Trading in the core and Walrasian price in a random exchange market

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Abstract

We study a random matching economy, where the participants have Cobb-Douglas utility functions. At each time period a pair of participants is selected and may choose to trade two goods. Under the appropriate symmetry conditions, depending on the relation between the initial distribution of endowments and the agents preferences, we show that the sequence of bilateral prices converges to the Walrasian price for this economy. Additionally, we study the effect of an asymmetry in the preferences on the difference between the bilateral price and the Walrasian price for this economy. We extend this model by associating a selfishness factor to each participant in this market. This brings up a game alike the prisoner‘s dilemma. We discuss the effect of the selfishness on the increase in utility.

Keywords: Bilateral price, Walrasian price, random exchange market, Cobb-Douglas, General equilibrium theory

This working paper, adapted from [6, 9], studies the problem of providing strategic foundations of general equilibrium theory, which has been a long standing problem of crucial importance in economic theory. The main objective of this strand of thought is to provide a market game, which is realistic enough to describe the behaviour of agents in real market situations, such that the equilibrium of this game approaches under certain conditions the competitive equilibrium for the same market. A particularly fruitful way of pursuing this line of research is through the use of dynamic matching games, in which agents meet randomly, and exchange rationally, according to local rules. Such attempts started with the work of Edgeworth in 1881 (see [5]) and were further advanced by a number of researchers [1, 2, 3, 4, 7, 10, 11, 12, 13].

In [6, 9], we studyied conditions under which the equilibrium of a market game, defined by a random matching game (see Binmore and Herrero [4]), approaches the equilibrium of a fully competitive Walrasian model. The random matching game, consists of agents, paired at random, who exchange goods at the bilateral Walras equilibrium price, determined by their Coob-Douglas utility functions [8, 14]. We provided some results [6, 9] on the expectation of the limiting bilateral price p_∞, and how this compares to the Walrasian price. In particular, under some rather general symmetry conditions on the initial endowments of the agents and distribution of initial preferences, we show that the expectation of the logarithm of p_∞ equals the logarithm of the Walrasian price for the same initial endowments of the agents. Hence, even though the agents meet and trade myopically in random pairs, they somehow "self-organize" and the expected limiting price equals that of a market where a central planner announces prices and all the agents conform to them through utility maximization, as occurs in the Walrasian model. The main reason why organizing behaviour is observed is the symmetry in the endowments and preferences of the agents that poses global constraints in the market, in the sense that it enforces each agents to have a *mirror*, or a *dual agent*. We aim to extend these results, by studying the effect of an asymmetry in the agent's preferences. Furthermore, we will associate a selfishness factor to each participant in this market. This will bring up a game alike the prisoner's dilemma, where trade may occur to a point in the core different from the bilateral equilibrium, with advantage to the more selfish participant, or trade may not even be allowed. We discuss the effect of the selfishness on the increase in utility of the participants.

1 The Walrasian model

We look at a pure exchange economy $(\Im, X_i, \succeq_i, w_i)$ where $\Im$ is the population of agents, each of them characterized by a consumption set $X_i \in R^2_+$ and the agents preferences $\succeq_i$. So, an exchange economy in which some given amounts of goods X and Y are distributed among N agents (agent i owns an initial endowment x_i, y_i of good X and Y respectively) is considered. Note that the initial endowments $(x_i,\, y_i) \in int(X_i)$. We consider agents A_i with preferences that can be described by Cobb-Douglas type utility functions $U_i(x_i, y_i) = x_i^{\alpha_i}\, y_i^{1-\alpha_i}$ where the utility function is determined by the preference α_i of the agent A_i.

