Within-group heterogeneity in endogenous-policy contests[☆]

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Abstract

We analyze the selection of a public policy by a committee of heterogeneous agents (challengers) who subsequently will confront a group of status-quo defenders in a contest. We study how intra-group heterogeneity among challengers generates forces that affect their policy choice and show how they interact with the inter-group effects of this choice. As a result of this interaction, the policy selected by the challengers can be either more moderated or polarized than what would be selected in the absence of the succeeding struggle. We study this setting under different specifications of the contest success function and alternative rules to internally decide the policy among challengers.

Keywords: political processes; conflict; group contests; endogenous claims; intra-group heterogeneity
JEL Classification: D72, D74, C72

1. Introduction

The choice of the common stance that a group will defend in a subsequent confrontation against the opponents' position is a natural situation in a political environment: A party must internally choose an alternative (either a policy or a candidate) to face the opponents' choice in a succeeding dispute. But this can also appear in conflicts among industries, lobbies or interest groups. When groups are composed by heterogeneous members, the initial choice must solve a double-conflict: First, it must balance the internal tensions among groups' members and second, it should be a good competitor against the other groups' choices. In this

[☆] We acknowledge financial support from the Spanish Ministerio de Economia y Competitividad and FEDER through grant ECO2015-67901-P (MINECO/FEDER)

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study, we analyze the interplay between intra and inter-group forces in determining this strategic choice when the succeeding confrontation is modeled as a contest.

More specifically, in our setting there is a set of agents with single-peaked preferences over a one-dimensional public policy. These agents would be organized into two groups: Defenders of the status quo and challengers. Without loss of generality, it is assumed that challengers are at the left of the policy space. Their members have heterogeneous preferences on policies while status-quo defenders are homogeneous, they all defend the extreme-right policy. A two-stage game is played: First, challengers set a common target-policy. In this respect, we consider two alternative settings: (*i*) an exogenously determined representative selects this policy and (*ii*) the target-policy is collectively selected by the members of the challenging group by simple majority. Second, a contest between challengers and status-quo defenders determines which of the two policies is finally implemented. In our main setting we shall assume that the winning probability of a group is determined by a contest success function (hereafter CSF) homogeneous of degree zero that depends on the relative size of the group's aggregated effort. Agents select their effort individually and non-cooperatively in this second stage and the cost-effort function is convex.

Our analysis with heterogeneous agents contributes to the literature initiated by Epstein and Nitzan (2004, 2007). They show that, in a contest among two groups with homogeneous agents, if individuals' preferences are strictly concave and differentiable then any group would have incentives to sacrifice some utility by moderating the target-policy because this lessens competition in the contest stage. This moderation is known in the literature as *strategic restraint*.[1] The intuition behind this result is that the concavity of preferences causes that a slight moderation of a group target-policy reduces its members' stake just marginally but leads to a first-order impact on the opponents' stake. This produces an increase of the winning probability of the conceding group that overcomes its members' marginal utility loss due to moderation.[2] If the group members have heterogeneous preferences, a slight moderation of a group target-policy does not longer entail a marginal decrease of the aggregate stake of the conceding group. So, it is not clear whether strategic restraint will still arise in our setting: A moderation of the challenging policy causes a reduction of the stake of the status-quo defenders that leads them to lower their effort (inter-group effect), but also implies a modification of the aggregate stake of the challengers that leads them to vary their effort (intra-group effect). The analysis of the consequences of these two forces on the winning probability, the winning utility and the cost of effort will finally reveal whether challengers prefer

[1]Münster (2006) extends Epstein's and Nitzan's (2004) analysis to an all-pay auction contest. Many papers also follow from that seminal paper to address environmental issues as Heyes (1997), Liston-Heyes (2001) or Friehe (2013). Strategic restraint is also studied in voting contexts with policy motivated candidates, e.g. Lindbeck and Weibull (1993).

[2]Cardona and Rubí-Barceló (2016) show that with under-tent shaped or linear preferences, moderation is not obtained and groups would claim their most preferred policies.

moderation or polarization.

In order to disentangle the intra and inter-group forces that affect the strategic choice of the target-policy, we first study the case where the effort of the status-quo defenders is not chosen strategically, so that a modification of the challenging policy only affects the winning probability through the efforts of the challengers.[3] Note that having non-strategic opponents in the setting of Epstein and Nitzan (2004) would imply that there is nothing to win from moderation; so that parties would stick to their preferred policy. Hence, any strategic policy moderation or polarization of the challenging group in this first setting (with non-strategic opponents) would capture the new forces due to *intra-group* heterogeneity. Our result shows that the optimal target-policy of any agent lies in between her most-preferred policy and the maximizer of the gross aggregate stake: Agents renounce to part of their utility in exchange for a larger winning probability. Thus, in case that a representative can select the challenging policy of her group, the intra-group effect would yield moderation only if the preferred policy of the representative is more moderated than the policy maximizing the gross aggregate stake. Likewise, in the alternative setting where simple majority determines the group decision, it is shown that under quadratic preferences the median's preferred policy prevails, and that the selected target-policy (the Condorcet winner) lies in between the median's peak and the mean of the peaks (*i.e.*, the policy maximizing the gross aggregate stake). So again, moderation will only arise if the preferred policy of the median is more moderated than the policy maximizing the gross aggregate stake.

Second, we allow the defenders of the status quo to strategically select their efforts in the contest. In this case, the interaction of the above commented intra-group effect with the inter-group forces would determine the policy choice. In this case, we show that moderation is always profitable. That is, any agent prefers a target-policy in between the status quo and her most preferred policy. That is, the positive inter-group effects of moderation dominate any possible negative intra-group effect. If the group decision is made by simple majority in this case, we show that under quadratic preferences any Condorcet winner policy must be (weakly) more moderated than the median's optimal policy. Interestingly, this result is independent on the sizes of the groups. Thus, although moderation might reduce the winning utility of many challengers and increase the utility of a few status-quo defenders, a moderation of the challenging policy would always be profitable.

Finally, our results are extended into two directions: (i) we consider linear (instead of convex) costs of effort. Unlike the main framework, non-extreme agents will be non-active in the contest of this setting but results are essentially the same. And (ii) we analyze an alternative CSF, the linear-difference form, to show how

[3] In this case, the choice of a particular policy may be interpreted as the choice of an optimal internal (non-transferable utility) sharing rule.

part of our results crucially depend on this element of the setting. In particular, we demonstrate that polarization will arise from the interaction between intra and inter-group forces when the most preferred policy of the representative is sufficiently moderated and the number of status-quo defenders is low enough to limit the inter-group benefits of a policy moderation. This result also extends to the case in which the target-policy is selected by simple majority.

As the choice of a particular policy entails an internal distribution of efforts and therefore utilities, in our model the selected target-policy can be interpreted as a sharing rule, which certainly affects the aggregate effort of the group. From this viewpoint, our study can be related to the literature analyzing the effects of the internal sharing rule on the outcome of contests (e.g., Nitzan and Ueda, 2011 and 2016; Kolmar and Wagener, 2013 or Balart *et al.*, 2016). However, we differ from this literature because we consider the inter-group forces commented above. In an other respect, the selection of a target-policy can be interpreted as the choice of a representative like in the context of delegation (Baik and Kim, 1997, Schoonbeek, 2004 or Baik, 2007) but in our framework delegation is partial: The delegate sets the target-policy but efforts are individually chosen by the members of the group.

The two-stage selection of policy and effort of our setting is closely related to valence models of political competition (Groseclose, 2001; Aragonès and Palfrey, 2002; Aragonès and Xefteris, 2012). In particular, to those where valence is endogenously determined (Hirsch, 2016; Ashworth and Bueno de Mesquita, 2009; Herrera *et al.*, 2008; Meirowitz, 2008; Schofield, 2006). We could interpret total effort exerted in the contest as 'campaign valences', in the sense of Carter and Patty (2015), which increase party's probability of winning the election. The sources of polarization or moderation of policy platforms present in those papers differ from ours. The first main difference is that polarization in our setting is a consequence of intra-party forces. Heterogeneity among party members is not considered in previous literature, where a candidate rather than a party selects the policy platform. Only Schofield (2006) recognizes the role activists have in pulling equilibrium policy toward the extreme, without explicitly accounting for the distribution of party members. Concerning the inter-party forces, still many other differences can be found. In our setting, groups are policy motivated parties à la Wittman, in opposition to the Downsian party approach in Ashworth and Bueno de Mesquita (2009) and Mierowiotz (2008). As in our paper campaign spending attracts impressionable voters, moderation is a consequence of the strategic interaction of effort choices in the contest game (strategic restraint), rather than a consequence of more or less uncertainty in elections (Wittman, 1983 and Calvert, 1985). In a setting with ideological voters, Herrera *et al.* (2008) also found that a policy moderation decreases campaigning costs and increments the winning probability. However, this increment comes from narrowing the gap between the proposed policy and the expected median voter, and not from an strategic interaction with the opponent.

The next section presents the basic model and the equilibrium when the effort level of the status quo defenders is either fixed or endogenous. In Section 3 we discuss the policy choice of the group when decisions involve simple-majority voting. In Section 4 we analyze the results under two alternative specifications: a constant marginal cost of effort and a CSF of linear-difference form. We present some numerical examples in Section 5. Section 6 concludes.

2. The model

A set of players $N \subset [0, \tilde{x}]$ with cardinality n must choose the target-policy $x \in [0, 1]$ they will defend in a subsequent contest against the status quo alternative $y = 1$ defended by group M. The preferences of each agent $j \in N$ over public policies are given by $u_j(x) = 1 - \theta(|x - j|)$, with $\theta(0) = 0$, $\theta'(0) = 0$, $\theta'(z) > 0$ for $z > 0$ and $\theta'' > 0$ where, with some abuse of notation, $j \in N$ refers to both the agent and her most preferred policy (her peak).[4] Additionally, we assume $\tilde{x} < 1/2$ so that $y = 1$ is the worst possible policy for all agents in N. The winning probability of group N in the contest over public policies depends on the sum of individual efforts of its members (let a_j be the effort of agent $j \in N$), and it is denoted by $p(A, B)$ where $A = \sum_{j \in N} a_j$ is the aggregate effort of group N and B is the aggregate effort of group M. With the complementary probability, the status quo alternative $y = 1$ is implemented. We assume that $p(A, B)$ is homogeneous of degree 0 so that we can write $p(A, B) = f(A/B)$, satisfying (additionally) $f' > 0$ and $f'' < 0$. Efforts are costly. We assume that agents' preferences over public policies and costs of effort are separable so that the utility function is

$$v_j(x, a_j) = u_j(x) - a_j^2/2, \text{ for all } j \in N.$$

To determine the policy of the group we solve the game backwards. First, for any x, we find the Nash equilibrium of the contest game, where agents select their efforts individually. Second, we determine the target-policy selected by the group. For now, we assume that the group has a representative $r \in N$ who has full authority to select such a policy. In Section 3 we endogenize the selection of this representative.

