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LOCAL MARKET STRUCTURE IN A HOTELLING TOWN

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Abstract. We develop a theoretical framework to study the location-price competition in a Hotelling-type network game, extending the Hotelling model, with linear transportation costs, from a line (city) to a network (town). We show the existence of a pure Nash equilibrium price if, and only if, some explicit conditions on the production costs and on the network structure hold. Furthermore, we prove that the local optimal localization of the firms are at the cross-roads of the town.

1. **Introduction.** Since the seminal work of Hotelling [13], the model of spatial competition has been seen by many researchers as an attractive framework for analyzing oligopoly markets (see [1, 7, 12, 16, 17, 18, 19, 20, 21, 22, 25, 6, 27]).

Hotelling [13] presented a city represented by a line segment where a uniformly distributed continuum of consumers buy a single commodity. The consumers have to support linear transportation costs when buying the commodity in one of the two firms of the city. The firms compete in a two-staged location-price game, where simultaneously choose their location and afterwards set their prices in order to maximize their profits. Hotelling concluded that firms would agglomerate at the center of the line, an observation referred as the "Principle of Minimum Differentiation". In 1979, D'Aspremont et al. [2] showed that the "Principle of Minimum Differentiation" is invalid, since there was no (pure) price equilibrium solution for all possible locations of the firms, in particular when they are not far enough from each other.

Other models have been developed where the line in the Hotelling model is replaced by other topologies as for example in the Salop Model [25], where the line is replaced by the circle, or in the Spokes model [6] (see, also, [15, 14]).

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In this work, we study the Hotelling town model (see [19, 26]). The Hotelling town model extends the Hotelling model from a line (city) to a network (town). Trying to mimic a real town, the roads of the town are the edges of the network, the crossroad are the vertices with degree higher than two and the ends of no-exit roads are the vertices with degree one. The firms are spread over the town and the consumers are uniformly distributed along the roads (similar size houses). The roads can have different lengths (market sizes) and the firms can have different unitary productions costs (firm's heterogeneity).

There is also a vast literature in network games (see, for instance [4, 9, 8, 10]). Usually, following the modeling methodology common in social network analysis, these studies locate firms and consumers at nodes and the edges are used to establish relevant information among the agents. The Hotelling town model, presented here, is different because the consumers are assumed uniformly distributed along the edges of the network and not at the nodes.

A price strategy consists in associating to each firm a selling price of the commodity. As in the original Hotelling model, the expenditure of a consumer that chooses to buy in a shop consists in the sum of the price practiced by that shop plus the transportation cost that is proportional to the minimal distance between his house (position at the network) and the shop.

The firms compete in a two-stage location-price game, where simultaneously choose their location and afterwards set their prices in order to maximize their profits; and each consumer will buy in the shop that will minimize its expenditure.

In the price subgame, the main goal is to compute the price strategy that has the following two essential economic properties: (i) any small deviation of a price of a firm provokes a decrease in its own profit (local strategic optimum); and (ii) all firms have non-empty market (local market structure). Property (i) stabilizes the prices because the firms do not have an incentive to do small changes in their prices. Property (ii) stabilizes the set of competing firms because every firm has a non-empty market (positive profit) and so, does not go to bankruptcy, and a sufficiently high entry cost can avoid new firms to appear in the town. We call a price strategy satisfying these two properties a local market optimum price strategy.

We prove that the price subgame has a local market optimum price strategy if, and only if, some explicit conditions on the production costs and road lengths hold. We show that if there is a local market optimum price strategy then it is unique. Furthermore, we introduce the weak bounded costs condition that gives a simple bound on the maximum difference between the production costs and on the maximum difference between the road lengths in terms of the transportation cost and the minimal road length. The weak bounded costs condition is a simple sufficient condition for the existence of the local market optimum price strategy.

