Vote Trading under Complete Information

Dimitrios Xefteris[*]   Nicholas Ziros[†]

March, 2017

Abstract

We study two-party elections considering that: a) prior to the voting stage voters are free to trade votes for money according to the rules of the Shapley-Shubik strategic market games; and b) voters' preferences –both ordinal rankings and cardinal intensities– are public information. While under plurality rule no trade occurs, under a power-sharing system (voters' utilities are proportionally increasing in the vote share of their favorite party) *full-trade is always an equilibrium* (two voters –the strongest supporter of each party– buy the votes of all others). Notably, this equilibrium implements *proportional justice* with respect to the two buyers: the ratio of the parties' vote shares is equal to the ratio of the preference intensities of the two most opposing voters.

JEL: C72, D72, P16

Keywords: Vote trading; Complete information; Strategic market games; Power sharing; Proportional justice.

---

[*]Department of Economics, University of Cyprus, P.O. Box 20537, Nicosia 1678, Cyprus (e-mail: xefteris.dimitrios@ucy.ac.cy)

[†]Department of Economics, University of Cyprus, P.O. Box 20537, Nicosia 1678, Cyprus (e-mail: n.ziros@ucy.ac.cy).

1 Introduction

Vote trading is a common practice in bodies of collective decision making because it allows voters to express their preference intensities over alternatives and can be seen in various forms, such as exchange of supports for different proposals (logrolling) or exchange of ballots for money. The theoretical investigation of vote markets has attracted the interest of scholars from many disciplines not only for their frequent use but also for their distinct attributes. However, the relevant literature has failed to provide definite answers to many central questions and hence there are no general conclusions about the properties of vote trading.[1]

A number of researchers have identified that some difficulties with respect to vote trading are due to issues of information. For example, Piketty (1994) argues that complete information can be responsible for the nonexistence of an equilibrium with active vote trading, as there are no incentives for trading if individual (conflicting) preferences are publicly known. However, Casella et al. (2012) claim that the nonexistence of equilibrium with active vote trading remains a problem even under uncertainty about the preferences of other voters. In this paper we present a simple vote-trading model that differs from earlier approaches in various ways, so as to offer new insights on the effects of information on vote trading. Most works in the literature feature incomplete information and, in specific, consider either that there is uncertainty about both the ordinal and cardinal preferences of other voters (e.g., Casella et al., 2012; Xefteris and Ziros, 2017), or that ordinal preferences are publicly known but the intensities of these preferences are private information (e.g., Casella et al., 2014; Casella and Turban, 2014). We, on the other hand, consider the complete information assumption as the starting point of our analysis. Our approach is motivated by the fact that in most legislatures or committees it is not realistic to assume uncertainty about the preferred policy alternatives of their members. In addition in most cases not only their preferred alternatives but also the intensities of these preferences are publicly known, which also determine the vote buying or selling incentives when vote trading is allowed. Hence, we believe that our complete information approach is relevant in many bodies of group decision making.

---

[1] See Philipson and Snyder (1996); Casella et al. (2012) for a detailed exposition of the issue.

Our approach also employs alternative rules of exchange, as vote trading is conducted via the market mechanism of strategic market games (introduced in Shubik, 1973; Shapley and Shubik, 1977), which maps agents' actions to prices and allocations. Hence, we have a non-cooperative game in strategic form which allows us: a) to use Nash equilibrium as a solution concept and b) not to impose any price-taking hypothesis as the standard approaches on vote markets (e.g., Philipson and Snyder, 1996; Casella et al., 2012). In particular, we study a two-party election in which prior to the voting stage individuals are free to trade votes for money, if they find it profitable to do so. That is, an individual can offer her vote in exchange for money or can place a monetary bid in exchange for votes. In this setup, the price of a vote is endogenously determined by the actions of vote traders, while the distribution mechanism allocates the supplied votes to vote buyers in proportion to their bids and accordingly distributes monetary bids to those who chose to sell their votes.

In this framework we study whether vote trading with perfect information can be applied in different electoral systems. Initially we argue that under the simple plurality rule the unique equilibrium involves all players abstaining from vote trading. Then, we move on to examine whether an alternative electoral system, which avoids some deficiencies associated with plurality rule (e.g., severe discontinuities in the outcome function), can guarantee a generic existence of an equilibrium with vote trading. To this end we consider a power-sharing system, in which the decision-making power is distributed among the two competing parties in proportion to their vote shares. Similar frameworks have been extensively employed in the political economics literature[2], but, to the best of our knowledge, only Xefteris and Ziros (2017) have studied vote trading in such systems.[3] In such a setup the whole distribution of votes is crucial for the determination of policies and a voter's utility is proportionally increasing in the vote share of her favorite party.

