Fair Utilitarianism*

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Abstract

Utilitarianism plays a central role in economics, but there is a gap between theory, where it is dominant, and applications, where monetary criteria are often used. For applications, a key difficulty for utilitarianism remains to define how utilities should be measured and compared across individuals. Drawing on Harsanyi's approach (Harsanyi, 1955) involving choices in risky situations, we introduce a new normalization of utilities that is the only one ensuring that: 1) a transfer from a rich to a poor is welfare enhancing, and 2) populations with more risk averse people have lower welfare. We embed these requirements in a new characterization of utilitarianism and study some implications of this "fair utilitarianism" for risk sharing and collective risk aversion.

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1 Introduction

Utilitarianism played a key role in the development of economic theory and provided key concepts such as utility and welfare used to analyze individual behaviors and social situations. It remains the normative basis for many economic policy judgements in the academic literature (see, e.g., Riley, 2008; Kaplow, 2008), in spite of some recent criticism.[1] However, in spite of the long domination of utilitarianism, one issue that is most important for concrete applications has never been really settled: What practical method should be used to measure individual utility in an interpersonally comparable way? For lack of a fully applicable recipe, utilitarianism has often been used in a rather abstract way and the economic policy literature has often assumed individual utility functions to be identical. This assumption is however unrealistic, as the empirical literature has documented that there is, in particular, substantial heterogeneity in risk aversion in the population.[2]

Four specific proposals have been made, however. One proposal, due to Edgeworth (1881) and recently revived by Argenziano and Gilboa (2015), is to consider that "just noticeable differences" are of equal value across individuals. A second proposal has emerged from the literature on subjective well-being. It consists in using answers to survey questions about happiness or satisfaction as direct measures of utility (see, e.g., Kahneman, Wakker and Sarin, 1997; Layard, Mayraz and Nickell, 2008). Both the first and second approaches rely on the comparability of individual perceptions. But such a comparability is far from obvious and remains much debated.[3]

[1]Utilitarianism is now contested as insufficiently flexible to incorporate important values that prevail in the public debate, such as fairness and desert. See Rawls (1971), Sen (1985) and the subsequent literature, as well as Roemer (1996), Mankiw (2010), Fleurbaey and Maniquet (2011) and Suzumura (2016).

[2]See Kimball, Sahm and Shapiro (2009), Chiappori and Paiella (2011), Schulhofer-Wohl (2011) and Fossen and Glocker (2014) for evidence of preference heterogeneity. Recently, in prominent reviews of public economics, Kaplow (2008), Boadway (2012) and Piketty and Saez (2012) have criticized the assumption that utility functions are identical.

[3]Critical discussions of the "just noticeable differences" approach can be found in Arrow (1951), Svensson (1985) and Hammond (1991). Critical discussions of the happiness approach

A third proposal is based on Harsanyi's (1953) *impartial observer theorem* and has been further discussed in Harsanyi (1977). It assumes that, behind a veil of ignorance, individual have extended preferences on prospects of outcomes and identities, so that they can compare their wellbeing with different preferences. A key assumption of Harsanyi is that different deliberators behind the veil of ignorance will have the same extended preferences.[4] This assumption has been much disputed (see for instance Adler, 2012, 2016, for recent discussions and criticisms of Harsanyi's approach).

A fourth proposal to measure individual utilities is related to the previous one, and relies on Harsanyi's (1955) *aggregation theorem*. This theorem, which involves the Pareto principle applied in risky situations, characterizes utilitarianism as a sum of Von Neumann – Morgenstern (henceforth VNM) individual utilities. In that context, the proposal is to scale the utility functions by setting the utility of a "good" option (generally the best) equal to one and the utility of a "bad" option (generally the worst) equal to zero. This has been called "relative utilitarianism" by Dhillon and Mertens (1999). The 0-1 normalized utilities have appeared in the literature many times,[5] and they have been criticized almost as often.[6]

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can be found in Sen (1985), Nussbaum (2008), Graham (2009), Adler (2012), Fleurbaey and Blanchet (2013).

[4]To justify this, Harsanyi argues that deliberators are able, through "empathic projection", to experience the actual wellbeing levels of people in different circumstances. Adler (2012) criticizes this position and proposes an alternative method using "sympathetic projection".

[5]The literature is often vague about the origins of the idea. Arrow (1951) attributes the idea of 0-1 normalized utility functions to (apparently oral communication by) Abraham Kaplan. Luce and Raiffa (1957, p. 34) refer to the normalization without citing any source. Hammond (1991) cites Isbell (1959) and Schick (1971). Dhillon and Mertens (1999) and Sprumont (2013) refer to Kalai and Smorodinsky (1975) — who equalize these normalized utilities instead of maximizing their sum. Karni (1998) does not refer to earlier sources, and Segal (2000) cites Cao (1981).

Observe that the standard gamble on which the proof of existence of a VNM utility function usually relies (Von Neumann and Morgenstern, 1953, in the appendix written for the 1946 edition) already produces such a 0-1 normalization (between two arbitrary options). There remains a leap before such normalization can be used for interpersonal comparisons, but nevertheless, one could argue that the idea has been around since the very advent of VNM utilities.

[6]Arrow (1951), Luce and Raiffa (1957) and Hammond (1991), quoted above, are all critical, as well as Sen (1970), Jeffrey (1971), and Hausman (1995).

In this paper, we build on this fourth proposal and propose an alternative normalization of utilities in Harsanyi's (1955) theorem to obtain a "local" or "marginal" variant of relative utilitarianism. Our proposal consists in equalizing marginal utilities at the poverty line. We offer two ways of justifying this third approach. One argument relies on the idea that social priority should be greater for any individual below the poverty line than for any individual above it. Another argument involves the idea that more risk averse individuals have a lower utility level both below and above the poverty line — comparisons of levels, as opposed to differences, do matter for utilitarianism in the evaluation of changes in the composition of the population. We also show that, in light of such considerations, the 0-1 normalization appears questionable and leads to unappealing consequences.

We do not claim that our proposal is unquestionably the best, but we would like to argue that it deserves consideration, and we find it surprising that it is not more prominent in the literature — as a matter of fact, we have not found any trace of it. In the next section, we compare the 0-1 normalization of relative utilitarianism to the new normalization of utilities that we propose. In Section 3 we introduce a formal framework for our analysis, which involves risk on consumption and on the composition of the population and makes it possible to invoke Harsanyi's aggregation theorem. In Section 4, a first argument in terms of priority around the poverty line is shown to characterize the utility normalization we propose. In Section 5, an argument involving utility levels and population changes is analyzed and yields another characterization. In Section 6, we discuss some implications of our fair utilitarian approach for risk sharing and collective risk aversion. We show that all individuals must have a consumption level above the poverty level whenever this is feasible, and none should consume more than the poverty level otherwise. It is known that an increase in agents' relative risk aversion does not necessarily increases collective risk aversion (Mazzocco, 2004; Jouini, Napp and Nocetti, 2013). In the case of fair utilitarian criteria and individuals having constant relative risk aversion (CRRA) utility functions, we show that collective risk aversion always increases around the poverty line. At low levels (resp. high levels) of per capita consumption, collective risk aversion also increases when the risk aversion of the most risk-averse (resp. least risk averse) agent increases. Section 7 discusses policy implications, focusing on the example of health insurance. An Appendix contains the proofs of our results.

2 Comparing utility normalizations

Let x denote an allocation of resources, and $u_i(x_i)$ be the utility of person i with this allocation, where u_i represents individual preferences over resources and x_i are the resources allocated to i. The utilitarian social welfare function W has the well-known form:

$$W(x) = \sum_i u_i(x_i). \tag{1}$$

Suppose that u_i is also a VNM utility function, so that its concavity represents individual i's risk aversion. The 0-1 normalization of utilities sets $u_i(\underline{x}) = 0$ and $u_i(\bar{x}) = 1$ for specific consumption bundles $\underline{x}, \bar{x}$. It has different implications depending on how the good and bad options are chosen. The most popular approach has been to take the worst option as the bad one, $\underline{x}$, and the best option as the good one, $\bar{x}$, thus ensuring that utility ranges over the interval $[0, 1]$. Another approach, proposed by Adler (2012, 2016), consists in picking the worst option as the bad option $\underline{x}$, and a poverty level as the good option $\bar{x}$.

The literature quoted in the introduction has focused its criticism of the 0-1 normalization on the fact it involves a dubious assumption that the good and the bad option mean the same thing for different people. However, the advocates of this normalization (e.g., Isbell, 1959, and Segal, 2000) have often argued that the normalization did not make such an assumption, but posited, as a matter of fairness, that the best option should be treated as equivalent for everyone in terms of social evaluation, and similarly for the worst option. The debate can of course continue about whether it is really fair to posit such an equivalence when individuals do enjoy these options differently (Luce and Raiffa, 1957, take the example of small gambles over $1 for a rich and a poor person). Another criticism of relative utilitarianism in its worst-best form (see, e.g., Arrow, 1951)

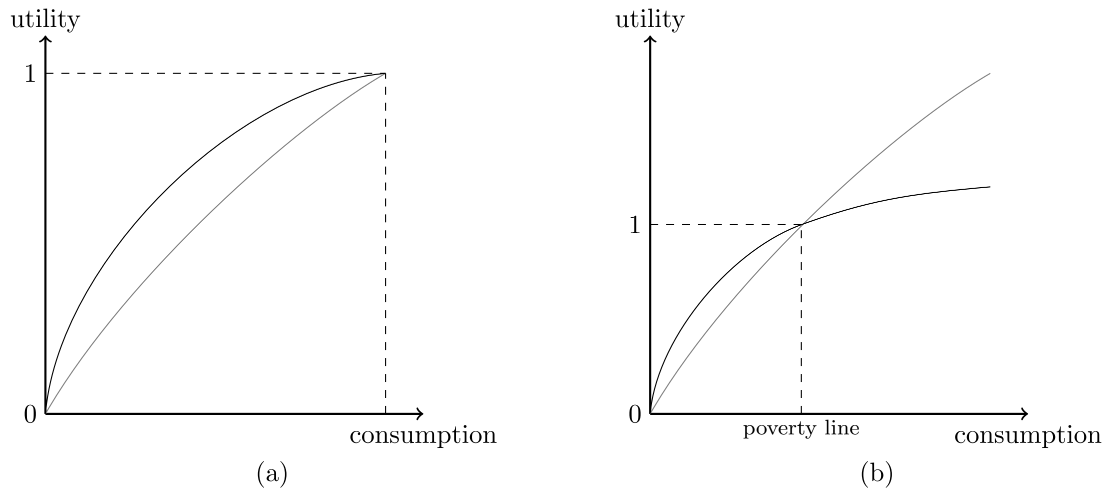

Figure 1: 0-1 normalizations of VNM utility functions

points out that the ranking of prospects may not be invariant to adding feasible options that change the best and worst outcomes for individuals. In an economic context (where x is a distribution of income or consumption), the outcome space is the set of positive real numbers, so that there is no well-defined best outcome and, under non-satiation, individuals have no maximum utility, which is also problematic for the 0-1 normalization.