As in Epstein and Nitzan (2004), this setting will allow to analyze how the succeeding confrontation against an opponent affects the strategic choice of a public policy (inter-group effect). But additionally, this choice can also be affected by the heterogeneity of the group (intra-group effect), as the present framework aims to show. In order to isolate the latter effect from the former, we will start by setting a fixed $B = \bar{B}$.[5]

[4] Many of our results use a quadratic specification of the utility: $\theta(|x - j|) = (x - j)^2$.

[5] An alternative interpretation of this model with a fixed B is a standard public good provision where $p(A, B)$ is the 'size' of the public good and $u_j(x) - u_j(1)$ is the valuation of agent j for a public good of size 1 when located at x.

2.1. Non-strategic opponent (NSO)

In the contest stage, for any $x \in [0,1]$, agent $j \in N$ chooses a_j to maximize

$$f\left(\left(A_{-j}+a_j\right)/\bar{B}\right) D_j\left(x\right) + u_j\left(1\right) - a_j^2/2$$

where $A_{-j} = A - a_j$ and $D_j\left(x\right) = u_j\left(x\right) - u_j\left(1\right)$ denotes the stake of agent $j \in N$. Hence, as efforts are aggregated linearly, the optimal effort level a_j^* satisfies

$$f'\left(\left(A_{-j}+a_j^*\right)/\bar{B}\right) \frac{1}{\bar{B}} D_j\left(x\right) - a_j^* \equiv 0. \tag{1}$$

Adding up the FOCs of all players, we implicitly obtain the equilibrium aggregate effort of the group $A\left(x\right) = A^*$ as

$$f'\left(A^*/\bar{B}\right) \frac{1}{\bar{B}} D_N\left(x\right) - A^* \equiv 0. \tag{2}$$

where $D_N = \sum_{j \in N} D_j\left(x\right)$.[6]

Note that, in equilibrium

$$a_j^*/A^* = \frac{D_j\left(x\right)}{D_N\left(x\right)} \text{ and } a_j^*/a_i^* = \frac{D_j\left(x\right)}{D_i\left(x\right)}$$

for all $i, j \in N$. Due to the strict convexity of effort costs, all agents will exert a positive effort. In this sense, there is no free-riding. Nevertheless, exerting effort generates positive externalities on the other members of the group.

Defining $\bar{Q}\left(x\right) = A^*/\bar{B}$ and given the homogeneity of p, we can implicitly write the (equilibrium) aggregate efforts of group N as $A^* = A\left(x\right) = f'\left(\bar{Q}\left(x\right)\right)\left(1/\bar{B}\right) D_N\left(x\right)$ and therefore

$$\bar{Q}\left(x\right) = f'\left(\bar{Q}\left(x\right)\right) \frac{1}{\bar{B}^2} D_N\left(x\right). \tag{3}$$

Differentiating (3) with respect to x yields

$$f''\left(\bar{Q}\left(x\right)\right) \frac{1}{\bar{B}^2} D_N\left(x\right) \bar{Q}'\left(x\right) + f'\left(\bar{Q}\left(x\right)\right) \frac{1}{\bar{B}^2} D_N'\left(x\right) - \bar{Q}'\left(x\right) = 0,$$

so that

$$\bar{Q}'\left(x\right) = \frac{\bar{Q}\left(x\right)}{1 - f''\left(\bar{Q}\left(x\right)\right)\left(1/\bar{B}^2\right) D_N\left(x\right)} \frac{D_N'\left(x\right)}{D_N\left(x\right)} = \frac{f'\left(\bar{Q}\left(x\right)\right)\bar{Q}\left(x\right)}{f'\left(\bar{Q}\left(x\right)\right) - f''\left(\bar{Q}\left(x\right)\right)\bar{Q}\left(x\right)} \frac{D_N'\left(x\right)}{D_N\left(x\right)}$$

As $f'\left(Q\right) > 0$ and $f''\left(Q\right) \leq 0$, it can be concluded that

$$\bar{Q}'\left(x\right) \geq 0 \Longleftrightarrow D_N'\left(x\right) \geq 0.$$

[6] As $f''\left(A/\bar{B}\right) < 0$ it is immediate that the solution is unique.

Note that $D_N(x)$ is the sum of concave functions in x, which is (uniquely) maximal at $\overline{x}$ satisfying $\sum_{j\in N} u'_j(x) = 0$.[7] Therefore, moving the target-policy towards $\overline{x}$ increases the aggregate effort of group N and, given that M is non-strategic, increments its winning probability.

A part from this effect on the winning probability, there are two additional aspects that affect the optimal policy choice of any particular agent $j \in N$. One is the utility she would obtain from the implementation of that policy and the other is the cost of the effort she would exert in the equilibrium of the subsequent contest. Regarding the latter effect it can be seen that a_j^* must decrease in x when moving from j towards $\overline{x}$.[8] Consequently, on the one hand, any agent j would benefit from choosing a target-policy closer to $\overline{x}$ in terms of both the winning probability and the cost of effort but, on the other, j would be damaged by moving the target-policy away from her peak. The following proposition shows that the interaction among these three forces leads the optimal policy of agent j, x_j^*, to lie in between j and $\overline{x}$. Unless otherwise stated, all proofs are in the Appendix.

Proposition 1. *If $j \neq \overline{x}$ then $\left(x_j^* - j\right)\left(x_j^* - \overline{x}\right) < 0$. Otherwise, $x_j^* = j = \overline{x}$.*

In a setting in which the representative agent r is exogenously fixed, the intra-group forces caused by the heterogeneity of group N push the representative to move the target-policy away from her peak towards $\overline{x}$ because this generates two positive effects on the representative, say (i) a cost-effort saving and (ii) a stronger incentive of her group members' to engage in rent-seeking efforts and, consequently, a higher winning probability, that overcome the reduction of her gain from winning the contest. Thus, the optimal target-policy of group N can be either more moderated or extreme than the policy that would be selected if there were no contest, *i.e.* the peak of its representative agent. Specifically, there is moderation when $r < \overline{x}$ and polarization when $r > \overline{x}$.

Note that if the members of group M could strategically select their efforts, these intra-group forces would interact with the inter-group effect that is analyzed in Epstein and Nitzan (2004) or Cardona and Rubí-Barceló (2016). We address the analysis of this interaction next.

2.2. A strategic opponent (SO)

When B can be strategically selected by the members of group M, the group N representative's optimal target-policy balances the intra-group forces described above and the inter-group effects on the opponents' incentives to exert efforts in the

[7] It is immediate that when preferences are quadratic ($u(x) = 1 - (x-j)^2$) then $\overline{x}$ is the mean of the peaks.

[8] As $f''(Q) < 0$, it is immediate from (1) that $Q'(x) > 0$ and $D'_i(x) < 0$ imply that a_j^* must also decrease in x when moving from j towards $\overline{x}$.

contest; a moderation of the target-policy of group N would decrease the stake of the opponents and thus decreases their incentives to bid against it. Therefore, when $r < \overline{x}$ intra and inter-group forces are aligned to induce a moderation of the target-policy with respect to the representative's peak r but when $r > \overline{x}$ the two forces have opposite directions, thus the resulting effect requires the analysis developed in this section.

For simplicity, we assume that M is formed by m identical members with preferences represented by $u_j(x) = 1 - \theta(1-x)$ for all $x \in [0,1]$. For any x, any $j \in M$ chooses effort b_j that maximizes

$$\begin{aligned} V_j(x) &= f(A/B)\, u_j(x) + [1 - f(A/B)]\, u_j(1) - b_j^2/2 \\ &= [1 - f(A/B)]\, D_j(x) + u_j(x) - b_j^2/2, \end{aligned}$$

where $A = \sum_{i\in N} a_i$, $B = \sum_{i\in M} b_i$ and $D_j(x) = u_j(1) - u_j(x) > 0$. Hence,

$$f'(A/B)\, \frac{A}{B^2} D_j(x) - b_j^* \equiv 0.$$

Aggregating for all agents $j \in M$ yields

$$f'(A/B)\, \frac{A}{B^2} D_m(x) - B^* \equiv 0. \tag{4}$$

Using (1), (2) and (4), we can express the equilibrium efforts A^*, B^* and a_j^* as a function of $Q(x) \equiv A(x)/B(x)$ as follows

$$A^* = A(x) = [f'(Q(x))\, Q(x)\, D_N(x)]^{1/2}, \tag{5}$$

$$B^* = B(x) = [f'(Q(x))\, Q(x)\, D_M(x)]^{1/2} \text{ and} \tag{6}$$

$$a_j^* = a_j(x) = \left[f'(Q(x))\, Q(x)\, D_j(x)\left(\frac{D_j(x)}{D_N(x)}\right)\right]^{1/2}, \tag{7}$$

where

$$Q(x) = \left(\frac{D_N(x)}{D_M(x)}\right)^{1/2} \tag{8}$$

is uniquely defined. Last equation puts forward the importance of group sizes in determining the winning probability. Nevertheless, as the next result shows, the variation of this probability due to a change in the target-policy of group N is not affected by the relative sizes of the groups. A target-policy moderation induces a positive inter-group effect on the winning probability that overcomes the negative intra-group effect that is generated when $r > \overline{x}$ (when $r < \overline{x}$ both effects push towards an increase of the winning probability). Thus, moderating the target-policy unequivocally increases the winning probability of group N.

Lemma 1. $Q'(x) > 0$.

Consequently, the confluence of intra and inter-group forces produces a positive effect of a target-policy moderation on the winning probability of group N. And this occurs even when n/m is very high, so that a policy moderation would imply a huge reduction of the aggregate stake of group N and a small reduction of the aggregate stake of the opposite group. In general, the representative's incentives to moderate the target-policy should be interpreted as a consequence of the interaction among the following three effects. By moving the target-policy, the representative affects (i) her utility for winning the contest, (ii) as the stakes are also modified, all agents' incentives to engage in rent-seeking efforts and, consequently, the winning probability of her group, and (iii) her equilibrium costs of effort. The previous result shows that effect (ii) is always positive in case of a moderation. Moreover, the representative's utility from winning the contest is also increased by a moderation from x only if this implies moving the target-policy towards the representative's peak. Thus, effect (i) is positive if $x < r$ and negative, otherwise. The interaction of those two effects with the third one explains the next result.

