On the Coalitional Stability of Monopoly Power in Differentiated Bertrand and Cournot Oligopolies

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Abstract

In this article we revisit the classic comparison between Bertrand and Cournot competition in the presence of a cartel of firms that faces outsiders acting individually. This competition setting enables to deal with both non-cooperative and cooperative oligopoly games. We concentrate on industries consisting of symmetrically differentiated products where firms operate at a constant and identical marginal cost. First, while the standard Bertrand-Cournot rankings still hold for Nash equilibrium prices, we show that the results may be altered for Nash equilibrium quantities and profits. Second, we define cooperative Bertrand and Cournot oligopoly games with transferable utility on the basis of their non-cooperative foundation. We establish that the core of a cooperative Cournot oligopoly game is strictly included in the core of a cooperative Bertrand oligopoly game when the number of firms is lower or equal to 25. Otherwise the cores cannot be compared. Moreover, we focus on the aggregate-monotonic core, a subset of the core, that has the advantage to select point solutions satisfying both core selection and aggregate monotonicity properties. We succeed in comparing the aggregate-monotonic cores between Bertrand and Cournot competition regardless of the number of firms.

Keywords: Bertrand; Cournot; Differentiated oligopoly; Cartel; Nash equilibrium; Core; Aggregate-monotonic core

1 Introduction

The purpose of this article is to revisit the classic comparison between Bertrand and Cournot competition in the presence of a cartel of firms. We concentrate on industries consisting of symmetrically differentiated products represented by Shubik's demand system (Shubik 1980), each one produced by a single firm. Furthermore, we assume that firms operate at a constant and identical marginal cost. While cartel members maximize their joint profit by correlating their strategies and play as a multiproduct firm, other firms, called outsiders, are

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supposed to act independently. The main interest of this competition setting is to examine the two most well-known solution concepts in non-cooperative and cooperative games, namely, the Nash equilibrium (Nash 1950) and the core (Gillies 1953).

In oligopoly theory, a well-known result is that Bertrand competition is more competitive and efficient than Cournot competition. More properly speaking, Bertrand competition yields lower prices and profits and higher quantities, consumer surplus, and welfare than Cournot competition. Singh and Vives (1984) have first established these standard Bertrand-Cournot rankings which have been extended by Cheng (1985), Vives (1985) and Okuguchi (1987). Some years later the limitations of these results have been pointed out by Dastidar (1997) exploiting cost asymmetries, and Häckner (2000), and Amir and Jin (2001) using product differentiation. Other limitations have been put forward by, among others, Lofaro (2002) with incomplete information on costs, Miller and Pazgal (2001) in environments with strategic managerial delegation, and Pal (2015) including networks externalities in the latter approach.

To date, the literature comparing Bertrand and Cournot competition has exclusively focused on environments where all firms maximize their profits individually. In the first part of this article, merely assuming that a cartel of firms has been formed and faces outsiders acting individually, we provide new limitations of the standard Bertrand-Cournot rankings discussed above. More accurately, while the standard Bertrand-Cournot rankings still hold for Nash equilibrium prices, the results may be altered for Nash equilibrium quantities and profits. Indeed, Bertrand competition yields higher quantities for cartel members than Cournot competition but each outsider may raise or reduce its production depending on the quantity change of cartel members. As a consequence, outsiders still earn lower profits in Bertrand than in Cournot competition but the cartel joint profit may be larger in Bertrand competition when the number of firms is sufficiently large. In spite of these results, we show that the standard Bertrand-Cournot rankings on profits always hold when the number of firms is lower or equal to 25 which corresponds, in practice, to the majority of differentiated oligopolies with symmetric costs.

In economic welfare analysis, it is a well-established and old idea that monopoly power can negatively affect social welfare. One of the main sources of monopoly power is collusion between firms which has long been the focus of much theoretical and empirical work. While tacit horizontal agreements have traditionally been modeled by means of repeated games (Friedman 1971 and Abreu 1986, 1988), formal collusion[1] has more recently been analyzed in the framework of cooperative oligopoly games with transferable utility, henceforth oligopoly TU-games. Besides the set of players, a TU-game consists of a characteristic function assigning to each subset of players, called coalition, a real number which represents the worth that these players can obtain by agreeing to cooperate. In oligopolies, since the decision of a cartel as well as its joint profit depend on the behaviors of outsiders, the determination of the worth that a coalition can obtain requires to specify how such outsiders act. An appropriate approach, called the γ-approach, is proposed by Hart and Kurz (1983) and, more specifically, by Chander and Tulkens (1997). It consists in considering a competition setting

[1] An example of a cartel with formal collusion is California's Raisin Administrative Committee created in 1949 as a result of the Agricultural Marketing Agreement Act of 1937.

in which cartel members face outsiders acting individually. The worth of any coalition is then determined by the joint profit it obtains at any Nash equilibrium in the underlying normal form oligopoly game.[2]

An appropriate set-valued solution for oligopoly TU-games that deals with the stability of monopoly power is the core. Given a payoff vector in the core, the grand coalition, i.e., the cartel comprising all firms, could form and distribute its worth as payoffs to its members in such a way that no coalition can contest this sharing by breaking off from the grand coalition. In oligopoly TU-games, the stability of monopoly power sustained by the grand coalition is then related to the non-emptiness of the core. Balancedness is a necessary and sufficient condition for the core to be non-empty (Bondareva 1963, Shapley 1967). Until now, the cores of Bertrand and Cournot oligopoly TU-games have been independently studied by Zhao (1999), Norde et al. (2002), Driessen and Meinhardt (2005), Lardon (2010, 2012) and Lekeas and Stamatopoulos (2014) among others. In the second part of this article, we aim to build bridges between the cores of Bertrand and Cournot oligopoly TU-games. More precisely, based on the previous analysis on Nash equilibrium profits of cartel members, we establish that the core of a Cournot oligopoly TU-game is strictly included in the core of a Bertrand oligopoly TU-game when the number of firms is lower or equal to 25. Otherwise the cores cannot be compared. Furthermore, we prove that the core of Cournot oligopoly TU-games is non-empty which has not been established before under product differentiation. Afterwards, we focus on the aggregate-monotonic core, a subset of the core, introduced by Calleja et al. (2009). Whenever the core is non-empty, the aggregate-monotonic core selects point solutions in the core that satisfy aggregate monotonicity property, proposed by Meggido (1974). Roughly speaking, this natural property requires that the payoff of each player does not decrease if the worth of the grand coalition grows. We prove that the aggregate-monotonic core of a Cournot oligopoly TU-game is strictly included in the aggregate-monotonic core of a Bertrand oligopoly TU-game regardless of the number of firms. Ultimately, most of our results advocate that it is easier for firms to collude in Bertrand than in Cournot competition.

The remainder of the article is organized as follows. In Section 2, we introduce the non-cooperative and cooperative models of differentiated Bertrand and Cournot oligopolies. Section 3 compares Nash equilibrium prices, quantities and profits in normal form Bertrand and Cournot oligopoly games in the presence of a cartel of firms. Section 4 is devoted to the comparison of the cores and the aggregate-monotonic cores between Bertrand and Cournot oligopoly TU-games. Section 5 gives some concluding remarks on the difficulty to extend the analysis from symmetric to asymmetric product differentiation or costs. Lastly, Section 6 is the appendix where proofs of some results are presented.

[2]Initially, the first two approaches which permit to convert a normal form game into a TU-game, called the α and β-approaches, are suggested by Aumann (1959). They consist in computing the max-min and the min-max payoffs of each coalition respectively. However, these two approaches are not the most appropriate with regard to the rational behaviors of firms in oligopolies as discussed by Lardon (2012).

2 Bertrand and Cournot models

In this section, we first define normal form Bertrand and Cournot oligopoly games by taking into account the possibility for some firms to cooperate. Then, we introduce the general approach of TU-games as well as the solution concepts of the core and the aggregate-monotonic core. Finally, we convert normal form oligopoly games into oligopoly TU-games for both competition types.

2.1 Normal form Bertrand and Cournot oligopoly games with a single partnership

We consider a set of firms $N = \{1, 2, \ldots, n\}$ where $n \geq 3$ in a differentiated oligopoly, each producing a different variety of goods. Each producer $i \in N$ operates at a constant marginal and average cost of $c \in \mathbb{R}_+$.

In Bertrand competition, the environment of each producer $i \in N$ is described by his brand demand function, $D_i : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$, derived from Shubik's demand system (Shubik 1980), and given by:

$$D_i(p_1, \ldots, p_n) = V - p_i - r\left(p_i - \frac{1}{n}\sum_{j=1}^{n} p_j\right) \tag{1}$$

where $p_i \geq 0$ is the price charged by firm i, $V > c$ is the intercept of demand[3] and $r > 0$ is the substitutability parameter. The quantity demanded of firm i's good depends on its own price p_i and on the difference between p_i and the average price in the industry $\sum_{j=1}^{n} p_j/n$. When r is close to zero, products become unrelated, and when r approaches infinity, they become homogeneous. Profits for the ith producer in terms of prices, $\pi_i^B : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$, are expressed as:

$$\pi_i^B(p_1, \ldots, p_n) = (p_i - c)D_i(p_1, \ldots, p_n) \tag{2}$$

In Cournot competition, each producer $i \in N$ is associated with an inverse brand demand function, $P_i : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$, obtained by inverting Shubik's demand system (1), and given by:

$$P_i(q_1, \ldots, q_n) = V - q_i + \frac{r}{(1+r)}\left(q_i - \frac{1}{n}\sum_{j=1}^{n} q_j\right) \tag{3}$$

where $q_i \geq 0$ is the quantity produced by firm i. The market price of firm i's good depends on its own quantity q_i and on the difference between q_i and the average quantity in the industry $\sum_{j=1}^{n} q_j/n$. Note that inverting Shubik's demand system does not change the intercept V which will make the comparative analysis in the next section easier. Profits for the ith producer in terms of quantities, $\pi_i^C : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$, are expressed as:

$$\pi_i^C(q_1, \ldots, q_n) = (P_i(q_1, \ldots, q_n) - c)q_i \tag{4}$$

[3]The condition $V > c$ ensures that equilibrium quantities will be positive.