---

[2]See, for instance, Lijphart (1984), Austen-Smith and Banks (1988), Ortuño-Ortín (1997), Alesina and Rosenthal (2000), Llavador (2006), Sahuguet and Persico (2006), Herrera et al. (2014), Iaryczower and Mattozzi (2013), Matakos et al. (2016) among others.

[3]That paper considered incomplete information regarding voters' preferences –which, as explained above, enhances the prospects of equilibrium existence–, symmetric uncertainty –in the sense that no party is expected to be supported by a more voters than the other– and restricted strategy spaces –in the sense that vote buyers were not allowed to bid any arbitrary monetary amount. In this paper, we consider instead perfect information, arbitrary voters' preferences and unrestricted strategy spaces: each voter is free to bid any monetary amount she deems best.

With respect to our results we provide a full characterization of Nash equilibria under the power-sharing electoral rule. Apart from the no-trade equilibrium we show that, for every generic preference profile, there exists a unique full-trade equilibrium. In this equilibrium only two players, the strongest supporter of each party, are buying votes whereas all the other players prefer to sell their votes. We moreover, show that partial-trade equilibria might non-generically exist. That is, depending on the precise preference profile we might additionally have equilibria in which trade occurs, but not among all players. In these equilibria, again, only the strongest supporter of each party buys votes, some players sell their votes while the rest -with preference intensities within a party-specific interval- prefer to refrain from vote trading and simply vote for their preferred party during the elections. Hence, in all equilibria with active trading the competition between two vote buyers determines in a large degree the final vote shares of the two parties. It should be noted that similar results with respect to the number of vote traders have been obtained by the means of alternative equilibrium concepts and institutional settings in Casella et al. (2012), Casella et al. (2014) and Casella and Turban (2014), where the two voters with the highest valuations buy votes and all other voters sell their votes.[4]

Concerning the welfare properties of vote trading, the earlier literature has produced both positive (for example, Buchanan and Tullock, 1962) and negative (for example, Riker and Brams, 1973) results, focusing on Benthamite/utilitarian criteria. More recently, Casella et al. (2012), Casella et al. (2014), and Casella and Turban (2014) showed that vote trading is welfare decreasing when compared to plurality rule without vote trading, in the sense that it implements less frequently the alternative that maximizes the sum of individual cardinal utilities. On the other hand, Xefteris and Ziros (2017), in an incomplete information variant of the current framework, proved that vote trading is welfare improving because when vote trading is allowed all players' expected utility is larger compared to the case where vote trading is prohibited. The welfare analysis of vote trading under complete information is orthogonal to all these results, since in certain cases, vote trading leads to a larger social utility compared to simple voting, and in some

---

[4]In Casella et al. (2014) and Casella and Turban (2014) these two vote buyers are necessarily one from each party, whereas in Casella et al. (2012) there is no such restriction.

others not.

In fact vote trading under complete information in power-sharing systems is found to implement a different welfare optimum: it achieves *proportional justice in policy* with respect to the two buyers. That is, the ratio of the parties' vote shares is equal to the ratio of the preference intensities of their strongest supporters. The origins of proportional justice with respect to a distributional problem involving two individuals may be traced back to Aristotle and it has been recently studied by Broome (1984, 1991) and Segal (2006), in a more standard economics' context.[5] This result is arguably of independent interest as, to the best of our knowledge, this is the first known mechanism that takes into account only each party's stronger supporter and is "fairly biased" –in the context of Segal (2006)– towards the one with the most intense preferences. Of course this welfare analysis holds specifically for our unique full-trade equilibrium, and does not extend to other outcomes possibilities. But since in our complete information environment, the full-trade equilibrium is the unique one that exists for every generic preference profile, it is the only reasonable candidate for a comprehensive welfare analysis: other equilibria might only deliver insights for merely a fraction of possible preference distributions.

The remainder of the paper is organized as follows. In Section 2 we develop the model, in Section 3 we present the results and in Section 4 we discuss the welfare properties of the full-trade equilibrium. Some further comments follow in Section 5.