Proposition 2. *The optimal target-policy of the representative x_r^* is strictly greater than* $\min\{\overline{x}, r\}$. *Moreover, under quadratic preferences $x_r^* > r$.*

This Proposition shows that when $x \leq \min\{\overline{x}, r\}$, the two positive effects (i) and (ii) of a target-policy moderation are not offset by the effect (iii). However, when $x \in [\overline{x}, r]$ it can only be proved that (iii) will not offset the positive effects of moderation (i) and (ii) for quadratic preferences but not in general.[9] That is, for the general set of preferences, a target-policy moderation in this case can be not profitable for the representative because it would involve a sufficiently high increase of her equilibrium effort. Finally, when $x \in [r, \overline{x}]$ a target-policy moderation implies that effect (i) is negative, so this moderation will only take place if this effect is sufficiently small.

This result reinforces Epstein and Nitzan (2004) as strategic restraint still arises after adding the forces caused by intra-group heterogeneity to the inter-group effects analyzed in that paper. Moreover, this is independent of the groups' sizes. When the representative is relatively extreme ($r < \overline{x}$), both intra and inter-group effects induce moderation. Thus, the optimal target-policy would be unequivocally larger than the representative's peak. Nevertheless, when the representative is relatively moderated ($r > \overline{x}$), intra-group forces would lead to polarization, so in that case the interaction with an strategic opponent is key for having moderation. The comparison between optimal target policies in the NSO and SO cases is tricky because it would crucially depend on the exogenous value $\bar{B}$. This may generate counter-intuitive situations in which, for certain values of $\bar{B}$, the selected target-policy of $r \in N$ is more polarized in the SO case than in the NSO case.

[9] We are not conclusive at this point, as we do not have any example where this happens. Although $a_r(x)/A(x)$ increases in those cases, it is usually the case that $a_r(x)$ decreases as well.

3. Endogenous representative

Until now, we considered that there is an exogenously determined representative of group N that selects the target-policy of the group. Now, we assume that the target-policy is collectively selected by the group members. One reasonable assumption is to consider that the group would choose a Condorcet winner policy, when it exists.[10] The existence of Condorcet winners is guaranteed when the utilities of any two individuals satisfy single-crossing (*Representative Voter Theorem*, Rothstein, 1991) or when all have single-peaked preferences (*Median Voter Theorem*, Black, 1958). Although we are not able to show that agents' preferences satisfy single-peakedness, single-crossing is obtained in the NSO case when preferences are quadratic. Thus, in those cases, the Condorcet winner will be the optimal target-policy of the median agent.[11]

Proposition 3. *(NSO case) Under quadratic preferences, the optimal target-policy of the median player is the Condorcet winner.*

Thus, given Proposition 1, the Condorcet winner policy will be more moderated (polarized) than the median player's peak when this peak is lower (higher) than $\overline{x}$.

When the effort of the opponent group is strategically selected by its members the existence of a Condorcet winner and the prevalence of the median player's preference are unknown, although all our numerical examples suggest that the agents' preferences over policies are single-peaked. Next result shows that a target-policy that is selected under a majority voting rule would never be more extreme than the median's player optimal policy.

Proposition 4. *(SO case) Under quadratic preferences, any Condorcet winner policy must be larger than or equal to the median's optimal target-policy.*

Consequently, given the second part of Proposition 2, a contest following the policy choice would unequivocally involve a target-policy moderation with respect to the median player's peak.

4. Further specifications

4.1. Linear costs

The case of linear costs is particularly illustrative because, in any heterogeneous group, only the extreme agents would exert a positive effort. Without loss of gener-

[10]Condorcet winners are a robust prediction of the group's decision, particularly "for situations in which people can act in concert, with various subsets of people coordinating their actions to form coalitions [...] for unilaterally insuring an improvement in the welfare of all of its members"(Ordeshook, 1980).

[11]Nitzan and Ueda (2016) consider a representative that maximizes the aggregate surplus of the group.

ality, we assume that there is an agent at 0.[12] Hence, the effort of a group positively depends on the stake of its extreme player. Notice that A solves

$$f'(A/B)\frac{1}{B}D_0(x) - 1 = 0. \tag{9}$$

In the NSO case (where $B = \bar{B}$), implicit differentiation of (9) gives

$$\bar{Q}'(x) = \frac{A'(x)}{\bar{B}} = -\frac{f'\left(A/\bar{B}\right)D_0'(x)}{f''\left(A/\bar{B}\right)D_0(x)} < 0 \text{ for all } x > 0.$$

Hence, the optimal target-policy of $j \in N$, $j \neq 0$, solves

$$\begin{aligned} 0 &= f'(Q(x))Q'(x)D_j(x) + f(Q(x))D_j'(x) \\ &= D_j(x)\left[f(Q(x))\frac{D_j'(x)}{D_j(x)} - \frac{\left(f'\left(A/\bar{B}\right)\right)^2}{f''\left(A/\bar{B}\right)}\frac{D_0'(x)}{D_0(x)}\right], \end{aligned}$$

implying $D_0'(x) \cdot D_j'(x) < 0$. That is, $x_j^* \in (0, j)$ for all $j \neq 0$. Therefore, any non-extreme representative $r \in N$ will select a target-policy that is more polarized than her own peak. As explained before, this results from the trade-off between maximizing the utility in case of winning (this occurs at $x = r$) and maximizing the winning probability (in this case, this happens at the peak of the extreme player $x = 0$). Regarding agent 0, her optimal target-policy is $x_0^* = 0$ as $V_0'(x) < 0$ for all $x > 0$.

When the members of M choose their effort strategically, B will depend on the stake of the extreme agent(s). In our setting, since all agents in M are located at 1, the effort will depend on the stake D_1. So, in the SO case, B satisfies

$$f'(A/B)\frac{A}{B^2}D_1(x) = 1,$$

so that

$$Q(x) = \frac{A(x)}{B(x)} = \frac{D_0(x)}{D_1(x)} < 1 \Longrightarrow A(x) = f'(Q(x))Q(x)D_0(x).$$

The optimal target-policy of any $j \in N$, $j \neq 0$, satisfies

$$\begin{aligned} 0 &= f'(Q(x))Q'(x)D_j(x) + f(Q(x))D_j'(x) \\ &= f'(Q(x))Q(x)\left(\frac{D_0'(x)}{D_0(x)} - \frac{D_1'(x)}{D_1(x)}\right)D_j(x) + f(Q(x))D_j'(x). \end{aligned}$$

[12]If that was not the case, we could make a normalization to locate the most extreme agent at 0.

As $\frac{D_0'(x)}{D_0(x)} - \frac{D_1'(x)}{D_1(x)} > 0$, this implies that $x_j^* > j$ for all j. That is, any representative of group N will choose a target-policy more moderated than her own peak. As in the quadratic-cost case, the inter-group forces push the representative to moderate the target-policy in order to increase the winning probability of her group. The difference with respect to the previous case is that now moderation does not alter the cost of effort of any representative $r > 0$, as all the effort will be exerted by the extreme player(s). Thus, moderation is beneficial for any $r > 0$ because this generates an increase of the winning probability that always offsets the utility loss from choosing a less preferred target-policy.

In case that the target-policy is collectively selected by the group members, we can proceed as in Section 3 to conclude that the selected target-policy cannot be more polarized than the median's optimal policy. This is implied by the following result:

Lemma 2. *For $z > x$,*

$$V_d(z) \geq V_d(x) \Longrightarrow V_l(z) > V_l(x) \ \text{for all } l > d \neq 0,$$

where d denotes the median player.

Proof. In the Appendix, Lemma 6 shows that for any $i < j$ and $z > x$ we have that

$$\frac{D_j(z)}{D_j(x)} > \frac{D_i(z)}{D_i(x)}.$$

Let us assume that $V_d(z) - V_d(x) \geq 0$, *i.e.* $f(Q(z))\,D_d(z) - f(Q(x))\,D_d(x) \geq 0$. Then, for all $l > d$

$$\frac{f(Q(x))}{f(Q(z))} \leq \frac{D_d(z)}{D_d(x)} < \frac{D_l(z)}{D_l(x)}.$$

Hence,

$$V_l(x) = f(Q(x))\,D_l(x) < f(Q(z))\,D_l(z) = V_l(z).$$

□

Consequently, x_d^* would be preferred to any lower policy for any agent $l \geq d$ when $d \neq 0$.[13] Therefore, as in the case with quadratic costs, any Condorcet winner policy must be larger than or equal to the median's optimal policy. Moreover, since $x_j^* > j$ for all j as showed above, it can be concluded that the Condorcet winner target-policy would be more moderated than the median's peak.

[13]If $d = 0$ then it is easy to see that the Condorcet winner target-policy would be x_0^*.

4.2. Linear-difference Contest Success Function

Now we consider an alternative CSF. In particular, $p_N(A, B) = 1/2 + s(A - B)$, for appropriate $s > 0$.[14] Cardona and Rubí-Barceló (2016) analyzed this setting for homogeneous groups and linear preferences and found that the equilibrium target policies were affected by the group size. In the present setting with non-linear preferences and heterogeneous groups a target-policy moderation is going to affect the agents' stakes differently, so the extension of those results is not immediate. In this section, the analysis is going to be restricted to the case of quadratic preferences.

By backward induction, we start from the contest stage. There, the optimal effort level a_j^* satisfies

$$sD_j - a_j^* = 0.$$

Thus, $A^* = sD_N$. Then, the indirect utility function of agent j is

$$V_j(x) = p_N(x)D_j(x) + u_j(1) - (sD_j(x))^2/2.$$

From here we analyze the NSO and SO cases in turn. In the NSO case (where $B = \bar{B}$), we obtain the following result.

Proposition 5. *(NSO case) Under the linear-difference CSF, either* $\left(x_j^* - j\right)\left(x_j^* - \overline{x}\right) < 0$ *or* $x_j^* = \overline{x} = j$*, for any* $j \in N$.

