Dynamic contribution to a Public Good with Constant Returns to Scale Technology - *Work in progress*

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Abstract

We consider a dynamic contribution game in which a group of agents collaborate to complete a public project. The agents exert efforts over time and get rewarded upon completion of the project, once the cumulative effort reaches a pre-specified level. The cost of effort is linear. We explicitly derive the cooperative solution and a noncooperative Markov-perfect Nash equilibrium. We characterize the set of socially efficient projects, i.e., projects that cooperative groups find worth completing. Comparing with the Markov-perfect Nash equilibrium, we find that non-cooperative groups give up large socially efficient projects and take too much time to complete the others.

1 Introduction

Since the seminal work of Olson (1965), the problem of free riding in groups has been mostly analyzed in static settings. However, many voluntary contributions to public projects have a dynamic and recurring feature (Fershtman and Nitzan, 1991). In this paper, we consider a differential game in which players contribute to a joint project generating utility only after completion. Our model can be used to describe collaborative situations such as search teams, R&D joint ventures or funding discrete public goods. We show that noncooperative groups fail to carry out some socially optimal projects and procrastinate on the projects which they complete. We also design a balanced incentive scheme to correct this failure.

Our model ties into the literature on free-riding in groups, showing that self-interested individual members have too little incentives to further their common interests (Olson, 1965). In static settings, free-riding incentives are found with continuous public goods (Olson, 1965; Cornes and Sandler, 1984, 1985; Bergström and al, 1986), but may vanish with discrete public goods (Palfrey and Rosenthal, 1984; Bagnoli and Lipman, 1989; Nitzan and Romano, 1990).[1] In dynamic settings, free-riding occurs with both continuous public goods (Fershtman and Nitzan, 1991) and discrete public goods (Georgiadis, 1994; Kessing, 2007: Yildirim, 2006). However, the players' contributions become strategic complements with discrete public goods, which partially mitigates the incentives to free-ride (Georgiadis, 1994; Yildirim, 2006; Kessing, 2007).

In this paper, we study a game of dynamic contributions to a discrete public good in continuous time. To the best of my knowledge, the closest papers in the literature are Admati and Perry (1991), Yildirim (2006), Kessing (2007), Georgiadis (2014) and Rouillon (2016).

Admati and Perry (1991) consider a game in discrete time in which *two* players alternate in contributing a discrete public good. Using strictly convex cost functions, they can show that some socially desirable projects may not be completed in equilibrium. Besides technical differences (i.e., here, any number of players contributing simultaneously in continuous time), the present paper complements Admati and Perry (1991), by fully characterizing the inefficiency arising in equilibrium using a *linear cost function*.[2]

Yildirim (2006) solves a game in discrete time in which players simultaneously make binary contributions to further a joint project, which is completed only after a given number of steps where at least one player contributed. The players' costs of contributing are private information and drawn periodically from a common distribution. In equilibrium, no matter the number of steps required before completion, each player contributes with a strictly positive, but socially too small probability. The model of Yildirim (2006) differs dramatically from the one used below, in particular in the technology considered for producing the public good (respectively, the maximum versus the sum of individual contributions).[3]

Kessing (2007) sets a dynamic game of discrete public good completed by private contributions. Using quadratic cost functions, Kessing (2007) calculates

[1] Nitzan and Romano (1990) argue that a discrete public good induces a discontinuity in the players' payoffs which corrects the free-riding incentives.

[2] Admati and Perry (1991) mention the linear cost as a limit case in their Lemma 4.1. However, their statement of the equilibrium and associated proof is irrelevant here, due to the differences between their model and the one considered here.

[3] Subgames in Yildirim (2006) are coordination games, instead of public good games in this paper.

explicitly the socially optimal and the Markov-perfect Nash equilibrium contribution paths and shows that in equilibrium, some socially profitable projects are not carrried out and the completion time is too large. The present paper checks the conclusions of Kessing (2007) in a different setting, by using a *linear cost function* instead of a quadratic cost function.