Let 2^N denotes the power set of N. We consider the situation in which a subset of firms $S \in 2^N \backslash \{\emptyset\}$ form a cartel while outsiders continue to act independently. The size of cartel S is denoted by $s = |S|$. From now on, in order to facilitate reading, we will use index i to denote any cartel member and index j to refer to any outsider. The profit of any cartel $S \in 2^N \backslash \{\emptyset\}$ is defined as the sum of the profits of its members for both competition types, i.e.:

$$\sum_{i \in S} \pi_i^B(p_1, \ldots, p_n) \text{ and } \sum_{i \in S} \pi_i^C(q_1, \ldots, q_n) \tag{5}$$

While cartel members behave as a multiproduct firm by the signature of a binding agreement which enables them to correlate their strategies (prices or quantities), outsiders are assumed to act independently and aim to maximize their individual profit.

2.2 TU-games and core solution concepts

Generally speaking, a cooperative game with transferable utility or, for short, a TU-game consists of a set of players N and a characteristic function $v : 2^N \longrightarrow \mathbb{R}$ with the convention that $v(\emptyset) = 0$. Subsets of N are called coalitions, and the number $v(S)$ is the worth of coalition S that these members can obtain by agreeing to cooperate. We denote by G the set of TU-games.

A natural property of TU-games that will interest us is symmetry. A TU-game $(N, v) \in G$ is symmetric if there exists a function $f : \mathbb{N} \longrightarrow \mathbb{R}$ such that for every coalition $S \in 2^N \backslash \{\emptyset\}$, $v(S) = f(s)$. In words, the worth of any coalition S only depends on its size and not on the identity of its members.

In a TU-game $(N, v) \in G$, every player $i \in N$ may receive a payoff $x_i \in \mathbb{R}$. A vector $x = (x_1, \ldots, x_n)$ is a payoff vector. For any coalition $S \in 2^N \backslash \{\emptyset\}$ and any payoff vector $x \in \mathbb{R}^n$, we define $x(S) = \sum_{i \in S} x_i$. Given a TU-game $(N, v) \in G$, a payoff vector $x \in \mathbb{R}^n$ is efficient if $x(N) = v(N)$, i.e., the worth of the grand coalition is fully distributed among players. The set of efficient payoff vectors is denoted by $X(N, v)$. A single-valued solution is a function σ which assigns to every TU-game $(N, v) \in G$ a payoff vector $\sigma(N, v) \in X(N, v)$. A payoff vector $x \in \mathbb{R}^n$ is acceptable if for every coalition $S \in 2^N \backslash \{\emptyset\}$, $x(S) \geq v(S)$, i.e., the payoff vector provides a total payoff to the members of coalition S that is at least as great as its worth. The core (Gillies 1953) of a TU-game $(N, v) \in G$, denoted by $C(N, v)$, is the set of efficient payoff vectors that are acceptable, i.e.:

$$C(N, v) = \{x \in \mathbb{R}^n : \forall S \in 2^N \backslash \{\emptyset\},\ x(S) \geq v(S) \text{ and } x(N) = v(N)\} \tag{6}$$

Given a payoff vector in the core, the grand coalition could form and distribute its worth as payoffs to its members in such a way that any coalition cannot contest this sharing by breaking off from the grand coalition.

According to the Bondareva-Shapley theorem (Bondareva 1963, Shapley 1967), balancedness property is a necessary and sufficient condition to guarantee the non-emptiness of the core. Let $\mathcal{B} \subseteq 2^N \backslash \{\emptyset\}$ be a family of coalitions and denote by $\mathcal{B}_i = \{S \in \mathcal{B} : i \in S\}$ the subset of those coalitions of which player i is a member. Then $\mathcal{B}$ is said to be a balanced family of coalitions if for every $S \in \mathcal{B}$ there exists a balancing weight $\lambda_S \in \mathbb{R}_+$ such that $\sum_{S \in \mathcal{B}_i} \lambda_S = 1$ for all $i \in N$. We denote by $\Lambda(N)$ the set of balanced collections and $\Lambda^*(N)$ the subset of those collections not containing the grand coalition. A TU-game $(N, v) \in G$ is balanced if for every balanced collection $\mathcal{B} \in \Lambda(N)$ it holds that:

$$\sum_{S \in \mathcal{B}} \lambda_S v(S) \leq v(N).$$

The Bondareva-Shapley theorem establishes that a TU-game $(N, v) \in G$ has a non-empty core if and only if it is balanced. Furthermore, a single-valued solution σ is said to satisfy the core selection property if whenever the TU-game is balanced, then $\sigma(N, v) \in C(N, v)$.

Another natural property is aggregate monotonicity, introduced by Meggido (1974). A single-valued solution σ is said to satisfy the aggregate monotonicity property if for any two TU-games $(N, v), (N, v') \in G$, with $v(S) = v'(S)$ for any $S \subset N$ and $v(N) < v'(N)$, it holds that $\sigma(N, v) \leq \sigma(N, v')$, where $\leq$ is the weak inequality for $\mathbb{R}^n$, i.e., $x \leq y$ if $x_i \leq y_i$ for all $i \in N$. Roughly speaking, this property requires that the payoff of each player does not decrease if the worth of the grand coalition grows.
We now introduce the notion of root game that will be necessary for the definition of the aggregate-monotonic core. Given a TU-game $(N, v) \in G$, the associated root game, denoted by (N, v_R), is defined as $v_R(N) = \min_{x \in \mathbb{R}^n} \{x(N) : \forall S \subset N,\, x(S) \geq v(S)\}$ and $v_R(S) = v(S)$ for any $S \subset N$. The root game coincides with the original one except for the grand coalition. Instead, we take the minimum level of efficiency in order to get balancedness. Hence an alternative formula for the worth of the grand coalition in the root game is the following:

$$v_R(N) = \max_{\mathcal{B} \in \Lambda^*(N)} \sum_{S \in \mathcal{B}} \lambda_S v(S) \tag{7}$$

Note that $v(N) \geq v_R(N)$ if and only if $C(N, v) \neq \emptyset$. The aggregate-monotonic core (Calleja et al. 2009) of a TU-game $(N, v) \in G$, denoted by $AC(N, v)$, is defined as:

$$AC(N, v) = C(N, v_R) + (v(N) - v_R(N)) \cdot \Delta_n \tag{8}$$

where Δ_n denotes the unit-simplex, i.e., $\Delta_n = \{x \in \mathbb{R}^n_+ : x(N) = 1\}$.
The aggregate-monotonic core is well-defined since (N, v_R) is balanced. We observe that it results from two sequential steps. First, it selects an element in the core of the root game (N, v_R). Second, it consists in adding a non-negative or a non-positive vector to go back to the initial level of efficiency of (N, v). Whenever the core is non-empty, it holds that $AC(N, v) \subseteq C(N, v)$. Calleja et al. (2009) have proved that the aggregate-monotonic core is the subset of $X(N, v)$ which any single-valued solution σ should pick up to satisfy both core selection and aggregate monotonicity properties.

2.3 Bertrand and Cournot oligopoly TU-games

Based on the two previous subsections, we now define Bertrand and Cournot oligopoly TU-games following the γ-approach (Hart and Kurz 1983, Chander and Tulkens 1997) which is appropriate in oligopolies. The worth of any coalition is then determined by the total profit of its members at any Bertrand (Cournot, respectively)-Nash equilibrium.[4] Given a set of firms N, the Bertrand and Cournot oligopoly TU-games, denoted by (N, v^B) and (N, v^C) respectively, are defined for any coalition $S \in 2^N \backslash \{\emptyset\}$ as:

$$v^B(S) = \sum_{i \in S} \pi_i^B(p_s^*, \tilde{p}_s),$$

and

$$v^C(S) = \sum_{i \in S} \pi_i^C(q_s^*, \tilde{q}_s),$$

where $(p_s^*, \tilde{p}_s)$ and $(q_s^*, \tilde{q}_s)$ are the unique Bertrand and Cournot-Nash equilibria[5] respectively with the understanding that each cartel member $i \in S$ charges a price p_s^* and produces a quantity q_s^*, and each outsider $j \in N \backslash S$ charges a price $\tilde{p}_s$ and produces a quantity $\tilde{q}_s$. Since products are symmetrically differentiated and firms operate at a constant and identical marginal cost both Nash equilibria only depend on the size s of coalition S. As a consequence, identical parties (cartel members or outsiders) earn identical profits for both competition types. Hence the worth of any coalition S can be expressed as either $v^B(S) = s\pi_i^B(p_s^*, \tilde{p}_s)$ or $v^C(S) = s\pi_i^C(q_s^*, \tilde{q}_s)$ where $i \in S$ is a representative cartel member. It follows from these remarks that Bertrand and Cournot oligopoly TU-games $(N, v^B) \in G$ and $(N, v^C) \in G$ are symmetric.
When the grand coalition forms, cartel members behave as a multiproduct monopoly maximizing its total profit. In oligopolies, we ascertain that any efficient payoff vector in the core permits to stabilize the monopoly power into the grand coalition. Furthermore, the aggregate-monotonic core selects a subset of those payoff vectors for which if the profit of multiproduct monopoly grows, no cartel member can suffer from it.