We give an explicit closed formula and an explicit series expansion formula for the local market optimum price strategy. The series expansion formula shows explicitly how the local market optimum price strategy of a firm depends on the production costs, road market sizes and firms locations. Furthermore, the influence of a firm in the local market optimum price strategy of other firm decreases exponentially fast with the distance between the firms.

Assuming that the firms might not know the entire network, we introduce the idea of space bounded information that determines how far from its location each firm knows the network structure in terms of the production costs, node degrees and road sizes. We show that each firm is able to compute an approximation of its own local market optimum price strategy that improves exponentially fast with the space bounded information knowledge of the firm.

We say that a price strategy has the profit degree growth property if the profits of the firms increase with the degree of the nodes in the neighborhoods in which they are located. We give an example where the local market optimum price strategy does not have the profit degree growth property. Hence, we introduce the degree-bound condition, that is similar to the weak bounded condition, and we prove, under this condition, that the local market optimum price strategy has the profit degree growth property.

For the price-subgame, we note that a Nash equilibrium price with the local market structure property is a local market optimum price strategy, but a local market optimum price strategy might not be a Nash equilibrium price. However, we prove that the local market optimum price strategy is a Nash equilibrium price if, and only if, some explicit conditions on the production costs and road lengths hold. Furthermore, we introduce the strong bounded condition that in comparison with the weak bounded condition has the additional feature of depending also on the maximum node degree of the network. The strong bounded costs condition is a simple sufficient condition for the existence of the Nash equilibrium price. We note that this condition avoids that firms can get too close, and so D'Aspremont et al. [2] objection to the existence of a Nash equilibrium price does not occur.

For the location-subgame, we assume that, for every location allowed by the weak bounded condition, the firms choose the corresponding local market optimum price strategy for the price-subgame. A localization strategy for the firms in the network is for every firm to choose his position in the Hotelling town. A local optimal localization strategy is achieved when small perturbations in location do not result in improved profits. Similarly to the original Hotteling model in the line, we prove that the firms do not prefer to be located at the ends of no-exit roads. However, in contrast with the original Hotteling model in the line, we prove that the firms prefer maximum differentiation, in the sense that they prefer to be located at the crossroads of the network. This result is observed in real towns because the owners of the shops usually prefer to have them located at crossroads than along the roads.

2. **Hotelling town.**

The *Hotelling town* model (see [19]) consists of a network of *consumers* and *firms*. The consumers (*buyers*) are located along the *edges* (*roads*) of the network. Every road has two endpoints (*vertices*). For simplicity of the model, we assume that in a neighborhood of every vertex is located a single firm (*shop*). The *degree* k of the vertex is given by the number of incident edges. If the degree k is greater that 2 then the vertex is a crossroad of k roads; if the degree k is equal to 2 then the vertex is a junction between two roads; and if k is equal to 1 the vertex is in the end of a road with no exit. Every consumer will buy one unit of the commodity from only one firm in the network and each firm will charge its customers the same price for the commodity.

2.1. **Prices and profits.**

A Hotelling town *price strategy* $\mathbf{P}$ consists of a vector whose coordinates are the prices p_i of each firm F_i. Every firm F_i is located at a position y_i in a neighborhood of a vertex $i \in V$. A consumer located at a point x of the network who decides to buy at firm F_i *spends*

$$E(x; i, \mathbf{P}) = p_i + t\, d(x, y_i)$$

the price p_i charged by the firm F_i plus the *transportation cost* that is proportional t to the minimal distance measured in the network between the position y_i of the firm F_i and the position x of the consumer. Given a price strategy $\mathbf{P}$, the consumer will choose to buy in the firm $F_{v(x,\mathbf{P})}$ that minimizes his expenditure

$$v(x, \mathbf{P}) = argmin_{i \in V} E(x; i, \mathbf{P}),$$

where V is the set of all vertices of the Hotelling town. Hence, for every firm F_i, the *market*