Georgiadis (2014) constructs a game in which players contribute a discrete public project which progresses stochastically (Brownian motion). In equilibrium, no matter the cumulative effort required to complete it, the players' contributions are always strictly positive, but socially insufficient.[4] The present paper departs from Georgiadis (2014), by considering a *linear cost function* (instead of isoelastic and convex cost functions) and by assuming that the state of the project follows a deterministic process. This allows us to solve the game explicitly and derive clearcut properties.[5]

Another contribution of this paper with respect to the ones surveyed above is the definition of an economic mechanism to implement the socially optimal contribution path.

The remainer of the paper is organized as follows. Section 2 sets out the model. Section 3 analyzes the cooperative solution. Section 4 characterizes a Markov-perfect Nash equilibrium. Section 5 discusses some normative implications of our results.

2 The model

Consider a group composed of n agents who can collaborate to complete a joint project. Each agent i exerts an instantaneous effort $q_i(t) \leq \bar{q}$. The unit cost of effort is c. The progression of the project is represented by the state $x(t)$, which evolves according to the ordinary differential equation

$$\dot{x}(t) = -\sum_{i=1}^{n} q_i(t), \; x(0) = \ell. \tag{1}$$

The initial state ℓ shall be interpreted as the length of the project. The project is completed at time T, when $x(T) = 0$ occurs for the first time. Each agent i then receives a reward b [6]. The agents are assumed to discount time at the common rate δ.

Each agent i's problem is to choose an individual effort path $q_i(.)$ to maximize

[4] This may be surprising at first glance. It can be explained because, as the project progresses stochastically, there always is some positive probability that it goes to completion. Hence, exerting an infinitesimal effort always induce a positive marginal expected benefit.

[5] Georgiadis (2014) also adresses interesting design issues (i.e., team size, rewards), from the perspective of a residual claimant of the project. These are beyond the scope of this paper.

[6] In general, the individual reward b may depend on the size n of the group (i.e., $b = \beta(n)$). For the sake of simplicity, we do not make this apparent in the notations.

$$be^{-\delta T}-\int_0^T cq_i\left(t\right)e^{-\delta t}dt. \quad (2)$$

The team's problem is to find a vector of individual effort paths $q_i\left(t\right)$, for all i, to maximize

$$\sum_{i=1}^{n}\left(be^{-\delta T}-\int_0^T cq_i\left(t\right)e^{-\delta t}dt\right). \quad (3)$$

3 Optimal policy

We determine here the socially optimal path of individual efforts and discuss its properties.

The team's problem is to find a path $q_i\left(t\right)$, for all i, and a completion time T, to maximize

$$\sum_{i=1}^{n}\left(be^{-\delta T}-\int_0^T cq_i\left(t\right)e^{-\delta t}dt\right)$$

subject to

$$\dot{x}\left(t\right)=-\sum_{i=1}^{n}q_i\left(t\right),\ x\left(0\right)=\ell \text{ and } x\left(T\right)=0.$$

Proposition 1 below characterizes the optimal solution of the cooperative team.

Proposition 1. A cooperative team undertakes and completes a public project if and only if $\ell<\ell^o\equiv n\left(\bar{q}/\delta\right)\ln\left(1+\left(b\delta\right)/\left(c\bar{q}\right)\right)$. The cooperative individual effort is $q_i=f\left(x\right)$, where $f\left(x\right)$ is equal to $\bar{q}$ if $x<\ell^o$ and 0 if $x\geq\ell^o$ an, meaning that a cooperative team completes any project that it undertakes as quickly as possible.

Proof. For all x, let $J\left(x\right)$ be equal to the maximized discounted payoff function of the team. Assuming differentiability, it satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$\delta J\left(x\right)=\max\left\{-\sum_{i=1}^{n}cq_i-J'\left(x\right)\sum_{i=1}^{n}q_i;\forall i,0\leq q_i\leq\bar{q}\right\}, \text{ for all } x,$$

and the boundary conditions

$$J\left(0\right)=nb \text{ and } \lim_{x\to\infty}J\left(x\right)=0.$$

The proof is constructive. We display a value function $V(x)$, satisfying the HJB equation and the boundary conditions, with $q_i = f(x)$, for all i, the corresponding optimal control.