2   The model

We consider a committee of $n > 2$ voters and two parties (or policy alternatives), L and R. Voters fall into two types depending on their ordinal preferences, $t_i \in \{L, R\}$, where $t_i = L$ if $L \succ R$ and $t_i = R$ if $R \succ L$ for voter i. Hence we have two sets of voters with cardinality $n_L \geq 1$ and $n_R \geq 1$ respectively, where $n_L + n_R = n$. Each voter i is also characterized by her distinct intensity parameter $w_i > 0$ and let us denote with $\bar{w}^L$, $\bar{w}^R$ the valuations of the each party's strongest supporter. All voters have one vote each and

---

[5] We should note that we consider implementation in the limit: we demonstrate that when the number of voters becomes arbitrarily large, the full-trade equilibrium is such that the ratio of the vote shares of the two parties converges to the ratio of the preference intensities of the two buyers.

concerning their monetary endowments we assume that they are significantly large (i.e., no individual faces liquidity constraints).[6]

The timing of the game is as follows: initially vote trading takes place; next players cast the amount of votes they have after the vote-trading stage in order to maximize their utilities; finally the payoffs of all players are computed.

Vote trading takes place in a trading post where each voter chooses whether to offer her whole vote for sale, $q_i \in \{0, 1\}$, or whether to place a monetary bid, $b_i \geq 0$, for purchase of votes, with the restriction that a voter is not allowed to be active in both sides of the market. Hence, the strategy set of voter i is $S_i = \{(b_i, q_i) : b_i \geq 0, q_i \in \{0, 1\}, b_i q_i = 0\}$.

Given a strategy profile $(b, q) \in \prod_{i \in I} S_i$ let B, Q denote aggregate bids and offers of all voters and B^T, Q^T denote aggregate bids and offers of all voters of type $T \in \{L, R\}$. In addition, for each i define B_{-i}, Q_{-i} as aggregate bids and offers of all voters other than voter i and $B^{t_i}_{-i}$, $Q^{t_i}_{-i}$ as aggregate bids and offers of all voters of type t_i other than voter i.

For a strategy profile with $BQ \neq 0$ the price of a vote is given by the fraction

$$p = \frac{B}{Q}.$$

The amount of votes, x_i, that a voter ends up with after the vote-trading stage and her net monetary transfers, m_i, are

$$(x_i, m_i) = \begin{cases} (1 + b_i/p, -b_i) & \text{if she is a vote buyer,} \\ (0, p) & \text{if she is a vote seller,} \\ (1, 0) & \text{if she chooses not to trade.} \end{cases}$$

The interpretation of this allocation rule is that the amount of votes offered for sale is distributed among vote buyers in proportion to their bids, whereas the monetary bids are equally distributed among vote sellers. In our framework, votes are perfectly divisible

---

[6]This is a standard assumption in the vote-trading (e.g., Casella et al., 2014; Casella and Turban, 2014) and the vote-buying literature (e.g., Lalley and Weyl, 2016; Goeree and Zhang, 2016).

and, hence, a trader might end up having a non-integer number of votes. This is perfectly legitimate in our framework as, both under plurality rule and in a power-sharing system, all that matters is the share and not the actual number of votes that each alternative receives.

The vote shares $v_L \in [0,1]$ and $v_R = 1 - v_L$ of the two parties after vote trading are given by

$$v_L = \frac{1}{n}\left(n_L - Q^L + B^L/p\right)$$

and

$$v_R = \frac{1}{n}\left(n_L - Q^R + B^R/p\right).$$

Under plurality rule the utility of voter i after the election is given by

$$u_i = \theta \times w_i + m_i, \text{ where } \theta = \begin{cases} 1 & \text{if } v_{t_i} > 1/2, \\ \frac{1}{2} & \text{if } v_{t_i} = 1/2, \\ 0 & \text{if } v_{t_i} < 1/2. \end{cases}$$

In case of a power-sharing system the utility of voter i after the election is given by

$$u_i = v_{t_i} \times w_i + m_i.[7]$$

Given that the behavior of players in the voting stage is completely unambiguous (i.e., casting all votes that one has to her preferred party is her dominant strategy), we essentially have an one-shot game and an equilibrium is defined as a profile of pure strategies $(b,q) \in \prod_{i \in I} S_i$ that forms a Nash equilibrium.