$$M(i, \mathbf{P}) = \{x : v(x, \mathbf{P}) = i\}$$

consists of all consumers who minimize their expenditures by opting to buy in firm F_i. The *road market size* $l_{i,j}$ of a road $R_{i,j}$ is the Lebesgue measure (or length) of the road $R_{i,j}$, because the buyers are uniformly distributed along the roads. The *market size* $S(i, \mathbf{P})$ of the firm F_i is the Lebesgue measure of $M(i, \mathbf{P})$. The Hotelling town *production cost* $\mathbf{C}$ is the vector whose coordinates are the production costs c_i of the firms F_i. The Hotelling town *profit* $\Pi(\mathbf{P}, \mathbf{C})$ is the vector whose coordinates

$$\pi_i(\mathbf{P}, \mathbf{C}) = (p_i - c_i)\, S(i, \mathbf{P})$$

are the *profits* of the firms F_i.

2.2. **Local market network structure.** The *local firms* of a consumer located at a point x in a road $R_{i,j}$ with vertices i and j are the firms F_i and F_j. For every vertex i let N_i be the set of all neighboring vertices j for which there is a road $R_{i,j}$ connecting the vertices. A price strategy $\mathbf{P}$ determines a *local market structure* if every consumer buys from one of his local firms, i.e.

$$M(i, \mathbf{P}) \subset \bigcup_{j \in N_i} R_{i,j}.$$

If a price strategy $\mathbf{P}$ determines a local market structure then for every road $R_{i,j}$ there is one consumer located at a point $\mathbf{x}_{i,j} \in R_{i,j}$ who is *indifferent* to the local firm from which he going to buy his commodity, i.e. $E(x; i, \mathbf{P}) = E(x; j, \mathbf{P})$.

2.3. **Inner network structure.** The Hotelling town *admissible market size* $\mathbf{L}$ is the vector whose coordinates are the *admissible local firm market sizes*

$$L_i = \frac{1}{k_i} \sum_{j \in N_i} l_{i,j}.$$

The Hotelling town *neighboring market structure* $\mathbf{K}$ is the matrix whose coordinates are (i) $k_{i,j} = k_i^{-1}$, if there is a road $R_{i,j}$ between the firms F_i and F_j; and (ii) $k_{i,j} = 0$, if there is not a road $R_{i,j}$ between the firms F_i and F_j.

2.4. **Nash equilibrium network price.** The *candidate Nash equilibrium with a local market structure* is

$$\mathbf{P}^L = \frac{1}{2}\left(\mathbf{1} - \frac{1}{2}\mathbf{K}\right)^{-1}(\mathbf{C} + t\,\mathbf{L}) = \sum_{m=0}^{\infty} 2^{-(m+1)} \mathbf{K}^m (\mathbf{C} + t\,\mathbf{L}). \tag{1}$$

where $\mathbf{1}$ is the identity matrix. We note that $\mathbf{K}$ is a stochastic matrix and so has spectrum radium one.

Equation (1) gives an explicit closed formula and an explicit series expansion formula for the the candidate Nash equilibrium with a local market structure. The series expansion formula shows explicitly how the candidate Nash equilibrium with a local market structure of a firm depends on the production costs, road market sizes and firms locations. Furthermore, the influence of a firm in the the candidate

Nash equilibrium with a local market structure of other firm decreases exponentially fast with the distance between the firms.

2.5. **Heterogeneous network measures.** Let c_M (resp. c_m) be the maximum (resp. minimum) production cost of the Hotelling town

$$c_M = \max\{c_i : i \in V\} \quad \text{and} \quad c_m = \min\{c_i : i \in V\}.$$

Let l_M (resp. l_m) be the maximum (resp. minimum) road length of the Hotelling town

$$l_M = \max\{l_e : e \in E\} \quad \text{and} \quad l_m = \min\{l_e : e \in E\},$$

where E is the set of all edges of the Hotelling town. Let

$$\Delta(c) = c_M - c_m \quad \text{and} \quad \Delta(l) = l_M - l_m.$$

Let k_M (resp. k_m) be the maximum (resp. minimum) node degree of the Hotelling town

$$k_M = \max\{k_i : i \in V\} \quad \text{and} \quad k_m = \min\{k_i : i \in V\}$$

3. **Local best response price strategy.** In the following sections, for simplicity of exposition, we will assume that every firm is located at the corresponding node, i.e. $y_i = i$. In the section 10, we extend all the results to the general case where the firms are not necessarily located at the nodes, but can choose their locations in the neighbourhoods of the nodes. Furthermore, in the section 10, we give the proofs of all the results presented through the paper.