3 Comparative analysis of Nash equilibrium prices, quantities and profits

In this section, we first derive from the maximization of profits given by (2), (4) and (5) the reaction functions of any cartel member and any outsider for both competition types. Then, we proceed to a comparative analysis of Nash equilibrium prices, quantities and profits involving a cartel of firms. Since products are symmetrically differentiated and firms operate at a constant and identical marginal cost, any Bertrand (Cournot, respectively)-Nash equilibrium implies that identical parties (cartel members or outsiders) must choose identical prices (quantities, respectively) denoted by p_i (q_i, respectively) for each cartel member, and p_j (q_j, respectively) for each outsider. Given a coalition $S \in 2^N \backslash \{\emptyset, N\}$, this will permit us to represent the reaction functions into simple two-dimensional diagrams in

[4]A Nash equilibrium involving a subset of players correlating their strategies is also called a partial agreement equilibrium.

[5]This uniqueness result is proved in Section 3.

price space (quantity space, respectively) where the vertical dimension indicates the price p_i charged (quantity q_i produced, respectively) by any cartel member $i \in S$, and where the horizontal dimension indicates the price p_j charged (quantity q_j produced, respectively) by any outsider $j \in N \backslash S$. Furthermore, we assume that $c = 0$. This is without loss of generality as we can perform the transformations $\bar{V} = V - c$, $\bar{p}_i = p_i - c$ and $\bar{p}_j = p_j - c$ in price space.

3.1 Reaction functions

In Bertrand competition, denote by $R_I^B(p_j)$ the price charged by each cartel member for any given price p_j charged by each outsider. This reaction function derived from the maximization of the joint profit $\sum_{i \in S} \pi_i^B(p_1, \ldots, p_n)$ given by (5) is upward sloping:

$$R_I^B(p_j) = \frac{nV + r(n-s)p_j}{2(n + r(n-s))} \tag{9}$$

Denote by $R_O^B(p_i)$ the price charged by each outsider for any given price p_i charged by each cartel member. This reaction function derived from the maximization of the profit $\pi_j^B(p_1, \ldots, p_n)$, $j \in N \backslash S$, given by (2), is upward sloping:

$$R_O^B(p_i) = \frac{nV + rsp_i}{2n + r(n+s-1)} \tag{10}$$

Both curves have slopes less than one and intersect at the unique Bertrand-Nash equilibrium $(p_s^*, \tilde{p}_s)$. In Figure 1, this Bertrand-Nash equilibrium occurs at B where reaction functions $R_I^B(p_j)$ and $R_O^B(p_i)$ intersect.

In Cournot competition, denote by $R_I^C(q_j)$ the production of each cartel member for any given quantity q_j produced by each outsider. This reaction function derived from the maximization of the joint profit $\sum_{i \in S} \pi_i^C(q_1, \ldots, q_n)$ given by (5) is downward sloping:

$$R_I^C(q_j) = \frac{n(1+r)V - r(n-s)q_j}{2(n+rs)} \tag{11}$$

Denote by $R_O^C(q_i)$ the quantity reaction of each outsider for any given quantity q_i produced by each cartel member. This reaction function derived from the maximization of the profit $\pi_j^C(q_1, \ldots, q_n)$, $j \in N \backslash S$, given by (4), is downward sloping:

$$R_O^C(q_i) = \frac{n(1+r)V - rsq_i}{2n + r(n-s+1)} \tag{12}$$

Both curves intersect at the unique Cournot-Nash equilibrium $(q_s^*, \tilde{q}_s)$. In Figures 2, 3 and 4, this Cournot-Nash equilibrium occurs at C where reaction functions $R_I^C(q_j)$ and $R_O^C(q_i)$ intersect.

3.2 Comparative analysis of Nash equilibrium prices

In order to make price comparison, we study reaction functions of both competition types in price space. In Cournot competition, Shubik's demand system given by (1) permits to express the reaction function of any cartel member given by (11) in price space as:

$$\overline{R}_I^C(p_j) = \frac{(n+rs)(nV + r(n-s)p_j)}{2n^2(1+r) + nr^2 s - r^2 s^2} \tag{13}$$

and the reaction function of any outsider given by (12) as:

$$\overline{R}_O^C(p_i) = \frac{(n+r)(nV + rsp_i)}{n^2(2+r) + nr(s+1) + r^2 s} \tag{14}$$

In price space, both curves intersect at the unique Cournot-Nash equilibrium in terms of prices $(P_i(q_s^*, \tilde{q}_s), P_j(q_s^*, \tilde{q}_s))$ with the understanding that each cartel member and each outsider sells its products at price $P_i(q_s^*, \tilde{q}_s)$ and $P_j(q_s^*, \tilde{q}_s)$ respectively. In Figure 1, this Cournot-Nash equilibrium occurs at C where reaction functions $\overline{R}_I^C(p_j)$ and $\overline{R}_O^C(p_i)$ intersect.

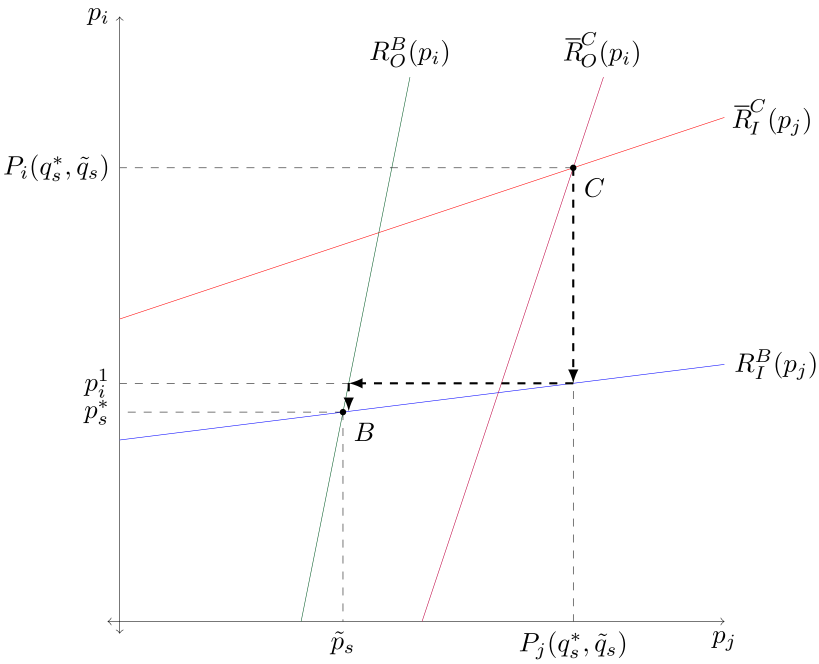

Figure 1: Dynamic adjustment process in price space from Cournot to Bertrand competition

In price space, both the y-intercepts and the slopes of the reaction functions in Cournot competition are higher than those of reaction functions in Bertrand competition (the proofs are given in Subsection 6.1 in the appendix). One conclusion immediately follows from these geometrical properties.

Proposition 3.1 *All prices are larger in Cournot than in Bertrand competition.*

Hence the standard Bertrand-Cournot rankings on prices still hold. The dynamic adjustment process from Cournot to Bertrand competition can be described as follows (Figure

1). In Bertrand competition, at the unique Cournot-Nash equilibrium in terms of prices $(P_i(q_s^*, \tilde{q}_s), P_j(q_s^*, \tilde{q}_s))$, any cartel has negative marginal revenue and therefore must reduce price from $P_i(q_s^*, \tilde{q}_s)$ to p_i^1 which is its best response to $P_j(q_s^*, \tilde{q}_s)$ charged by each outsider. Then, because reaction functions are upward sloping (due to strategic complementarity of the price strategies) each outsider will react to this new price configuration by reducing its own price. In response, the cartel further reduces price and so until reaching the unique Bertrand-Nash equilibrium $(p_s^*, \tilde{p}_s)$.

3.3 Comparative analysis of Nash equilibrium quantities

In order to make quantity comparison, we study reaction functions of both competition types in quantity space. In Bertrand competition, Shubik's inverse demand system given by (3) permits to express the reaction function of any cartel member given by (9) in quantity space as:

$$\overline{R}_I^B(q_j) = \frac{(n + r(n-s))((1+r)nV - r(n-s)q_j)}{2n^2(1+r) + nr^2s - r^2s^2} \tag{15}$$

and the reaction function of any outsider given by (10) as:

$$\overline{R}_O^B(q_i) = \frac{(n(1+r) - r)((1+r)nV - rsq_i)}{n^2(2+3r+r^2) - nr(1+r)(s+1) + r^2s} \tag{16}$$

In quantity space, both curves intersect at the unique Bertrand-Nash equilibrium in terms of quantities $(D_i(p_s^*, \tilde{p}_s), D_j(p_s^*, \tilde{p}_s))$ with the understanding that the quantity demanded of each cartel member and each outsider are $D_i(p_s^*, \tilde{p}_s)$ and $D_j(p_s^*, \tilde{p}_s)$ respectively. In Figures 2, 3 and 4 this Bertrand-Nash equilibrium occurs at B where reaction functions $\overline{R}_I^B(q_j)$ and $\overline{R}_O^B(q_i)$ intersect.

In quantity space, both the y-intercepts and the absolute value of the slopes of the reaction functions in Bertrand competition are higher than those of reaction functions in Cournot competition (the proofs are given in Subsection 6.2 in the appendix). Unlike Nash equilibrium prices, these geometrical properties don't permit to compare Nash equilibrium quantities. However, we analytically establish that the quantity change of each cartel member is negative from Bertrand to Cournot competition.