---

[7]This formulation of voters' preferences is perfectly compatible with other papers studying power-sharing systems (see, for example, Herrera et al., 2014; Iaryczower and Mattozzi, 2013).

3 Main Results

Let us first discuss the difficulties that arise in our vote-trading model under plurality rule. Proposition 1 highlights the problem of nonexistence of equilibrium with active trading in a majoritarian environment.

Proposition 1 *No trade is the only equilibrium under plurality rule.*

Proof. We can straightforwardly claim that no trade is always an equilibrium, as abstaining from trade is the best response of an individual when all other voters choose not to trade.

Now let us show that any equilibrium with vote trading must involve vote buyers of both types. Suppose, on the contrary, that there are only type L vote buyers. In such an eventuality a vote buyer of type L will always deviate by reducing her bid, hence no equilibrium involves vote buyers of only one type.

With vote trading two cases arise as possible outcomes; in Case 1 there is a tie between the two alternatives, that is both parties have the same number of votes, whereas in Case 2 one of the two alternatives wins. Case 1 cannot be an equilibrium outcome, as in such an eventuality a vote buyer will be willing to slightly increase her bid, which in turn will result in her favorite alternative acquiring a majority position and thus to a substantial increase in her payoff.

In Case 2, suppose that party L is the plurality winner after the vote-trading stage. Then a vote buyer of type R will deviate to selling her vote. In other words, there will be no vote buyers of type R, but, on the contrary, all type R individuals will be willing to sell their votes. Similarly, if party R is the plurality winner after the vote-trading stage a type L vote buyer will deviate to selling her vote. Hence, there cannot be an equilibrium with vote trading resulting in a party being the plurality winner.

Hence, under plurality rule no equilibrium involves vote trading. ∎

We now turn our attention to the power-sharing systems where, apart from no trade, we show that equilibria involving trade always exist. We start by characterizing the behavior of players in such equilibria (Lemma 1 and Lemma 2) and then we establish that they generically exist (Proposition 1).

Lemma 1 *In a power-sharing system, an equilibrium with vote trading is such that exactly two voters –the strongest supporter of each party– buy votes.*

Proof. Notice that if in an equilibrium we have vote trading, it must be the case that at least a player sells her vote and at least one player bids a positive monetary amount: i.e., $BQ > 0$. First, we show that in an equilibrium (b, q) with vote trading only one voter from each party buys votes. Consider a voter i of type L with valuation w_i who buys $b_i > 0$ votes in some equilibrium (b, q). This individual faces the following –constrained– problem

$$\max_{b_i \geq 0} u_i = \frac{1}{n} \left(n_L - Q^L_{-i} + (b_i + B^L_{-i}) \frac{Q^L_{-i} + Q^R}{b_i + B^L_{-i} + B^R} \right) w_i - b_i,$$

which is well-behaved in $b_i \in [0, +\infty)$.[8] By solving this problem we get that b_i must be such that $b_i = -B^L_{-i} - B^R + \left(\frac{w_i Q B^R}{n} \right)^{1/2}$. Hence, for the equilibrium profile (b, q), the total bids of type L individuals can be written as

$$B^L = -B^R + \left(\frac{w_i Q B^R}{n} \right)^{1/2}. \tag{1}$$

Now assume that another voter j of type L, with valuation $w_j \neq w_i$, also buys votes $b_j > 0$ in equilibrium (b, q). Solving the maximization problem for this voter one can derive that b_j is such that $b_j = -B^L_{-j} - B^R + \left(\frac{w_j Q B^R}{n} \right)^{1/2}$ and hence the corresponding total bids of type L individuals are

$$B^L = -B^R + \left(\frac{w_j Q B^R}{n} \right)^{1/2}. \tag{2}$$

---

[8]We use the term constrained since the presented maximization does not consider the possibility of selling one's vote, and we consider that it is well behaved in b_i in the sense that, as long as at least one other player sells her vote, it is well defined, differentiable, and concave in $[0, +\infty)$.

However in order for the expressions (1) and (2) to hold at the same time we need $w_j = w_i$, which contradicts our assumption that each voter of type L is characterized by a distinct preference intensity. Hence, there is no equilibrium (b, q) with two or more buyers of type L. Similarly, in equilibrium (b, q), only one individual of type R may be buying votes.

