\frac{G_RG_{lR}}{G_l}-G_{RR}\right)\frac{ff'}{\beta^2}-G_l\right\}d\beta-G_ld\rho.\end{aligned} \tag{A.16}$$

From (9), (A.13), and $f''\leq 0$, the coefficient of dL_R in (A.16) is positive. Also, from (A.12), the coefficient of dp is positive if $\alpha_1>\alpha_2$. Therefore, $dL_R/dp>0$ holds under Assumption 1.

(ii) In light of (8) and (A.3), (17) can be rewritten as

$$\rho+\beta=\frac{G_R}{G_l}f'(L_R)=\frac{(\alpha_1pY_1+\alpha_2Y_2)\beta f'(L_R)(L-L_R)}{\{(1-\alpha_1)pY_1+(1-\alpha_2)Y_2\}f(L_R)}, \tag{A.17}$$

from which we have:

$$\begin{aligned}\lim_{p\to 0}\frac{G_R}{G_l}f'(L_R)&=\frac{\alpha_2}{1-\alpha_2}\frac{\beta f'(L_R)(L-L_R)}{f(L_R)},\\\lim_{p\to\infty}\frac{G_R}{G_l}f'(L_R)&=\frac{\alpha_1}{1-\alpha_1}\frac{\beta f'(L_R)(L-L_R)}{f(L_R)}.\end{aligned}$$

Under Assumption 1, it holds that $\lim_{p\to 0}\frac{G_R}{G_l}f'(L_R)<\lim_{p\to\infty}\frac{G_R}{G_l}f'(L_R)$. In addition, both $\lim_{p\to 0}\frac{G_R}{G_l}f'(L_R)$ and $\lim_{p\to\infty}\frac{G_R}{G_l}f'(L_R)$ are decreasing in L_R:

$$\frac{d[\beta f'(L_R)(L-L_R)/f(L_R)]}{dL_R}=\frac{\beta\{[(L-L_R)f''-f']f-(L-L_R)(f')^2\}}{f^2}<0.$$

Therefore, the solution for L_R that satisfies (A.17) in each limiting case (i.e., $p\to 0$ and $p\to\infty$) is uniquely determined, as illustrated in Figure A1. □

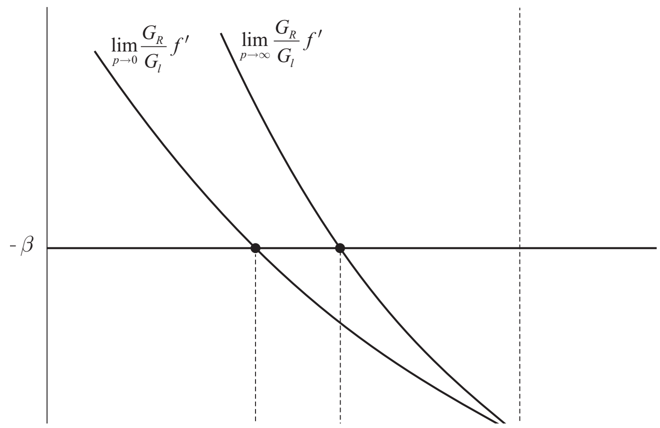

Figure A1: Determination of $\overline{L}_R$ and $\underline{L}_R$

A.3 Proof of Theorem 1

Let us define the left-hand side of (20) by $\zeta_L(p)$ and the right-hand side by $\zeta_R(p)$. It is easily verified that

$$\lim_{p\to 0}\zeta_L(p)=\infty,\quad \lim_{p\to\infty}\zeta_L(p)=0,$$

$$\lim_{p\to 0}\zeta_R(p)=0,\quad \lim_{p\to\infty}\zeta_R(p)=\left[\frac{f(\bar{L}_R)}{\beta}\right]^{\alpha_1}(L-\bar{L}_R)^{1-\alpha_1}+\left[\frac{f(\bar{L}_R^*)}{\beta^*}\right]^{\alpha_1}(L^*-\bar{L}_R^*)^{1-\alpha_1}>0.$$