We observe that, for every road $R_{i,j}$, there is an *indifferent buyer* located at a distance

$$0 < x_{i,j} = \frac{p_j - p_i + t\, l_{i,j}}{2\,t} < l_{i,j} \tag{2}$$

of firm F_i if, and only if, $|p_i - p_j| < t\, l_{i,j}$. Thus, a price strategy $\mathbf{P}$ determines a local market structure if, and only if, $|p_i - p_j| < t\, l_{i,j}$ for every road $R_{i,j}$.

The firms F_i and F_j (or vertices i and j) are *neighbors* if there is a road $R_{i,j}$ with end nodes i and j. Let N_i the set of all vertices that are neighbors of the vertex i and, so, k_i is the cardinality of the set N_i that is equal to the degree of the vertex i. If the price strategy determines a local market structure then $S(i, \mathbf{P}) = \sum_{j \in N_i} x_{i,j}$ and

$$\pi_i(\mathbf{P}, \mathbf{C}) = (p_i - c_i)\, S(i, \mathbf{P}) = \frac{p_i - c_i}{2\,t} \sum_{j \in N_i} (p_j - p_i + t\, l_{i,j}). \tag{3}$$

Given a pair of price strategies $\mathbf{P}$ and $\mathbf{P}^*$ and a firm F_i, we define the price vector $\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*)$ whose coordinates are $\tilde{p}_i = p_i^*$ and $\tilde{p}_j = p_j$, for every $j \in V \setminus \{i\}$. Let $\mathbf{P}$ and $\mathbf{P}^*$ be price strategies that determine local market structures. The price strategy $\mathbf{P}^*$ is a *local best response* to the price strategy $\mathbf{P}$, if for every $i \in V$ the price strategy $\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*)$ determines a local market structure and

$$\frac{\partial \pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*), \mathbf{C})}{\partial \tilde{p}_i} = 0 \quad \text{and} \quad \frac{\partial^2 \pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*), \mathbf{C})}{\partial \tilde{p}_i^2} < 0.$$

Recall from subsection 2.3 the definitions of the admissible market size $\mathbf{L}$ and of the neighboring market structure $\mathbf{K}$.

Lemma 3.1. *Let $\boldsymbol{P}$ and $\boldsymbol{P}^*$ be price strategies that determine local market structures. The price strategy $\boldsymbol{P}^*$ is the local best response to price strategy $\boldsymbol{P}$ if, and only if,*

$$\boldsymbol{P}^* = \frac{1}{2}\,(\boldsymbol{C} + t\,\boldsymbol{L}) + \frac{1}{2}\,\boldsymbol{K}\,\boldsymbol{P} \tag{4}$$

and the price strategies $\tilde{\boldsymbol{P}}(i, \boldsymbol{P}, \boldsymbol{P}^*)$ determine local market structures for all $i \in V$.

4. **Local market optimum price strategy.** We prove that the price subgame has a local market optimum price strategy if, and only if, some explicit conditions on the production costs and road lengths hold. We show that if there is a local market optimum price strategy then it is the candidate Nash equilibrium with a local market structure $\mathbf{P}^L$ introduced in subsection 2.4. We introduce the *weak bounded* WB costs condition that gives a simple bound on the $\Delta(c)$ and $\Delta(l)$ in terms of the transportation cost t and the minimal road length l_m of the network. We prove that if the WB holds then $\mathbf{P}^L$ is the price subgame local market optimum price strategy.