Next we argue that there can be no equilibrium with active trading if there is only one buyer. In other words, in an equilibrium (b, q) with vote trading there must be one buyer of each type. Indeed, consider an equilibrium (b, q) such that there is only one buyer of type L and no buyer of type R. Then, from the above expressions we get that the type L buyer is submitting a zero bid, which leads to a contradiction.

Now we proceed to show that in an equilibrium (b, q) with vote trading, the two vote buyers must be the strongest supporters of each party. Suppose, on the contrary, that a voter i with valuation $w_i < \bar{w}^L$ is the only vote buyer of type L. In an equilibrium (b, q) with vote trading in which i is a vote buyer, it must be true that $b_i = -B^R + \left(\dfrac{w_i Q B^R}{n} \right)^{1/2}$. But given this profile of strategies (b, q), the best response bid of the strongest supporter of type L will be $\bar{b} = -b_i - B^R + \left(\dfrac{\bar{w}^L Q B^R}{n} \right)^{1/2} = \left(\dfrac{\bar{w}^L Q B^R}{n} \right)^{1/2} - \left(\dfrac{w_i Q B^R}{n} \right)^{1/2} > 0$, that is, she also submits a positive bid for purchase of votes, which is a contradiction to the fact that there can be no equilibrium with two buyers of the same type. Hence, in an equilibrium (b, q) with vote trading only the individual of type L with the highest valuation buys votes. Similarly, only the strongest supporter of type R buys votes.

So if an equilibrium (b, q) with vote trading exists it must be such that exactly two voters –the strongest supporter of each party– buy votes. ∎

Let us note that in an equilibrium (b, q) with vote trading, it must hold that $\dfrac{B^L}{B^R} = \dfrac{\bar{w}^L}{\bar{w}^R}$. Indeed, the equilibrium bids of type L and type R are $B^L = \dfrac{Q(\bar{w}^L)^2 \bar{w}^R}{n(\bar{w}^L + \bar{w}^R)^2}$ and $B^R = \dfrac{Q\bar{w}^L(\bar{w}^R)^2}{n(\bar{w}^L + \bar{w}^R)^2}$; and the total bids are $B = \dfrac{Q\bar{w}^L \bar{w}^R}{n(\bar{w}^L + \bar{w}^R)}$.

Now we turn our attention to the other side of the market and we examine the behavior of vote sellers in a full-trade equilibrium. That is, in an equilibrium in which every player either sells her vote or places monetary bids to acquire more votes.

Lemma 2 *In a full-trade equilibrium of a power-sharing system, an individual i of type T chooses to sell her vote if and only if $w_i < \bar{w}^T$.*

Proof. This lemma is in fact a trivial corollary of Lemma 1. If individual i of type L is characterized by an intensity parameter $w_i < \bar{w}^L$ then, by Lemma 1, she is not placing monetary bids to acquire more votes. If we are in a full-trade equilibrium and i is not buying votes then she must be selling her vote. This establishes the "if" direction. To establish also the "only if" direction, notice that if individual i of type L, sells her vote in a full-trade equilibrium, she must be characterized by an intensity parameter $w_i < \bar{w}^L$, as, by Lemma 1, we know that the the strongest supporter of type L never sells votes in any equilibrium with vote trading, including the full-trade one. ∎

The next Proposition proves the existence of a unique full-trade equilibrium for every possible parameter values. In such an equilibrium only the two preference intensities of the players who decided to buy votes actually shape the voting outcome.

Proposition 2 *In a power-sharing system, for any distribution of intensity parameters there exists a unique full-trade equilibrium and it is such that exactly two voters –the strongest supporter of each party– buy votes and all other players offer their vote for sale.*

Proof. From Lemma 1 we have established that in an equilibrium (b, q) with vote trading only the two players with the most intense preferences, one from each party, buy votes. In a full-trade equilibrium (where $Q = n - 2$) the equilibrium bids of the two vote buyers are
$B^L = \dfrac{(n-2)}{n} \dfrac{(\bar{w}^L)^2 \bar{w}^R}{(\bar{w}^L + \bar{w}^R)^2}$, $B^R = \dfrac{(n-2)}{n} \dfrac{\bar{w}^L (\bar{w}^R)^2}{(\bar{w}^L + \bar{w}^R)^2}$ yielding $B = \dfrac{(n-2)}{n} \dfrac{\bar{w}^L \bar{w}^R}{(\bar{w}^L + \bar{w}^R)}$.