Because both $\zeta_L(p)$ and $\zeta_R(p)$ are continuous in p, it follows that there exist at least one solution for p in $(0,\infty)$. Differentiating $\zeta_L(p)$ and $\zeta_R(p)$, respectively, and rearranging in light of (A.15), we have

$$\zeta_L'(p)=\frac{\gamma}{p^2}\left\{p\left(\frac{\rho}{\beta}G_l\psi'+\frac{\rho^*}{\beta^*}G_l^*\psi^{*\prime}\right)-(Y_2+Y_2^*)\right\}, \tag{A.18}$$

$$\zeta_R'(p)=G_{pp}+G_{pp}^*+\left(\frac{\rho+\beta}{\beta}\frac{G_lG_{pR}}{G_R}-G_{pl}\right)\psi'+\left(\frac{\rho^*+\beta^*}{\beta^*}\frac{G_l^*G_{pR}^*}{G_R^*}-G_{pl}^*\right)\psi^{*\prime}. \tag{A.19}$$

From (9), (A.12), and Lemma 1, it follows that $\zeta_R'(p)>0$. Therefore, that if $\zeta_L'(p)<\zeta_R'(p)$ holds at the equilibrium price such that $\zeta_L(p)=\zeta_R(p)$, the equilibrium is unique. From (A.18) and (A.19), we have

$$\begin{aligned}\zeta_L'(p)-\zeta_R'(p)=&\left(\frac{\gamma}{p}\frac{\rho}{\beta}G_l-\frac{\rho+\beta}{\beta}\frac{G_lG_{pR}}{G_R}+G_{pl}\right)\psi'+\left(\frac{\gamma}{p}\frac{\rho^*}{\beta^*}G_l^*-\frac{\rho^*+\beta^*}{\beta^*}\frac{G_l^*G_{pR}^*}{G_R^*}+G_{pl}^*\right)\psi^{*\prime}\\&-\frac{\gamma^2}{p}(Y_2+Y_2^*)-(G_{pp}+G_{pp}^*).\end{aligned} \tag{A.20}$$

Let us assume that the two countries are identical: $L=L^*$, $\beta=\beta^*$, and $\rho=\rho^*$. Then, the two countries share identical GDP function, and thus the world market-clearing condition $\zeta_L(p)=\zeta_R(p)$ implies that $\gamma/p=G_p/G$. Substituting this into the first term in the right-hand side of (A.20), we have

$$\left(\frac{\gamma}{p}\frac{\rho}{\beta}G_l-\frac{\rho+\beta}{\beta}\frac{G_lG_{pR}}{G_R}+G_{pl}\right)\psi'=\left\{\frac{\rho}{\beta}\left(\frac{G_p}{G}-\frac{G_{pR}}{G_R}\right)G_l+\left(G_{pl}-\frac{G_lG_{pR}}{G_R}\right)\right\}\psi'.$$

From (A.9), (A.12), and Lemma 1, the above expression is shown to be negative, and so is the second term in (A.20). Therefore, $\zeta_L'(p)<\zeta_R'(p)$ and thus there exists a unique solution p_{ss} in the symmetric case.

We next turn to the stability of the steady state. The dynamic system of the world economy is described as

$$\begin{aligned}\dot{R}&=f(L_R)-\beta R,\\ \dot{R}^*&=f(L_R^*)-\beta^*R^*,\\ \dot{\theta}&=(\rho+\beta)\theta-\frac{G_R(p,R,L-L_R)}{G(p,R,L-L_R)},\\ \dot{\theta}^*&=(\rho^*+\beta^*)\theta^*-\frac{G_R(p,R^*,L^*-L_R^*)}{G(p,R^*,L^*-L_R^*)},\\ 0&=\frac{\gamma}{p}\{G(p,R,L-L_R)+G(p,R^*,L^*-L_R^*)\}-G_p(p,R,L-L_R)-G_p(p,R^*,L^*-L_R^*).\end{aligned}$$