We would like to introduce another set of worries, having to do with the heterogeneity of risk attitudes, and the implications of the 0-1 normalization for two types of social decisions: 1) the allocation of consumption across individuals (which depends on marginal utilities), and 2) the addition of new members to the population (which depends on utility levels).

Consider Figure 1. On panel (a), the normalization spans the worst-best interval. On panel (b), utility is equal to 1 at the poverty line. The figures show the VNM utility functions of two individuals with different levels of risk aversion, when resources are measured by a level of consumption.

On panel (a), the more risk averse person (black curve) has greater marginal utility up to some point and then lower marginal utility. Depending on the exact shape of the curves, the relative priority of the two agents, when considering allocations of resources between them, may be reversed at any point in the interval, and this point may vary for different pairs of individuals. As a consequence, it may happen that for one pair of individuals, the more risk averse has greater priority for most of the interval, whereas for another pair, the less risk averse has greater priority for most of the interval. This is odd, since it seems that greater risk aversion should modify the relative priority for the allocation of resources in a more systematic way.

Now, if one considers adding a new person to the society at a given level of consumption, on panel (a) the more risk averse is everywhere considered better-off in utility, and therefore it is always better to add more risk averse rather than less risk averse agents. This is not very intuitive, since one rather tends to view risk aversion as a burden.

On panel (b), a highly risk averse person (black curve) who is just below the poverty line has less priority for the allocation of resources than a less risk averse person who is just above the poverty line. It seems odd to penalize the risk averse in this way. In contrast, when the problem is to add new members to the society, a highly risk averse person is preferred to a less risk averse person when they are at the same level of consumption below the poverty line, but the preference is reversed when they are above. It is hard to make sense of this reversal around the poverty line. In particular, it seems counterintuitive to deem the more risk averse person's utility to be higher than a less risk averse person's when hit by the bad luck of poverty.

Figure 2 illustrates the alternative normalization that we propose in this paper. The level and slope of everyone's utility function at the poverty line is the same. This produces a very clear pattern for the two social decisions discussed before. For the allocation of resources, the more risk averse individuals have greater priority below the poverty line and lower priority above the poverty line. This makes sense if one views risk aversion as triggering a greater loss in utility under bad luck and a lower gain under good luck. Moreover, with the proposed normalization, every individual below the poverty line has greater priority than every individual above the poverty line, independently of risk aversion, which

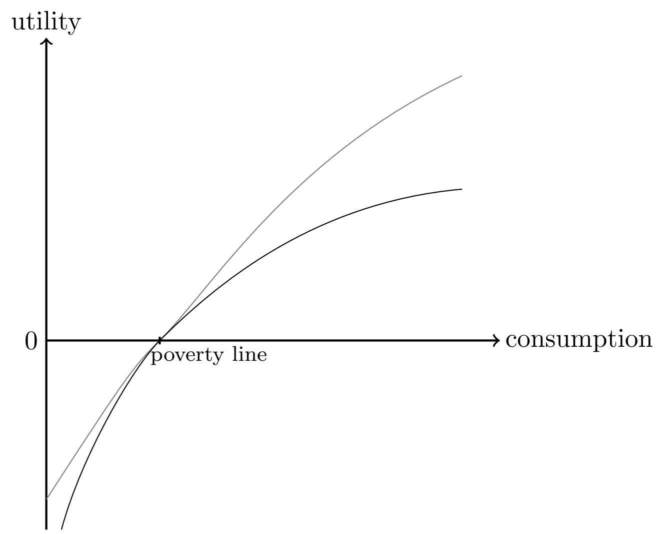

Figure 2: Normalizing the level and slope of VNM utility functions at the poverty level

seems consistent with the standard understanding of a poverty line as identifying those who should be targeted the most in social policy.

As far as adding new members to the population is concerned, this normalization treats risk aversion as a burden and therefore systematically prefers to add less risk averse rather than more risk averse newcomers to the population.

Observe that this new normalization can be viewed as a local variant of the $0-1$ normalization, since it is the limit of a 0-1 normalization when the good and bad options come closer together and the corresponding utility levels are kept in proportion.

We propose this normalization for utilitarianism, but note that non-utilitarian criteria can also use it. For instance, the generalized utilitarian (also called "prioritarian", see Adler, 2012) criterion that maximizes a sum of concave transforms of normalized utilities, with the same transform for all individuals, can also satisfy the fairness and the anti-risk-aversion principles —though not the Pareto principle ex ante.

In the literature, two kinds of axioms have been proposed to justify the 0-1 normalization. First, impartiality axioms stipulate that it is indifferent to favor one individual or the other if the utility gains are the same (in a specific sense, see Karni, 1998; Segal, 2000). Second, restricted independence axioms, inspired by Arrow's Independence of Irrelevant Alternatives, refer to changes in available intermediate options (Dhillon and Mertens, 1999; Sprumont, 2013). In this paper, we suggest two new principles that reflect the considerations presented in this section about relative priorities for the allocation of resources and for the addition of new agents.

3 The framework

We develop a framework making possible to consider that diverse populations consisting of people with different risk preferences may exist. We do so to be able to represent our intuitions regarding the addition of more or less risk averse new agents to a population. Such a framework is also needed for comparing social welfare or living standards across populations with different sizes and preferences (for instance across time and space, see Sen, 1976, or Fleurbaey and Tadenuma, 2014).

We let $\mathbb{N}$ denote the set of positive integers, $\mathcal{N}$ the set of non-empty finite subsets of $\mathbb{N}$, $\mathbb{R}$ the set of real numbers, and $\mathbb{R}_{++}$ the set of positive real numbers. For a set D and any $n \in \mathbb{N}$, D^n is the n-fold Cartesian product of D. Also, for two sets D and E, D^E denotes the set of mappings from E into D.

The set of *potential* individuals who may or may not exist is $\mathbb{N}$. In an alternative, only a population $N \in \mathcal{N}$ exists. Each potential individual $i \in \mathbb{N}$ is associated with a preference ordering, so that considering different populations N makes it possible to study the impact of changes in individual preferences. We study the allocation of resources among the individuals composing the population that exist. To simplify, we assume that a single good, labelled as consumption, has to be distributed. It can be a summary statistics of all the resources available (for instance an equivalent consumption level).

We let $X = \bigcup_{N \in \mathcal{N}} (\mathbb{R}_+)^N$ denote the set of possible alternatives when at least one individual exists. An alternative, then, is a function assigning a positive consumption level to each individual living in that particular alternative, where N is the set of such individuals. Hence the size of the population may vary from one alternative to another and it is important to keep track of the population in an alternative. For any $x \in X$, we let $N(x) \in \mathcal{N}$ be the set of individuals in the alternative and $n(x) = |N(x)|$ be the number of individuals in the alternative. We assume that there always are finitely many people in any alternative.

We also assume that it is not always known for sure what the final allocation of consumption will be nor what set of individuals will eventually exist. To model this, we assume that there exists an infinite set of states of the world S, with typical element $s \in S$. We denote Σ a σ-algebra over S, so that (S, Σ) is a measurable space. There is a probability measure P on the measurable space (S, Σ). We assume that the measure space (S, Σ, P) satisfies a property of convex-rangedness:

> For any $A \in \Sigma$ and $\kappa \in [0, 1]$, there exists $A' \in \Sigma$ such that $A' \subset A$ and $P(A') = \kappa P(A)$.

A prospect f is a function from S to X, which is supposed to be Σ-measurable. For $s \in S$, $f(s)$ is therefore the alternative induced by the prospect x in state s. We restrict attention to simple prospects that are finitely-valued, in the sense that there exists a finite partition $(A_1, \cdots, A_m)$ for which $f(s) = f(s')$ for all $s, s' \in A_k$, all $k = 1, \cdots, m$. We denote F the set of all such prospects. For any random variable K, i.e., any Σ-measurable function $K : S \to \mathbb{R}$, its expected value is $\mathbb{E}\big[K\big] = \int_S K(s) dP(s)$.

Although our framework seems similar to Savage's (1954) model of decision under uncertainty (our prospects are similar to Savage's acts), we want to model a situation of risk. Indeed, we assume that all individuals consider P as the correct belief function, which is an objective probability measure. For our purpose, this framework is more convenient than the usual framework for objective risk that deals with lotteries, because it makes it easy to express individual prospects and redistribution of resources.

For an alternative $x \in X$, whenever $i \in N(x)$, $x_i \in \mathbb{R}_+$ denotes the consumption of individual i. For a prospect $f \in F$, whenever $i \in N\big(f(s)\big)$, $f_i(s)$

denotes the consumption of individual i in state of the world $s \in S$. We denote $A_i(f) = \big\{s \in S \mid i \in N\big(f(s)\big)\big\}$ the event in which individual i exists and $p_i(f) = P\big(A_i(f)\big)$ the probability that he exists. We also let f_i denote the mapping $f_i \in \mathbb{R}_+^{A_i(f)}$ determining individual i's consumption in each state of the world in which i exists, under the social prospect f. The mapping f_i represents i's individual prospect.