For all $0 \leq x < \ell^o$, define

$$V(x) = nb\left(\left(1 + \frac{c\bar{q}}{b\delta}\right)e^{-\frac{\delta}{n\bar{q}}x} - \frac{c\bar{q}}{b\delta}\right)$$

and

$$f(x) = \bar{q}.$$

We show below that $V(x)$ satisfies the HJB equation

$$\delta V(x) = \max\left\{-\sum_{i=1}^{n} cq_i - V'(x)\sum_{i=1}^{n} q_i; \forall i, 0 \leq q_i \leq \bar{q}\right\} \tag{4}$$

and and the boundary conditions

$$V(0) = nb \text{ and } \lim_{x\to\ell^o} V(x) = 0,$$

with $q_i = f(x)$, for all i, the corresponding optimal control.

By differentiation, we obtain

$$V'(x) = -\frac{b\delta}{\bar{q}}\left(1 + \frac{c\bar{q}}{b\delta}\right)e^{-\frac{\delta}{n\bar{q}}x}$$

and therefore

$$-c - V'(x) = -c + \frac{b\delta}{\bar{q}}\left(1 + \frac{c\bar{q}}{b\delta}\right)e^{-\frac{\delta}{n\bar{q}}x}.$$

This expression is strictly positive for all $0 \leq x < \ell^o$.[7] This implies that the right-hand side of (4) is (strictly) increasing in q_i and thus is maximum when $q_i = f(x) = \bar{q}$, for all i. Then, substituting $q_i = f(x)$, for all i, $f(x) = \bar{q}$ and $V'(x) = -\frac{b\delta}{\bar{q}}\left(1 + \frac{c\bar{q}}{b\delta}\right)e^{-\frac{\delta}{n\bar{q}}x}$, we obtain

$$\max\left\{-\sum_{i=1}^{n} cq_i - V'(x)\sum_{i=1}^{n} q_i; \forall i, q_i \geq 0\right\} = \delta nb\left(\left(1 + \frac{c\bar{q}}{b\delta}\right)e^{-\frac{\delta}{n\bar{q}}x} - \frac{c\bar{q}}{b\delta}\right),$$

which proves that $V(x)$ satisfies (4). The boundary conditions are trivially verified.

For all $x \geq \ell^o$, it is trivial to show that $V(x) = 0$ satisfies the HJB equation and the boundary conditions, with $q_i = f(x) = 0$, for all i, the corresponding optimal control. □

[7] Because it is decreasing in x and is equal to zero when $x = \ell^o$, by definition of ℓ^o.

4 Markov-perfect Nash equilibrium.

In this section, we derive a Markov-perfect Nash equilibrium and discuss its properties.

A (stationary) Markovian strategy for individual i is a function s_i, associating project states x with agent i's efforts $q_i = s_i(x)$. A vector $S = (s_i)_{i=1}^n$ is called a strategic profile. It is said to be feasible if there exists a unique absolutely continuous state trajectory $x(\cdot)$ satisfying (1), with $q_i(t) = s_i(x(t))$, for all i and t, and if the corresponding agents' objectives (2), for all i, are well defined (Dockner et al., 2000).

For all feasible strategic profile $S = (s_i)_{i=1}^n$ and initial state ℓ, let

$$V_i(S, \ell) = be^{-\delta T} - \int_0^T cq_i(t)\, e^{-\delta t} dt$$

with

$$\dot{x}(t) = -\sum_{i=1}^n q_i(t),\ x(0) = \ell \text{ and } x(T) = 0,$$

$$(q_i(t))_{i=1}^n = (s_i(x(t)))_{i=1}^n.$$

A (stationary) Markov-perfect Nash equilibrium is a feasible vector $S^* = (s_i^*)_{i=1}^n$ such that, for all i, s_i and ℓ, $V_i(S^*, \ell) \geq V_i((S^*/s_i), \ell)$, with $(S^*/s_i) = \left(s_1^*, ..., s_{i-1}^*, s_i, s_{i+1}^*, ..., s_n^*\right)$ a feasible strategic profile.