Proposition 3.2 *Quantity produced by each cartel member is larger in Bertrand than in Cournot competition.*

The proof is given in Subsection 6.4 in the appendix. We observe that the same conclusion does not hold for the quantity change of each outsider. For example, on the basis of a demand intercept $V = 1000$, a number of firms $n = 25$ and a substitutability parameter $r = 2$ we compare the quantity change of each outsider from Bertrand to Cournot competition by distinguishing three illustrative cases:[6]

- when $s = 2$, the quantity change $\tilde{q}_s - D_j(p_s^*, \tilde{p}_s) \simeq 736 - 745 = -9$ is negative. Since the

[6]The expressions of Nash equilibrium quantities produced by each cartel member and each outsider in Bertrand and Cournot competition are given by (19), (20), (22) and (23) in the appendix.

two cartel members have incentive to act as price-taking profit maximizer in Cournot competition, their low quantity change $q_s^* - D_i(p_s^*, \tilde{p}_s) \simeq 709 - 735 = -26$ does not significantly impact on price for all outsiders. With this low increase in price, each outsider has negative marginal revenue and must decrease its quantity too.
- when $s = 10$, the quantity change $\tilde{q}_s - D_j(p_s^*, \tilde{p}_s) \simeq 774 - 767 = 70$ is positive. Since the ten cartel members have incentive to act as price-making profit maximizer in Cournot competition, there is a significant quantity change $q_s^* - D_i(p_s^*, \tilde{p}_s) \simeq 575 - 657 = -82$. Each outsider has then a positive marginal revenue and will take advantage of this price raising by increasing its own quantity.
- when $s = 24$, the quantity change $\tilde{q}_s - D_j(p_s^*, \tilde{p}_s) \simeq 944 - 956 = -12$ is negative. Since the cartel has quasi-monopoly power, the quantity produced by each cartel member in Bertrand competition is close to its optimal production in Cournot competition. Hence, the quantity change $q_s^* - D_i(p_s^*, \tilde{p}_s) \simeq 501 - 513 = -12$ is low. As in the first case, the unique outsider will respond by decreasing its quantity.
Thus, the quantity change of each outsider from Bertrand to Cournot competition is not monotonic with respect to the size of the cartel which leads to distinguish two complementary cases in quantity space depending on whether each outsider decides to reduce (Figure 2) or raise (Figure 3) its own quantity.

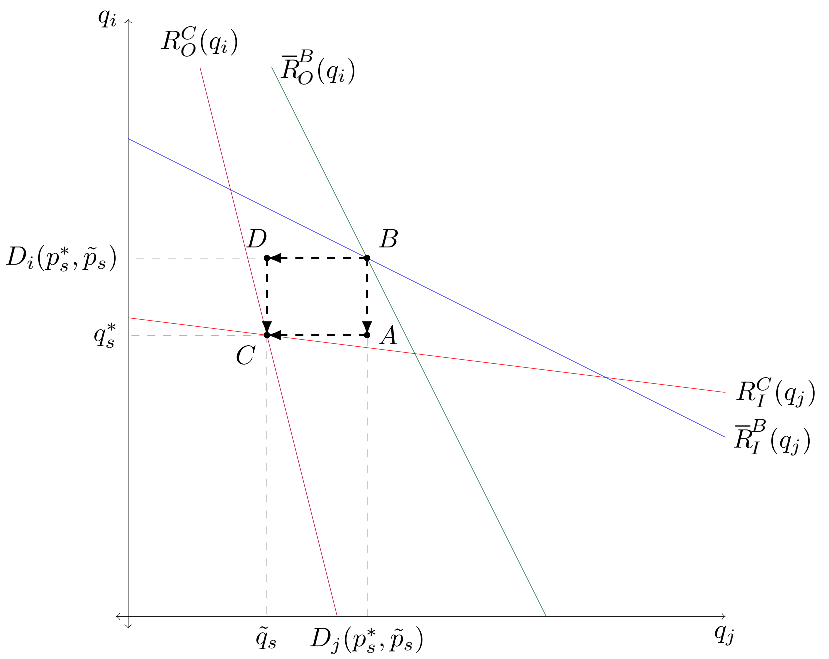

Figure 2: Nash equilibrium quantities when each outsider reduces its quantity from Bertrand to Cournot competition

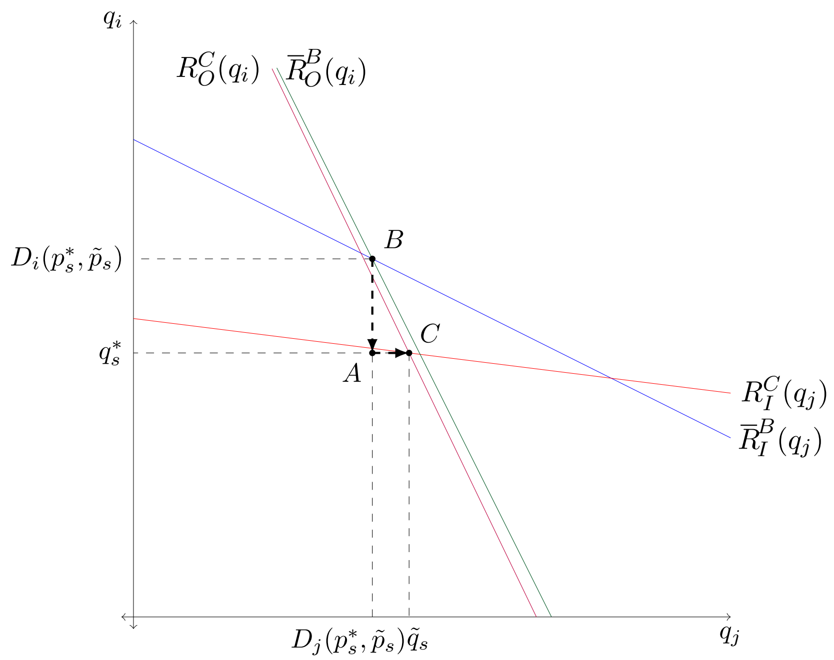

Figure 3: Nash equilibrium quantities when each outsider raises its quantity from Bertrand to Cournot competition

In light of the above example, we identify typical oligopoly cases in which each outsider raises its quantity from Bertrand to Cournot competition.

Proposition 3.3 *Quantity produced by each outsider is smaller in Bertrand than in Cournot competition providing that we consider a cartel of size* $s = n/k$ *for some* $k > 1$[7] *and the number of firms* n *is sufficiently large.*

Proof: Consider the asymptotic case of an industry containing an infinite number of firms $n \to \infty$ with a cartel of significant size $s \to \infty/k = \infty$ as illustrated in Figure 4. As a consequence, the asymptotic reaction functions of any cartel member $\overline{R}_I^B(q_j)$ and $R_I^C(q_j)$ differ for any $k > 1$.[8] By contrast, since each outsider acts as a price-taking profit maximizer its asymptotic reaction functions $\overline{R}_O^B(q_i)$ and $R_O^C(q_i)$ are equal (see footnote 8). In quantity space, decompose the quantity change from B to C into two stages. First, it follows from Proposition 3.2 that each cartel member reduces its quantity from $D_i(p_s^*, \tilde{p}_s)$ to q_s^* to achieve A which raises price for all outsiders. Second, in response each atomic outsider must increase its own quantity from $D_j(p_s^*, \tilde{p}_s)$ to $\tilde{q}_s$. ■

[7] Rational number k can be interpreted as the weight of the cartel in the industry. For example, $k = 2$ means that the cartel represents half of the total number of firms in the industry. This permits to study asymptotically when $n \to \infty$ the behaviors of cartel members and outsiders.

[8] The expressions of asymptotic reaction functions are given in Subsection 6.3 in the appendix. Furthermore, it is proved that both the y-intercepts and the absolute value of the slopes of asymptotic reaction function $\overline{R}_I^B(q_j)$ are higher than those of reaction function $R_I^C(q_j)$.

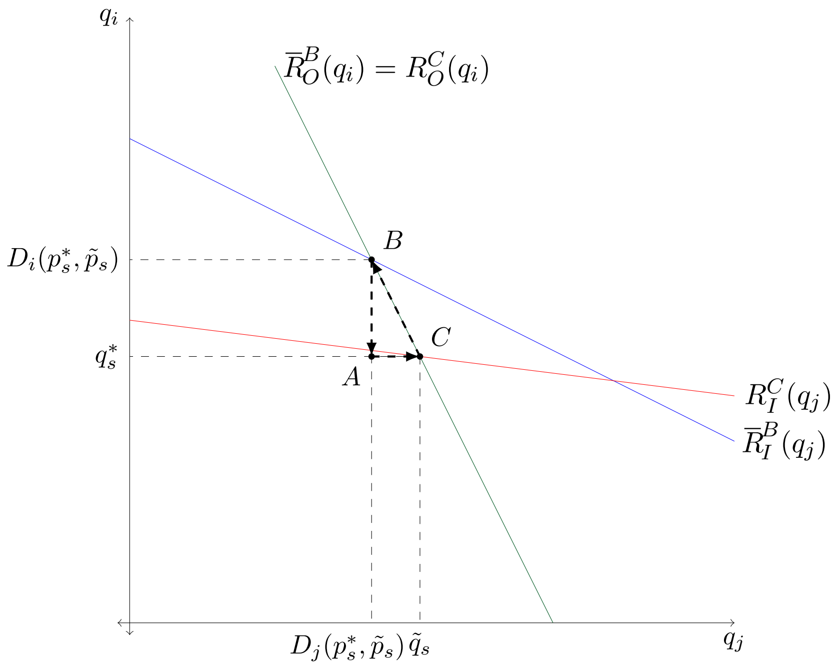

Figure 4: Asymptotic Nash equilibrium quantities

3.4 Comparative analysis of Nash equilibrium profits

Although the comparison of Nash equilibrium profits according to the type of product differentiation has already been drawn (Häckner 2000) the influence of the market structure on the profit change of firms from Bertrand to Cournot competition remains to be determined.