For a population $N \in \mathcal{N}$, let F_N denote the set of prospects f such that, for every $s \in S$, $N\big(f(s)\big) = N$. These are the prospects such that the same individuals are present in all states of the world, so that there is no risk on individual preferences. By convention, for any $x \in X$, we let x denote the prospect $f \in F$ such that $f(s) = x$ for all $s \in S$. These are sure prospects yielding the same allocation in all states of the world. Similarly, let $X \subset F$ also denote the set of sure prospects and $X_N \subset F_N$ be the set of sure prospects such that the population is N.

It remains to specify how individuals rank different prospects and alternatives. We retain the prominent assumption in the literature dealing with risk, namely, we assume that individuals are expected utility maximizers. One difficulty is that, for a given social prospect, an individual $i \in \mathbb{N}$ may not exist in all states of the world. We hence need to define individual preferences conditional on their existence in an event $A \in \Sigma$. To do so, for any $A \in \Sigma$ we let $\Sigma_A = \{B \in \Sigma, B \subset A\}$, which is a sigma-algebra over A. For any Σ_A-measurable function $K : A \to \mathbb{R}$, its conditional expected value is $\mathbb{E}_A\big[K\big] = \frac{1}{P(A)} \int_A K(s)dP(s)$. Hence, the set $\mathbb{R}_+^A$ of mappings that are Σ_A-measurable represents individual prospects conditional on existence in A. We then assume that individual preferences are conditional expected utilities:

> For each $i \in \mathbb{N}$ and $A \in \Sigma$, individual i's preferences are represented by a complete reflexive and transitive relation $\succsim_i^A$ over $\mathbb{R}_+^A$. Furthermore, there exists an increasing, continuously differentiable and concave function $u_i : \mathbb{R}_{++} \to \mathbb{R}$ such that, for all $f_i, g_i \in \mathbb{R}_{++}^A$,
>
> $$f_i \succsim_i^A g_i \Longleftrightarrow \mathbb{E}_A\big[u_i \circ f_i\big] \geq \mathbb{E}_A\big[u_i \circ g_i\big].$$

In order to stick to the Utilitarian tradition and in particular to Harsanyi's (1955) approach, we also assume that the social observer is an expected utility maximizer:

> The social ordering $\succsim$ is a complete reflexive and transitive relation over F. Furthermore, there exists a continuous function $U : X \to \mathbb{R}$ such that, for all $f, g \in F$:
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> $$f \succsim g \Longleftrightarrow \mathbb{E}\big[U(f)\big] \geq \mathbb{E}\big[U(g)\big].$$

4 Fair social choice under risk for fixed populations

Let us first consider the case in which there is no uncertainty about the composition of the population, so that N is fixed. Then, the well-known Pareto principle can be expressed in the following way.[7]

Same-population Pareto. For all $N \in \mathcal{N}$, for all $f, g \in F_N$, if $f_i \succsim_i^S g_i$ for all $i \in N$, then $f \succsim g$. If furthermore $f_i \succ_j^S g_i$ for some $j \in N$, then $f \succ g$.

As discussed in Section 2, we also want social orderings to satisfy principles of fairness regarding relative priorities for the allocation of resources. In this respect, a standard principle in the literature is the Pigou-Dalton principle that describes social-welfare-enhancing transfers of resources. It is however known that the Pigou-Dalton principle may conflict with the Pareto principle (Fleurbaey and Trannoy, 2003; Brun and Tungodden, 2004). We thus focus on a weak fairness requirement that applies only when transfers are made from a non-poor to a poor.

Pigou-Dalton transfer for the poor. There exists a poverty level $z_p \in \mathbb{R}_{++}$ such that, for all $N \in \mathcal{N}$, for all $x, y \in X_N$, if there exist $i, j \in N$ such that

1. $y_i < x_i < z_p < x_j < y_j$;
2. $x_i - y_i = y_j - x_j$;

[7] Our version of the Pareto principle is a version of the so-called "Strong Pareto" principle.

3. $x_k = y_k$ for all $k \in N \setminus \{i, j\}$;

then $x \succsim y$.

Let $\succsim_N$ denote the restriction of the social welfare ordering $\succsim$ to prospects in F_N. The next proposition characterizes fair utilitarian criteria.

Theorem 1 *For all $N \in \mathcal{N}$, the social ordering $\succsim_N$ satisfies Same-population Pareto and Pigou-Dalton transfer for the poor if and only if there exists a poverty level $z_p \in \mathbb{R}_+$ such that, for all $f, g \in F_N$,*

$$f \succsim_N g \Longleftrightarrow \mathbb{E}\left[\sum_{i \in N} \frac{u_i(f_i)}{u_i'(z_p)}\right] \geq \mathbb{E}\left[\sum_{i \in N} \frac{u_i(g_i)}{u_i'(z_p)}\right].$$

Fair utilitarian criteria normalize utilities so that marginal utility is the same for all individuals at the poverty level. As noted in Section 2, they also has the following attractive property: The more risk averse a person is, the greater marginal utility it has in poverty and the lower marginal utility it has out of poverty. In other words, a greater risk aversion implies that one gets greater social priority in poverty and less priority out of poverty.

Observe that this result does not say anything about utility levels, since only marginal utility matters for utilitarianism dealing with fixed populations. The question of utility levels comes up in the next section.

5 Fair criteria for populations of variable sizes and compositions

As discussed in Section 2, it seems intuitive to deem a population facing a risk to be less well-off if it contains more risk averse individuals. To make such judgements, we must be able to compare populations with different risk attitudes. In this section, we incorporate uncertainty about the preferences or existence of people composing the population into our analysis. This also makes more general evaluations possible, allowing for instance a possible uncertainty about what the preferences of people will be in the future.

However, this also raises the issue of how the Pareto principle should be applied. Indeed, individual preferences are normatively relevant only for states of the world in which the preference bearers exist. We thus introduce a general version of the Pareto principle, such that individual preferences are respected whenever individuals live in the same states of the world in the two prospects that are being compared.

Pareto. For all $f, g \in F$, if for all $i \in \mathbb{N}$, $A_i(f) = A_i(g) := A_i$, and if $f_i \succsim_i^{A_i} g_i$ for all $i \in \mathbb{N}$ such that $P(A_i) > 0$, then $f \succsim g$. If furthermore $f_i \succ_i^{A_i} g_i$ for some $j \in \mathbb{N}$ such that $P(A_j) > 0$, then $f \succ g$.

The next two principles describe how populations of different sizes and composition are being compared. The first principle introduces the notion of a critical level of consumption, i.e., a level such that adding a person at this level is a matter of social indifference.

Critical level. There exists $z_c \in \mathbb{R}_{++}$ such that for all $x, y \in X$, if $N(y) = N(x) \cup \{j\}$, with $j \in \mathbb{N} \setminus N(x)$, $y_i = x_i = z_c$ for all $i \in N(x)$ and $y_j = z_c$ then $x \sim y$.

Our critical-level principle is related to other principles proposed in the literature on population ethics (see Blackorby, Bossert and Donaldson, 2005, for a presentation of this literature). The main difference is that our principle is expressed in terms of resource and not in terms of welfare. Our principle is weak because it does not require the critical-level to be the same for all possible allocations, instead it posits that there is a level at which population expansion is indifferent. The level of z_c can be thought of as a subsistence level below which life is not worth living, so that a smaller population is preferable when everyone is below this level, whereas a greater population is desirable when everyone is at a level above z_c.

A second principle compares populations of same size facing an egalitarian prospect, i.e., a prospect in which income varies across states but individuals always end up in equal positions. The principle represents the argument made in

Section 2 that risk aversion is a burden that reduces social welfare. The reason why this should be so is that a more risk-averse individual always has a lower certainty-equivalent, and is therefore willing to pay more to avoid the aggregate risk on income. Thus, when facing a given egalitarian risky prospect, a population with less risk-averse agents should be considered better off.

To state our principle, we need the following definition. We say that an individual $j \in \mathbb{N}$ is more risk averse than another individual $i \in \mathbb{N}$ if there exists a continuously differentiable, increasing and concave function ψ such that $u_j = \psi \circ u_i$.

Dominance of more risk averse populations. For all $f, g \in F$, if there exists a population $N \in \mathcal{N}$ and two individuals $i, j \in \mathbb{N} \setminus N$, with j more risk averse than i, such that:

1. $N\big(f(s)\big) = N \cup \{i\}$ and $N\big(g(s)\big) = N \cup \{j\}$ for all $s \in S$,
2. $f_i(s) = g_j(s) = f_k(s) = g_k(s)$ for all $k \in N$ and $s \in S$,

then $f \succsim g$.

To make sure that the above principle is not empty, we make the following assumption, that guarantees the existence of an individual that is comparable to others in terms of risk aversion. To simplify the proof, we also assume that utilities are linearly independent (otherwise, there may be some indeterminacy of the weights).

There exists an individual $i \in \mathbb{N}$ such that, for all $j \in \mathbb{N} \setminus \{i\}$, either i is more risk averse than j, or j is more risk averse than i.

Furthermore, for any $N \in \mathcal{N}$, the utility functions $(u_i)_{i \in N}$ are linearly independent.

It is then possible to characterize the class of fair utilitarian criteria for variable populations.

Theorem 2 *The social ordering $\succsim$ satisfies Pareto, Critical-level and Dominance of more risk averse populations if and only if there exists a critical level $z_c \in \mathbb{R}_+$ such that, for all $f, g \in F$,*

$$f \succsim g \Longleftrightarrow \sum_{i \in \mathbb{N}} p_i(f) \times \mathbb{E}\left[\frac{u_i(f_i) - u_i(z_c)}{u_i'(z_c)}\right] \geq \sum_{i \in \mathbb{N}} p_i(g) \times \mathbb{E}\left[\frac{u_i(g_i) - u_i(z_c)}{u_i'(z_c)}\right].$$

A noticeable consequence of Theorem 2 is that these fair utilitarian criteria satisfy Pigou-Dalton transfer for the poor, although it is not an assumption of their characterization. In addition, the income poverty level is shown to be exactly equal to the critical-level. This result comes from the fact that the normalization of marginal utilities at z_c is the only normalization that guarantees that, given that utility levels are equalized at z_c (which is implied by Critical level), a more risk averse individual is worse-off than a less risk averse individual when facing a small risk around z_c (see Fig. 2).