Proposition 2 and 2' below characterize the equilibrium solution of the non-cooperative team, separating the cases of large and small groups.

Proposition 2. Large groups. Asume that $n \geq 1 + (b\delta)/(c\bar{q})$. A non-cooperative team undertakes and completes a public project if and only if $\ell < \ell^* \equiv b/c$. The equilibrium individual effort is $q_i = g(x)$, where $g(x)$ is equal to $\delta(b/c - x)/(n-1)$ if $x < \ell^*$ and 0 if $x \geq \ell^*$. As $g(x) < f(x)$, a non-cooperative team takes to much time completing the projects that it undertakes.

Proof. Assume that $n \geq 1 + (b\delta)/(c\bar{q})$. For all x, let $J_i(x)$ be equal to the maximized discounted payoff function of agent i, given that the others play the strategy $s_j^* = g(\cdot)$. Assuming differentiability, it satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$\delta J_i(x) = \max\left\{-cq_i - J_i'(x)(q_i + (n-1)g(x));\, 0 \leq q_i \leq \bar{q}\right\}, \text{ for all } x,$$

and the boundary conditions

$$J_i(0) = b \text{ and } \lim_{x\to\infty} J_i(x) = 0.$$

The proof is constructive. We display a value function $v(x)$, satisfying the HJB equation and the boundary conditions, with $q_i = g(x)$ the corresponding equilibrium strategy.

For all $0 \leq x < \ell^*$, define

$$v(x) = b\left(1 - \frac{c}{b}x\right)$$

and

$$g(x) = \frac{\delta}{n-1}\left(\frac{b}{c} - x\right).$$

We show below that $v(x)$ satisfies the HJB equation

$$\delta v(x) = \max\{-cq_i - v'(x)(q_i + (n-1)g(x)); 0 \leq q_i \leq \bar{q}\} \tag{5}$$

and the boundary conditions

$$v(0) = b \text{ and } \lim_{x \to \ell^*} v(x) = 0,$$

with $q_i = g(x)$ the corresponding optimal control.

By differentiation, we obtain

$$v'(x) = -c$$

and therefore

$$-c - v'(x) = 0.$$

This implies that the right-hand side of (5) is constant for all q_i and thus, in particular, is maximum when $q_i = g(x) = \delta(b/c - x)/(n-1)$.[8] Then, substituting $q_i = g(x)$, $g(x) = \delta(b/c - x)/(n-1)$ and $v'(x) = -c$, we obtain

$$\max\{-cq_i - v'(x)(q_i + (n-1)g(x)); 0 \leq q_i \leq \bar{q}\} = \delta b\left(1 - \frac{c}{b}x\right),$$

which implies that $v(x)$ satisfies (5). The boundary conditions are easily verified.

For all $x > \ell^*$, it is trivial to show that $v(x) = 0$ satisfies the HJB equation and the boundary conditions, with $q_i = g(x) = 0$, for all i, the corresponding optimal control. □

Proposition 2'. Small groups. Asume that $n < 1 + (b\delta)/(c\bar{q})$. A non-cooperative team undertakes and completes a public project if and only if $\ell < \ell^{**} \equiv n(\bar{q}/\delta)[\ln(1 + (b\delta)/(c\bar{q})) - \ln(n) + 1 - 1/n]$. There exists $0 < \bar{\ell} < \ell^{**}$ such that equilibrium effort $q_i = g(x)$, where $g(x)$ is equal to $\bar{q}$ if $x < \bar{\ell}$, $(n-1)\frac{c\bar{q}}{\delta} - c\left(x - \bar{\ell}\right)$ if $\bar{\ell} \leq x < \ell^{**}$, and 0 otherwise. Again, as $g(x) \leq f(x)$, a non-cooperative team takes to much time completing the projects that it undertakes.