The Walrasian general equilibrium model assumes that consumers are passive price takers. They regard a given set of prices as parameters in determining their optimal net demands and supplies. The equilibrium price is such that the market clears. Then the consumers change their endowments by the allocations determined by the equilibrium price. A mechanism that leads to the equilibrium price can be achieved, for instance, through an auctioneer who collects all the offers and demands for each good and adjusts the price vector to clear the market. Let $\mathcal{A} = \{A_1, A_2, ..., A_N\}$ be a collection of N agents. The agent A_i starts with a preference α_i and an initial endowment (x_i, y_i). The Walrasian equilibrium price $p(\omega_A)$ is given by

$$p(\omega_A) = \frac{\sum_{i=1}^{N} \alpha_i y_i}{\sum_{i=1}^{N} (1-\alpha_i) x_i} \tag{1}$$

The bilateral trade occurs when $N = 2$, and the Walrasian equilibrium price is called the bilateral equilibrium price. The bilateral trade is the well known scenario analyzed in the Edgeworth box diagram (see Figure 1). The horizontal axis represents the amount of good X and the vertical represents the amount of good Y of participant i. The point (x_i+x_j, y_i+y_j) is the vertex opposite to the origin. The horizontal and vertical lines starting at the opposite vertex are the axes representing the amounts of good X and Y, respectively, of participant j. We represent in the Edgeworth box the indifference curves for both participants passing through the point corresponding to the initial endowments of both participants. The core is the curve where the indifference curves of both participants are tangent and such that the utilities of both participants are greater or equal to the initial ones. The bilateral price determines a segment of allocations that pass through the point corresponding to the initial endowments. The interception of this segment with the core determines the new allocations $(\hat{x}_i, \hat{y}_i) = (\alpha_i\,(x_i + 1/py_i)\,, (1-\alpha_i)(y_i + px_i))$ of the two participants.

2 The p-statistical duality

We introduce the concept of *duality* in the market for the agents. Statistical duality for the agents guarantees that the prices observed in the random matching Edgeworthian economy coincide in expectation with those of the Walrasian economy.

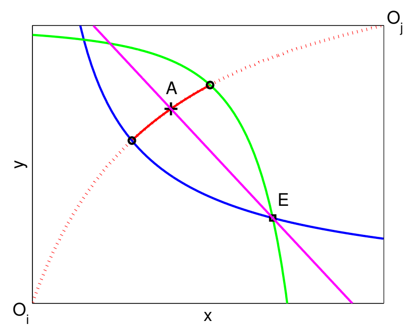

Figure 1: Edgeworth Box with the indifference curves for participant i (blue convex curve) and j (green concave curve). The red curve is the core and the red dots represent the contract curve. The slope of the pink segment line is the bilateral equilibrium price. The interception point (A) of the core with the pink segment line determines the new allocations and the square (E) marks the initial endowments. Reproduced from [9]

We assume that a collection of agents is completely characterized by their preferences α, and their endowments (x, y) in the 2 goods. We define a probability distribution function $f(\alpha, x, y)$, on (α, x, y) space which provides the probability that an agent has preferences in $(\alpha, \alpha + d\alpha) \times (x, x + dx) \times (y, y + dy)$. We assume that the probability distribution has compact support, and the support in (x, y) is bounded away from zero.

Definition 1 *We say that a market satisfies the* **p - statistical duality** *condition if the probability function has the symmetry property*

$$f(\alpha, x, y) = f\left(1 - \alpha, \frac{y}{p}, p\,x\right)$$

where $p \in \mathbb{R}^+$.

The p-statistical duality property means that each agent with characteristics (α, x, y) has a mirror agent with characteristics $(1 - \alpha, y/p, p\,x)$ with the same probability under f. The class of probability functions $f(\alpha, x, y)$ of the form $f_1(\alpha)f_2(x, y)$ with the property that $f_1(\alpha) = f_1(1 - \alpha)$ and $f_2(x, y) = f_2(y/p, p\,x)$ satisfies the p-statistical duality. A common probability function f_2, satisfying the above condition, is the uniform distribution. Another common example of a probability function satisfying the p-statistical duality is used in Corollary 1, below, and determines the most well known matching technology used in random matching games with N agents.