6 Optimal risk sharing and social risk aversion

In this section, we explore some of the implications of fair utilitarian criteria. To do so, we consider a simple static economy with no production and a given population N of size $n \in \mathbb{N}$.[8] We assume that the only risk is a risk on per capita endowment in the economy, described by a finitely-valued random variable ω on the measure space (S, Σ, P). Thus per capita endowment in state $s \in S$ is $\omega(s)$. A prospect f is *feasible* if for all $s \in S$:

$$\tfrac{1}{n} \sum_{i \in N} f_i(s) \leq \omega(s).$$

6.1 Fair risk sharing

We want to compare the fair optimal risk sharing rule with the optimal risk sharing rule using 0-1 normalized utilitarian criteria. Let us first look at fair op-

[8]We will compare different populations of the same size, but we only consider situations where there is no risk on the composition of the group.

timal risk sharing, defined as the solution of the following first-best optimization problem:

$$\begin{aligned} \max_{f \in F_N} \quad & \sum_{i \in N} \mathbb{E}\left[\frac{u_i(f_i) - u_i(z_c)}{u_i'(z_c)}\right] \\ s.t. \quad & \tfrac{1}{n}\sum_{i \in N} f_i(s) \leq \omega(s) \quad \forall s \in S. \end{aligned} \tag{2}$$

Another optimal risk sharing rule is the solution of the same optimization problem but using a 0-1 normalized utilitarian criterion:

$$\begin{aligned} \max_{f \in F_N} \quad & \sum_{i \in N} \mathbb{E}\left[\frac{u_i(f_i) - u_i(0)}{u_i(\bar{x}) - u_i(0)}\right] \\ s.t. \quad & \tfrac{1}{n}\sum_{i \in N} f_i(s) \leq \omega(s) \quad \forall s \in S, \end{aligned} \tag{3}$$

where $\bar{x}$ is either the best available option ($n \times \sup_{s \in S} \omega(s)$), like in "relative utilitarianism" (Dhillon and Mertens, 1999), or an exogenous poverty line, as suggested by Adler (2012).

Let us denote $f_i^*\big(\omega(s)\big)$ the fair optimal risk sharing rule, defined as the solution of problem (2), and $f_i^{**}\big(\omega(s)\big)$ the solution of problem (3). The next proposition contrasts the two risk sharing rules. It highlights that, while the standard 0-1 normalized approach can generate large inequalities, the fair risk sharing rule induces bounds on what agents may receive.

Proposition 1

1. *The optimal allocation $f_i^*\big(\omega(s)\big)$ for problem (2) is such that:*
   - *(a) For all $s \in S$ and all $i, j \in N$, if i is more risk-averse than j and $\omega(s) \leq z_c$, then $f_j^*\big(\omega(s)\big) \leq f_i^*\big(\omega(s)\big) \leq z_c$.*
   - *(b) For all $s \in S$ and all $i, j \in N$, if i is more risk-averse than j and $\omega(s) \geq z_c$, then $f_j^*\big(\omega(s)\big) \geq f_i^*\big(\omega(s)\big) \geq z_c$.*

2. *For any risk on per capita endowment, and for any* $0 < \varepsilon < 1$*, there exists a profile of preferences such that the optimal allocation for problem (3) yields* $f_i^{**}\big(\omega(s)\big) \geq (1-\varepsilon)n\omega(s)$ *for some individual* i *in the population and* $f_k^{**}\big(\omega(s)\big) \leq \varepsilon\omega(s)$ *for all other individuals* k*.*

The first part of Proposition 1 shows that the fair utilitarian sharing necessarily imposes bounds to the inequality in the allocation of resources. When resource are low, so that it is impossible to provide everyone with a consumption level above the poverty line, then nobody should have a consumption level above the line. It can be viewed as a fairness requirement that poverty should be shared by everyone in poor societies. Moreover, when resources are low, the more risk averse agents should have relatively more. On the other hand, when resource are sufficiently large, so that it is possible to raise everyone's consumption above the poverty line we should do so. Again, this can be viewed as a fairness requirement that prosperity should be shared. In this case, the more risk averse agents should have relatively less.

The first part of Proposition 1 also implies that, when $\omega(s) = z_p$, the optimal consumption level is perfectly egalitarian (at z_p for all agents). The existence of such an efficient and egalitarian optimum was assumed by Jouini, Napp and Nocetti (2013). It is an implication of fair utilitarian criteria.

The second part of Proposition 1 shows that the risk sharing rule derived from the 0-1 normalized utilitarian criteria can create large inequalities, where one individual gets almost all the aggregate endowment $n\omega(s)$, while the others have a consumption level close to zero. In particular, even when there are enough resources to bring everyone above the poverty level, it may be optimal to have only one individual well above the poverty level while others remain in deep poverty. This seems to be a quite extreme distributive conclusion, which is avoided whatever the preferences of the population by the fair utilitarian rule.

6.2 Collective risk aversion

One might think that populations composed of more risk averse agents are less prone to take risks when acting collectively. However, Mazzocco (2004) and Jouini, Napp and Nocetti (2013) showed that is not always the case, even when there are only two agents with Constant Relative Risk Aversion (CRRA) utility functions. In this section, we study how the risk aversion associated with a fair utilitarian criterion changes when the composition of population $N \in \mathcal{N}$ changes.

Let us assume that all (potential) individuals have utility functions of the following specific Harmonic Absolute Risk Aversion (HARA) form:[9] for all $i \in \mathbb{N}$, there exists $\alpha_i \in \mathbb{R}_{++}$ such that $u_i(x_i) = (\kappa + x_i)^{1-\alpha_i}/\big(1-\alpha_i\big)$ for all $x_i \in \mathbb{R}_{++}$, where κ is a positive real number. Parameter α_i determines whether individual i is more or less risk averse. Indeed, relative risk aversion of individual i at the consumption level $x_i \in \mathbb{R}_+$ is $R_i(x_i) = -\big[x_i u_i''(x_i)\big]/u_i'(x_i) = \alpha_i/(1+\kappa/x_i)$. At any consumption level $x_i \in \mathbb{R}_+$, an individual whose utility has a higher parameter α_i is more averse to risk.

For any population $N \in \mathcal{N}$ and for any level of per capita resource $z \in \mathbb{R}_+$, let us denote:

$$\begin{aligned} W_N(z) := \quad & \max_{x \in \mathbb{R}^n} \quad \sum_{i \in N} \frac{u_i(x_i) - u_i(z_c)}{u_i'(z_c)} \\ & s.t. \qquad \tfrac{1}{n} \sum_{i \in N} x_i \leq z. \end{aligned}$$

$W_N(z)$ is the indirect social welfare for the fair utilitarian criterion when per capita endowment is equal to z.

To characterize collective risk attitudes, we need to study the level of risk aversion associated to function W_N. The *collective relative risk aversion* for a population N at the per capita resource level $z \in \mathbb{R}_{++}$ is $R_N(z) := -\big[zW_N''(z)\big]/W_N'(z)$.

[9]HARA utility functions have the form $u_i(x_i) = \frac{\zeta}{1-\alpha_i}(\eta_i + x_i/\alpha_i)^{1-\alpha_i}$, with $\alpha_i, \zeta > 0$. Our assumption is to set $\eta_i = \kappa/\alpha_i$ and $\zeta = 1/\alpha_i$ so that comparative risk aversion is determined only by parameter α_i. The CRRA can be seen as a limit case when $\kappa \to 0$. The advantage of our formulation is to have well-defined utility functions on $\mathbb{R}_+$, which is not the case of CRRA utility functions when $\alpha_i > 1$ and $x_i = 0$.

The next proposition highlights situations in which we are sure that an increase in an agent's relative risk aversion increases collective relative risk aversion. To present the result, denote, for any $N \in \mathcal{N}$, $\overline{\alpha_N} := \max_{i \in N} \alpha_i$ and $\underline{\alpha_N} := \min_{i \in N} \alpha_i$. Also denote $\underline{z_N} := (\kappa/(z_c + \kappa))^{\underline{\alpha_N}/\overline{\alpha_N}} (z_c + \kappa) - \kappa$. The level of resources $\underline{z_N}$ guarantees that the optimal sharing rule induces strictly positive consumption levels for all individuals.

Proposition 2 *For all $N \in \mathcal{N}$ and all $i, j \in \mathbb{N} \setminus N$, if individual i is more risk averse than individual j then collective relative risk aversion is higher in population $N \cup \{i\}$ than in population $N \cup \{j\}$ at $z \geq \underline{z_M}$ (where $M = N \cup \{i, j\}$) for the following values of z:*

1. $e^{-\rho} z_p - (1 - e^{-\rho})\, \kappa \leq z \leq e^{\rho} z_p + (e^{\rho} - 1)\kappa$*, with* $\rho = \left[\frac{\overline{\alpha_M}}{\underline{\alpha_M}} - 1\right]^{-1}$.
2. $z \leq z_p$ *(resp.* $z \geq z_p$*), provided i and j is more risk averse than any individual in N (resp. less risk averse than any individual in N).*

Proposition 2 implies that, around the poverty level, an increase in an agent's relative risk aversion always increases collective relative risk aversion (Case 1). This is true for a large interval around the poverty level when there is not too much heterogeneity in relative risk tolerance and when the most risk averse agents is not too risk averse. In particular, a marginal change from a perfectly homogenous population always changes collective risk aversion in the expected direction.

Proposition 2 also provides sufficient conditions for changes in the risk aversions of individuals at the extremes of the preference profile to affect collective risk aversion in the expected direction — either the most risk averse agents when resources are low, or the least risk averse agent when resources are high.

7 Policy implications: Health insurance

Let us briefly examine how the normalization proposed in this paper would bear on policy issues. It is interesting to focus on the example of the impact of preference heterogeneity in the case of health insurance. While traditionally health insurance has been analyzed as simply aiming to insure against random shocks on health expenditures (Arrow, 1963; Zeckhauser, 1970), it has become increasingly recognized in the empirical literature that state-dependent preferences may be more appropriate to describe what happens to individuals hit by health events that are durable or chronic (see however Arrow, 1974, for an early analysis of the impact of health state-dependent utilities). Health, then, does not only affect expenditures, but also the utility function itself.