The type L vote buyer is willing to deviate from this strategy to selling her vote if

$$\left(\frac{(n-1)\bar{w}^L + \bar{w}^R}{n(\bar{w}^L + \bar{w}^R)} \right) \bar{w}^L - \frac{(n-2)}{n} \frac{(\bar{w}^L)^2 \bar{w}^R}{(\bar{w}^L + \bar{w}^R)^2} < \frac{(n-2)}{n(n-1)} \frac{\bar{w}^L (\bar{w}^R)^2}{(\bar{w}^L + \bar{w}^R)^2}$$

resulting in $(n-1) \left(\bar{w}^R + (n-1)\bar{w}^L \right)^2 < 0$, which is impossible. Hence, the vote buyer of type L is not willing to deviate to selling her vote. Similarly, the vote buyer of type R is also not willing to deviate to selling her vote. Straightforwardly, none of them wishes to deviate to any other bidding amount in $[0, +\infty)$ given the concavity of

the maximization problem $\max\limits_{b_i \geq 0} u_i$, and hence the posited strategies are their unique best responses.

Let us now turn our attention to vote sellers. In a full-trade equilibrium an individual i of type L with intensity parameter $w_i < \bar{w}^L$ sells her vote if

$$\frac{1}{n}\left(n_L - (Q_{-i}^L + 1) + \frac{B^L}{B}(Q_{-i} + 1)\right)w_i + \frac{B}{Q_{-i} + 1} > \frac{1}{n}(n_L - Q_{-i}^L + \frac{B^L}{B}Q_{-i})w_i,$$

which reduces to $w_i < \dfrac{nB^2}{B^R(Q_{-i} + 1)}$. Substituting for the best response bids, we can calculate that in a full-trade equilibrium ($Q = Q_{-i} + 1 = n - 2$) individual i sells her vote if $w_i < \bar{w}^L$, which is always the case. That is, the best response of such an individual is to sell her vote. Similarly, we get that an individual i of type R sells her vote if $w_i < \bar{w}^R$, which is again always the case. So we established that all individuals, other than the two individuals with highest valuations for each party, are selling their votes. Hence, all voters participate in vote trading. ∎

It should be stressed here that as far as the comparative statics of the equilibria of the game are concerned we notice that the bids of vote buyers are increasing in the total number of voters and in the valuations $\bar{w}^L, \bar{w}^R$. In the Appendix we characterize all non-generic equilibria of our game and discuss their properties.[9]

4 Welfare analysis

"The just [...] involves at least four terms; for the persons for whom it is in fact just are two, and the things in which it is manifested, the objects distributed, are two. [...] and the ratio between one pair is the same as that between the other pair; for there is a similar distinction between the persons and between the things." Aristotle (Nicomachean Ethics, V.3)

Concerning social welfare the adoption of the standard utilitarian approach of maximizing total welfare does not yield definite results as to whether our vote-trading approach

---

[9]These equilibria exhibit partial vote trading –the strongest supporter of each party buys votes, some players sell their votes and others prefer to refrain from trading and just vote– and exist only for particular preference profiles.

produces equilibrium outcomes that are welfare increasing when compared to simple voting in power-sharing system. It is clear that voting alone yields positive vote shares for both parties, whereas the Benthamite/utilitarian optimum dictates having one party (the party with the greater sum of supporters' valuations) getting all votes. In order for the full-trade equilibrium outcome to be closer to the utilitarian optimum than the simple voting outcome, a necessary condition is that the voter with the highest overall valuation is a voter of the party with the greater sum of individual valuations. Consequently both positive and negative results are likely to be produced by our vote-trading framework.

However, one can discuss the properties of this full-trade equilibrium in terms of other –non-utilitarian– welfare optima. To this end, we identify an alternative welfare criterion that is based on Aristotle's notion of proportional justice and suggests an outcome that takes into account only the valuations of the two most opposing voters (i.e., the strongest supporter of each party).

Definition 1 (Proportional Justice) *The ratio of vote shares of the two parties must be equal to the ratio of the preference intensities of the two most opposing voters.*

The following result exhibits that as the population becomes arbitrarily large the full-trade equilibrium yields vote shares for the two alternatives that satisfy the proportional justice.