Notice that L_R is dependent on R, θ, and p, and given (12), the following derivatives are obtained as follows:

$$\frac{\partial L_R}{\partial R}=\frac{G_{lR}-\theta f'G_R}{\theta Gf''-\theta f'G_l+G_{ll}}=\frac{G_{lR}-\frac{G_lG_R}{G}}{\theta Gf''-\theta f'G_l+G_{ll}}>0,$$

$$\frac{\partial L_R}{\partial\theta}=-\frac{f'G}{\theta Gf''-\theta f'G_l+G_{ll}}>0,$$

$$\frac{\partial L_R}{\partial p}=\frac{G_{lp}-\theta f'G_p}{\theta Gf''-\theta f'G_l+G_{ll}}=\frac{G_{lp}-\frac{G_lG_p}{G}}{\theta Gf''-\theta f'G_l+G_{ll}}>0\quad\text{if}\quad\alpha_1>\alpha_2.$$

Assuming that the two countries are identical and linearizing the dynamic system around the symmetric steady state, we obtain

$$\begin{bmatrix} \dot{R} \\ \dot{R}^* \\ \dot{\theta} \\ \dot{\theta}^* \\ 0 \end{bmatrix} = \begin{bmatrix} f'\frac{\partial L_R}{\partial R} - \beta & 0 & f'\frac{\partial L_R}{\partial \theta} & 0 & f'\frac{\partial L_R}{\partial p} \\ 0 & f'\frac{\partial L_R}{\partial R} - \beta & 0 & f'\frac{\partial L_R}{\partial \theta} & f'\frac{\partial L_R}{\partial p} \\ \delta_R & 0 & \delta_\theta & 0 & \delta_p \\ 0 & \delta_R & 0 & \delta_\theta & \delta_p \\ \chi_R & \chi_R & \chi_\theta & \chi_\theta & \chi_p \end{bmatrix} \begin{bmatrix} R - \bar{R} \\ R^* - \bar{R}^* \\ \theta - \bar{\theta} \\ \theta^* - \bar{\theta}^* \\ 0 \end{bmatrix}, \tag{A.21}$$

where

$$\begin{aligned}
\delta_R &\equiv -\frac{1}{G}\left\{G_{RR} - \frac{G_R^2}{G} + \left(\frac{G_R G_l}{G} - G_{Rl}\right)\frac{\partial L_R}{\partial R}\right\}, \\
\delta_\theta &\equiv \rho + \beta - \frac{1}{G}\left(\frac{G_R G_l}{G} - G_{Rl}\right)\frac{\partial L_R}{\partial \theta}, \\
\delta_p &\equiv -\frac{1}{G}\left\{G_{Rp} - \frac{G_R G_p}{G} + \left(\frac{G_R G_l}{G} - G_{Rl}\right)\frac{\partial L_R}{\partial p}\right\}, \\
\chi_R &\equiv \frac{G_p G_R}{G} - G_{pR} + \left(G_{pl} - \frac{G_p G_l}{G}\right)\frac{\partial L_R}{\partial R}, \\
\chi_\theta &\equiv \left(G_{pl} - \frac{G_p G_l}{G}\right)\frac{\partial L_R}{\partial \theta}, \\
\chi_p &\equiv 2\left\{-\frac{Y_1 Y_2}{pG} - G_{pp} + \left(G_{pl} - \frac{G_p G_l}{G}\right)\frac{\partial L_R}{\partial p}\right\}.
\end{aligned}$$

Let us denote the above matrix by J and the corresponding eigenvalues as z, which is determined by the characteristic equation

$$\Omega(z) \equiv \left| J - z \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right| = 0.$$