[8] We verify that $0 \leq g(x) \leq \bar{q}$ for all $0 \leq x < \ell^*$.

Proof. Asume that $n < 1 + (b\delta) / (c\bar{q})$. For all x, let $J_i(x)$ be equal to the maximized discounted payoff function of agent i, given that the others play the strategy $s_j^* = g(\cdot)$. Assuming differentiability, it satisfies the Hamilton-Jacobi-Bellman equation

$$\delta J_i(x) = \max\{-cq_i - J_i'(x)(q_i + (n-1)g(x)); q_i \geq 0\}, \text{ for all } x,$$

and the boundary conditions

$$J_i(0) = b \text{ and } \lim_{x\to\infty} J_i(x) = 0.$$

The proof is constructive. We display a value function $v(x)$, satisfying the HJB equation and the boundary conditions, with $q_i = g(x)$ the corresponding equilibrium strategy.

Define the value function

$$v(x) = \begin{cases} b\left(\left(1 + \frac{c\bar{q}}{b\delta}\right)e^{-\frac{\delta}{n\bar{q}}x} - \frac{c\bar{q}}{b\delta}\right) & \text{if } 0 \leq x < \bar{\ell} \\ c(\ell^{**} - x) & \text{if } \bar{\ell} \leq x < \ell^{**} \\ 0 & \text{if } x \geq \ell^{**} \end{cases}$$

and the corresponding optimal control

$$g(x) = \begin{cases} \bar{q} & \text{if } 0 \leq x < \bar{\ell} \\ \frac{\delta}{n-1}(\ell^{**} - x) & \text{if } \bar{\ell} \leq x < \ell^{**} \\ 0 & \text{if } x \geq \ell^{**} \end{cases}$$

where[9]

$$\bar{\ell} = n\frac{\bar{q}}{\delta}\left(\ln\left(1 + \frac{b\delta}{c\bar{q}}\right) - \ln(n)\right) \text{ and } \ell^{**} = n\frac{\bar{q}}{\delta}\left(\ln\left(1 + \frac{b\delta}{c\bar{q}}\right) - \ln(n) + 1 - 1/n\right).$$

We show below that $v(x)$ satisfies the HJB equation

$$\delta v(x) = \max\{-cq_i - v'(x)(q_i + (n-1)g(x)); 0 \leq q_i \leq \bar{q}\} \tag{6}$$

and the boundary conditions

$$v(0) = b \text{ and } \lim_{x\to\ell^*} v(x) = 0,$$

with $q_i = g(x)$ the corresponding optimal control.

For all $0 \leq x < \bar{\ell}$, we obtain by differentiation

$$v'(x) = -\frac{b\delta}{n\bar{q}}\left(1 + \frac{c\bar{q}}{b\delta}\right)e^{-\frac{\delta}{n\bar{q}}x}.$$

[9] Remark that $\ell^{**} = \bar{\ell} + (n-1)\bar{q}/\delta$.

Therefore,

$$-c - v'(x) = -c + \frac{b\delta}{n\bar{q}}\left(1 + \frac{c\bar{q}}{b\delta}\right)e^{-\frac{\delta}{n\bar{q}}x}.$$

This expression is (strictly) positive for all $0 \leq x < \bar{\ell}$.[10] This implies that the right-hand side of (6) is (strictly) increasing in q_i and thus is maximum when $q_i = g(x) = \bar{q}$. Then, substituting $q_i = g(x)$, $g(x) = \bar{q}$ and $v'(x) = -\frac{b\delta}{n\bar{q}}\left(1 + \frac{c\bar{q}}{b\delta}\right)e^{-\frac{\delta}{n\bar{q}}x}$, we obtain

$$\max\{-cq_i - v'(x)(q_i + (n-1)g(x)); 0 \leq q_i \leq \bar{q}\} = \delta b\left(\left(1 + \frac{c\bar{q}}{b\delta}\right)e^{-\frac{\delta}{n\bar{q}}x} - \frac{c\bar{q}}{b\delta}\right),$$

which implies that $v(x)$ satisfies (6). The boundary condition $v(0) = b$ is immediately verified.