There are two ways in which the utility function on income or expenditures may be health state-dependent. First, it may take the form $v(h) \times u(x)$, where h is health and x is consumption. This is the form of dependence studied for instance in Viscusi and Evans (1990), Finkelstein, Luttmer and Notowidigdo (2009) and Finkelstein, Luttmer and Notowidigdo (2013). The three studies find that, at any level of income, marginal utility is increasing in health, which means that $v(.)$ is an increasing function. The authors conclude that people should not be fully insured against health shocks, since a utilitarian planner would assign less resources to individuals with a lower marginal utility.

To reach this conclusion, they consider that the function $v(h) \times u(x)$ is the correct interpersonally comparable measure of individual welfare to be used in the utilitarian formula (Eq. (1)) with uniform weights. In Viscusi and Evans (1990) and Finkelstein, Luttmer and Notowidigdo (2009), they use a revealed preference argument to measure marginal utilities, by looking at choices where individuals anticipate their utility in the two states. The authors argue that individuals are thus able to compare wellbeing with different health status, and that we should accept their way of comparing welfare (this is similar to the third proposal mentioned in the introduction, where people behind a veil of ignorance can compare their wellbeing with different 'identities'). In Finkelstein, Luttmer and Notowidigdo (2013), they use happiness data and take them at face value, as direct measures of wellbeing. This corresponds to the second proposal mentioned in the introduction of this paper: to rely on subjective wellbeing.

If one questions the comparability of subjective wellbeing or the ability of individuals to compare welfare in different health states, a solution discussed in the present paper is to consider the non-comparable ordinal risk preferences and impose a specific normalization based on some principles. For the type of state dependence embodied in the form $v(h) \times u(x)$, VNM utilities differ by a multiplicative constant. In that case, both the 0-1 normalization and the normalization proposed in the present eliminate the impact of $v(h)$ and treat individuals identically, independently of their health state. Then, full insurance remains the objective of a utilitarian policy and is only hindered by incentive and market design issues.

A second form of health state-dependence is when health affects risk aversion, for instance when utility takes the form

$$\frac{x^{1-v(h)}-1}{1-v(h)}.$$

In that case, if $v(.)$ is a decreasing function, then healthy individuals are less risk averse on income or consumption. There is evidence that this may be the case (Schurer, 2015; Decker and Schmitz, 2016). Facing such a phenomenon, fair utilitarianism does recommend to give incomplete health expenditure coverage to populations above the poverty threshold, and an over-compensating coverage to individuals below the poverty threshold. The 0-1 normalization does not imply such clearcut conclusions, because the details then depend on exactly how the utility function is transformed by health.

Whether it is reasonable to penalize the sick people for their lower marginal utility — whether it is directly recorded in happiness scores or induced by normalizing utility for their greater risk aversion — is of course a normative issue. Note that, when the VNM utility function is defined on the positive real space and marginal utility for any normalization tends to 0 (for instance when people have CRRA or CARA preferences), the more risk averse person eventually has lower marginal utility, whatever normalization is adopted. Thus, if sick people are more risk averse, penalizing sick people at high income levels may be a generic feature of utilitarian criteria. One can perhaps intuitively think that it is harsh on sick people to impose the double penalty of the unpleasantness of sickness and a reduction in non-medical consumption. This points to how controversial utilitarianism can be, as an approach that is willing to penalize lower *marginal* utility even when it is associated with lower *total* utility.

If we want to consider not only insurance but also the prevention of health risks (or self-protection, using the label by Ehrlich and Becker, 1972), what matters will not only be marginal utilities but utility levels. Indeed, the willingness to decrease the probability of a negative health shock depends on the utility loss of being sick compared to being in good health. Viscusi and Evans (1990) highlighted that, if the health shock decreases the utility level, and not only the marginal utility level, this will have an impact on people willingness to accept risky jobs through the implicit value of statistical injury. Compared to the 0-1 normalization, our approach has the nice implication that, if injuries or health shocks are increasing risk aversion, we are collectively willing to pay a positive amount of money to reduce health risks. This is because, more risk averse people always have a lower utility (see Fig. 2). On the contrary, with the 0-1 normalization, utility of more risk averse people is always (or sometimes, depending on how the good option $\bar{x}$ is chosen) above the utility of less risk averse people (see Fig. 1). Hence, we may want to *increase* the probability of accidents or health shocks. This seems counter-intuitive and provides an argument in favor of our normalization and of the principle of Dominance of more risk averse populations: Risk aversion is a burden that should perhaps be recognized as such and that justifies a differential treatment based on individual risk attitudes.

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Appendix

Proof of Theorem 1

Proving that social ordering $\succsim$ characterized in Theorem 1 satisfies Same-population Pareto and Pigou-Dalton transfer for the poor is straightforward. We show that only this social ordering satisfies the two principles.

Step 1: The social VNM function is an affine combination of individuals ones

Let first remark that social choices on F_N can be translated as choices on the set of simple lotteries over X_N. To do so, define $\Delta(X_N)$ the set of simple lotteries, that is mapping $p : X_N \to [0,1]$ such that $\sum_{x \in X_N} p(x) = 1$ and $p(x) > 0$ for a finite number of elements in X_N. For any prospect $f \in F_N$, there exists a finite partition $(A_1, \cdots, A_m)$ for which, whatever $k = 1, \cdots, m$, $f(s) = f(s') \neq f(s'')$ for all $s, s' \in A_k$ and $s'' \in S \setminus A_k$. Define $p^f \in \Delta(X_N)$ such that, for any $x \in X$, $p^f(x) = P(A_k)$ if $f(s) = x$ for all $s \in A_k$ and $p^f(x) = 0$ otherwise. Conversely, for any $p \in \Delta(X_N)$, we can define an associated prospect $f^p \in F_N$ such that $f^p(s) = x$ for all $s \in A_k$, with $P(A_k) = p(x)$ when $p(x) > 0$.

Hence, we can define the following functions on $\Delta(X_N)$:

1. For all $i \in N$, for all $p \in \Delta(X_N)$, $V_i(p) = \mathbb{E}\big[u_i(f_i^p)\big]$.

2. for all $p \in \Delta(X_N)$, $V_0(p) = \mathbb{E}\big[U(f^p)\big]$.

Each V_i is linear in the sense of Fishburn (1984) and, by Same-population Pareto, for all $p, q \in \Delta(X_N)$, $V_0(p) \geq V_0(q)$ whenever $V_i(p) \geq V_i(q)$ for all $i \in N$ (with a strict inequality if $V_j(p) > V_j(q)$ for some $j \in N$). Hence, by Theorem 2 in Fishburn (1984), there exist positive real numbers $\left(\alpha_i^N\right)_{i\in N} \in \mathbb{R}^N_{++}$ and a real number β^N such that

$$V_0(p) = \sum_{i\in N} \alpha_i^N V_i(p) + \beta^N.$$

The weights $\left(\alpha_i^N\right)_{i\in N} \in \mathbb{R}^N_{++}$ are furthermore unique because utility is self-centered (it depends only on individual consumption), so that the utility functions u_i are linearly independent.

Given the correspondence between preferences over prospects and preferences over lotteries, this implies that:

$$U(x) = \sum_{i\in N} \alpha_i^N u_i(x_i) + \beta^N, \tag{4}$$

where U is the function used to compute the social expected utility.

Step 2: Characterizing the weights on utilities

Assume now that there are two individuals $i, j \in N$ such that $\alpha_i^N u_i'(z_p) < \alpha_j^N u_j'(z_p)$, where z_p is the poverty level. By continuity of u_i' and u_j', this implies that there exist $z < z_p < z'$ such that $\alpha_i^N u_i'(z) < \alpha_j^N u_j'(z')$. For any $\varepsilon \in \mathbb{R}_{++}$ such that $0 < \varepsilon < max\{z' - z_p, z_p - z\}$, define $x, y \in X$ in the following way:

1. $y_i = z$, $x_i = z + \varepsilon$, $x_j = z' - \varepsilon$ and $y_j = z'$;
2. $x_k = y_k$ for all $k \in N \setminus \{i, j\}$.

By Step 1, $x \succeq_N y \iff \alpha_i^N u_i(z+\varepsilon) + \alpha_j^N u_j(z'-\varepsilon) > \alpha_i^N u_i(z) + \alpha_j^N u_j(z')$. But, by concavity of the functions u_i and u_j, we have $u_i(z+\varepsilon) - u_i(z) < u_i'(z)\,\varepsilon$ and $u_j(z') - u_j(z'-\varepsilon) > u_j'(z')\,\varepsilon$. We thus have

$$\alpha_i^N\big(u_i(z+\varepsilon) - u_i(z)\big) < \alpha_i^N\; u_i'(z)\,\varepsilon < \alpha_j^N\; u_j'(z')\,\varepsilon < \alpha_j^N\big(u_j(z') - u_j(z'-\varepsilon)\big)$$

this contradicts $x \succ_N y$ and hence Pigou-Dalton transfer for the poor.

Therefore, there must exists $\kappa^N > 0$ such that, for all $i \in N$, $\alpha_i^N \ u_i'(z_p) = \kappa^N$. This implies that $\alpha_i^N = \frac{\kappa^N}{u_i'(z_p)}$ for all $i \in N$. By expected utility and Eq. (4), this means that for all $f, g \in F_N$,

$$f \succsim_N g \Longleftrightarrow \mathbb{E}\left[\sum_{i \in N} \frac{u_i(f_i)}{u_i'(z_p)}\right] \geq \mathbb{E}\left[\sum_{i \in N} \frac{u_i(g_i)}{u_i'(z_p)}\right].$$

Proof of Theorem 2

Proving that social ordering $\succsim$ characterized in Theorem 2 satisfies Pareto, Critical-level and Dominance of more risk averse populations is straightforward. We show that only this social ordering satisfies the three principles.