Proposition 3.4 *Each outsider earns lower profits in Bertrand than in Cournot competition. Each cartel member earns lower profits in Bertrand than in Cournot competition providing that each outsider reduces its quantity from Bertrand to Cournot competition.*

Proof: In quantity space, decompose the quantity change from B to C into two stages as illustrated in Figures 2 and 3. First, each cartel member reduces its quantity from $D_i(p_s^*, p_j^*)$ to q_s^* to achieve A. By gross substitutability, this raises price for all outsiders, and hence benefits them. Second, facing q_s^*, each outsider reduces its quantity (Figure 2) or raises it (Figure 3) to $\tilde{q}_s$ to achieve C which is profit-maximizing.
When each outsider reduces its quantity from Bertrand to Cournot competition, a similar argument permits to conclude that each cartel member earns larger profits by decomposing the quantity change from B to C via D (Figure 2). ■

The number of firms turns out to be a key parameter in order to ensure larger profits to cartel members in Cournot than in Bertrand competition for at least two reasons. First, when n is small each outsider has an incentive to act as a price-making profit maximizer by reducing its quantity (Figure 2). Second, even if a small number of outsiders raise their quantity (Figure 3), this does not cause a substantial damage on the profit of each cartel member. For example, when $V = 1000$, $n = 15$, $s = 12$ and $r = 2$, although the quantity change of each outsider $\tilde{q}_s - D_j(p_s^*, \tilde{p}_s) \simeq 862 - 860 = 2$ is positive, the profit change of each cartel member $\pi_i^C(q_s^*, \tilde{q}_s) - \pi_i^B(p_s^*, \tilde{p}_s) \simeq 225985 - 224000 = 1985$ also remains positive.[9]

Proposition 3.5 *If $n \leq 25$, then each cartel member earns higher profits in Cournot than in Bertrand competition.*

This result is proved by resorting to iterative computations detailed in Subsection 6.4 in the appendix. It suggests to distinguish two oligopoly types: oligopoly of small or medium size ($n \leq 25$) for which the standard Bertrand-Cournot rankings on profits still hold and oligopoly of large size ($n > 25$) for which one asymptotic conclusion can be established.

Proposition 3.6 *If we consider a cartel of size $s = n/k$ for some $k > 1$ and the number of firms n is sufficiently large, then each cartel member earns higher profits in Bertrand than in Cournot competition.*

This result is analytically proved in Subsection 6.4 in the appendix. To get the intuition behind this result, note that for any $k > 1$ and $n \to \infty$, outsiders become enough to cause a substantial damage on the profit of each cartel member by increasing their quantity from Bertrand to Cournot competition (Proposition 3.3). For example, when $V = 1000$, $n = 50$, $k = 2$ (hence there are 25 outsiders) and $r = 2$, the profit change $\pi_i^C(q_s^*, \tilde{q}_s) - \pi_i^B(p_s^*, \tilde{p}_s) \simeq 200485 - 200152 = 333$ is positive (see footnote 9). However, with the same parameters V, k and r if the number of firms increases from 50 to 100 (hence there are 50 outsiders) the profit change $\pi_i^C(q_s^*, \tilde{q}_s) - \pi_i^B(p_s^*, \tilde{p}_s) \simeq 199423 - 199797 = -374$ becomes negative as predicted by Proposition 3.6.

4 Comparison of the cores and the aggregate-monotonic cores

In this section, based on the previous analysis on Nash equilibrium profits of cartel members, we first establish the result on the cores. Then, we prove that the core of Cournot oligopoly TU-games is non-empty. Finally, we proceed to the comparison of the aggregate-monotonic cores.

First, the maximizations of the joint profit of the grand coalition $\sum_{i \in N} \pi_i^B(p_1, \ldots, p_n)$ and $\sum_{i \in N} \pi_i^C(q_1, \ldots, q_n)$ given by (5) lead to the same worth since both problems are perfectly dual, i.e., $v^B(N) = v^C(N)$. Moreover, it follows from Proposition 3.5 that $v^B(S) = s\pi_i^B(p_s^*, \tilde{p}_s) < s\pi_i^C(q_s^*, \tilde{q}_s) = v^C(S)$ when $n \leq 25$. Hence we deduce from (6) the following result.

Corollary 4.1 *For any $n \leq 25$, the core of (N, v^C) is strictly included in the core of (N, v^B).*

[9] The expressions of Nash equilibrium profits of each cartel member in Bertrand and Cournot competition are given by (21) and (24) in the appendix.

This result highlights that it is easier for firms to collude in Bertrand than in Cournot competition when $n \leq 25$ which corresponds, in practice, to the majority of differentiated oligopolies with symmetric costs. Otherwise Proposition 3.6 suggests that the cores of Bertrand and Cournot oligopoly TU-games cannot be compared. For example, on the basis of a demand intercept $V = 10$, a number of firms $n = 50$ and a substitutability parameter $r = 28$, we want to establish that $C(N, v^C) \not\subseteq C(N, v^B)$. The worth of the grand coalition is given by $v^B(N) = v^C(N) = 1250$ (see footnote 9). Now, consider coalition $\overline{S} = \{1, 2, \dots, 41\}$ and payoff vector $x = ((\frac{305}{41})_{i=1}^{41}, (105)_{i=42}^{50}) \in \mathbb{R}^n$. Hence any player in $\overline{S}$ obtains the lowest payoff according to x. It follows from $x(N) = 1250$ that $x \in X(N, v^B) = X(N, v^C)$. The worth of coalition $\overline{S}$ is given by either $v^B(\overline{S}) = \frac{498688919375}{1633129744} \simeq 305,3578$ or $v^C(\overline{S}) = \frac{7122110000}{23551609} \simeq 302,4044$. Note that $v^B(\overline{S}) > v^C(\overline{S})$ as predicted by Proposition 3.6. Furthermore, we deduce from $x(\overline{S}) = 305 < v^B(\overline{S})$ that $x \notin C(N, v^B)$. It remains to show that $x \in C(N, v^C)$. To this end, we distinguish two complementary cases. First, consider any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $S \subseteq \overline{S}$. Then, it holds that:

$$\begin{aligned} x(S) - v^C(S) &= \frac{305s}{41} - \frac{237568s(28s+50)10^4}{(2800(s+51) + 784(52-s)s + 10^4)^2} \\ &= \frac{5s(146461s^4 - 16278094s^3 + 395208569s^2 + 1041879300s + 1758552500)}{41(49s^2 - 2723s - 9550)^2}, \end{aligned}$$

which is non-negative for any $s \leq 41$. Since any player in $\overline{S}$ obtains the lowest payoff according to x, it follows from the symmetry of $(N, v^C) \in G$ that $x(S) - v^C(S) \geq 0$ for any $S \in 2^N \backslash \{\emptyset\}$ such that $s \leq 41$. Second, consider any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $S \supseteq \overline{S}$. Then, it holds that:

$$\begin{aligned} x(S) - v^C(S) &= 305 + 105(s-41) - \frac{237568s(28s+50)10^4}{(2800(s+51) + 784(52-s)s + 10^4)^2} \\ &= \frac{5(50-s)(-50421s^4 + 5003684s^3 - 99354409s^2 - 824884550s - 1459240000)}{(49s^2 - 2723s - 9550)^2}, \end{aligned}$$

which is non-negative for any $s \geq 42$. By the same argument as above, the symmetry of $(N, v^C) \in G$ implies that $x(S) - v^C(S) \geq 0$ for any $S \in 2^N \backslash \{\emptyset\}$ such that $s \geq 42$. Thus, we conclude that $x \in C(N, v^C)$.

Then, we aim to establish that the core of Cournot oligopoly TU-games is non-empty. We first need the following lemma.

Lemma 4.2 *In Cournot competition, when $s < n$, the profit of each cartel member attains its maximum at $s = 1$ or at $s = n - 1$. Furthermore, when $n \geq 5$, this profit is maximum at $s = n - 1$, i.e., for any $s \in \{1, \dots, n-2\}$, $\pi_i^C(q_{n-1}^*, \tilde{q}_{n-1}) > \pi_i^C(q_s^*, \tilde{q}_s)$.*

This result is analytically proved in Subsection 6.4 in the appendix.

Proposition 4.3 *The core of any Cournot oligopoly TU-game $(N, v^C) \in G$ is non-empty.*

Proof: It follows from (7) that a Cournot oligopoly TU-game $(N, v^C) \in G$ is balanced, and so has a non-empty core, if and only if $v^C(N) \geq v^C_R(N)$ where (N, v^C_R) is the root game of (N, v^C). Furthermore, we deduce from Lemma 4.2 that the worth of the grand coalition $v^C_R(N)$ is obtained either at the balanced collection $\{\{k\} : k \in N\}$ or at the balanced collection $\{N\backslash\{k\} : k \in N\}$. Hence, it holds that:

$$v^C_R(N) = n \max\{\pi^C_i(q^*_1, \tilde{q}_1), \pi^C_i(q^*_{n-1}, \tilde{q}_{n-1})\} \tag{17}$$

It remains to show that $\pi^C_i(q^*_n, \tilde{q}_n) \geq \max\{\pi^C_i(q^*_1, \tilde{q}_1), \pi^C_i(q^*_{n-1}, \tilde{q}_{n-1})\}$. Using the expression of Nash equilibrium profit of any cartel member in Cournot competition (see footnote 9), one gets:

$$\begin{aligned}\pi^C_i(q^*_n, \tilde{q}_n) - \pi^C_i(q^*_1, \tilde{q}_1) &= \frac{(Vr(n-1))^2}{4(r(1+n)+2n)^2} \\ &> 0,\end{aligned}$$

and

$$\begin{aligned}\pi^C_i(q^*_n, \tilde{q}_n) - \pi^C_i(q^*_{n-1}, \tilde{q}_{n-1}) &= \frac{V^2r^2((5n^2-14n+9)r^2+(8n^3-16n^2+4n)r+8n^3-12n^2)}{4(3r^2(n-1)+4n^2(1+r))^2} \\ &> 0.\end{aligned}$$

Hence we conclude that $v^C(N) \geq v^C_R(N)$ which is equivalent to $C(N, v^C) \neq \emptyset$. ■

This result establishes that there always exists an efficient payoff vector which permits to stabilize the monopoly power in Cournot competition.