3 Random Matching Edgworthian Economies

Each agent A_i starts with a preference α_i and a set of initial endowments $(x_i(0), y_i(0))$. N agents are picked up randomly with or without replacement according to a given probability distribution f satisfying the p-statistical duality. Hence, the initial choice of agents is a random event which will be denoted by $\omega_{\mathcal{A}}$. We can define the random variable $\mathcal{A}(\omega_A) = \{A_1, A_2, \cdots, A_N\}$ which is the initial choice of agents that will participate in the market. Let $p(\omega_{\mathcal{A}})$ be the Walrasian equilibrium price of the market for this collection of agents $A(\omega_{\mathcal{A}})$ of N agents. Let $\mathbb{E}[\ln(p(\omega_{\mathcal{A}}))]$ be the expectation of the logarithm of the Walrasian equilibrium price $p(\omega_{\mathcal{A}})$ computed with respect to the initial consumption bundles of the agents over all the initial collections $\mathcal{A}(\omega_{\mathcal{A}})$ of N agents. Now, let us consider that the collection $A(\omega_A)$ of agents trade in random pairs chosen with the same probability, and after the t trade they end up with a consumption bundle $(x_i(t), y_i(t))$ which are traded in the bilateral equilibrium price $p(t)$, given by the formula (1) with $N = 2$ and with x_i, x_j, y_i, y_j substituted by $x_i(t-1), x_j(t-1), y_i(t-1), y_j(t-1)$. On each trade only two randomly chosen agents i, j exchange goods, and the consumption bundles of all the other agents $k \neq i, j$ remain unchanged, i.e. $(x_k(t-1), y_k(t-1)) = (x_k(t), y_k(t))$. Having initially chosen the group of agents $\mathcal{A} = \{A_1, \cdots, A_N\}$, denote by ω_r the infinite sequence of pairs $\omega_r = (\omega_r(1), \omega_r(2), \cdots)$ where $\omega_r(t)$ is the pair $(i(t), j(t))$, $i(t) \neq j(t)$, corresponding to the pair of agents $(A_{i(t)}, A_{j(t)})$ that have been randomly chosen to trade at time t. A full run of the game is the sequence $\omega_{\mathcal{A}}\omega_r$ that is an initial choice of agents and an infinite sequence of random matchings. A finite time run of the game is the sequence $\omega_{\mathcal{A}}\omega_r|t$ where $\omega_r|t$ is the restriction of ω_r for the first t random matchings.

Let $p(t, \omega_{\mathcal{A}}\omega_v)$ be the bilateral price of the trade at time t determined by the finite run $\omega_{\mathcal{A}}\omega_v$. The bilateral equilibrium price, determines the unique point in the core such that the market "locally" clears. In some sense, the agents behave in a myopic way, interacting only in pairs, and not forseeing the future interactions or keeping memory of their past encounters. Let $\mathbb{E}[\ln(p(t, \omega_{\mathcal{A}}\omega_v))]$ be the expected value of the logarithm of $p(t, \omega_{\mathcal{A}}\omega_v)$ over all initial collections $A(\omega_{\mathcal{A}})$ of agents and over the first t random meetings.

By [7] the limiting price $p(\omega_{\mathcal{A}}\omega_r) = \lim_{t\to\infty} p(t,\omega_{\mathcal{A}}\omega_r)$ exists almost surely and it is a random variable depending on the actual game of the play, that is the exact order of the random pairing of the agents. Let $\mathbb{E}[\ln(p(\omega_{\mathcal{A}}\omega_r))]$ be the expected value of the logarithm of the limiting price $p(\omega_{\mathcal{A}}\omega_r)$ over all the possible distributions of the agents $\mathcal{A}(\omega_{\mathcal{A}})$ and over all random matchings ω_r depending on the actual game of the play, that is the exact order of the random pairing of the agents.