By the first step of the proof of Theorem 1 (Eq. (4)), Pareto (which implies Same-population Pareto) implies that for all $x \in X$:

$$U(x) = \sum_{i \in N(x)} \alpha_i^{N(x)} u_i(x_i) + \beta^{N(x)}, \tag{5}$$

where the weights $\left(\alpha_i^N\right)_{i \in N} \in \mathbb{R}_{++}^N$ are unique positive numbers and $\beta^{N(x)}$ is a real number.

Now consider any $i \in \mathbb{N}$ and any $N \in \mathcal{N}$ such that $i \in N$. Let $A, B, C \in \Sigma$ such that A, B, C form a partition of S and $P(A) = P(B) = P(C) = 1/3$. For any $z_1, z_2, z_3, z_4 \in \mathbb{R}_+$, define $f, g \in F$ in the following way:

For all $s \in A$, $N(f(s)) = N(g(s)) = N$, $f_j(s) = z_1$ and $g_j(s) = z_3$ for all $j \in N$;

For all $s \in B$, $N(f(s)) = N(g(s)) = \{i\}$, $f_i(s) = z_2$ and $g_i(s) = z_4$;

For all $s \in C$, $N(f(s)) = N(g(s)) = N \setminus \{i\}$, $f_j(s) = z_3$ and $g_j(s) = z_1$ for all $j \in (N \setminus \{i\})$.

Denote $\alpha_i := \alpha_i^{\{i\}}$, that is α_i is the multiplicative constant transforming individual utility into social utility U when the population is reduced to the singleton

$\{i\}$. Because the social ordering is an expected utility and by Eq. (5),

$$f \succsim g \iff \frac{1}{3}\Big[\alpha_i^N u_i(z_1) + \varphi(z_1)\Big] + \frac{1}{3}\alpha_i u_i(z_2) \geq \frac{1}{3}\Big[\alpha_i^N u_i(z_3) + \varphi(z_3)\Big] + \frac{1}{3}\alpha_i u_i(z_4),$$

where $\varphi(z) = \sum_{j \in N}(\alpha_j^N - \alpha_j^{N\setminus\{i\}})u_j(z)$. But by Pareto, it is also the case that $f \succsim g \iff \frac{1}{2}u_i(z_1) + \frac{1}{2}u_i(z_2) \geq \frac{1}{2}u_i(z_3) + \frac{1}{2}u_i(z_4)$, because individuals other than i are indifferent (they face a fifty-fifty chance of receiving either z_1 or z_3). The reasoning is true for any $z_1, z_2, z_3, z_4 \in \mathbb{R}_+$, so there must exist a continuous increasing function Ψ such that for all $z_1, z_2 \in \mathbb{R}_+$:

$$\frac{1}{3}\Big[\alpha_i^N u_i(z_1) + \varphi(z_1)\Big] + \frac{1}{3}\alpha_i u_i(z_2) = \Psi\bigg(\frac{1}{2}u_i(z_1) + \frac{1}{2}u_i(z_2)\bigg). \tag{6}$$

Denote G, H the real-valued functions such that $G(v_1) = \frac{2\alpha_i^N}{3}v_1 + \frac{1}{3}\phi \circ u_i^{-1}(2v_1)$ and $H(v_2) = \frac{2\alpha_i}{3}v_2$, and $V = \{(v_1, v_2) \in \mathbb{R}^2 | \exists (z_1, z_2) \in \mathbb{R}^2_{++}, v_1 = \frac{1}{2}u_i(z_1), v_2 = \frac{1}{2}u_i(z_2)\}$, a rectangular connected subset of $\mathbb{R}^2$. Eq.(6) implies that for any $(v_1, v_2) \in V$, $G(v_1) + H(v_2) = \Psi(v_1 + v_2)$. Given that Ψ, G and H are continuous and increasing, it must be the case that there exists $c \in \mathbb{R}_+$, $a, b \in \mathbb{R}$ such that $G(v_1) = cv_1 + a$ and $H(v_2) = cv_2 + b$ (Aczél, 1966, Chap. 3, Cor. 1). Because the u_i functions are linearly independent, $\phi \circ u_i^{-1}$ cannot be affine unless it is constant, which implies $\alpha_j^N = \alpha_j^{N\setminus\{i\}}$ for all $j \in N \setminus \{i\}$. And we need $\frac{2\alpha_i}{3} = \frac{2\alpha_i^N}{3} = c$ so that $\alpha_i = \alpha_i^N$.

To sum up, there exist weights $(\alpha_i)_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}_{++}$ and real numbers $\beta^N \in \mathbb{R}$ such that, for all $x \in X$:

$$U(x) = \sum_{i \in N(x)} \alpha_i u_i(x_i) + \beta^{N(x)}.$$

By repeated application of Critical-level, there exists $z_c \in \mathbb{R}_{++}$ such that, for any $N \in \mathcal{N}$ and any $i \in N$, $\sum_{j \in N} \alpha_j u_j(z_c) + \beta^N = \alpha_i u_i(z_c) + \beta^{\{i\}}$. For the special case where $N = \{1, i\}$, this yields $\alpha_i u_i(z_c) + \beta^{\{i\}} = \alpha_1 u_1(z_c) + \beta^{\{1\}}$. Hence, for all $N \in \mathcal{N}$, $\sum_{j \in N} \alpha_j u_j(z_c) + \beta^N = \alpha_1 u_1(z_c) + \beta^{\{1\}}$ and $\beta^N = -\sum_{i \in N} \alpha_j u_j(z_c) + \alpha_1 u_1(z_c) + \beta^{\{1\}}$.

To sum up, and denoting $\Lambda = \alpha_1 u_1(z_c) + \beta^{\{1\}}$, for all $f, g \in F$,

$$\begin{aligned}
f \succsim g \iff & \mathbb{E}\Big[\sum_{i \in N(f)} \alpha_i u_i(f_i) + \beta^{N(f)}\Big] \geq \mathbb{E}\Big[\sum_{i \in N(g)} \alpha_i u_i(g_i) + \beta^{N(g)}\Big] \\
\iff & \mathbb{E}\Big[\sum_{i \in N(f)} \alpha_i\big(u_i(f_i) - u_i(z_c)\big) + \Lambda\Big] \geq \mathbb{E}\Big[\sum_{i \in N(g)} \alpha_i\big(u_i(g_i) - u_i(z_c)\big) + \Lambda\Big] \\
\iff & \mathbb{E}\Big[\sum_{i \in N(f)} \alpha_i\big(u_i(f_i) - u_i(z_c)\big)\Big] \geq \mathbb{E}\Big[\sum_{i \in N(g)} \alpha_i\big(u_i(g_i) - u_i(z_c)\big)\Big] \\
\iff & \mathbb{E}\Big[\sum_{i \in N(f)} \tilde{u}_i(f_i)\Big] \geq \mathbb{E}\Big[\sum_{i \in N(g)} \tilde{u}(g_i)\Big], \qquad (7)
\end{aligned}$$

with $\tilde{u}_i(x_i) = \alpha_i\big(u_i(x_i) - u_i(z_c)\big)$. Function $\tilde{u}_i$ is such that $\tilde{u}_i(z_c) = 0$ and $\tilde{u}'_i(x_i) = \alpha_i u'_i(x_i)$.

By assumption, there exist an individual i such that any other individual is either more risk averse or less risk averse. Let $j \in \mathbb{N}$ be less risk averse than i (the reasoning is similar when j is more risk averse than j). Let $N \in \mathcal{N}$ be such that $i, j \notin N$. Assume that $\tilde{u}'_j(z_c) \neq \tilde{u}'_i(z_c)$:

Case 1: $\tilde{u}'_i(z_c) > \tilde{u}'_j(z_c)$. Then there exists $z > z_c$ such that $\tilde{u}_i(z) > \tilde{u}_j(z)$, because $\tilde{u}_i(z_c) = \tilde{u}_j(z_c)$. Let $x, y \in X^e$ be such $N\big(x\big) = N \cup \{i\}$ $N\big(y\big) = N \cup \{j\}$, and $x_i = y_i = z$ for all $i \in N$. Then, $x \succ y$ (by Eq. (7)), because $\sum_{k \in N(x)} \tilde{u}_k(x_k) = \sum_{k \in N} \tilde{u}_k(z) + \tilde{u}_i(z) > \sum_{k \in N} \tilde{u}_k(z) + \tilde{u}_j(z) = \sum_{k \in N(y)} \tilde{u}_k(y_k)$. But this contradicts Dominance of more risk averse populations given that i is more risk averse than j.

Case 2: $\tilde{u}'_i(z_c) < \tilde{u}'_j(z_c)$. Then there exists $z < z_c$ such that $\tilde{u}_i(z) > \tilde{u}_j(z)$, because $\tilde{u}_i(z_c) = \tilde{u}_j(z_c)$. Thus, by the same reasoning as above, $x \succ y$, which is a contradiction of Dominance of more risk averse populations.

Hence, for all $j \in \mathbb{N} \setminus \{i\}$, we must have $\tilde{u}'_j(z_c) = \tilde{u}'_i(z_c)$. Denote $K = \tilde{u}'_i(z_c)$. We obtain that for all $j \in \mathbb{N}$ $\tilde{u}'_j(z_c) = \alpha_j u'_j(z_c) = K$, so that $\alpha_j = K/u'_j(z_c)$. Hence, by Eq. (7) and the definition of functions $\tilde{u}_i$, for all $f, g \in F$:

$$f \succsim g \iff \mathbb{E}\left[\sum_{i \in N(f)} \frac{u_i(f_i) - u_i(z_c)}{u'_i(z_c)}\right] \geq \mathbb{E}\left[\sum_{i \in N(g)} \frac{u_i(g_i) - u_i(z_c)}{u'_i(z_c)}\right].$$

Proof of Proposition 1

Proof of part 1

Let us first focus on interior solutions of Problem (2). Given that the utility utility functions u_i are concave, the first-order conditions are both necessary and sufficient for an optimum. These conditions are that, for all $s \in S$ and all $i \in N$,

$$\frac{u_i'\big(f_i^*(\omega(s))\big)}{u_i'(z_c)} = \lambda(s), \tag{8}$$

where $\lambda(s) > 0$ is the multiplier associated with the resource constraint. This implies for all $s \in S$ and all $i, j \in N$,

$$\frac{u_i'\big(f_i^*(\omega(s))\big)}{u_i'(z_c)} = \frac{u_j'\big(f_j^*(\omega(s))\big)}{u_i'(z_c)}. \tag{9}$$

Assume that $\omega(s) = z_p$. Because of the resource constraint $\frac{1}{n}\sum_{i\in N} f_i(s) \leq z_p$, we cannot have $f_i^*(z_p) \geq z_p$ for all $i \in N$ and $f_j^*(z_p) > z_p$ for some $j \in N$. On the other hand, $f_i^*(z_p) \leq z_p$ for all $i \in N$ and $f_j^*(z_p) < z_p$ for some $j \in N$ would be inefficient (we can increase the consumption of j without violating the resource constraint), and thus not optimal. There are only two possibility for an optimum:

1. $f_i^*(z_p) = z_p$ for all $i \in N$;
2. there exists $i, j \in N$ such that $f_i^*(z_p) < z_p < f_j^*(z_p)$.