After tedious calculations, the characteristic equation can be rewritten as $\Omega(z) = 2\Omega_1(z)\Omega_2(z) = 0$, where

$$\begin{aligned}
\Omega_1(z) \equiv\ & z^2 - \rho z + \left(f'\frac{\partial L_R}{\partial R} - \beta\right)\delta_\theta - f'\frac{\partial L_R}{\partial \theta}\delta_R, \\
\Omega_2(z) \equiv\ & \left\{\left(G_{pl} - \frac{G_p G_l}{G}\right)\frac{\partial L_R}{\partial p} - \left(\frac{Y_1 Y_2}{pG} + G_{pp}\right)\right\} z^2 \\
& + \left\{\left(G_{pl} - \frac{G_p G_l}{G}\right)\left[\frac{\partial L_R}{\partial p}\left(f'\frac{l}{R} + \beta - \delta_\theta\right) + \frac{\partial L_R}{\partial \theta}\delta_p\right] + \left(\frac{Y_1 Y_2}{pG} + G_{pp}\right)\rho\right\} z \\
& + \left(G_{pl} - \frac{G_p G_l}{G}\right)\left(\frac{\partial L_R}{\partial \theta}\delta_p - \frac{\partial L_R}{\partial p}\delta_\theta\right) f'\left(\frac{l}{R} + \frac{\beta}{f'}\right) \\
& - \left(\frac{Y_1 Y_2}{pG} + G_{pp}\right)\left[\left(f'\frac{\partial L_R}{\partial R} - \beta\right)\delta_\theta - f'\frac{\partial L_R}{\partial \theta}\delta_R\right].
\end{aligned}$$

The constant term of $\Omega_1(z)$ can be rewritten as

$$\begin{aligned}
& \left(f'\frac{\partial L_R}{\partial R} - \beta\right)\delta_\theta - f'\frac{\partial L_R}{\partial \theta}\delta_R \\
&= \frac{f}{R}\left(\frac{f' L_R}{f}\frac{\partial L_E}{\partial R}\frac{R}{L_R} - 1\right)(\rho + \beta) + \frac{1}{G}\frac{\partial L_R}{\partial \theta}\left(f' G_{RR} - \beta G_{Rl} - \rho\frac{G_R G_l}{G}\right),
\end{aligned} \tag{A.22}$$

which is unambiguously negative because of the concavity of f. This indicates that the two solutions to $\Omega_1(z) = 0$ have opposite signs. Next, the z^2–term of $\Omega_2(z)$ is negative:

$$\begin{aligned}
&\left(G_{pl} - \frac{G_p G_l}{G}\right)\frac{\partial L_R}{\partial p} - \left(\frac{Y_1 Y_2}{pG} + G_{pp}\right) \\
&= \frac{1}{\theta G f'' - \theta f' G_l + G_{ll}}\left(G_{pl} - \frac{G_p G_l}{G}\right)^2 - \left(\frac{Y_1 Y_2}{pG} + G_{pp}\right) < 0.
\end{aligned}$$

Moreover, since the sign of (A.22) is negative and

$$\begin{aligned}
&\left(G_{pl} - \frac{G_p G_l}{G}\right)\left(\frac{\partial L_R}{\partial \theta}\delta_p - \frac{\partial L_R}{\partial p}\delta_\theta\right) \\
&= \left(G_{pl} - \frac{G_p G_l}{G}\right)\left\{\frac{\partial L_R}{\partial \theta}\frac{1}{G}\left(G_{Rp} - \frac{G_R G_p}{G}\right) + (\rho + \beta)\frac{\partial L_R}{\partial p}\right\} > 0,
\end{aligned}$$

the constant term of $\Omega_2(z)$ is positive. Therefore, the two solutions to $\Omega_2(z) = 0$ also have opposite signs. To sum up, there are two positive characteristic roots and two negative roots. Because there are two state variables, R and R^*, it follows that the steady state is a saddle point.