Lemma 3 *The full-trade equilibrium implements proportional justice when $n \to \infty$.*

Proof. The vote share of the alternative L in the full-trade equilibrium is $v_L = \dfrac{(n-1)\bar{w}^L + \bar{w}^R}{n(\bar{w}^L + \bar{w}^R)} = \dfrac{\bar{w}^L}{(\bar{w}^L + \bar{w}^R)} + \dfrac{\bar{w}^R - \bar{w}^L}{n(\bar{w}^L + \bar{w}^R)}$ and hence $\lim_{n\to\infty} v_L = \dfrac{\bar{w}^L}{(\bar{w}^L + \bar{w}^R)}$ and $\lim_{n\to\infty} v_R = 1 - \dfrac{\bar{w}^L}{(\bar{w}^L + \bar{w}^R)} = \dfrac{\bar{w}^R}{(\bar{w}^L + \bar{w}^R)}$. Therefore, for $n \to \infty$ we have $\dfrac{v_L}{v_R} = \dfrac{\bar{w}^L}{\bar{w}^R}$. ∎

5   Concluding remarks

This paper studies a simple vote market with strategic players in order to exhibit that many difficulties with respect to vote trading are not solely due to complete information;

it's the combination of market mechanisms, electoral systems and levels of information that creates such obstacles. We provide clear-cut results about the existence of equilibrium involving trade in a power-sharing system as opposed to the simple plurality rule where no trade occurs. Moreover, under a power-sharing system *full-trade* is always an equilibrium, which establishes the willingness of voters to participate in market institutions that complement elections. Notably, this equilibrium implements proportional justice in policy and hence our setup serves as an example of how this alternative notion of welfare optimality can be applied in issues of political economics.

6 Appendix

In this part of the paper we focus on partial-trade equilibria, which are non-generic as they exist only for particular preference profiles. In these equilibria, the strongest supporter of each party buys votes, some players sell their votes and others prefer to vote without participating in vote trading. The next lemma exhibits that in a partial-trade equilibrium a player who chooses to abstain from vote trading must have a valuation within a certain party-specific interval.

Lemma 4 *In a power-sharing system in any equilibrium (b, q) with $m < n - 2$ vote sellers, for an individual i of type T who chooses not to trade it must be the case that $w_i \in \left(\frac{m}{m+1} \bar{w}^T, \bar{w}^T \right)$.*

Proof. From Lemma 1 we have established that in any equilibrium (b, q) with vote trading only the two players with the most intense preferences, one from each party, buy votes. For the case of $m < n - 2$ vote sellers, their equilibrium bids are $B^L = \frac{m(\bar{w}^L)^2 \bar{w}^R}{n(\bar{w}^L + \bar{w}^R)^2}$, $B^R = \frac{m \bar{w}^L (\bar{w}^R)^2}{n(\bar{w}^L + \bar{w}^R)^2}$ yielding $B = \frac{m}{n} \frac{\bar{w}^L \bar{w}^R}{(\bar{w}^L + \bar{w}^R)}$.

Let us now turn our attention to vote sellers. Consider an individual j of type L with intensity parameter $w_j < \bar{w}^L$ who sells her vote in a partial-trade equilibrium and all other players know it (that is, $Q = Q_{-j} + 1 = m$). From Lemma 1 this type L individual will never deviate to buying votes. Moreover, she is willing to deviate to abstaining from vote trading only if

$$\frac{1}{n} \left(n_L - (Q_{-j}^L + 1) + \frac{B^L}{B}(Q_{-j} + 1) \right) w_j + \frac{B}{Q_{-j} + 1} < \frac{1}{n}(n_L - Q_{-j}^L + \frac{B^L}{B} Q_{-j}) w_j,$$

which reduces to $w_j > \frac{nB^2}{B^R(Q_{-i} + 1)}$. Substituting for the best response bids, we can calculate that individual j deviates to abstaining from vote trading if $w_j > \bar{w}^L$, which is a contradiction to our assumption about her preferences. That is, individual j never deviates from her strategy. Similarly, we get that a type R vote seller will never deviate.