Proof. Differentiating the indirect utility function of agent j yields

$$V_j'(x) = p_N' D_j + \left(p_N - s^2 D_j\right) D_j',$$

where $p_N = 1/2 + s\left(sD_N - \bar{B}\right)$. Additionally, the existence of an interior equilibrium for any x and in particular for $x = 1$ (the case that implies a lower p_N) requires that $p_N(1) = \frac{1}{2} - s\bar{B} \geq 0$. Given that $D_{-j} = \sum_{i \in N, i \neq j} D_i(x) \geq 0$, this implies that

$$p_N - s^2 D_j = \frac{1}{2} + s\left(sD_N - \bar{B}\right) - s^2 D_j = \frac{1}{2} + s\left(sD_{-j} - \bar{B}\right) \geq 0$$

Consequently, $V_j'(x) = 0$ implies that $p_N' D_j' < 0$ in equilibrium, where $p_N' = s^2 D_N'$. Since D_j and D_N are uniquely maximal at j and $\overline{x}$, respectively, the statement of the proposition follows.[15] □

[14] In general, probabilities in the linear difference-form CSF are such that

$$p_N(A, B) = \max\{0, \min[1/2 + s(A - B), 1]\}.$$

For some values of s, a pure-strategy equilibrium fails to exist. Che and Gale (2000) characterize the set of mixed strategy equilibria in contests between two players and with linear costs. To our knowledge there is no characterization of such equilibria in our setting. In this paper, we will focus on pure-strategy equilibria. Hence, restricting the parameter set would be required to guarantee existence.

[15] Notice that the proof is valid not only for quadratic preferences.

This result is identical to Proposition 1 and also the forces that explain it. Consequently, the optimal target-policy of group N can imply either a moderation or a polarization with respect to the peak of its representative agent r depending on whether $r < \overline{x}$ or $r > \overline{x}$, respectively.

In the SO case, the inter-group forces come into play and interact with the intra-group forces that are also acting in the NSO case. Following similar steps, we obtain that $B^* = sD_M$. Thus, $p_N = 1/2 + s^2 (D_N - D_M)$ and $p_N' = s^2 (D_N' - D_M')$. Let $\hat{x} = \frac{n\overline{x}+m}{n+m}$. Then

Proposition 6. *(SO case) Under quadratic preferences and the linear-difference CSF, $x_j^* > j$ iff $j < \hat{x}$ and $x_j^* < j$ iff $j > \hat{x}$, for any $j \in N$.*

Proof. Since $D_N' = 2n(\overline{x} - x)$ and $D_M' = 2m(x-1)$,

$$p_N' = 2s^2 (n\overline{x} - x(n+m) + m).$$

So, $p_N' > 0 \Leftrightarrow x < \hat{x}$ and $p_N' < 0 \Leftrightarrow x > \hat{x}$.

Notice that $V_j'(x) = p_N' D_j + D_j' (p_N - s^2 D_j)$. Under quadratic preferences $D_j(x) = (1-x)(1+x-2j)$ and $D_j'(x) = 2(j-x)$, so $V_j'(j) = p_N' D_j(j)$. Since $D_j(j) > 0$ for any $j < 1/2$ and $x < 1$, the statement of the proposition follows. □

Therefore, unlike the result in the main setting (under a CSF homogeneous of degree zero), the previous result shows that under a linear-difference CSF the representative r will optimally choose a target-policy more moderated than her peak only when she is sufficiently extreme, *i.e.* when $r < \hat{x}$. Otherwise, there will be polarization. Notice that the threshold $\hat{x}$ is the weighted average between $\overline{x}$ and 1, *i.e.* the two policies that maximize D_N and D_M, respectively, where weights are the relative sizes of each group. So, for a given r, the target-policy will be more polarized than the representative's peak only when $r > \overline{x}$ and the relative size of the opposite group (m) is so low that $r > \hat{x}$. Intuitively, in a contest against a sufficiently small group M, the positive inter-group effects of a target-policy moderation that lower the equilibrium effort of M are not enough to compensate the negative consequences of this moderation.

When there is not a representative but the target-policy is collectively selected by the group members, we will proceed as before by analyzing first the NSO case. Here, we can establish the following preliminary result:

Lemma 3. *(NSO case) Under quadratic preferences and the linear-difference CSF, the indirect utility is single-peaked.*

Proof. The indirect utility function can be written as, $V_j(x) = D_j G + u_j(1)$, where $G = P_N - \frac{s^2}{2} D_j \geq 0$ and $D_j \geq 0$. Thus,

$$V_j'(x) = D_j' G + D_j \frac{\partial G}{\partial x}.$$

At any interior optimum we must have,

$$D_j' \frac{\partial G}{\partial x} \leq 0,$$

where $D_j' \frac{\partial G}{\partial x} = 2s^2(j-x)(D_N' - \frac{1}{2}D_j') = 2s^2(j-x)(2n(\bar{x}-x)-(j-x))$. The second partial derivative is,

$$V_j''(x) = D_j''G + 2D_j' \frac{\partial G}{\partial x} + \frac{\partial^2 G}{\partial x^2},$$

where $\frac{\partial^2 G}{\partial x^2} = -s^2(2n-1) \leq 0$, as $D_j'' = -2 < 0$, and we know that at any interior optimum $D_j' \frac{\partial G}{\partial x} \leq 0$, then the interior optimum is actually a maximum. □

This result guarantees the existence of a Condorcet winner policy (*Median Voter Theorem*, Black, 1958). The following result characterizes the location of this policy.

Proposition 7. *(NSO case) Under quadratic preferences and the linear-difference CSF, the Condorcet winner optimal target-policy must be larger (or lower) than the median's peak d if $d < \overline{x}$ (or $d > \overline{x}$).*

Proof. Lemma 3 guarantees the existence of a Condorcet winner policy, say x_w^*. Let us consider that $d < \overline{x}$. By Proposition 5, $x_j^* > \overline{x}$ for any $j > \overline{x}$ and $x_j^* > d$ for any $j \in (d, \overline{x})$. Therefore, x_w^* cannot be lower than or equal to d. A symmetric argument proves that $x_w^* < d$ when $d > \overline{x}$. □

Consequently, although we cannot say in this setting that the optimal target-policy of the median player will be the Condorcet winner as in the case with a CSF homogeneous of degree zero, the following conclusion is exactly the same as in the main setting: the Condorcet winner optimal policy will be more moderated (or polarized) than the median's peak when this peak is lower (or higher) than $\overline{x}$.

When we consider the interplay of intra and inter-group forces in the SO case, we proceed as in the previous case to obtain the following result:

Proposition 8. *(SO case) Under quadratic preferences and the linear-difference CSF, the Condorcet winner optimal target-policy must be larger (or lower) than or equal to the median's peak d if $d < \hat{x}$ (or $d > \hat{x}$).*

Therefore, unlike the result in Proposition 4 that shows that any Condorcet winner will imply a moderation with respect to the median's peak when the CSF is homogeneous of degree zero, the previous result shows that under a linear-difference CSF the Condorcet winner policy can imply a polarization when the median d is sufficiently moderate $d > \hat{x}$, where this threshold $\hat{x}$ depends positively on the relative size of the opposite group. Thus, the bigger is the group of status-quo defenders the more moderated needs to be the median of the challengers to obtain polarization.

5. Numerical examples

Next we show some results from the numerical simulations performed under the two alternative CSFs (a Tullock function, as a special case of a CSF homogeneous of degree zero, and the linear-difference form CSF) in the Strategic Opponent case (SO). All numerical examples are performed assuming quadratic preferences.

It is worth noting that for convex costs of effort, we obtained a positive monotonicity between agents' peaks and their chosen target-policies in all simulations. This implies that, under majority voting, the Condorcet-winner policy of group N (x_w^*) will be equal to the optimal target-policy of the median member of the group (x_d^*).

All results are independent to the distributions of voters' preferred policies provided that the same base parameters $(n, m, \bar{x}, d, s)$ prevail. Therefore if we understand concessions as a non-necessarily neutral redistribution of stakes, we find that conditional on such parameters, equilibrium aggregate effort, policy and therefore rival's aggregate effort is neutral to such a redistribution. In consequence, the equilibrium obtained under the economy $(n, m, \bar{x}, d, s)$ characterizes the equilibria of all the economies sharing those parameters irrespective of the distribution of preferred policies.

Homogeneous CSF

We start from the SO case with a Tullock CSF of the form $p = \sum_{j \in N} a_j / (\sum_{j \in N} a_j + \sum_{i \in M} a_i)$. Table 1 illustrates the above-commented positive monotonicity, meaning that if $k > j$, then $x_k^* > x_j^*$. As stated in Proposition 2, for quadratic preferences every j will moderate regardless of his relative position with respect to $\bar{x}$ and irrespective of group's sizes.

Table 1: j's preferred policy

	$j = 0$	$j = 1/10$	$j = 1/4$	$j = 1/3$	$j = 1/2$
$n = 25, m = 2$	0.1600	0.2203	0.3254	0.3903	0.5304
$n = 15, m = 2$	0.1889	0.2436	0.3414	0.4028	0.5375
$n = 5, m = 2$	0.2682	0.3075	0.3848	0.4368	0.5570
$n = 5, m = 6$	0.3313	0.3637	0.4291	0.4741	0.5810

Nevertheless, the magnitude of the concession does depend upon relative group sizes. Indeed, Table 2 shows that, under majority voting, the larger is n relative to m the closer will be the Condorcet winner policy (that in this case coincides with the median optimal target-policy, $x_w^* = x_d^*$) to the median's peak (d). It also illustrates that group N equilibrium winning probability is larger the larger is n with respect to m.

Table 2: Equilibrium results ($d = 0.25, \bar{x} = 0.2367$)

	x_w^*	Prob.	$(A; B; Q)$	$\hat{x}$
$n = 25, m = 2$	0.3254	0.7989	(1.5193; 0.3824; 3.9737)	0.2932
$n = 15, m = 2$	0.3414	0.7587	(1.253; 0.3985; 3.1441)	0.3265
$n = 5, m = 2$	0.3848	0.6581	(0.7943; 0.4127; 1.9244)	0.4548
$n = 5, m = 6$	0.4291	0.5415	(0.8230; 0.6968; 1.1811)	0.6530

Homogeneous CSF. Convex vs linear cost

In case of a linear cost function results are qualitatively the same. Agents always moderate regardless of the relative group sizes, meaning that the inter-group effect dominates. But now, only extreme agents exert some effort so they have more incentives to moderate than others. This breaks the positive monotonicity found for the homogeneous CSF with convex costs. This is illustrated by the examples in Figure 1, where $m = 1$. In Figure 1a, $n = 3$, the cost function is convex and the Condorcet winner policy is equal to 0.25, corresponding to the optimal target-policy of the agent $j = 1/100$. In the case of linear costs, Figure 1b (also with $n = 3$) shows that the Condorcet winner policy is 0.5, corresponding to the optimal target-policy of the extreme agent ($j = 0$). So, moderation is more pronounced in this case because the extreme agent must exert all the effort of her group. When $n = 5$, Figures 1c and 1d show that there is more moderation under linearity than under convexity of effort costs: Under convex costs the Condorcet winner policy (at 0.36) coincides with the optimal policy of the median whereas under linear costs the Condorcet winner policy (at 0.48) is more moderated than the optimal policy of the median agent.