Third, we compare the aggregate-monotonic cores between Bertrand and Cournot competition. To this end, we need the following result.

Theorem 4.4 (Deneckere and Davidson 1985) *In Bertrand competition, the profit of each cartel member is strictly increasing with respect to s, i.e., for any* $s \in \{1, \ldots, n-1\}$, $\pi^B_i(p^*_{s+1}, \tilde{p}_{s+1}) > \pi^B_i(p^*_s, \tilde{p}_s)$.

Theorem 4.5 *The aggregate-monotonic core of* (N, v^C) *is strictly included in the aggregate-monotonic core of* (N, v^B).

Proof: First, we determine the aggregate-monotonic cores of $(N, v^B) \in G$ and $(N, v^C) \in G$ respectively. It follows from Theorem 4.4 that the worth of the grand coalition $v^B_R(N)$ is obtained at the balanced collection $\{N\backslash\{k\} : k \in N\}$. Hence, it holds that:

$$v^B_R(N) = n\pi^B_i(p^*_{n-1}, \tilde{p}_{n-1}) \tag{18}$$

where (N, v_R^B) is the root game of (N, v^B). Moreover, since oligopoly TU-games (N, v^B) and (N, v^C) are symmetric, the cores of their associated root games $C(N, v_R^B)$ and $C(N, v_R^C)$ respectively are singletons.[10] We deduce from (17) and (18) that:

$$C(N, v_R^B) = \{(\pi_i^B(p_{n-1}^*, \tilde{p}_{n-1}))_{i=1}^n\},$$

and

$$C(N, v_R^C) = \{(\max\{\pi_i^C(q_1^*, \tilde{q}_1), \pi_i^C(q_{n-1}^*, \tilde{q}_{n-1})\})_{i=1}^n\}.$$

It follows from (8) that:

$$AC(N, v^B) = \{(\pi_i^B(p_{n-1}^*, \tilde{p}_{n-1}))_{i=1}^n + (v^B(N) - n\pi_i^B(p_{n-1}^*, \tilde{p}_{n-1})) \cdot \Delta_n\},$$

and

$$\begin{aligned} AC(N, v^C) = \{&(\max\{\pi_i^C(q_1^*, \tilde{q}_1), \pi_i^C(q_{n-1}^*, \tilde{q}_{n-1})\})_{i=1}^n \\ &+(v^C(N) - n\max\{\pi_i^C(q_1^*, \tilde{q}_1), \pi_i^C(q_{n-1}^*, \tilde{q}_{n-1})\}) \cdot \Delta_n\}, \end{aligned}$$

where Δ_n denotes the unit-simplex.
Second, since $v^B(N) = v^C(N)$ it is sufficient to show that $\pi_i^C(q_{n-1}^*, \tilde{q}_{n-1}) > \pi_i^B(p_{n-1}^*, \tilde{p}_{n-1})$ in order to establish that $AC(N, v^C) \subset AC(N, v^B)$. Using the expressions of Nash equilibrium profits of any cartel member in Bertrand and Cournot competition (see footnote 9), one gets:

$$\begin{aligned} \pi_i^C(q_{n-1}^*, \tilde{q}_{n-1}) - \pi_i^B(p_{n-1}^*, \tilde{p}_{n-1}) &= \frac{V^2 n r^3 (n-1)(2+r)}{(3r^2(n-1) + 4n^2(1+r))^2} \\ &> 0, \end{aligned}$$

which concludes the proof. ■

Finally we verify that the aggregate-monotonic core may be strictly included in the core for both competition types. For example, when $N = \{1, 2, 3, 4\}$, $V = 9$ and $r = 2$, the worth of any coalition is given in the following table (see footnote 9).

[10] On the set of balanced TU-games, the core of a root game (N, v_R) coincides with the contraction core introduced by Gonzalez and Lardon (2016). For any symmetric TU-game, Gonzalez and Lardon (2016) have proved that the contraction core is a singleton.

s	1	2	3	4
$v^B(S)$	$\dfrac{2268}{121} \simeq 18,74$	$\dfrac{54675}{1444} \simeq 37,86$	$\dfrac{1093500}{18769} \simeq 58,26$	81
$v^C(S)$	$\dfrac{3240}{169} \simeq 19,17$	$\dfrac{19683}{512} \simeq 38,44$	$\dfrac{1102248}{18769} \simeq 58,73$	81

The aggregate-monotonic cores of $(N, v^B) \in G$ and $(N, v^C) \in G$ are given by:

$$AC(N, v^B) = \left\{ x \in \mathbb{R}^n : \forall i \in N,\ x_i \geq \frac{364500}{18769} \simeq 19,42 \text{ and } x(N) = 81 \right\},$$

and

$$AC(N, v^C) = \left\{ x \in \mathbb{R}^n : \forall i \in N,\ x_i \geq \frac{367416}{18769} \simeq 19,57 \text{ and } x(N) = 81 \right\},$$

respectively. Now, consider payoff vectors $x_B = (\frac{364499}{18769}, \frac{1155790}{56307}, \frac{1155790}{56307}, \frac{1155790}{56307})$ and $x_C = (\frac{367415}{18769}, \frac{1152874}{56307}, \frac{1152874}{56307}, \frac{1152874}{56307})$. We can verify that $x_B \in C(N, v^B)$ but $x_B \notin AC(N, v^B)$, and $x_C \in C(N, v^C)$ but $x_C \notin AC(N, v^C)$.

5 Concluding remarks

Throughout this work we have revisited the classic comparison between Bertrand and Cournot competition. First, we have shown that merely assuming the formation of a cartel of firms or, equivalently, the existence of a multiproduct firm in an industry are sufficient to alter the standard Bertrand-Cournot rankings on quantities and profits. Second, comparing the cores and the aggregate-monotonic cores of Bertrand and Cournot oligopoly TU-games, we have proved that in most cases it is easier for firms to collude in Bertrand than in Cournot competition.

Although our analysis is restricted to industries with symmetric product differentiation and costs, we argue that it becomes a very difficult task to compare asymmetric Nash equilibrium prices, quantities and profits both analytically and geometrically. Moreover, without symmetry assumptions Bertrand and Cournot oligopoly TU-games are not symmetric anymore which makes the set of linear inequalities in (6) much more difficult to analyze even with a small number of firms. For example, when $N = \{1, 2, 3\}$, $r = 2$, and constant marginal costs of firms 1, 2 and 3 are given by $c_1 = 0$, $c_2 = 2\delta$, and $c_3 = 4\delta$ respectively, where $0 \leq \delta < (1/8)V$,[11] the worth of any coalition in Bertrand and Cournot competition is given in the following table.

[11] This condition ensures that equilibrium quantities are positive.

S	$v^B(S)$	$v^C(S)$
$\{1\}$	$\dfrac{21(4V+7\delta)^2}{1600}$	$\dfrac{45(2V+3\delta)^2}{784}$
$\{2\}$	$\dfrac{21(V-2\delta)^2}{100}$	$\dfrac{45(V-2\delta)^2}{196}$
$\{3\}$	$\dfrac{21(4V-23\delta)^2}{1600}$	$\dfrac{45(2V-11\delta)^2}{784}$
$\{1,2\}$	$\dfrac{160V^2-40\delta V+547\delta^2}{363}$	$\dfrac{56V^2-28\delta V+185\delta^2}{121}$
$\{1,3\}$	$\dfrac{2(80V^2-320\delta V+1409\delta^2)}{363}$	$\dfrac{2(28V^2-112\delta V+475\delta^2)}{121}$
$\{2,3\}$	$\dfrac{160V^2-1240\delta V+2947\delta^2}{363}$	$\dfrac{56V^2-420\delta V+969\delta^2}{121}$
$\{1,2,3\}$	$\dfrac{3(V^2-4\delta V+12\delta^2)}{4}$	$\dfrac{3(V^2-4\delta V+12\delta^2)}{4}$

Calculating the difference between $v^C(S)$ and $v^B(S)$ for any $S \in 2^N \backslash \{\emptyset, N\}$ leads to:

$$v^C(\{1\}) - v^B(\{1\}) = \frac{3(512V^2 - 1208\delta V - 3307\delta^2)}{78400},$$

$$v^C(\{2\}) - v^B(\{2\}) = \frac{24(V-2\delta)^2}{1225},$$

$$v^C(\{3\}) - v^B(\{3\}) = \frac{3(512V^2 - 2888\delta V + 53\delta^2)}{78400},$$

$$v^C(\{1,2\}) - v^B(\{1,2\}) = \frac{4(2V^2 - 11\delta V + 2\delta^2)}{363},$$

$$v^C(\{1,3\}) - v^B(\{1,3\}) = \frac{8(V-2\delta)^2}{363},$$

and

$$v^C(\{2,3\}) - v^B(\{2,3\}) = \frac{4(2V^2 - 5\delta V - 10\delta^2)}{363}.$$

Hence, for any $S \in 2^N \backslash \{\emptyset, N\}$, it holds that $v^C(S) - v^B(S) > 0$ which implies that $C(N, v^C) \subset C(N, v^B)$. Thus, the result on the cores given by Corollary 4.1 still holds in this example. The analysis of the aggregate-monotonic cores becomes much more difficult too, in part because Bertrand and Cournot oligopoly TU-games with asymmetric costs make the minimum level of efficiency given by (7) hard to compute.