Theorem 1 (p-statistical duality fixed point) *Assume a market consisting of a finite number of agents, such that p-statistical duality holds for the initial endowments, then*

$$\mathbb{E}_t[\ln(p(t,\omega_{\mathcal{A}}\omega_v)] = \mathbb{E}[\ln(p(\omega_{\mathcal{A}}))] = \ln(p), \text{ for all } t \in \{1,2,\ldots,+\infty\}.$$

Furthermore,

$$\mathbb{E}[\ln(p,\omega_{\mathcal{A}}\omega_v)] = \ln(p)$$

In Theorem 1, the advantage of using the logarithm of the price is that if we consider the other good to be the enumeraire, the absolute value of the logarithm of the price keeps the same and just the sign of the value of the logarithm of the price changes. In particular, Theorem 1 is a fixed point theorem for the expected value $\mathbb{E}[\ln(p(t,\omega_{\mathcal{A}}\omega_v)]$ that is proven to be a fixed point along time t.

4 Matching games with dual agents

A relevant and well known example of an economy with the p-statistical duality property is an economy where with probability 1 we start with a sample of $N = 2\,M$ agents where M agents have characteristics

$$(a_i, x_i, y_i)\ , i = 1,\cdots,M\ ,$$

and the remaining M agents have characteristics

$$(a_{i+M}, x_{i+M}, y_{i+M}) = (1-a_i, y_i/p, p\,x_i)\ , i = 1,\cdots,M\ .$$

In other words, in this economy, each agent has a dual agent, i.e. agent i is dual to agent $i+M$ where $i = 1,\cdots,M$.

Corollary 1 *Assume a market consisting of a finite number $N = 2M$ of agents, such that M agents have characteristics $(a_i, x_i, y_i), i = 1, \cdots, M$, and the remaining M agents have characteristics*

$$(a_{i+M}, x_{i+M}, y_{i+M}) = (1 - a_i, y_i/p, p\, x_i) \ , i = 1, \cdots, M \ , \textit{then}$$

$$\mathbb{E}[\ln(p(\omega_{\mathcal{A}}\omega_v)] = \ln(p(\omega_{\mathcal{A}})) = \ln(p), \ \textit{for all} \ t \in \{1, 2, \ldots, +\infty\}.$$

Furthermore,

$$\mathbb{E}[\ln(p(\omega_{\mathcal{A}}\omega_v))] = \ln(p)$$

where $\mathbb{E}$ is the expectation over all possible runs of the game.

5 Trade deviating from the equilibria

The *selfishness model* is similar to the Edgeworth model in which we introduce a new parameter: the selfishness of the participants. If two non selfish participants meet they will trade in the point of the core determined by their bilateral equilibrium price, as in the Edgeworth model. However, if a self-

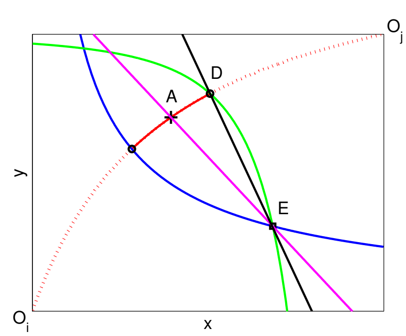

Figure 2: Edgeworth Box with the indifference curves for the selfish participant i (blue convex curve) and non selfish participant j (green concave curve). The red curve is the core and the red dots represent the contract curve. Slope between the pink segment line and the black segment line represent prices that give advantage to the selfish participant. Reproduced from [6]

ish participant meets a non selfish participant, they will trade in a point of the core between the point determined by their bilateral equilibrium price and the interception of the core with the indifference curve of the non selfish participant, see Figure 2. Finally, if both participants are selfish they are penalized by not being able to trade. This is similar to the prisoner's dilemma, where two non cooperative players are penalized, a non cooperative player has a better payoff than a cooperative player, and two cooperative players have a better payoff than when they meet a non cooperative player but still worse than the payoff of the non cooperative player.