Linear-difference form CSF

Table 3 reflects the positive monotonicity that appears in all our simulations under convex costs of effort, even when the CSF is of linear-difference form.

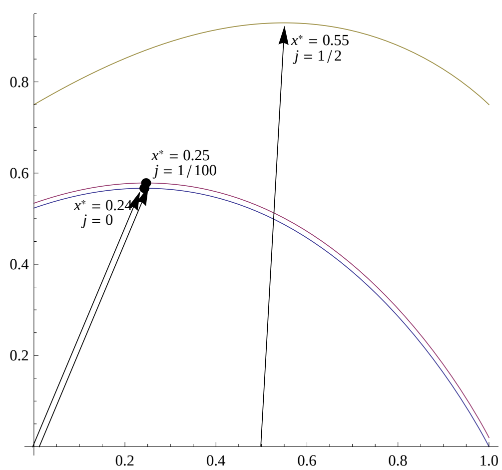
Table 3: j's preferred policy

	$j = 0$	$j = 1/10$	$j = 1/4$	$j = 1/3$	$j = 1/2$
$n = 15, m = 2$	0.1745	0.2088	0.2796	0.3311	0.4622
$n = 5, m = 2$	0.1987	0.2356	0.3103	0.3631	0.4932
$n = 5, m = 10$	0.4665	0.4756	0.4974	0.5158	0.5753

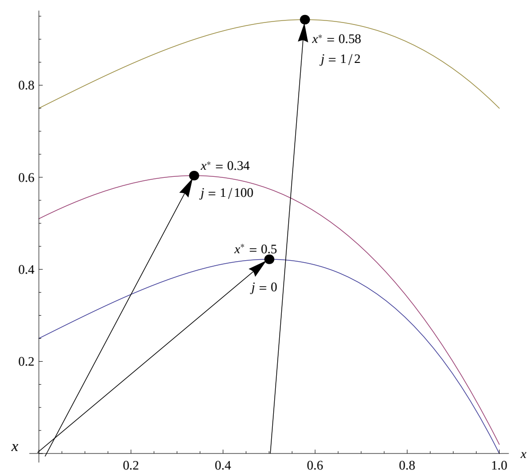
Under this CSF, intra-group effects are stronger and agent j polarizes her position with respect to her preferred policy whenever $j > \hat{x}$. In Table 4, the median's preferred policy equals $0.25 < \hat{x}$ so, the model predicts that the Condorcet winner policy will be more moderated than 0.25. Again, the monotonicity in our example implies that $x_w^* = x_d^*$. This table also shows that the magnitude of this moderation decreases as the relative size of group N increases.

Figure 1: Convex vs Linear cost ($m = 1$)

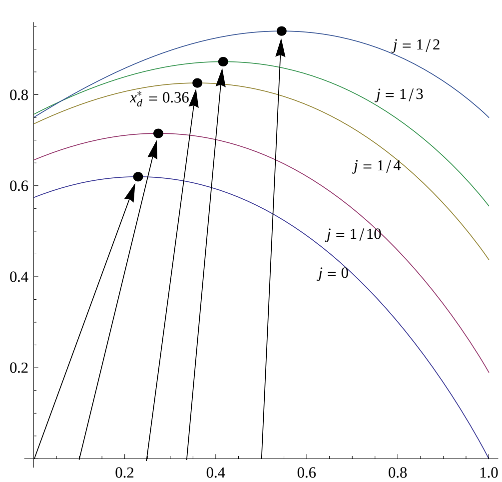
(a) $Convex, \bar{x} = 0.17$



(b) $Linear, \bar{x} = 0.17$



(c) $Convex, \bar{x} = 0.2367$



(d) $Linear, \bar{x} = 0.2367$

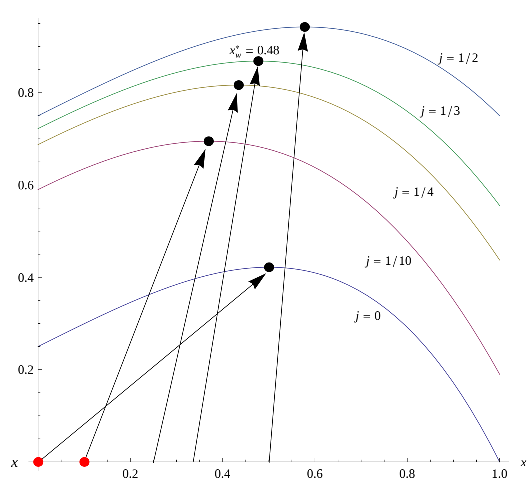


Table 4: Equilibrium results ($Median = 0.25, \bar{x} = 0.2367, s = 0.25$)

	x_w^*	Prob.	$(A; B; Q)$	$\hat{x}$
$n = 15, m = 2$	0.2796	0.9797	(2.1781; 0.2595; 8.3941)	0.3265
$n = 5, m = 2$	0.3103	0.62093	(0.7216; 0.2378; 3.0337)	0.4548
$n = 5, m = 10$	0.4974	0.50294	(0.6434; 0.6316; 1.0186)	0.7456

Table 5 presents an example where $\hat{x} < d$, so that, according to Proposition 8 the target-policy chosen by the median member is more polarized than her peak. Again, monotonicity implies $x_w^* = x_d^*$. Notice also that the further away is $\hat{x}$ from the median's peak the more polarized will be the target-policy chosen by the median.

Table 5: Equilibrium results ($d = 0.25, n = 15, m = 1, s = 0.2$)

	x_w^*	Prob.	$(A; B; Q)$	$\hat{x}$
$\bar{x} = 0.1798$	0.2444	0.8783	(2.0058; 0.1142; 17.5652)	0.2310
$\bar{x} = 0.16$	0.2391	0.8964	(2.0980; 0.1158; 18.117)	0.2125

6. Conclusion

We studied the contest between two groups of agents when one of them tries to change the status quo. Previous to the contest challengers must set their target-policy. As showed in Epstein and Nitzan (2004), altering this target-policy affects the challengers' utility through three different channels: (i) their equilibrium effort, (ii) their equilibrium winning probability and (iii) their equilibrium winning utility. The novelty is that in our setting part of these effects are due to the heterogeneity among challengers: We showed that when the probability of implementing the selected policy depends only on the efforts of the group, so the strategic effect acting in Epstein and Nitzan (2004) is neutralized, then the choice of the optimal target-policy solves the trade-off between maximizing the winning utility of the representative (at her most preferred policy) and maximizing her winning probability (at the policy where the aggregate stake of the group in the contest is maximized).[16] As a result, the optimal policy might be either more polarized or moderated than the representative agent's peak, depending on the location of this peak with respect to the policy that maximizes the aggregate stake. When the efforts of the opposite group are strategically chosen and the CSF is homogeneous of degree zero, we showed that the inter-group positive effects of moderation offset the intra-group effects due to heterogeneity, so that the optimal policy is always more moderated than the representative's peak. This is independent of the groups' sizes and of how this moderation alters the aggregate surplus of the group. These results hold either with convex or linear costs of effort. However, when the CSF has the linear-difference form, the optimal policy can be either more moderated or polarized than the representative's peak. Polarization will arise when the representative of the challenging group is sufficiently moderate and the group defending the status-quo is sufficiently small because in this case the inter-group positive effects of moderation will not offset the intra-group forces. The paper also analyzes the case where the challenging group has no representative and the target-policy is collectively selected. In this case, the heterogeneity among challengers imply that there is no unanimous consent on the best policy to lobby for in the subsequent contest. The results are similar to those obtained in the main setting of the paper: any Condorcet winner policy must always imply a moderation with respect to the median's peak under a CSF homogeneous of degree zero either with convex or linear costs of effort but not under the linear-difference CSF. In this case, the Condorcet winner policy might be more polarized than the median's peak.

[16] We can interpret this setting as a moral hazard problem in the context of a public good provision with non-transferable utility where the size of the public good depends on the aggregate effort of the group members.

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Appendix A. Proofs

Proof of Proposition 1. Given $\bar{Q}(x)$ and $a_j^* = a_j(x)$ for all $j \in N$, which are uniquely defined (from (1) and (2)), we can write the indirect utility function of agent $j \in N$ as

$$V_j(x) = f\left(\bar{Q}(x)\right) D_j(x) + u_j(1) - \frac{a_j^2(x)}{2}.$$

Differentiating this function yields

$$V_j'(x) = f'\left(\bar{Q}(x)\right) D_j(x)\bar{Q}'(x) + f\left(\bar{Q}(x)\right) D_j'(x) - a_j(x)\frac{\partial a_j(x)}{\partial x}$$

Using (1), we obtain

$$\frac{\partial a_j(x)}{\partial x} = f''\left(\bar{Q}(x)\right)\frac{1}{\bar{B}} D_j(x)\bar{Q}'(x) + f'\left(\bar{Q}(x)\right)\frac{1}{\bar{B}} D_j'(x)$$

Thus,

$$\begin{aligned}
V_j'(x) &= f'\left(\bar{Q}(x)\right) D_j(x)\bar{Q}'(x) + f\left(\bar{Q}(x)\right) D_j'(x) \\
&\quad - a_j(x)\left[f''\left(\bar{Q}(x)\right)\frac{1}{\bar{B}} D_j(x)\bar{Q}'(x) + f'\left(\bar{Q}(x)\right)\frac{1}{\bar{B}} D_j'(x)\right] \\
&= a_j(x)\bar{B}\bar{Q}'(x) + f\left(\bar{Q}(x)\right) D_i'(x) \\
&\quad - a_j(x)\left[f''\left(\bar{Q}(x)\right)\frac{1}{\bar{B}} D_j(x)\bar{Q}'(x) + f'\left(\bar{Q}(x)\right)\frac{1}{\bar{B}} D_j'(x)\right] \\
&= a_j(x)\bar{B}\left[1 - f''\left(\bar{Q}(x)\right)\frac{1}{\bar{B}^2} D_j(x)\right]\bar{Q}'(x) \\
&\quad + \left[f\left(\bar{Q}(x)\right) - a_j(x) f'\left(\bar{Q}(x)\right)\frac{1}{\bar{B}}\right] D_j'(x)
\end{aligned}$$

From the concavity of $f(Q)$ and the fact that $f(Q) \in [0,1]$, it is immediate that $f(Q) > f'(Q)Q$ for all $Q > 0$. Hence,

$$f\left(\bar{Q}(x)\right) - f'\left(\bar{Q}(x)\right)\frac{a_j(x)}{\bar{B}} > f\left(\bar{Q}(x)\right) - f'\left(\bar{Q}(x)\right)\bar{Q}(x) > 0.$$