To finish, our work indicates that much more remains to be explored in understanding the collusive behaviors of firms in oligopolies. Following in the footsteps of this work, it is possible to extend our analysis from symmetric to asymmetric product differentiation or costs assuming a restricted number of firms. Such extensions will be a significant step to help social planner as well as competition authorities to make optimal decisions.

6 Appendix

6.1 Geometrical properties of the reaction functions in price space

- Comparison of the y-intercepts of $R_I^B(p_j)$ and $\overline{R}_I^C(p_j)$ given by (9) and (13) respectively:

$$\frac{(n+rs)nV}{2n^2(1+r)+nr^2s-r^2s^2}-\frac{nV}{2(n+r(n-s))}=\frac{nVr^2s(n-s)}{2(n+r(n-s))(2n^2(1+r)+nr^2s-r^2s^2)} > 0.$$

- Comparison of the slopes of $R_I^B(p_j)$ and $\overline{R}_I^C(p_j)$ given by (9) and (13) respectively:

$$\frac{(n+rs)r(n-s)}{2n^2(1+r)+nr^2s-r^2s^2}-\frac{r(n-s)}{2(n+r(n-s))}=\frac{r^3(n-s)^2s}{2(n+r(n-s))(2n^2(1+r)+nr^2s-r^2s^2)} > 0.$$

- Comparison of the y-intercepts of $R_O^B(p_i)$ and $\overline{R}_O^C(p_i)$ given by (10) and (14) respectively:

$$\frac{(n+r)nV}{n^2(2+r)+nr(s+1)+r^2s}-\frac{nV}{2n+r(n+s-1)} = \frac{nV(n-1)r^2}{(2n+r(n+s-1))(n^2(2+r)+nr(s+1)+r^2s)} > 0.$$

- Comparison of the slopes of $R_O^B(p_i)$ and $\overline{R}_O^C(p_i)$ given by (10) and (14) respectively:

$$\frac{(n+r)rs}{n^2(2+r)+nr(s+1)+r^2s}-\frac{rs}{2n+r(n+s-1)} = \frac{r^3(n-1)s}{(2n+r(n+s-1))(n^2(2+r)+nr(s+1)+r^2s)} > 0.$$

6.2 Geometrical properties of the reaction functions in quantity space

- Comparison of the y-intercepts of $R_I^C(q_j)$ and $\overline{R}_I^B(q_j)$ given by (11) and (15) respectively:

$$\frac{(n+r(n-s))(1+r)nV}{2n^2(1+r)+nr^2s-r^2s^2}-\frac{(1+r)nV}{2(n+rs)}=\frac{nVr^2(1+r)(n-s)s}{2(n+rs)(2n^2(1+r)+nr^2s-r^2s^2)} > 0.$$

- Comparison of the absolute value of the slopes of $R_I^C(q_j)$ and $\overline{R}_I^B(q_j)$ given by (11) and (15) respectively:

$$\frac{(n+r(n-s))r(n-s)}{2n^2(1+r)+nr^2s-r^2s^2}-\frac{r(n-s)}{2(n+rs)}=\frac{r^3(n-s)^2s}{2(n+rs)(2n^2(1+r)+nr^2s-r^2s^2)}$$
$$>0.$$

- Comparison of the y-intercepts of $R_O^C(q_i)$ and $\overline{R}_O^B(q_i)$ given by (12) and (16) respectively:

$$\frac{(n(1+r)-r)(1+r)nV}{n^2(2+3r+r^2)-nr(1+r)(s+1)+r^2s}-\frac{nV(1+r)}{2n+r(n-s+1)}$$
$$=\frac{nV(n-1)r^2(1+r)}{(2n+r(n-s+1))(n^2(2+3r+r^2)+nr(-rs-s-r-1)+r^2s)}>0.$$

- Comparison of the absolute value of the slopes of $R_O^C(q_i)$ and $\overline{R}_O^B(q_i)$ given by (12) and (16) respectively:

$$\frac{(n(1+r)-r)rs}{n^2(2+3r+r^2)-nr(1+r)(s+1)+r^2s}-\frac{rs}{2n+r(n-s+1)}$$
$$=\frac{r^3(n-1)s}{(2n+r(n-s+1))(n^2(2+3r+r^2)+nr(-rs-s-r-1)+r^2s)}>0.$$

6.3 The asymptotic reaction functions in quantity space

By substituting s by n/k into (11), (12), (15) and (16), and taking the limit $n\to\infty$, the cartel asymptotic reaction functions are expressed as:

$$R_I^C(q_j)=\frac{(1+r)kV-r(k-1)q_j}{2(k+r)},$$

and

$$\overline{R}_I^B(q_j)=\frac{(r(k-1)+k)((1+r)kV-r(k-1)q_j)}{(k-1)r^2+2k^2r+2k^2}.$$

The asymptotic reaction functions of any outsider are given by:

$$R_O^C(q_i)=\overline{R}_O^B(q_i)=\frac{(1+r)kV-rq_i}{2k+(k-1)r}.$$

- Comparison of the y-intercepts of $R_I^C(q_j)$ and $\overline{R}_I^B(q_j)$ in the asymptotic case:

$$\frac{(r(k-1)+k)(1+r)kV}{(k-1)r^2+2k^2r+2k^2}-\frac{(1+r)kV}{2(k+r)}=\frac{(k-1)kr^2(1+r)V}{2(k+r)((k-1)r^2+2k^2r+2k^2)}>0.$$

- Comparison of the absolute value of the slopes of $R_I^C(q_j)$ and $\overline{R}_I^B(q_j)$ in the asymptotic case:

$$\frac{(r(k-1)+k)(k-1)r}{(k-1)r^2+2k^2r+2k^2}-\frac{r(k-1)}{2(k+r)}=\frac{(k-1)^2r^3}{2(k+r)((k-1)r^2+2k^2r+2k^2)}>0.$$

6.4 Proofs of Propositions 3.2, 3.5 and 3.6, and Lemma 4.2

Proof of Proposition 3.2: In quantity space, the intersections of reaction functions $R_I^C(q_j)$ and $R_O^C(q_i)$, and $\overline{R}_I^B(q_j)$ and $\overline{R}_O^B(q_i)$ given by (11), (12), (15) and (16) respectively, provide Nash equilibrium quantities produced by each cartel member in Bertrand and Cournot competition respectively:

$$D_i(p_s^*,\tilde{p}_s)=\frac{V(2n(1+r)-r)(r(n-s)+n)}{4n^2+r(n(6n-2(s+1))+r(n-s)(2n+s-2))} \tag{19}$$

and

$$q_s^*=\frac{nV(2n+r)(1+r)}{4n^2+2rn(1+n)+rs((n+2)r+2n)-r^2s^2}.$$

Calculating the difference between these two quantities leads to:

$$D_i(p_s^*,\tilde{p}_j)-q_s^*=\frac{Vr^2(n-s)p(r)}{AB},$$

where $A>0$ and $B>0$ denote the denominators of $D_i(p_s^*,\tilde{p}_s)$ and q_s^* respectively, and $p(r)$ is defined as:

$$\begin{aligned}p(r)&=r^2(2n((n-s)(s-1)+1)+s(s-2))\\&\quad+r2n(s(2n-s)+n(s-1)+1)\\&\quad+4n^2s\\&>0,\end{aligned}$$

which concludes the proof. ∎

A similar argument to the one in the proof of Proposition 3.2 permits to determine Nash equilibrium quantities produced by each ousider in Bertrand competition:

$$D_j(p_s^*,\tilde{p}_s)=\frac{V(2n(1+r)-rs)(r(n-1)+n)}{4n^2+r(n(6n-2(s+1))+r(n-s)(2n+s-2))} \tag{20}$$

Proof of Proposition 3.5: In price space, the intersection of reaction functions $R_I^B(p_j)$ and $R_O^B(p_i)$ given by (9) and (10) respectively, provides Nash equilibrium prices charged by each cartel member:

$$p_s^*=\frac{V\big(2n(1+r)-r\big)n}{2\big(2n+r(n+s-1)\big)\big(n+r(n-s)\big)-r^2s(n-s)},$$

and by each outsider:

$$\tilde{p}_s = \frac{V\big(2n(1+r) - rs\big)n}{2\big(2n + r(n+s-1)\big)\big(n + r(n-s)\big) - r^2 s(n-s)}.$$

The profit of each cartel member at Bertrand-Nash equilibrium $(p_s^*, \tilde{p}_s)$ is expressed as:[12]

$$\pi_i^B(p_s^*, \tilde{p}_s) = \frac{V^2(2n(1+r) - r)^2 n(n + r(n-s))}{\big(4n^2 + 6n^2 r - 2nrs + 2n^2 r^2 - r^2 s^2 - 2nr - 2nr^2 + 2r^2 s - nr^2 s\big)^2} \tag{21}$$

In quantity space, the intersection of reaction functions $R_I^C(q_j)$ and $R_O^C(q_i)$ given by (11) and (12) respectively, provides Nash equilibrium quantities produced by each cartel member:

$$q_s^* = \frac{V(2n+r)(1+r)n}{4n^2 + 2rn(1+n) + rs((n+2)r + 2n) - r^2 s^2} \tag{22}$$

and by each outsider:

$$\tilde{q}_s = \frac{V(2n+rs)(1+r)n}{4n^2 + 2rn(1+n) + rs((n+2)r + 2n) - r^2 s^2} \tag{23}$$