Now let us now consider an individual i of type L with valuation $w_i < \bar{w}^L$ who does not sell her vote (plays $q_i = 0$) in a partial-trade equilibrium and all other players know it

(that is, $Q = Q_{-i} = m$). From Lemma 1 we have established that this type L individual will never deviate to buying votes. Moreover, she prefers not to deviate from this strategy to selling her vote if

$$\frac{1}{n}\left(n_L - (Q^L_{-i} + 1) + \frac{B^L}{B}(Q_{-i} + 1)\right)w_i + \frac{B}{Q_{-i} + 1} < \frac{1}{n}(n_L - Q^L_{-i} + \frac{B^L}{B}Q_{-i})w_i,$$

which reduces to $w_i > \dfrac{nB^2}{B^R(Q_{-i} + 1)}$. Substituting for the best response bids of the two buyers we derive that for this voter it must be the case that $w_i > \frac{m}{m+1}\bar{w}^L$ or $w_i \in (\frac{m}{m+1}\bar{w}^L, \bar{w}^L)$. Similarly, for an individual i of type R who abstains from vote trading in a partial-trade equilibrium it must be the case that $w_i \in (\frac{m}{m+1}\bar{w}^R, \bar{w}^R)$. $\blacksquare$

References

[1] Alesina, A., & Rosenthal, H. (2000). Polarized platforms and moderate policies with checks and balances. *Journal of Public Economics*, 75 (1), 1-20.

[2] Austen-Smith, D., and J. Banks. (1988). "Elections, coalitions, and legislative outcomes." *American Political Science Review,* 82 (2), 405-422.

[3] Broome, J. (1984). "Selecting people randomly." *Ethics,* 95 (1), 38–55.

[4] Broome, J. (1991). "V–fairness." *Proceedings of the Aristotelian Society,* 91, 87–102.

[5] Buchanan, J.M., and G. Tullock. (1962). *The Calculus of Consent: Logical foundations of constitutional democracy.* Ann Arbor: University of Michigan Press.

[6] Casella, A., A. Llorente-Saguer, and T. Palfrey. (2012). "Competitive equilibrium in markets for votes." *Journal of Political Economy,* 120 (4), 593-658.

[7] Casella, A., T. Palfrey, and S. Turban**.** (2014). "Vote trading with and without party leaders." *Journal of Public Economics,* 112, 115-128.

[8] Casella, A., and S. Turban. (2014). "Democracy undone. Systematic minority advantage in competitive vote markets." *Games and Economic Behavior,* 88, 47-70.

[9] Goeree, J.K., and J. Zhang. (2017). "One man, one bid." *Games and Economic Behavior,* 101, 151-171.

[10] Herrera, H., M. Morelli, and T. Palfrey. (2014). "Turnout and power sharing." *The Economic Journal,* 124 (574), F131-F162.

[11] Iaryczower, M., and A. Mattozzi. (2013). "On the nature of competition in alternative electoral systems." *The Journal of Politics,* 75 (3), 743-756.

[12] Lalley, S.P., & Weyl, E.G. (2016). "Quadratic voting." *SSRN,* 2003531.

[13] Lijphart, A. (1984). *Democracies: Patterns of Majoritarian and Consensus Government in Twenty-One Countries.* New Haven: Yale University Press.

[14] Llavador, H. (2006). "Electoral platforms, implemented policies, and abstention." *Social Choice and Welfare,* 27 (1), 55-81.

[15] Matakos, K., O. Troumpounis, and D. Xefteris. (2016). "Electoral rule disproportionality and platform polarization." *American Journal of Political Science*, 60 (4), 1026-1043.

[16] Ortuño-Ortín, I. (1997). "A spatial model of political competition and proportional representation." *Social Choice and Welfare,* 14 (3), 427-438.

[17] Philipson, T., and J. Snyder. (1996). "Equilibrium and efficiency in an organized vote market." *Public Choice,* 89 (3), 245-265.

[18] Piketty, T. (1994). "Information aggregation through voting and vote trading." MIT, unpublished.

[19] Riker, W.H., and S.J. Brams. (1973). "The paradox of vote trading." *The American Political Science Review,* 67 (4), 1235-1247.

[20] Sahuguet, N., and N. Persico. (2006). "Campaign spending regulation in a model of redistributive politics." *Economic Theory,* 28 (1), 95-124.

[21] Segal, U. (2006). "Fair bias." *Economics and Philosophy,* 22 (2), 213-229.

[22] Shapley, L., and M. Shubik. (1977). "Trade using one commodity as a means of payment." *Journal of Political Economy,* 85 (5), 937-968.

[23] Shubik, M. (1973). "Commodity money, oligopoly, credit and bankruptcy in a general equilibrium model." *Western Economic Journal,* 11 (1), 24-38.

[24] Xefteris, D., and N. Ziros. (2017). "Strategic vote trading in power-sharing systems." *American Economic Journal: Microeconomics* (forthcoming).