Therefore, as $1 - f''\left(\bar{Q}(x)\right)\frac{1}{\bar{B}^2} D_j(x) > 0$, we obtain that any optimal solution must satisfy $\bar{Q}'(x) D_j'(x) < 0$. Moreover, when $j < \overline{x}$ then (i) $\bar{Q}'(x) > 0$ and $D_j'(x) \geq 0$ for all $x \leq j$ and (ii) $\bar{Q}'(x) \leq 0$ and $D_j'(x) < 0$ for all $x \geq \overline{x}$. Similarly in cases where $j > \overline{x}$. Thus, the statement of the proposition follows. $\square$

Proof of Lemma 1. We show first that $\left|\frac{D_N'(x)}{D_N(x)}\right| < \left|\frac{D_M'(x)}{D_M(x)}\right|$. Given that

$$\begin{aligned}
\left|\frac{D_N'(x)}{D_N(x)}\right| &= \left|\frac{\sum_{j\in N} D_j'(x)}{\sum_{j\in N} D_j(x)}\right| \leq \frac{\sum_{j\in N}\theta'(|x-j|)}{\sum_{j\in N}\theta(|1-j|) - \sum\theta(|x-j|)} \\
\left|\frac{D_M'(x)}{D_M(x)}\right| &= \frac{m\theta'(|1-x|)}{m\theta(|1-x|)}
\end{aligned}$$

then

$$\left|\frac{D_N'(x)}{D_N(x)}\right| - \left|\frac{D_M'(x)}{D_M(x)}\right| \leq \frac{1}{K}\left\{\theta(|1-x|)\sum_{j\in N}\theta'(|x-j|) \right.$$
$$\left. -\theta'(|1-x|)\sum_{j\in N}\left[\theta(|1-j|)-\theta(|x-j|)\right]\right\}$$

where $K \equiv \left(\sum_{j\in N}\theta(|1-j|) - \sum_{j\in N}\theta(|x-j|)\right)\theta(|1-x|) > 0$. From the convexity of θ, we have that

$$\begin{aligned}\frac{\theta(|1-j|)-\theta(|x-j|)}{1-x} &> \theta'(|x-j|) \text{ for all } j\in N\text{, and} \\ \frac{\theta(|1-x|)}{1-x} &< \theta'(|1-x|) \text{ for all } j\in M.\end{aligned}$$

Hence,

$$\begin{aligned} & \theta(|1-x|)\sum\theta'(|x-j|) - \theta'(|1-x|)\sum\left[\theta(|1-j|)-\theta(|x-j|)\right] \\ < \; & \theta(|1-x|)\sum\theta'(|x-j|) - \theta'(|1-x|)(1-x)\sum\theta'(|x-j|) \\ < \; & \theta'(|1-x|)(1-x)\sum\theta'(|x-j|) - \theta'(|1-x|)(1-x)\sum\theta'(|x-j|) = 0\end{aligned}$$

Therefore, $\left|\frac{D_N'(x)}{D_N(x)}\right| < \left|\frac{D_M'(x)}{D_M(x)}\right|$.

Noting that

$$Q'(x) = \frac{1}{2}Q^{-1}(x)\left[\frac{D_N'(x)}{D_M(x)} - \frac{D_N(x)\,D_M'(x)}{D_M^2(x)}\right] = \frac{1}{2}Q(x)\left[\frac{D_N'(x)}{D_N(x)} - \frac{D_M'(x)}{D_M(x)}\right]$$

and that $D_M'(x) < 0$ the claim follows. □

Proof of Proposition 2. Differentiating $V_r(x) = f(Q(x))\,D_r(x) + u_r(1) - \frac{1}{2}a_r^2(x)$ where $a_r(x)$ is given by (7) we obtain

$$\begin{aligned} V_r'(x) &= f'(Q(x))\,Q'(x)\,D_r(x) + f(Q(x))\,D_r'(x) \\ &\quad -\frac{1}{2}f''(Q(x))\,Q(x)\,Q'(x)\left(\frac{D_r^2(x)}{D_N(x)}\right) - \frac{1}{2}f'(Q(x))\,Q'(x)\left(\frac{D_r^2(x)}{D_N(x)}\right) \\ &\quad -f'(Q(x))\,Q(x)\left(\frac{D_r(x)}{D_N(x)}\right)D_r'(x) + \frac{1}{2}f'(Q(x))\,Q(x)\left(\frac{D_r^2(x)}{D_N^2(x)}\right)D_N'(x).\end{aligned}$$

For $x \leq r$, $D_r'(x) \geq 0$. Moreover from the concavity of f, $f(Q(x)) > f'(Q(x))\,Q(x)$. Consequently,

$$f(Q(x))\,D_r'(x) - f'(Q(x))\,Q(x)\left(\frac{D_r(x)}{D_N(x)}\right)D_r'(x) \geq 0$$

Hence, using $f''(Q(x)) < 0$ and $Q'(x) > 0$ (see Lemma 1),

$$\begin{aligned}
V_r'(x) &> f'(Q(x))\,Q'(x)\,D_r(x) - \frac{1}{2}f'(Q(x))\,Q'(x)\left(\frac{D_r^2(x)}{D_N(x)}\right) \\
&\quad + \frac{1}{2}f'(Q(x))\,Q(x)\left(\frac{D_r^2(x)}{D_N^2(x)}\right)D_N'(x) \\
&= f'(Q(x))\,Q'(x)\,D_r(x)\left[1 - \frac{1}{2}\left(\frac{D_r(x)}{D_N(x)}\right)\right] \\
&\quad + \frac{1}{2}f'(Q(x))\,Q(x)\left(\frac{D_r^2(x)}{D_N^2(x)}\right)D_N'(x) \qquad \text{(A.1)}
\end{aligned}$$

It is immediate that this derivative is positive when $D_N'(x) \geq 0$. Thus, $x_r^* > \min\{\overline{x}_N, r\}$, implying $x_r^* > r$ when $\overline{x} \geq r$.

To complete the proof, let consider $\overline{x} < r$ and $x \in (\overline{x}, r]$ so that $D_N'(x) < 0$ and $D_r'(x) \geq 0$.

From (A.1), and using $Q'(x) = \frac{1}{2}Q(x)\left(\frac{D_N'(x)}{D_N(x)} - \frac{D_M'(x)}{D_M(x)}\right)$ we obtain

$$\begin{aligned}
V_r'(x) &> \frac{1}{2}Q(x)\left(\frac{D_N'(x)}{D_N(x)} - \frac{D_M'(x)}{D_M(x)}\right)f'(Q(x))\,D_r(x)\left[1 - \frac{1}{2}\left(\frac{D_r(x)}{D_N(x)}\right)\right] \\
&\quad + \frac{1}{2}f'(Q(x))\,Q(x)\left(\frac{D_r^2(x)}{D_N^2(x)}\right)D_N'(x)
\end{aligned}$$

Thus,

$$V_r'(x) > \frac{1}{4}Q(x)\,f'(Q(x))\,\frac{D_r(x)}{D_N^2(x)\,D_M(x)}R(x)$$

where

$$R(x) = D_N'(x)\,D_M(x)\left[2D_N(x) + D_r(x)\right] - D_N(x)\,D_M'(x)\left[2D_N(x) - D_r(x)\right].$$

Under quadratic preferences,

$$\begin{aligned}
\frac{R(x)}{2mn(1-x)^3} &= -(x - \overline{x}_N)(2n - 2r + x - 4n\overline{x} + 2nx + 1) \\
&\quad + (1 + x - 2\overline{x})(-x + 2r + 2n - 4n\overline{x} + 2nx - 1) \\
&= (1 + x - 2\overline{x})(2n - 4n\overline{x} + 2nx - [1 + x - 2r]) \\
&\quad - (x - \overline{x})(2n - 4n\overline{x} + 2nx + [1 + x - 2r]) \\
&= [(1 + x - 2\overline{x}) - (x - \overline{x})][2n - 4n\overline{x} + 2nx] \\
&\quad - [1 + x - 2r][(1 + x - 2\overline{x}) + (x - \overline{x})] \\
&= 2n(1 - \overline{x})(1 + x - 2\overline{x}) - (1 + 2x - 3\overline{x})(1 + x - 2r) \\
&> {}_{r>\overline{x}} 2n(1 - \overline{x})(1 + x - 2r) - (1 + 2x - 3\overline{x})(1 + x - 2r) \\
&= [2n(1 - \overline{x}) - (3 - 3\overline{x}) + 2 - 2x](1 + x - 2r) \\
&= [(2n - 3)(1 - \overline{x}) + 2(1 - x)](1 + x - 2r) > 0
\end{aligned}$$

Therefore $R(x) > 0$ implying $V_r'(x) > 0$ and the claim follows. $\square$

Proof of Proposition 3. We start by proving a preliminary result:

Lemma 4. *If $x \leq \min\{i, \overline{x}\}$ then $V_i'(x) > 0$ and $V_i'(x) < 0$ for all $x \geq \max\{i, \overline{x}\}$.*

Proof. When the opposite group is non strategic, the indirect utility function of an agent $i \in N$ is

$$V_i(x) = f\left(\bar{Q}(x)\right) D_i(x) - \frac{1}{2}\left[f'\left(\bar{Q}(x)\right) D_i(x) \frac{1}{\bar{B}}\right]^2$$

Hence,

$$\begin{aligned} V_i'(x) &= f'\left(\bar{Q}(x)\right) D_i(x) \bar{Q}'(x) - f'\left(\bar{Q}(x)\right) D_i(x) \frac{1}{\bar{B}^2} f''\left(\bar{Q}(x)\right) D_i(x) \bar{Q}'(x) \\ &\quad + f\left(\bar{Q}(x)\right) D_i'(x) - f'\left(\bar{Q}(x)\right) D_i(x) \frac{1}{\bar{B}^2} f'\left(\bar{Q}(x)\right) D_i'(x) \\ &= f'\left(\bar{Q}(x)\right) D_i(x) \bar{Q}'(x) \left[1 - \frac{1}{\bar{B}^2} f''\left(\bar{Q}(x)\right) D_i(x)\right] \\ &\quad + D_i'(x) \left[f\left(\bar{Q}(x)\right) - f'\left(\bar{Q}(x)\right) D_i(x) \frac{1}{\bar{B}^2} f'\left(\bar{Q}(x)\right)\right] \\ &= f'\left(\bar{Q}(x)\right) D_i(x) \bar{Q}'(x) \left[1 - \frac{1}{\bar{B}^2} f''\left(\bar{Q}(x)\right) D_i(x)\right] \\ &\quad + D_i'(x) \left[f\left(\bar{Q}(x)\right) - f'\left(\bar{Q}(x)\right) \frac{D_i(x)}{D_N(x)} \bar{Q}(x)\right] \end{aligned}$$