A price strategy $\mathbf{P}^*$ is a *local market optimum price strategy* if (i) $\mathbf{P}^*$ is the local best response to $\mathbf{P}^*$; and (ii) $\mathbf{P}^*$ determines a local market structure.

Recall from subsection 2.4 the definition of the candidate Nash equilibrium $\mathbf{P}^L$ with a local market structure.

Theorem 4.1. *If there is a local market optimum price strategy then it is the the candidate price strategy $\boldsymbol{P}^L$. Furthermore, there is a local market optimum price strategy if, and only if,*

$$|p_i^L - p_j^L| < t\,l_{i,j}$$

for all firms F_i and F_j that are neighbors.

Recall from subsection 2.5 the definition of heterogeneous network measures.

Definition 4.2. A Hotelling town satisfies the *weak bounded length and costs (WB)* condition, if

$$\Delta(c) + t\,\Delta(l) < t\,l_m. \tag{5}$$

Theorem 4.3. *If the Hotelling town satisfies the WB condition, then the candidate price strategy is the unique local market optimum price strategy. Furthermore, the local market optimum prices p_i^L are uniformly bounded*

$$t\,l_m + \frac{c_i + c_m}{2} \leq p_i^L \leq t\,l_M + \frac{c_i + c_M}{2} \tag{6}$$

and the local market optimum profits $\pi_i^L = \pi_i^L(\boldsymbol{P}, \boldsymbol{C})$ of firm F_i are uniformly bounded

$$\frac{k_i\,(2\,t\,l_m - \Delta(c))^2}{8\,t} \leq \pi_i^L = \frac{k_i\,(p_i^L - c_i)^2}{2\,t} \leq \frac{k_i\,(2\,t\,l_M + \Delta(c))^2}{8\,t}.$$

Let $a \in V$, $R_{b,c} \in E$ and $d \in V \setminus \{i\}$. The marginal rates of the local market optimum prices p_i^L are positive with respect to the production costs c_a, admissible local firm market sizes L_a, transportation costs t and road lengths $l_{b,c}$. The marginal rates of the local market optimum profits π_i^L are negative with respect to the production costs c_i and positive with respect to the production costs c_d, admissible local firm market sizes L_a, transportation costs t and road lengths $l_{b,c}$.

5. **Nash equilibrium price strategy.** For the price-subgame, we note that a Nash equilibrium price with the local market structure property is the local market optimum price strategy $\mathbf{P}^L$, but the local market optimum price strategy might not be a Nash equilibrium price. We introduce the *strong bounded SB* costs condition that gives a simple bound on the $\Delta(c)$ and $\Delta(l)$ in terms of the transportation cost t, the minimal road length l_m of the network and also on the maximum node degree k_M of the network. We prove that if the SB holds then $\mathbf{P}^L$ is the Nash equilibrium price.

The price strategy $\mathbf{P}^*$ is a *best response* to the price strategy $\mathbf{P}$, if

$$(\tilde{p}_i - c_i)\, S(i, \tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*)) \geq (p'_i - c_i)\, S(i, \mathbf{P}'_i),$$

for all $i \in V$ and for all price strategies $\mathbf{P}'_i$ whose coordinates satisfy $p'_i \geq c_i$ and $p'_j = p_j$ for all $j \in V \setminus \{i\}$. A price strategy $\mathbf{P}^*$ is a Hotelling town *Nash equilibrium* if $\mathbf{P}^*$ is the best response to $\mathbf{P}^*$.

Lemma 5.1. *In a Hotelling town satisfying the WB condition, if there is a Nash price* $\boldsymbol{P}^*$ *then* $\boldsymbol{P}^*$ *is unique and* $\boldsymbol{P}^* = \boldsymbol{P}^L$.