In case 2, by concavity of the utility functions, we would have

$$\frac{u_i'\big(f_i^*(\omega(s))\big)}{u_i'(z_c)} < \frac{u_i'(z_c)}{u_i'(z_c)} = 1 = \frac{u_j'(z_c)}{u_j'(z_c)} < \frac{u_j'\big(f_j^*(\omega(s))\big)}{u_i'(z_c)},$$

which is a violation of the first-order condition (9). Thus only case 1 is possible.

This proves that there exists an efficient and egalitarian allocation when $\omega(s) = z_p$. Then, in the case of an interior solution, we can proceed like in the proof of Proposition 3 in Jouini, Napp and Nocetti (2013), which is straightforward.

Let us now consider the possibility of a corner solution where $f_i^*(\omega(s)) = 0$ for some individual $i \in N$. For this to be the case, we would need $\frac{u_i'(0)}{u_i'(z_c)} \leq \lambda(s)$.

Assume that $\omega(s) < z_c$, we do have $f_i^*(\omega(s)) < z_c$, as required by statement (a) in Part 1 of the proof. Also, if i is more risk-averse than another individual j, we have by our normalization that $\frac{u_i'(0)}{u_i'(z_c)} > \frac{u_j'(0)}{u_j'(z_c)}$, and therefore that $\frac{u_j'(0)}{u_j'(z_c)} < \lambda(s)$. Thus $f_j^*(\omega(s)) = 0 \leq f_i^*(\omega(s))$, as required.

Assume that $\omega(s) \geq z_c$. By the same reasoning as above, we know that there exists an individual j such that $f_j^*(\omega(s)) \geq z_c$ and $\frac{u_j'\left(f_j^*(\omega(s))\right)}{u_j'(z_c)} = \lambda(s)$. But, by concavity of the utility functions, we necessarily have:

$$\lambda(s) = \frac{u_j'\left(f_j^*(\omega(s))\right)}{u_j'(z_c)} \leq 1 < \frac{u_i'(0)}{u_i'(z_c)} \leq \lambda(s),$$

which is contradiction. An corner solution is never possible in that case.

Proof of part 2

Assume that the population N is composed of an individual i with preferences $u_i(z) = 1 - e^{-\alpha z}$, for all $z \in \mathbb{R}$, and $n-1$ other individuals with the same preferences $u_k(z) = 1 - e^{-\theta\alpha z}$, for all $z \in \mathbb{R}$, where $\alpha > 0$ and $\theta > 0$.[10]

We will focus without loss of generality on cases with interior solutions. Given that the utility utility functions u_i are concave, Problem (3) is convex and the following first-order conditions are both necessary and sufficient for an optimum for all $j \in N$ and $s \in S$,

$$\frac{u_j'\left(f_j^{**}(\omega(s))\right)}{u_j(\bar{x}) - u_j(0)} = \lambda(s). \tag{10}$$

Given the utility function we have chosen, this yields:

$$\frac{\alpha e^{-\alpha f_i^{**}(\omega(s))}}{1 - e^{-\alpha\bar{x}}} = \frac{\theta\alpha e^{-\theta\alpha f_k^{**}(\omega(s))}}{1 - e^{-\theta\alpha\bar{x}}},$$

[10]We could take people with different preferences of the CARA type exhibited here, provided they have similar enough preferences.

for all $s \in S$ and $k \in N \setminus \{i\}$. This can be written in the following way:

$$f_i^{**}(\omega(s)) = \theta f_k^{**}(\omega(s)) - \frac{1}{\alpha} \ln \left(\theta \frac{1-e^{-\alpha \bar{x}}}{1-e^{-\theta \alpha \bar{x}}} \right).$$

For a given a $\theta > 0$, this means that for α high enough we have $f_i^{**}(\omega(s)) \approx \theta f_k^{**}(\omega(s))$ for all $s \in S$ and $k \in N \setminus \{i\}$. Obviously, for any $\epsilon > 0$, when θ is high enough, the resource constraints $\sum_{j \in N} f_j^{**}(\omega(s)) = n\omega(s)$ implies that $f_i^{**}\big(\omega(s)\big) \geq (1-\varepsilon) n\omega(s)$ and $f_k^{**}\big(\omega(s)\big) \leq \varepsilon\omega(s)$ for all $k \in N \setminus \{i\}$.

Proof of Proposition 2

Collective relative risk tolerance

Let N be any population in $\mathcal{N}$ with size $n = |N|$. Denote $x_i^{N*}(z)$ the optimal allocation sharing rule when per capita resources are equal to $z \in \mathbb{R}_{++}$. For any $i \in N$, we know that, for interior solutions, there exists $\lambda \in \mathbb{R}_{++}$ such that (see Equation 8): $\frac{u_i'\left(x_i^{N*}(z)\right)}{u_i'(z_c)} = \lambda$. Given the HARA form of the utility function u_i and defining $\tau_i = 1/\alpha_i$, this means that $x_i^{N*}(z) = e^{-\tau_i \lambda}(z_c + \kappa) - \kappa$. Parameter τ_i drives individual i's relative risk tolerance at the consumption level $x_i \in \mathbb{R}_+$ defined as $t_i(x_i) = -u_i'(x_i)/\left[x_i u_i''(x_i)\right] = \tau_i(1 + \kappa/x_i)$.

The resource constraint yields:

$$\frac{1}{n} \sum_{i \in N} e^{-\tau_i \lambda} = \frac{z+\kappa}{z_c+\kappa} .$$

Define implicitly function $\phi_N : \mathbb{R}_{++} \to \mathbb{R}$ by $\frac{1}{n} \sum_{i \in N} e^{-\tau_i \phi_N(y)} = y$ for all $y \in \mathbb{R}_{++}$. For any $z \in \mathbb{R}_{++}$, also define the function y such that $y(z) = \frac{z+\kappa}{z_c+\kappa}$. We obtain $x_i^{N*}(z) = e^{-\tau_i \phi_N(y(z))}(z_c + \kappa) - \kappa$.

Note that $\phi_N(y) \leq 0$ if $y \geq 1$ and $\phi_N(y) \geq 0$ if $y \leq 1$. Furthermore, noting $\overline{\tau_N} := \max_{i \in N} \tau_i$ and $\underline{\tau_N} := \min_{i \in N} \tau_i$, we have $\overline{\tau_N} = 1/\underline{\alpha_N}$ and $\underline{\tau_N} = 1/\overline{\alpha_N}$. And $e^{-\overline{\tau_N}\phi_N(y)} \geq y \geq e^{-\underline{\tau_N}\phi_N(y)}$ when $y \geq 1$ (because $\phi_N(y) \leq 0$), so that $-\frac{1}{\underline{\tau_N}} \ln(y) \leq \phi_N(y) \leq -\frac{1}{\overline{\tau_N}} \ln(y)$. Similarly, when $y \leq 1$, $-\frac{1}{\overline{\tau_N}} \ln(y) \leq \phi_N(y) \leq -\frac{1}{\underline{\tau_N}} \ln(y)$.

When $z \leq z_c$, we know by Proposition 1 (first part) that for all $i \in N$

$$x_i^{N*}(z) \geq e^{-\overline{\tau_N}\phi_N(y(z))}(z_c+\kappa)-\kappa,$$

because the most risk averse agent has less consumption in bad states. And, because $-\phi_N(y) \geq \frac{1}{\underline{\tau_N}}\ln(y)$ when $y \leq 1$, we obtain for all $i \in N$:

$$x_i^{N*}(z) \geq e^{\frac{\overline{\tau_N}}{\underline{\tau_N}}\ln(y(z))}(z_c+\kappa)-\kappa = \left(\frac{z+\kappa}{z_c+\kappa}\right)^{\frac{\overline{\alpha_N}}{\underline{\alpha_N}}}(z_c+\kappa)-\kappa.$$

When $z > \underline{z_N}$, we have $x_i^{N*}(z) > 0$ for all $i \in N$ and thus interior solutions. Note that, when $z \geq z_c$, we know by Proposition 1 (first part) that we necessarily have interior solutions.

The *collective relative risk tolerance* for a population $N \in \mathcal{N}$ at the per capita resource level $z \in \mathbb{R}_{++}$ is $t_N(z) := 1/R_N(z) = -\big[W_N'(z)\big]/zW_N''(z)$. Hara, Huang and Kuzmics (2007) proved the following result that relates the collective relative risk tolerance to the individual ones through the efficient sharing rule:

$$t_N(z) = \tfrac{1}{n}\sum_{i\in N}\tfrac{x_i^{N*}(z)}{z} \times t_i\big(x_i^{N*}(z)\big).$$

In the case of our HARA utility functions, $t_i(x_i) = \tau_i(1+\kappa/x_i)$. Using the optimal sharing rule for the fair utilitarian social ordering, we thus have:

$$t_N(z) = \tfrac{z_c+\kappa}{nz}\sum_{i\in N}\tau_i e^{-\tau_i\phi_N(y(z))}. \tag{11}$$

Change in relative risk aversion

Consider any population $N \in \mathcal{N}$, with size $n = |N|$, and any $i, j \in (\mathcal{N} \setminus N)$, such that individual i is more risk averse than individual j (so that $\tau_i < \tau_j$). Let $\tilde{\tau} \in \mathbb{R}_{++}$ be such that $\tau_j \geq \tilde{\tau} \geq \tau_i$: this would be the risk tolerance parameter of a fictitious individual, with relative risk aversion that is intermediate between the one of i and j.