Finally, we can check that $\Omega(0) \neq 0$. It follows that the implicit function theorem ensure us that even if the economic fundamentals of the two countries are slightly different with each other, the existence, uniqueness, and stability of the steady state are established (Chen et al., 2008). □

A.4 Signs of $\partial ED(p)/\partial L$ and $\partial ED(p)/\partial \beta$

Substituting the expression for $\partial\psi/\partial L$ into (22), we have

$$\begin{aligned}
\left.\frac{\partial ED(p)}{\partial L}\right|_{p=p_a} &= \frac{\left\{\left(G_{Rl} - \frac{G_R G_{ll}}{G_l}\right)\left(\frac{G_p G_R}{G} - G_{pR}\right) + \left(\frac{G_p G_l}{G} - G_{pl}\right)\left(\frac{G_R G_{lR}}{G_l} - G_{RR}\right)\right\}\frac{(f')^2}{\beta}}{\left(\frac{G_R G_{lR}}{G_l} - G_{RR}\right)\frac{(f')^2}{\beta} + \left(G_{lR} - \frac{G_R G_{ll}}{G_l}\right)f' - G_R f''} \\
&\quad - \frac{\left(\frac{G_p G_l}{G} - G_{pl}\right)G_R f''}{\left(\frac{G_R G_{lR}}{G_l} - G_{RR}\right)\frac{(f')^2}{\beta} + \left(G_{lR} - \frac{G_R G_{ll}}{G_l}\right)f' - G_R f''}. \qquad \text{(A.23)}
\end{aligned}$$

However, from (A.9), (A.10), (A.13), and (A.14), it is verified that the first term in the above equation becomes zero. Moreover, from (A.10), the term $\left(\frac{G_p G_l}{G} - G_{pl}\right)$ is positive under Assumption 1. Then, it follows that the sign of $\partial ED(p)/\partial L$ becomes nonnegative.

Substituting the expression for $\partial\psi/\partial\beta$ into (A.10), we have

$$\begin{aligned}
\left.\frac{\partial ED(p)}{\partial \beta}\right|_{p=p_a} &= -\frac{\left\{\left(G_{Rl} - \frac{G_R G_{ll}}{G_l}\right)\left(\frac{G_p G_R}{G} - G_{pR}\right) + \left(\frac{G_p G_l}{G} - G_{pl}\right)\left(\frac{G_R G_{lR}}{G_l} - G_{RR}\right)\right\}\frac{ff'}{\beta^2}}{\left(\frac{G_R G_{lR}}{G_l} - G_{RR}\right)\frac{(f')^2}{\beta} + \left(G_{lR} - \frac{G_R G_{ll}}{G_l}\right)f' - G_R f''} \\
&\quad + \frac{\left\{\frac{G_p G_l}{G} - G_{pl} - \frac{f'}{\beta}\left(\frac{G_p G_R}{G} - G_{pR}\right)\right\}G_l + \left(\frac{G_p G_R}{G} - G_{pR}\right)G_R\frac{ff''}{\beta^2}}{\left(\frac{G_R G_{lR}}{G_l} - G_{RR}\right)\frac{(f')^2}{\beta} + \left(G_{lR} - \frac{G_R G_{ll}}{G_l}\right)f' - G_R f''}. \qquad \text{(A.24)}
\end{aligned}$$

Again, the first term in the above equation becomes zero. Moreover, from (A.9) and (A.10), the terms $\left(\frac{G_p G_l}{G} - G_{pl}\right)$ and $\left(\frac{G_p G_R}{G} - G_{pR}\right)$ are positive and negative, respectively, under Assumption 1. Then, it follows that the sign of $\partial ED(p)/\partial\beta$ becomes positive.