Hence, the local market optimum price strategy $\mathbf{P}^L$ is the only candidate to be a Nash equilibrium price strategy. However, $\mathbf{P}^L$ might not be a Nash equilibrium price strategy because there can be a firm F_i that by decreasing his price is able to absorb markets of other firms in such a way that increases its own profit. Therefore, the best response price strategy $\mathbf{P}^{L,*}$ to the local market optimum price strategy $\mathbf{P}^L$ might be different from $\mathbf{P}^L$.

Let $\cup_{j\in N_i} R_{i,j}$ be the *1-neighbourhood* $\mathcal{N}(i,1)$ of a firm $i \in V$. Let $\cup_{j\in N_i} \cup_{k\in N_j} R_{j,k}$ be the *2-neighbourhood* $\mathcal{N}(i,2)$ of a firm $i \in V$.

Lemma 5.2. *In a Hotelling town satisfying the WB condition,*

$$M(i, \tilde{\boldsymbol{P}}(i, \boldsymbol{P}^L, \boldsymbol{P}^{L,*})) \subset \mathcal{N}(i,2)$$

for every $i \in V$.

Hence, a consumer $x \in R_{j,k}$ might not buy in its local firms F_j and F_k. However, the consumer $x \in R_{j,k}$ still has to buy in a firm F_i that is a neighboring firm of its local firms F_j and F_k, i.e. $i \in N_j \cup N_k$.

For every firm F_i and every $0 < p < p_i^L$, let $\hat{N}_i(p) \subset N_i$ be the set of all $j \in N_i$ such that

$$|p - p_j| < t\, l_{i,j}.$$

Let

$$\hat{S}_i(p) = \sum_{j\in \hat{N}_i(p)} \frac{p_j - p + t\, l_{i,j}}{2\, t} + \sum_{j\in N\setminus \hat{N}_i(p)} \sum_{k\in N_j} l_{i,j} + \frac{p_k - p + t\,(l_{j,k} - l_{i,j})}{2\, t}.$$

Theorem 5.3. *The local market optimum price strategy* $\boldsymbol{P}^L$ *is a Nash equilibrium price if, and only if,*

$$\hat{S}_i(p)\, p \leq \pi_i^L$$

for every firm F_i *and every* $0 < p < p_i^L$.

Recall from subsection 2.5 the definition of heterogeneous network measures.

Definition 5.4. A Hotelling town satisfies the *strong bounded length and costs (SB)* condition, if

$$\Delta(c) + t\Delta(l) \leq \frac{(2\, t\, l_m - \Delta(c))^2}{8\, t\, k_M\, l_M}. \tag{7}$$

The SB condition implies the WB condition, and so under the SB condition the only candidate to be a Nash equilibrium price strategy is the local market optimum price strategy $\mathbf{P}^L$. On the other hand, the condition

$$\Delta(c) + t\Delta(l) \leq t \, \min\left\{l_m, \frac{l_M}{8\, k_M}\right\}.$$

implies the SB condition.

Theorem 5.5. *If a Hotelling town satisfies the SB condition then there is a unique Hotelling town Nash equilibrium price strategy* $\boldsymbol{P}^* = \boldsymbol{P}^L$.

Hence, the Nash equilibrium price strategy for the Hotelling town satisfying the SB condition determines a local market structure, i.e. every consumer located at $x \in R_{i,j}$ spends less by shopping at his local firms F_i or F_j than in any other firm in the town and so the consumer at x will buy either at his local firm F_i or at his local firm F_j.

6. **Firm position stability.** For the location-subgame, we assume that, for every location allowed by the weak bounded condition, the firms choose the corresponding local market optimum price strategy for the price-subgame. A localization strategy for the firms in the network is for every firm F_i to choose his position in the neighborhood of its vertex i. A *local optimal localization strategy* is achieved when small perturbations in location do not result in improved profits. We prove that a Hotelling town network satisfying the WB condition and with the minimum node degree $k_m \geq 3$ has a local optimal localization strategy, whereby every firm F_i is located at the corresponding node i.