Let us fix $z \in \mathbb{R}_{++}$ the per capita endowment of the economy. We define function $\gamma^N_{y(z),\tilde{\tau}}(\ell)$ implicitly by:

$$\tfrac{1}{n+1}\left(\sum_{k\in N} e^{-\tau_k\gamma^N_{y(z),\tilde{\tau}}(\ell)}\right)+\tfrac{1}{n+1}e^{-(\tilde{\tau}+\ell)\gamma^N_{y(z),\tilde{\tau}}(\ell)}=y(z)$$

Differentiating with respect to ℓ (while keeping $y(z)$ constant), we obtain:

$$\tfrac{\partial\gamma^N_{y(z),\tilde{\tau}}}{\partial\ell}(\ell)=-\gamma^N_{y(z),\tilde{\tau}}(\ell)\tfrac{e^{-(\tilde{\tau}+\ell)\gamma^N_{y(z),\tilde{\tau}}(\ell)}}{\sum_{k\in N}\tau_k e^{-\tau_k\gamma^N_{y(z),\tilde{\tau}}(\ell)}+(\tilde{\tau}+\ell)e^{-(\tilde{\tau}+\ell)\gamma^N_{y(z),\tilde{\tau}}(\ell)}}.$$

We can also introduce the function $\xi^N_{y,\tilde{\tau}}(\ell)$ implicitly by:

$$\xi^N_{y(z),\tilde{\tau}}(\ell)=\tfrac{z_c+\kappa}{nz}\left(\sum_{k\in N}\tau_k e^{-\tau_k\gamma^N_{y(z),\tilde{\tau}}(\ell)}+(\tilde{\tau}+\ell)e^{-(\tilde{\tau}+\ell)\gamma^N_{y(z),\tilde{\tau}}(\ell)}\right).$$

Again differentiating with respect to ℓ, we obtain:

$$\begin{aligned}\tfrac{\partial\xi^N_{y(z),\tilde{\tau}}}{\partial\ell}(\ell) &= \tfrac{z_c+\kappa}{nz}\Bigg[-\tfrac{\partial\gamma^N_{y(z),\tilde{\tau}}}{\partial\ell}(\ell)\left(\sum_{k\in N}\tau_k^2 e^{-\tau_k\gamma^N_{y(z),\tilde{\tau}}(\ell)}+(\tilde{t}+\ell)^2e^{-(\tilde{\tau}+\ell)\gamma^N_{y(z),\tilde{\tau}}(\ell)}\right)\\ &\qquad -\gamma^N_{y(z),\tilde{\tau}}(\ell)(\tilde{\tau}+\ell)e^{-(\tilde{t}+\ell)\gamma^N_{y(z),\tilde{\tau}}(\ell)}+e^{-(\tilde{\tau}+\ell)\gamma^N_{y,\tilde{t}}(\ell)}\Bigg]\\ &= \Xi^N_{y,\tilde{\tau}}(\ell)\times\Bigg[\left(\sum_{k\in N}\Big[1+\gamma^N_{y(z),\tilde{\tau}}(\ell)\big(\tau_k-\tilde{\tau}-\ell\big)\Big]\tau_k e^{-\tau_k\gamma^N_{y(z),\tilde{\tau}}(\ell)}\right)\\ &\qquad +(\tilde{\tau}+\ell)e^{-(\tilde{\tau}+\ell)\gamma^N_{y(z),\tilde{\tau}}(\ell)}\Bigg],\end{aligned}$$

with $\Xi^N_{y,\tilde{\tau}}(\ell)=\frac{e^{-(\tilde{\tau}+\ell)\gamma^N_{y,\tilde{\tau}}(\ell)}(z_c+\kappa)}{nz\left(\sum_{k\in N}\tau_k e^{-\tau_k\gamma^N_{y,\tilde{\tau}}(\ell)}+(\tilde{\tau}+\ell)e^{-(\tilde{\tau}+\ell)\gamma^N_{y,\tilde{\tau}}(\ell)}\right)}>0.$

The quantity $\frac{\partial\xi^N_{y(z),\tilde{\tau}}}{\partial\ell}(0)$ measures the impact on collective relative risk tolerance of a small increase in the relative risk tolerance of a person with risk tolerance $\tilde{\tau}$ when the society is composed of this person and the population N, and when the per capita endowment of the economy is z. Clearly, if we have $\frac{\partial\xi^N_{y(z),\tilde{\tau}}}{\partial\ell}(0)\geq 0$ for all $\tau_j\geq\tilde{\tau}\geq\tau_i$, then collective relative risk aversion is higher in population

$N \cup \{i\}$ than in population $N \cup \{j\}$.

Given that $\Xi^N_{y,\tilde{\tau}}(0) > 0$ and

$$\frac{\partial \xi^N_{y(z),\tilde{\tau}}}{\partial \ell}(0) = \Xi^N_{y,\tilde{\tau}}(0) \times \left[\left(\sum_{k \in N} \left[1 + \gamma^N_{y(z),\tilde{\tau}}(0)\big(\tau_k - \tilde{\tau}\big) \right] \tau_k e^{-\tau_k \gamma^N_{y(z),\tilde{\tau}}(0)} \right) + \tilde{\tau} e^{-\tilde{\tau} \gamma^N_{y(z),\tilde{\tau}}(0)} \right],$$

a sufficient condition for collective relative risk aversion to be higher in population $N \cup \{i\}$ than in population $N \cup \{j\}$ is that $1 + \gamma^N_{y(z),\tilde{\tau}}(0)\big(\tau_k - \tilde{\tau}\big) \geq 0$ for all $k \in N$. Denote $\overline{\tau} = \overline{\tau_{N \cup \{i,j\}}}$ and $\underline{\tau} = \underline{\tau_{N \cup \{i,j\}}}$.

Case 1: $z \geq z_p$**.** If $z \geq z_p$, then $y = (z + \kappa)/(z_p + \kappa) \geq 1$ and for any population $M \in \mathcal{N}$ $\phi_M(y) \leq 0$ and $-\frac{1}{\underline{\tau_M}} \ln(y) \leq \phi_M(y) \leq -\frac{1}{\overline{\tau_M}} \ln(y)$. We have $\gamma^N_{y(z),\tilde{\tau}}(0) = \phi_M(y)$ where M is composed of population N and a fictitious individual with relative risk tolerance $\tilde{\tau}$ so that $\underline{\tau_M} \geq \underline{\tau}$ and $\overline{\tau_M} \leq \overline{\tau}$. Hence $-\frac{1}{\underline{\tau}} \ln(y) \leq \gamma^N_{y(z),\tilde{\tau}}(0) \leq -\frac{1}{\overline{\tau}} \ln(y) \leq 0$.

If $\tau_i \geq \tau_k$ for all $k \in N$, then $\tilde{\tau} \geq \tau_k$ and $\gamma^N_{y,\tilde{\tau}}(0)\big(\tau_k - \tilde{\tau}\big) \geq 0$ for all $k \in N$, which is sufficient to have $\frac{\partial \xi^N_{y(z),\tilde{\tau}}}{\partial \ell}(0) \geq 0$ for all $\tau_j \geq \tilde{\tau} \geq \tau_i$.

For all $k \in N$, $\gamma^N_{y(z),\tilde{\tau}}(0)\big(\tau_k - \tilde{\tau}\big) \geq \gamma^N_{y(z),\tilde{\tau}}(0)\big(\overline{\tau} - \underline{\tau}\big) \geq -\frac{\overline{\tau} - \underline{\tau}}{\overline{\tau}} \ln(y)$. If $\frac{z+\kappa}{z_p+\kappa} = y \leq e^{\frac{\underline{\tau}}{\overline{\tau} - \underline{\tau}}}$, then $1 + \gamma^N_{y(z),\tilde{\tau}}(0)\big(\tau_k - \tilde{\tau}\big) \geq 0$ for all $k \in N$, which is sufficient to have $\frac{\partial \xi^N_{y(z),\tilde{\tau}}}{\partial \ell}(0) \geq 0$ for all $\tau_j \geq \tilde{\tau} \geq \tau_i$.

Case 2: $z \leq z_p$**.** If $z \leq z_p$, then $y = (z + \kappa)/(z_p + \kappa) \leq 1$. Like above, we can show that $-\frac{1}{\underline{\tau}} \ln(y) \geq \gamma^N_{y,\tilde{t}}(0) \geq -\frac{1}{\overline{\tau}} \ln(y) \geq 0$.

If $\tau_j \leq \tau_k$ for all $k \in N$, then $\tilde{\tau} \leq \tau_k$ and $\gamma^N_{y,\tilde{\tau}}(0)\big(\tau_k - \tilde{\tau}\big) \geq 0$ for all $k \in N$, which is sufficient to have $\frac{\partial \xi^N_{y(z),\tilde{\tau}}}{\partial \ell}(0) \geq 0$ for all $\tau_j \geq \tilde{\tau} \geq \tau_i$.

For all $k \in N$, $\gamma^N_{y(z),\tilde{\tau}}(0)\big(\tau_k - \tilde{\tau}\big) \geq -\gamma^N_{y(z),\tilde{\tau}}(0)\big(\overline{\tau} - \underline{\tau}\big) \geq \frac{\overline{\tau} - \underline{\tau}}{\underline{\tau}} \ln(y)$. If $\frac{z+\kappa}{z_p+\kappa} = y \geq e^{-\frac{\underline{\tau}}{\overline{\tau} - \underline{\tau}}}$, then $1 + \gamma^N_{y(z),\tilde{\tau}}(0)\big(\tau_k - \tilde{\tau}\big) \geq 0$ for all $k \in N$, which is sufficient to have $\frac{\partial \xi^N_{y(z),\tilde{\tau}}}{\partial \ell}(0) \geq 0$ for all $\tau_j \geq \tilde{\tau} \geq \tau_i$.