A.5 Equilibrium in the simplified model

The GDP function in the simplified model is derived as

$$\begin{aligned}
G(p, R, l) &\equiv \max_{L_1, L_2}\left\{pR^\alpha L_1^{1-\alpha} + L_2 \quad \text{s.t.} \quad L_1 + L_2 = l\right\} \\
&= (1-\alpha)^{\frac{1-\alpha}{\alpha}}\alpha p^{\frac{1}{\alpha}}R + l. \qquad \text{(A.25)}
\end{aligned}$$

The second-order derivatives of the GDP function become

$$G_{pp} = (1-\alpha)^{\frac{1-\alpha}{\alpha}} \alpha p^{\frac{1-\alpha}{\alpha}} R, \quad G_{pR} = (1-\alpha)^{\frac{1-\alpha}{\alpha}} \alpha p^{\frac{1}{\alpha}}, \quad G_{pl} = G_{RR} = G_{Rl} = G_{ll} = 0.$$

Using (A.25) and $f(L_R) = L_R - L_R^2/2$, we obtain the steady-state conditions of the home country's optimal resource allocation:

$$L_R - \frac{L_R^2}{2} = \beta R,$$
$$\rho + \beta = q(1 - L_R),$$

where we used a variable transformation $q \equiv (1-\alpha)^{\frac{1-\alpha}{\alpha}} \alpha p^{\frac{1}{\alpha}}$. Solving the above equations, we have

$$L_R = 1 - \frac{\rho+\beta}{q}, \quad R = \frac{1}{2\beta}\left(1 - \frac{\rho+\beta}{q}\right)\left(1 + \frac{\rho+\beta}{q}\right). \tag{A.26}$$

Similarly for the foreign country, solving the optimal steady-state conditions

$$L_R^* - \frac{L_R^{*2}}{2} = \lambda\beta R^*,$$
$$\lambda(\rho + \beta) = q(1 - L_R^*),$$

we obtain

$$L_R^* = 1 - \frac{\lambda(\rho+\beta)}{q}, \quad R^* = \frac{1}{2\lambda\beta}\left\{1 - \frac{\lambda(\rho+\beta)}{q}\right\}\left\{1 + \frac{\lambda(\rho+\beta)}{q}\right\}. \tag{A.27}$$

The autarkic equilibrium condition in the home country is

$$\alpha\gamma(qR + 1 - L_R) = qR$$

Substituting (A.26) into the above equation and solving for q, we obtain

$$q_a = \sqrt{\frac{(\rho+\beta)\{(1+\alpha\gamma)\beta + (1-\alpha\gamma)\rho\}}{1-\alpha\gamma}}. \tag{A.28}$$

Similarly, in light of (A.27), the autarkic steady-state solution for q in the foreign country is derived as

$$q_a^* = \lambda\sqrt{\frac{(\rho+\beta)\{(1+\alpha\gamma)\beta + (1-\alpha\gamma)\rho\}}{1-\alpha\gamma}}. \tag{A.29}$$

The steady-state equilibrium condition under free trade, (20), is rewritten as

$$\alpha\gamma(qR + 1 - L_R) + \alpha\gamma(qR^* + 1 - L_R^*) = qR + qR^*$$

Substituting (A.26) and (A.27) into the above equation and solving for q, we obtain

$$q_{ss} = \sqrt{\frac{\lambda(\rho+\beta)\{(1+\alpha\gamma)\beta + (1-\alpha\gamma)\rho\}}{1-\alpha\gamma}}. \tag{A.30}$$

The steady-state welfare is

$$\begin{aligned} u &= \gamma \log C_1 + (1-\gamma)\log C_2 \\ &= \log(qR + 1 - L_R) - \alpha\gamma \log q + \Gamma, \end{aligned} \tag{A.31}$$

where $\Gamma \equiv \gamma\{\alpha\log\alpha + (1-\alpha)\log(1-\alpha)\} + \gamma\log\gamma + (1-\gamma)\log(1-\gamma)$. Substituting (A.26) and (A.28) into (A.31), we obtain the autarkic steady-state welfare as follows:

$$u_a = \frac{1-\alpha\gamma}{2}\log\frac{\rho+\beta}{1-\alpha\gamma} - \frac{1+\alpha\gamma}{2}\log\{(1+\alpha\gamma)\beta + (1-\alpha\gamma)\rho\} + \Gamma. \tag{A.32}$$