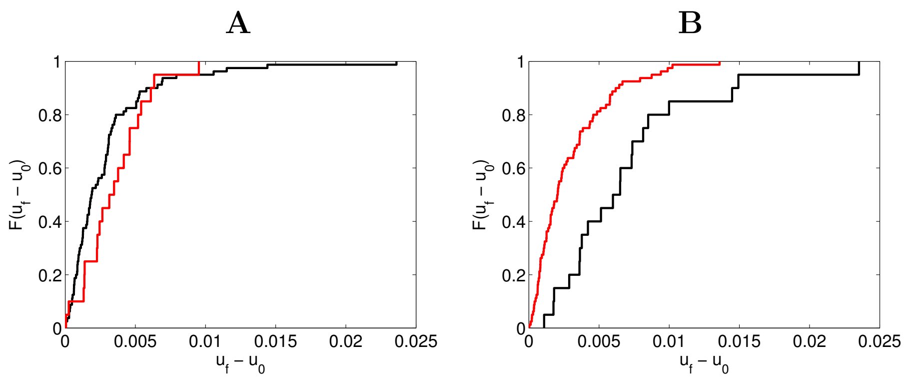

Figure 3: Cumulative distribution function of the variation of the utility (defined as $u_f - u_0$) for the less selfish participants (black) and for the more selfish participants (red), with $g_i \in \{0.25; 0.75\}$. **A**: Simulation with 20 more selfish participants and 80 less selfish participants; **B**: Simulation with 80 more selfish participants and 20 less selfish participants. Reproduced from [6]

We study the effect of the selfishness in the increase of the value of the utility of the participants. Let the variation of the utility function of a participant $u_f - u_0$ be the difference between the limit value of the utility function and the initial value of the utility function. We present, in Figure 3, two cumulative distribution functions of the variation of the utility functions one corresponding to the less selfish participants (black) and the other corresponding to the more selfish participants (red). This function indicates the proportion of participants that have variations of the utility function less than or equal to its argument. In Figure 3 (A) there are 20% of more selfish participants. We observe that the median of the variation of the utility function is higher for the more selfish participants. On the other hand, in

Figure 3 (B) there are 80% of more selfish participants, and we observe that the median of the variation of the utility function is lower for the more selfish participants. We notice that the strategy followed the minority is the one that provides a higher median variation in the utility function.

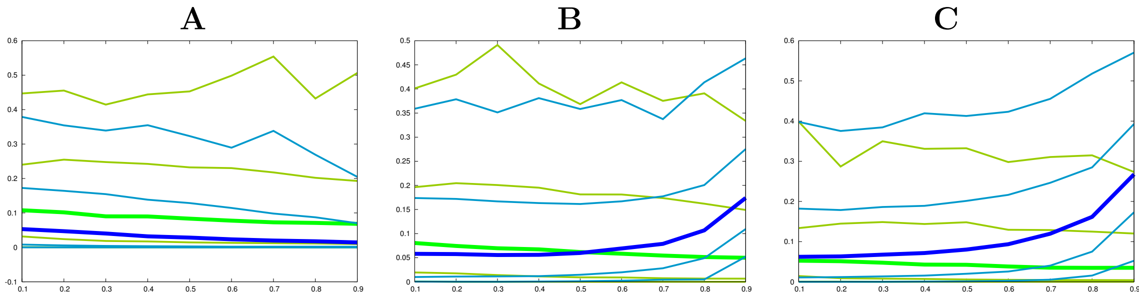

Figure 4: Variation of the utility $(u_f - u_0)$ for the less selfish participants (blue / cyan) and for the more selfish participants (green / yellow). Data from 100 simulations with 100 participants when the fraction of selfish skilled participants is $0.1, 0.2, \ldots, 0.9$. Each set of participants has lines for the minimum percentile, 5%, median (thick line), percentile 95% and maximum. **A**: Advantage to the more selfish participants when $g_i \in \{0; 1\}$; **B**: Advantage to the minority when $g_i \in \{0.25; 0.75\}$; **C**: Advantage to the less selfish participants when $g_i \in \{0.499; 0.501\}$. Reproduced from [6]