For all $\bar{\ell} \leq x < \ell^{**}$, we obtain by differentiation

$$v'(x) = -c$$

and therefore

$$-c - v'(x) = 0.$$

This implies that the right-hand side of (6) is constant for all q_i and thus, in particular, is maximum when $q_i = g(x) = \delta(\ell^{**} - x)/(n-1)$.[11] Then, substituting $q_i = g(x)$, $g(x) = \delta(\ell^{**} - x)/(n-1)$ and $v'(x) = -c$, we obtain

$$\max\{-cq_i - v'(x)(q_i + (n-1)g(x)); 0 \leq q_i \leq \bar{q}\} = \delta c(\ell^{**} - x),$$

which implies that $v(x)$ satisfies (6). The boundary conditions are easily verified.

For all $x > \ell^{**}$, it is trivial to show that $v(x) = 0$ satisfies the HJB equation and the boundary conditions, with $q_i = f(x) = 0$, for all i, the corresponding optimal control. □

5 Comparison

We investigate here the differences between the cooperative solution and the Markov-perfect Nash equilibrium determined previously.

Property 1. Free-riding and procrastination. Noncooperative groups (a) give up large projects that cooperative groups with same characteristics would find worth completing and (b) most of the time take too much time completing the others. Formally, we respectively show that: if $n < 1 + (b\delta)/(c\bar{q})$, then $\ell^o < \ell^*$ and $g(x) < f(x)$ for all $x < \ell^*$; if $n \geq 1 + (b\delta)/(c\bar{q})$, then $\ell^o < \ell^{**}$, and $g(x) \leq f(x)$ for all $\bar{\ell} < x < \ell^{**}$.

[10] Because it is decreasing in x and is equal to zero when $x = \bar{\ell}$, by definition of $\bar{\ell}$.

[11] Using $\ell^{**} - \bar{\ell} = (n-1)\bar{q}/\delta$, we verify that $0 \leq g(x) \leq \bar{q}$ for all $\bar{\ell} \leq x < \ell^{**}$.

Proof. We first demonstrate that

$$\ell^* < \ell^o \Leftrightarrow \frac{c\bar{q}}{b\delta} \ln\left(1 + \frac{b\delta}{c\bar{q}}\right) > \frac{1}{n}, \text{ for all } n \geq 1 + \frac{b\delta}{c\bar{q}}. \tag{7}$$

Clearly, it is sufficient to verify that (7) holds true when $n = 1 + \frac{b\delta}{c\bar{q}}$. In this case, we can rearrange it as

$$n \ln(n) - n + 1 > 0.$$

As the derivative of this expression is $\ln(n)$, which is positive given $n \geq 1$, it has a minimum equal to 0 when $n = 1$. This proves our assertion (7).

We now demonstrate that

$$\ell^{**} < \ell^o \Leftrightarrow n\frac{\bar{q}}{\delta}\left(\ln\left(1 + \frac{b\delta}{c\bar{q}}\right) - \ln(n) + 1 - 1/n\right) < n\frac{\bar{q}}{\delta}\ln\left(1 + \frac{b\delta}{c\bar{q}}\right), \text{ for all } n < 1 + \frac{b\delta}{c\bar{q}}.$$

It is easily to see that this inequality simplifies as

$$n \ln(n) - n + 1 > 0,$$

which we already know to always be true. □

6 Conclusion.

This paper has analyzed the problem of free-riding within groups, in case where the agents make contributions over time to complete a public good. It contributes to the literature in two ways. Firstly, considering a general technology for producing the public good, we still achieve to give an explicit characterization of the noncooperative path of contributions. We show that noncooperative groups fail to carry out some socially optimal projects and procrastinate on the projects which they complete. Secondly, we display an economic mechanism capable of implementing the socially optimal contribution path as a noncooperative equilibrium.

7 References.

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