Substituting (A.26) and (A.30) into (A.31), we obtain the steady-state welfare under free trade as follows:

$$\begin{aligned} u_{ss} = \frac{1-\alpha\gamma}{2}\log\frac{\rho+\beta}{1-\alpha\gamma} - \frac{1+\alpha\gamma}{2}\log\{\lambda[(1+\alpha\gamma)\beta+(1-\alpha\gamma)\rho]\} \\ + \log\left\{1+\frac{(\lambda-1)[(1+\alpha\gamma)\beta+(1-\alpha\gamma)\rho]}{2\beta}\right\} + \Gamma. \end{aligned} \tag{A.33}$$

In a similar manner, from (A.27), (A.29), (A.30), and (A.31), the steady-state welfare under autarky and free trade, respectively, in the foreign country are derived as

$$\begin{aligned} u_a^* &= \frac{1-\alpha\gamma}{2}\log\frac{\rho+\beta}{1-\alpha\gamma} - \frac{1+\alpha\gamma}{2}\log\{(1+\alpha\gamma)\beta+(1-\alpha\gamma)\rho\} - \alpha\gamma\log\lambda + \Gamma, \\ u_{ss}^* &= \frac{1-\alpha\gamma}{2}\log\frac{\rho+\beta}{1-\alpha\gamma} - \frac{1+\alpha\gamma}{2}\log\{\lambda[(1+\alpha\gamma)\beta+(1-\alpha\gamma)\rho]\} \\ &+ \log\left\{1+\frac{(\lambda-1)(1-\alpha\gamma)(\beta-\rho)}{2\beta}\right\} + \Gamma. \end{aligned} \tag{A.34}$$

References

[1] Abe, K., A Public Input As a Determinant of Trade, *Canadian Journal of Economics* 23 (1990), 400–407.

[2] Altenburg, L., Some Trade Theorems with a Public Intermediate Good, *Canadian Journal of Economics* 25 (1992), 310–332.

[3] Chen, B.-L., K. Nishimura, and K. Shimomura, Time Preference and Two-country Trade, *International Journal of Economic Theory* 4 (2008), 29–52.

[4] Ishizawa, S., Increasing Returns, Public Inputs, and International Trade, *American Economic Review* 78 (1988), 794–795.

[5] Long, N.V., *A Survey of Dynamic Games in Economics*, 2010, World Scientific.

[6] McMillan, J., A Dynamic Analysis of Public Intermediate Goods Supply in Open Economy, *International Economic Review* 19 (1978), 665–678.

[7] Manning, R. and J. McMillan, Public Intermediate Goods, Production Possibilities, and International Trade, *Canadian Journal of Economics* 12 (1979), 243–257.

[8] Meade, J.E., External Economies and Diseconomies in a Competitive Situation, *Economic Journal* 62 (1952), 54–67.

[9] Shimomura, K., Trade Gains and Public Goods, *Review of International Economics* 15 (2007), 948–954.

[10] Suga, N. and M. Tawada, International Trade with a Public Intermediate Good and the Gains from Trade, *Review of International Economics* 15 (2007), 284–293.

[11] Tawada, M., The Production Possibility Set with Public Intermediate Goods, *Econometrica* 48 (1980), 1005–1012.

[12] Tawada, M. and H. Okamoto, International Trade with a Pubic Intermediate Good, *Journal of International Economics* 17 (1983), 101–115.

[13] Tawada, M. and K. Abe, Production Possibilities and International Trade with a Public Intermediate Good, *Canadian Journal of Economics* 17 (1984), 232–248.

[14] Yanase, A. and M. Tawada, History-Dependent Paths and Trade Gains in a Small Open Economy with a Public Intermediate Good, *International Economic Review* 53 (2012), 303–314.