Note that, since f is concave and $f(\cdot) \in [0,1]$,

$$f\left(\bar{Q}(x)\right) - f'\left(\bar{Q}(x)\right) \frac{D_i(x)}{D_N(x)} \bar{Q}(x) > f\left(\bar{Q}(x)\right) - f'\left(\bar{Q}(x)\right) \bar{Q}(x) > 0.$$

Therefore, when $x \leq \min\{i, \overline{x}\}$ we have that $\bar{Q}'(x) > 0$ and $D_i'(x) > 0$ implying $V_i'(x) > 0$. Similarly, when $x \geq \max\{i, \overline{x}\}$ we have that $\bar{Q}'(x) < 0$ and $D_i'(x) < 0$ implying $V_i'(x) < 0$. $\square$

Let $d, j \in N$ and d denote the median player of N. We now define

$$\begin{aligned} H(x, j) &= V_d(x) - V_j(x) = \\ &\quad f\left(\bar{Q}(x)\right) \left[D_d(x) - D_j(x)\right] - \frac{1}{2}\left[\frac{f'\left(\bar{Q}(x)\right)}{\bar{B}}\right]^2 \left[D_d^2(x) - D_j^2(x)\right]. \end{aligned}$$

We first prove that: (i) $\frac{\partial H(x,j)}{\partial x} \leq 0$ for all $x \in [\overline{x}, d]$ iff $j \geq d$ and (ii) $\frac{\partial H(x,j)}{\partial x} \leq 0$ for all $x \in [d, \overline{x}]$ iff $j \leq d$.

When preferences are quadratic, $D_j(x) = (1-x)(1+x-2j)$ and $D_j'(x) = -2(x-j)$ for all $j \in N$. Thus,

$$D_d(x) - D_j(x) = 2(1-x)(j-d)$$

and

$$D_d'(x) - D_j'(x) = -2(j-d).$$

Also, $D_N(x) = n(1-x)(1+x-2\overline{x})$ and $D_N'(x) = -2n(x-\overline{x}) > 0$.

Differentiating $H(x,j)$ we obtain

$$\begin{aligned}
\frac{\partial H(x,j)}{\partial x} &= f'\left(\bar{Q}(x)\right)[D_d(x) - D_j(x)]\bar{Q}_0'(x)\left\{1 - \frac{f''\left(\bar{Q}(x)\right)}{B^2}[D_d(x)+D_j(x)]\right\} \\
&\quad + f\left(\bar{Q}(x)\right)\left[D_d'(x) - D_j'(x)\right] \\
&\quad - \left[\frac{f'\left(\bar{Q}(x)\right)}{B}\right]^2\left[D_d(x)D_d'(x) - D_j(x)D_j'(x)\right].
\end{aligned}$$

As $f'\left(\bar{Q}\right)\bar{Q} = f'\left(\bar{Q}\right)\left(A^*/\bar{B}\right) = \left(f'\left(\bar{Q}\right)/\bar{B}\right)^2 D_N(x)$ and

$$\bar{Q}'(x) = \frac{\bar{Q}(x)}{1 - f''\left(\bar{Q}(x)\right)\left(1/\bar{B}^2\right)D_N(x)}\frac{D_N'(x)}{D_N(x)},$$

the previous expression can be written as

$$\begin{aligned}
\frac{\partial H(x,j)}{\partial x} &= f'\left(\bar{Q}(x)\right)\bar{Q}(x)[D_d(x) - D_j(x)]W(x)\frac{D_N'(x)}{D_N(x)} \\
&\quad + f\left(\bar{Q}(x)\right)\left[D_d'(x) - D_j'(x)\right] \\
&\quad - \frac{f'\left(\bar{Q}(x)\right)\bar{Q}(x)}{D_N(x)}\left[D_d(x)D_d'(x) - D_j(x)D_j'(x)\right] \qquad \text{(A.2)}
\end{aligned}$$

where

$$W(x) = \frac{1 - \frac{f''\left(\bar{Q}(x)\right)}{\bar{B}^2}[D_d(x) + D_j(x)]}{1 - \frac{f''\left(\bar{Q}(x)\right)}{\bar{B}^2}D_N(x)} \in (0,1).$$

Note also that $f\left(\bar{Q}\right) \geq f'\left(\bar{Q}\right)\bar{Q}$.

Case 1: $x \in (\overline{x}, d)$ and $j > d$. As $D_d(x) > D_j(x) > 0$, $0 < D_d'(x) < D_j'(x)$ and

$D'_N(x) < 0$, we obtain

$$\begin{aligned}\frac{\partial H(x,j)}{\partial x} &< f'\left(\bar{Q}(x)\right)\bar{Q}(x)\Big\{\left[D_d(x) - D_j(x)\right] W \frac{D'_N(x)}{D_N(x)} + \left[D'_d(x) - D'_j(x)\right] \\ &\quad - \frac{D_d(x) D'_d(x) - D_j(x) D'_j(x)}{D_N(x)}\Big\} \\ &< f'\left(\bar{Q}(x)\right)\bar{Q}(x)\frac{1}{D_N(x)}\left\{D_N(x)\left[D'_d(x) - D'_j(x)\right]\right. \\ &\quad \left. - \left[D_d(x) D'_d(x) - D_j(x) D'_j(x)\right]\right\} \\ &< f'\left(\bar{Q}(x)\right)\bar{Q}(x)\frac{1}{D_N(x)}\left\{\left[D_d(x) + D_j(x)\right]\left[D'_d(x) - D'_j(x)\right]\right. \\ &\quad \left. - \left[D_d(x) D'_d(x) - D_j(x) D'_j(x)\right]\right\} \\ &= f'\left(\bar{Q}(x)\right)\bar{Q}(x)\frac{1}{D_N(x)}\left\{D_j(x) D'_d(x) - D_d(x) D'_j(x)\right\} < 0.\end{aligned}$$

Case 2: $x \in (\overline{x}, d)$ and $j < d$. In these cases, $D_d(x) < D_j(x) < 0$, $0 < D'_d(x)$, $D'_d(x) > D'_j(x)$ and $D'_N(x) < 0$. Thus, proceeding as before, we obtain

$$\frac{\partial H(x,j)}{\partial x} > f'\left(\bar{Q}(x)\right)\bar{Q}(x)\frac{1}{D_N(x)}\left\{D_j(x) D'_d(x) - D_d(x) D'_j(x)\right\} > 0.$$

Case 3: $x \in (d, \overline{x})$ and $j > d$. Now, $D_d(x) > D_j(x) > 0$, $D'_d(x) < 0$, $D'_d(x) < D'_j(x)$ and $D'_N(x) > 0$. Using (A.2) and $W < 1$, we get

$$\begin{aligned}\frac{\partial H(x,j)}{\partial x} &< \frac{f'\left(\bar{Q}(x)\right)\bar{Q}(x)}{D_N(x)}\left\{\left[D_d(x) - D_j(x)\right] D'_N(x) + D_N(x)\left[D'_d(x) - D'_j(x)\right]\right. \\ &\quad \left. - \left[D_d(x) D'_d(x) - D_j(x) D'_j(x)\right]\right\}\end{aligned}$$

Substituting the values for quadratic preferences, we obtain

$$\frac{\partial H(x,j)}{\partial x}\left(\frac{\bar{B}}{f'(\bar{Q})}\right)^2 < -2(1-x)(j-d)(2d - 3x - 1 - 4n\overline{x} + 3nx + n + 2j).$$

Since $\overline{x} \le \frac{1}{n}\left(\frac{n+1}{2}d + j + \frac{n-3}{2}\frac{1}{2}\right)$, we obtain

$$(2d - 3x - 1 - 4n\overline{x} + 3nx + n + 2j) \ge 2 - 3x - 2dn + 3nx - 2j > 0,$$

when $x > d$. So, as $j - d > 0$, this proves the claim.

Case 4: Proceeding as in Case 3, if $j < d$ then $D_d(x) - D_j(x) < 0$ and $D'_d(x) - D'_j(x) > 0$. Thus,

$$\frac{\partial H(x,j)}{\partial x}\left(\frac{B}{f'(\bar{Q})}\right)^2 > -2(1-x)(j-d)(2d - 3x - 1 - 4n\overline{x} + 3nx + n + 2j).$$

Since $(2d - 3x - 1 - 4n\overline{x} + 3nx + n + 2j) > 0$ and $(j - d) < 0$ the claim follows.

To complete the proof, we next show that the optimal target-policy of the median player beats any other alternative. We consider the case where $\overline{x} < d$; a similar argument will prove the statement for cases where $\overline{x} > d$.

By Lemma 4, any $x \in (0, \overline{x}) \cup (d, 1)$ is majority blocked, either by $\overline{x}$ or by d. By the previous result, for any $x, y \in [\overline{x}, d]$, we have that

1. $V_d(x) \geq V_d(y)$ for some $x > y \Longrightarrow V_j(x) > V_j(y)$ for all $j > d$
2. $V_d(x) \geq V_d(y)$ for some $x < y \Longrightarrow V_i(x) > V_i(y)$ for all $i < d$

Thus, the only Condorcet winner candidate is x_d^*. Moreover, no other $y \in [\overline{x}, d]$ gets the support of a majority. To prove that no $x \in (0, \overline{x}) \cup (d, 1)$ is majority preferred to x_d^* we proceed by contradiction. Suppose $x \in (0, \overline{x})$ is majority preferred. By Statement 1 in Lemma 4 $V_j(x_d^*) > V_j(\overline{x}) > V_j(x)$ for all $j > d$, which is a contradiction. Symmetrically, when $x \in (d, 1)$, $V_i(x_d^*) > V_i(d) > V_j(x)$ for all $i < d$. Then, the claim follows. $\square$

Proof of Proposition 4. We start by proving a preliminary result.