The profit of each cartel member at Cournot-Nash equilibrium $(q_s^*, \tilde{q}_s)$ is expressed as (see footnote 12):

$$\pi_i^C(q_s^*, \tilde{q}_s) = \frac{V^2((2n+r)^2 n(n+rs)(1+r))}{(4n^2 + 2rn(1+n) + rs((n+2)r + 2n) - r^2 s^2)^2} \tag{24}$$

Calculating the difference between the profits given by (21) and (24) leads to:

$$\pi_i^C(q_s^*, \tilde{q}_s) - \pi_i^B(p_s^*, \tilde{p}_s) = \frac{nr^3(2+r)(n-s)V^2 p(r)}{A^2 B^2},$$

where A^2 and B^2 denote the denominators of $\pi_i^C(q_s^*, \tilde{q}_s)$ and $\pi_i^B(p_s^*, \tilde{p}_s)$ respectively, and $p(r)$ is defined as:

$$\begin{aligned}
p(r) = \;& r^4(s(n-s)(2n+s-2)^2) \\
& + r^3 4n^2(n^2(-s^2+4s+1) + n(2s^3 - 3s^2 - 3s - 2) - s^4 - s^3 + 4s^2 - s + 1) \\
& + r^2 4n^2(4n^3 + n^2(-s^2+4s-3) + n(2s^3 - 3s^2 - 3s - 2) - s^4 - s^3 + 4s^2 - s + 1) \\
& + r(32n^5 - 32n^4) \\
& + 16n^5 - 16n^4.
\end{aligned}$$

It remains to study $p(r)$. Note that for any n and any $s < n$, it holds that $p(r) > 0$ when r is sufficiently small or sufficiently large. Two cases can occur:

[12] This expression is a special case of a more general expression of profits with asymmetric costs provided by Wang and Zhao (2010).

- for any $r > 0$, $p(r) > 0$, i.e., $p(r)$ has no positive root.
- there exists a root $r_1 > 0$ which implies that $p(r) < 0$ at the neighborhood of r_1.

A simple algorithm permitting to compute roots of $p(r)$ shows that no positive root appears for any $n \leq 25$ and any $s < n$.[13] ■

Proof of Proposition 3.6: By substituting s by n/k into (21) and (24), the difference between these two profits becomes:

$$\pi_i^C(q_s^*, \tilde{q}_s) - \pi_i^B(p_s^*, \tilde{p}_s) = \frac{(k-1)k^3r^3(r+2)V^2p(n)}{A^2B^2},$$

where $(An/k^2)^2$ and $(Bn/k^2)^2$ denote the denominators of $\pi_i^C(q_s^*, \tilde{q}_s)$ and $\pi_i^B(p_s^*, \tilde{p}_s)$ respectively, and $p(n)$ is defined as:

$$\begin{aligned} p(n) = \; & -n^4(4(r^3+r^2)(k^2-2k+1)) \\ & + n^3k(r^3(16k^2-12k-4) + r^2(16k^3+16k^2-12k-4) + 16k^3(2r+1)) \\ & + n^2(r^4(4k^3-3k-1) + r^3k^2(4k^2-12k+16) + r^2k^2(-12k^2-12k+16) \\ & \quad - 16k^4(2r+1)) \\ & + nk(r^4(-8k^2+4k+4) + r^3k^2(-8k-4) + r^2k^3(-8k-4)) \\ & + 4k^2r^2(r^2(k-1) + k^2(1+r)). \end{aligned}$$

Thus, there exists $\bar{n} > 0$ such that for any $n > \bar{n}$, it holds that $p(n) < 0$ which concludes the proof. ■

Proof of Lemma 4.2: First, for simplicity, we assume that the size s of cartel S is a real number in the interval $[1, n-1]$. Then differentiating $\pi_i^C(q_s^*, \tilde{q}_s)$ with respect to s leads to:

$$\frac{d}{ds}\pi_i^C(q_s^*, \tilde{q}_s) = \frac{V^2nr^2(1+r)(2n+r)^2(3rs^2 - (r(n+2)-2n)s - 2n)}{(4n^2 + 2rn(1+n) + rs((n+2)r+2n) - r^2s^2)^3}.$$

We aim to study the polynomial function of degree 2, $p : [1, n-1] \longrightarrow \mathbb{R}$, defined as:

$$p(s) = 3rs^2 - (r(n+2) - 2n)s - 2n.$$

The discriminant of $p(s)$ is given by:

$$\Delta = (n+2)^2r^2 + (16n - 4n^2)r + 4n^2,$$

and is positive for any $n \geq 3$ and any $r > 0$. Hence $p(s)$ has two distinct real roots:

$$s_{1,2} = \frac{r(n+2) - 2n \pm \sqrt{(n+2)^2r^2 + (16n-4n^2)r + 4n^2}}{6r}.$$

[13] The matlab program that we used in this proof is available to readers upon request. In a lexicographical order on (n, s), the first two positive roots $r_1 \simeq 14.91$ and $r_2 \simeq 22.26$ appear when $n = 26$ and $s = 11$. This implies that $p(r) < 0$ for any $r \in]r_1, r_2[$.

We want to prove that $s_1 < 0$ and $1 < s_2 < n-1$. We proceed in three steps.
First, we distinguish two cases. If $r(n+2) - 2n \geq 0$ then:

$$\begin{aligned} s_1 &= \frac{\sqrt{(r(n+2)-2n)^2} - \sqrt{(n+2)^2r^2 + (16n-4n^2)r + 4n^2}}{6r} \\ &= \frac{\sqrt{(n+2)^2r^2 + (-8n-4n^2)r + 4n^2} - \sqrt{(n+2)^2r^2 + (16n-4n^2)r + 4n^2}}{6r} \\ &< 0. \end{aligned}$$

Otherwise if $r(n+2) - 2n < 0$ then we can easily verify that $s_1 < 0$.
Second, it holds that:

$$s_2 - 1 = \frac{r(n-4) - 2n + \sqrt{(n+2)^2r^2 + (16n-4n^2)r + 4n^2}}{6r}.$$

We distinguish two cases. If $r(n-4) - 2n \leq 0$ then:

$$\begin{aligned} s_2 - 1 &= \frac{-\sqrt{(2n - r(n-4))^2} + \sqrt{(n+2)^2r^2 + (16n-4n^2)r + 4n^2}}{6r} \\ &= \frac{-\sqrt{(n-4)^2r^2 + (16n-4n^2)r + 4n^2} + \sqrt{(n+2)^2r^2 + (16n-4n^2)r + 4n^2}}{6r} \\ &> 0. \end{aligned}$$

Otherwise if $r(n-4) - 2n > 0$ then we can easily verify that $s_2 - 1 > 0$.
Third, it holds that:

$$n - 1 - s_2 = \frac{r(5n-8) + 2n - \sqrt{(n+2)^2r^2 + (16n-4n^2)r + 4n^2}}{6r}.$$

We distinguish two cases. If $n = 3$ then:

$$\begin{aligned} 2 - s_2 &= \frac{7r + 6 - \sqrt{25r^2 + 12r + 36}}{6r} \\ &= \frac{\sqrt{49r^2 + 84r + 36} - \sqrt{25r^2 + 12r + 36}}{6r} \\ &> 0. \end{aligned}$$

Otherwise if $n \geq 4$ then:

$$\begin{aligned} n - 1 - s_2 &\geq \frac{r(5n-8) + 2n - \sqrt{(n+2)^2r^2 + 4n^2}}{6r} \\ &> \frac{r(5n-8) + 2n - \sqrt{(n+2)^2r^2} - \sqrt{4n^2}}{6r} \\ &= \frac{4n-10}{6} \\ &> 0. \end{aligned}$$

It follows from the above three steps that $p(s)$ is strictly decreasing in the interval $[1, s_2]$ and strictly increasing in the interval $[s_2, n-1]$. This implies that $\pi_i^C(q_s^*, \tilde{q}_s)$ attains its maximum either at $s = 1$ or at $s = n-1$ which proves the first part of Lemma 4.2.
It remains to compare the two following profits of each cartel member derived from (24):

$$\pi_i^C(q_1^*, \tilde{q}_1) = \frac{V^2 n(1+r)(n+r)}{(2n + r(n+1))^2},$$

and

$$\pi_i^C(q_{n-1}^*, \tilde{q}_{n-1}) = \frac{V^2 n(1+r)(2n+r)^2(n + r(n-1))}{(4n^2(1+r) + 3r^2(n-1))^2}.$$

Calculating the difference between these two individual profits leads to:

$$\begin{aligned}\pi_i^C(q_{n-1}^*, \tilde{q}_{n-1}) - \pi_i^C(q_1^*, \tilde{q}_1) &= V^2 n(n-2) r^2 (1+r) \\ &\times \frac{(n^2 - 6n + 5)r^3 + (4n^3 - 16n^2 + 8n)r^2 + (4n^4 - 8n^3 - 4n^2)r + 4n^4 - 8n^3}{(2n + r(1+n))^2 (3r^2(n-1) + 4n^2(1+r))^2}.\end{aligned}$$

We can verify that $\pi_i^C(q_{n-1}^*, \tilde{q}_{n-1}) - \pi_i^C(q_1^*, \tilde{q}_1)$ is positive for any $n \geq 5$.[14] We conclude that for any $s \in \{1, \ldots, n-2\}$, $\pi_i^C(q_{n-1}^*, \tilde{q}_{n-1}) > \pi_i^C(q_s^*, \tilde{q}_s)$. ■

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[14]This difference is negative for $n \in \{3, 4\}$ and r sufficiently large. This is in line with Salant et al. (1983) who have observed that horizontal mergers may be disadvantageous to member firms.

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