When we compare different values assigned to the selfishness of the participants, we observe distinct behaviors. When $g_i = 0$, for the less selfish participants and $g_i = 1$ for the more selfish participants, the trade gives the most advantage possible to the more selfish participants (the final allocation is represented by point D in Figure 2). We see that the more selfish participants have a larger median increase in the utility (Figure 4a) for all fractions between 0.1 and 0.9 of more selfish participants. In the opposite case, when $g_i = 0.499$, for the less selfish participants and $g_i = 0.501$ for the selfish skilled participants, the trade gives a very small advantage to the more selfish participants (the final allocation is near point A in Figure 2). In this case, the more selfish participants have a smaller median increase in the utility (Figure 4c) for all fractions between 0.1 and 0.9 of more selfish participants, due to the impact of the penalization of no trade between them. If we consider $g_i = 0.25$, for the less selfish participants and $g_i = 0.75$ for the more selfish participants, the trade gives an intermediate advantage to the more selfish participants (the final allocation is a point in the core roughly midway between A and D in Figure 2). We observe that the group in minority has the advantage. Namely, for fractions of more selfish participants between 0.1 and 0.4, these have a higher median increase in the utility and for fractions of more selfish participants between 0.6 and 0.9, these have a lower median increase in the utility. For fractions of more selfish participants near 0.5, the median increase in the utility of the more selfish participants is similar to the less skilled participants (Figure 4b).

6 Conclusions

We presented a model of an Edgeworthian exchange economy where two goods are traded in a market place, where the agents preferences are characterized by the Cobb-Douglas utility function. Under symmetry conditions, prices converge to the Walrasian price. Furthermore, we presented a model where each participant has a selfishness factor. When studying the increase of the utility of each participant, we observed that, for some parameter values, it is better to be in minority. For instance, if there are more selfish participants, the increase of the value of their utilities is smaller than the increase of the value of the utilities of the non selfish participants.

Acknowledgments

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References

[1] C.D. Aliprantis, D. J. Brown and O.Burkinshaw, *Edgeworth equilibria*, Econometrica **55**, 1987, 1109-1137

[2] R. Aumann, *Correlated equilibrium as an expression of Bayesian rationality*, Econometrica **55**, 1987, 1-18

[3] R. Aumann and L. Shapley, *Values of non atomic games*, Princeton University Press, 1974

[4] K. Binmore and M. Herrero, *Matching and bargaining in dynamic markets*, Review of Economic Studies, **55**, 1988, 17-31

[5] F. Y. Edgeworth, *Mathematical psychics: An essay on the application of mathematics to the moral sciences*, London: Kegan Paul, 1881

[6] M. Ferreira, B. F. Finkenstädt, B. M. P. M. Oliveira, A. A. Pinto and A. N. Yannacopoulos, *A modified model of an Edgeworthian Economy.* Dynamics, Games and Science (Eds. M. M. Peixoto et al.), Springer Proceedings in Mathematics, 2011.

[7] D. Gale *Strategic Foundations of Perfect Competition*, Cambridge University Press, 2000

[8] P. J. Lloyd, *The origins of the von Thünen-Mill-Pareto-Wicksell-Cobb-Douglas function.* History of Political Economy **33**, 1-19, 2001.

[9] A. A. Pinto, M. Ferreira, B. F. Finkenstädt, B. M. P. M. Oliveira and A. N. Yannacopoulos, *On the convergence to Walrasian prices in random matching Edgeworthian economies.* CEJOR **20**, 485-495, 2012.

[10] A. McLennan and H. Schonnenschein, *Sequential bargaining as a non cooperative foundation for Walrasian equilibria*, Econometrica **59**, 1991, 1395-1424

[11] A. MasColell, *An equivalence theorem for a bargaining set*, Journal of Mathematical Economics, **18**, 1989, 129-139

[12] A. Rubinstein and A. Wolinsky, *Decentralized trading, strategic behaviour and the Walrasian outcome* Review of Economic Studies, **57**, 1990, 63-78

[13] M. Shubik, *Edgeworth's market games* , in Contributions to the theory of games IV, edited by R. D. Luce and A.W.Tucker, Princeton University Press, 1959

[14] M. Voorneveld, *From preferences to Cobb-Douglas utility*, SSE/EFI Working Paper Series in Economics and Finance No. 701., 2008