Lemma 5. *For any three agents $i, j, k \in N$ such that $i < j < k$ and two policies $x, y \in X$, if $V_j(x) > V_j(y)$, $V_i(x) < V_i(y)$ and $V_k(x) < V_k(y)$ then $y > x$.*

Proof. The three inequalities can be written as

$$\begin{aligned} f(Q(x)) D_j(x) - \frac{1}{2} a_j^2(x) &> f(Q(y)) D_j(y) - \frac{1}{2} a_j^2(y) \\ f(Q(x)) D_k(x) - \frac{1}{2} a_k^2(x) &< f(Q(y)) D_k(y) - \frac{1}{2} a_k^2(y) \\ f(Q(x)) D_i(x) - \frac{1}{2} a_i^2(x) &< f(Q(y)) D_i(y) - \frac{1}{2} a_i^2(y) \end{aligned}$$

Since $a_l^2(x) = \left[\frac{f'(Q(x))}{B(x)}\right]^2 D_l^2(x)$ for any $l \in N$ we have that

$$\begin{aligned} & [D_j(x) - D_i(x)] \left\{ f(Q(x)) - \frac{1}{2} \left[\frac{f'(Q(x))}{B(x)}\right]^2 [D_j(x) + D_i(x)] \right\} \\ > \; & [D_j(y) - D_i(y)] \left\{ f(Q(y)) - \frac{1}{2} \left[\frac{f'(Q(y))}{B(y)}\right]^2 [D_j(y) + D_i(y)] \right\} \end{aligned}$$

Given that $D_l(x) - D_r(x) = 2(1-x)(r-l)$ for any $l, r \in N$ we get

$$\begin{aligned} & (1-x) \left\{ f(Q(x)) - \frac{1}{2} \left[\frac{f'(Q(x))}{B(x)}\right]^2 [D_j(x) + D_k(x)] \right\} \\ > \; & (1-y) \left\{ f(Q(y)) - \frac{1}{2} \left[\frac{f'(Q(y))}{B(y)}\right]^2 [D_j(y) + D_k(y)] \right\} \end{aligned}$$

and

$$
\begin{aligned}
& (1-x)\left\{f(Q(x))-\frac{1}{2}\left[\frac{f'(Q(x))}{B(x)}\right]^{2}[D_j(x)+D_i(x)]\right\} \\
< \quad & (1-y)\left\{f(Q(y))-\frac{1}{2}\left[\left[\frac{f'(Q(y))}{B(y)}\right]^{2}[D_j(y)+D_i(y)]\right]\right\}
\end{aligned}
$$

Hence subtracting the previous inequalities,

$$
(1-x)\left[\frac{f'(Q(x))}{B(x)}\right]^{2}\{D_i(x)-D_k(x)\}>(1-y)\left[\frac{f'(Q(y))}{B(y)}\right]^{2}\{D_i(y)-D_k(y)\}
$$

Substituting $B^2(x)=f'(Q(x))\,Q(x)\,D_M(x)$ and $D_M(x)=m(1-x)^2$ we get

$$
\left[\frac{f'(Q(x))}{Q(x)(1-x)}\right]\{D_i(x)-D_k(x)\}>\left[\frac{f'(Q(y))}{Q(y)(1-y)}\right]\{D_i(y)-D_k(y)\}
$$

Using again $D_l(x)-D_r(x)=2(1-x)(r-l)$ for any $l,r\in N$ we have that

$$
\left[\frac{f'(Q(x))}{Q(x)}\right]>\left[\frac{f'(Q(y))}{Q(y)}\right]
$$

implying $Q(x)<Q(y)\Longleftrightarrow x<y$. □

Note that $d\in N$ denotes the median agent of group N. The previous result implies that x_d^* is preferred to any $x<x_d^*$ by (at least) any agent $l\geq d$, so any Condorcet winner must be greater than or equal to x_d^*. □

Lemma 6. *If $x<z$ and $i<j$ then $D_j(z)-D_j(x)>D_i(z)-D_i(x)$ and*

$$
\frac{D_j(x)}{D_j(z)}<\frac{D_i(x)}{D_i(z)}. \tag{A.3}
$$

Proof. To prove the statement we consider six possible cases depending on the relative position of x,z,i,j. In each case, we prove that: when $x<z$ and $i<j$ imply $D_j(z)-D_j(x)>D_i(z)-D_i(x)$. Therefore, as $D_j(x)<D_i(x)$ we get

$$
\frac{D_j(z)-D_j(x)}{D_j(x)}>\frac{D_i(z)-D_i(x)}{D_i(x)},
$$

which implies

$$
\frac{D_j(z)}{D_j(x)}>\frac{D_i(z)}{D_i(x)}.
$$

Note that $D_k(z)-D_k(x)=\theta(|k-x|)-\theta(|k-z|)$, where θ is strictly convex.

Case 1. $x < y \leq i < j$. In this case, we have that

$$\frac{\theta(|j-x|)-\theta(|j-z|)}{z-x} > \frac{\theta(|i-x|)-\theta(|i-z|)}{z-x}.$$

So the statement follows.

Case 2. $i < j \leq x < z$. Proceeding as in the previous case, we obtain

$$\frac{\theta(|z-i|)-\theta(|x-i|)}{z-x} > \frac{\theta(|z-j|)-\theta(|x-j|)}{z-x}.$$

Case 3. $x \leq i < j \leq z$. In these cases

$$\begin{aligned} \theta(|j-x|) &> \theta(|j-i|)+\theta(|i-x|) \\ \theta(|z-i|) &> \theta(|j-i|)+\theta(|z-j|) \end{aligned}$$

Hence, $\theta(|j-x|)-\theta(|j-z|) > \theta(|i-x|)+\theta(|z-i|)$.

Case 4. $x \leq i \leq z < j$. The statement follows directly, as the convexity of θ implies

$$\theta(|j-x|) > \theta(|j-z|)+\theta(|z-i|)+\theta(|i-x|) > \theta(|j-z|)+\theta(|i-x|)-\theta(|z-i|).$$

Case 5. $i \leq x \leq j < y$. Now, as before we obtain

$$\theta(|z-i|) > \theta(|x-i|)+\theta(|j-x|)+\theta(|z-j|) > \theta(|x-i|)-\theta(|j-x|)+\theta(|z-j|)$$

Case 6. $i < x \leq y < j$. In this cases,

$$\theta(j-x)-\theta(j-y) > 0 > \theta(x-i)-\theta(y-i)$$

□

Proof of Proposition 8. We start by proving an intermediate result.

Lemma 7. *Under quadratic preferences, the linear-difference CSF and a SO, the indirect utility is single-peaked.*

Proof. The indirect utility function of any agent $j \in N$ is

$$V_j(x) = (1-x)(1+x-2j)\,G \geq 0,$$

because

$$G = \frac{1}{2}\left[1-s^2(1-x)(1+x-2j)\right]+s^2(1-x)(n(1+x-2\overline{x})-m(1-x)) \geq 0.$$

Thus,

$$V_j'(x) = 2(j-x)G + (1-x)(1+x-2j)\frac{\partial G}{\partial x}.$$

Note that, as $j \leq 1/2$, at any interior optimum we must have $(j-x)\frac{\partial G}{\partial x} \leq 0$, where

$$\frac{\partial G}{\partial x} = -s^2(j - 2m - x - 2\overline{x}n + 2mx + 2nx)$$

The second partial derivative is

$$\begin{aligned} V_j''(x) &= -2G + 4(j-x)\frac{\partial G}{\partial x} + (1-x)(1+x-2j)\frac{\partial^2 G}{\partial x^2} \\ &= -2G + 4(j-x)\frac{\partial G}{\partial x} - s^2(2m+2n-1)(1-x)(1+x-2j) \end{aligned}$$

which is negative when $x \in [0,1]$. □

This result guarantees the existence of a Condorcet winner (*Median Voter Theorem*, Black, 1958). Next, we focus on locating the optimal policy of the Condorcet winner.

Lemma 8. *For any $x \in [0,1]$ and any two agents $i, k \in N$ such that $i < k$, $V_i'(x) < 0$ and $V_k'(x) < 0$ imply $V_j'(x) < 0$ for any $j \in (i,k)$.*

Proof. Considering quadratic preferences, for any $l \in N$, V_l' can be written as

$$V_l'(x) = \Phi l^2 + \Psi l + \Omega,$$

where Φ, Ψ and Ω are expressions that depend on $s, x, \overline{x}, m$ and n. Let $j = \alpha i + (1-\alpha)k$ and assume $\alpha V_i'(x) < 0$ and $(1-\alpha)V_k'(x) < 0$.

Hence,

$$\begin{aligned} & \alpha\Phi i^2 + \alpha\Psi i + \alpha\Omega + (1-\alpha)\Phi k^2 + (1-\alpha)\Psi k + (1-\alpha)\Omega \\ = \ & \alpha\Phi i^2 + (1-\alpha)\Phi k^2 + \Psi j + \Omega < 0 \end{aligned} \tag{A.4}$$

On the other hand,

$$V_j'(x) = \Phi(\alpha i + (1-\alpha)k)^2 + \Psi j + \Omega$$

Using (A.4), we obtain

$$V_j'(x) < \Phi(\alpha i + (1-\alpha)k)^2 - \alpha\Phi i^2 - (1-\alpha)\Phi k^2 = -\Phi\alpha(1-\alpha)(k-i)^2 < 0.$$

□

This implies that the Condorcet winner optimal policy can never be lower than x_d^*. Therefore, by Proposition 6 we can conclude that if $d < \hat{x}$ the optimal target-policy of the Condorcet winner is greater than d. Now, it remains to be shown that when $d > \hat{x}$ the Condorcet winner optimal target-policy will be lower than d. By contradiction let us assume that $x_w^* \geq d$. By Lemma 8 this can only happen when

$$V_0'(d) = p_N' D_0 + D_0' \left(P_N - s^2 D_0\right) > 0,$$

where 0 denotes the agent whose peak is at zero.[17] Given that $D_0'(d) < 0$ and $p_N'(d) < 0$ (since p_N is maximized at $\hat{x}$ and $d > \hat{x}$), necessarily $P_N(d) - s^2 D_0(d) < 0$. Under quadratic preferences this can be written as

$$\frac{1}{2} + s^2(1-d)(n(1+d-2\overline{x}) - (1+d) - m(1-d)) < 0$$

implying

$$\frac{1}{2} + s^2(n(1+d-2\overline{x}) - (1+d) - m(1-d)) < 0.$$

Additionally, the existence of an interior equilibrium for any x and in particular for $x = 0$ requires that

$$p_N(0) = \frac{1}{2} + s^2(n(1-2\overline{x}) - m) \geq 0.$$

The last two inequalities can both hold only when $d \leq 1/(n+m+1)$. Since $d > \hat{x}$ and $\hat{x} > 1/(n+m+1)$ we have the contradiction that concludes the proof. $\square$

[17] Without loss of generality, we assume that there is an agent at this point. If not, we could make the normalization to locate the most extreme agent at 0.