Consider that a firm F_i located a node i changes its location to a point y_i in a road $R_{i,j}$ at distance x for the node i. In the last section, we prove that there is an $\epsilon > 0$ such that, for every $x < \epsilon$, the price vector $\mathbf{P}(x; i, j)$ is the Nash equilibrium price strategy, given by the local market optimum price strategy, taking in account the new localization of the firm F_i. Let $\pi_i(x; i, j)$ denote the profit of firm F_i with respect to the price strategy $\mathbf{P}(x; i, j)$.

Definition 6.1. We say that a firm F_i is *node local stable* if there is $\epsilon_i > 0$ such that $\pi_i(0; i, j) > \pi_i(x; i, j)$ for every $0 < x < \epsilon_i$, with respect to the local market optimum price strategy. A Hotelling network is *firm position local stable* if every firm in the network is node stable.

Theorem 6.2. *A Hotelling town satisfying the WB condition and with* $k_m \geq 3$ *is firm position local stable.*

Hence, a Hotelling town network satisfying the WB condition and with $k_m \geq 3$ has a local market optimum price strategy, whereby every firm F_i is located at the corresponding node i.

In the subsection 10.4, we observe that firms F_i, with node degree $k_i = 1$, are node local unstable. Firms F_i, with $k_i = 2$, are node local unstable, except for networks satisfying special symmetric properties. Firms F_i, with $k_i = 3$, whose neighboring firms have nodes degree greater or equal to 3 are node local stable. Furthermore, firms F_i, with $k_i \geq 4$, whose neighboring firms have nodes degree greater or equal to 2 are node local stable.

7. **Space bounded information.** Assuming that the firms might not know the entire network, we introduce the idea of n-space bounded information. We say that a firm has *n-space bounded information*, if the firm knows the production costs of the other firms and the road lengths of the network up to n consecutive nodes of distance. Given a Hotelling town network satisfying the WB condition, every firm with n-space bounded information can compute a price $p_i(n)$ that estimates its own local market optimum price p_i^L, with exponential precision depending upon n. In addition, the firm can also estimate its profit with exponential precision depending upon n. Given $m+1$ vertices $x_0, \ldots, x_m$ with the property that there are roads $R_{x_0,x_1}, \ldots, R_{x_{m-1},x_m}$ the *(ordered) m path R* is

$$R = (R_{x_0,x_1}, \ldots, R_{x_{m-1},x_m}).$$

Let $\mathcal{R}(i,j;m)$ be the set of all m (ordered) paths $R = (R_{x_0,x_1}, \ldots, R_{x_{m-1},x_m})$ starting at $i = x_0$ and ending at $j = x_m$. Given a m order path $R = (R_{x_0,x_1}, \ldots, R_{x_{m-1},x_m})$, the corresponding *weight* is

$$k(R) = \prod_{q=0}^{m-1} k_{x_q,x_{q+1}}.$$

The matrix $\mathbf{K}^0$ is the identity matrix and, for $n \geq 1$, the coordinates of the matrix $\mathbf{K}^m$ are

$$k_{i,j}^m = \sum_{R \in \mathcal{R}(i,j;m)} k(R).$$

Definition 7.1. A Hotelling town has *n-space bounded information (n-I)* if for every $1 \leq m \leq n$, for every firm F_i and for every non-empty set $\mathcal{R}(i,j;m)$: (i) firm F_i knows the cost c_j and the average length road L_j of firm F_j; (ii) for every m path $R \in \mathcal{R}(i,j;m)$, firm F_i knows the corresponding weight $k(R)$.

The n *local market optimum price vector* is

$$\mathbf{P}(n) = \sum_{m=0}^{n} 2^{-(m+1)}\, \mathbf{K}^m \left(\mathbf{C} + t\, \mathbf{L}\right).$$

We observe that in a n-I Hotelling town, the firms might not be able to compute $\mathbf{K}$, $\mathbf{C}$ or $\mathbf{L}$. However, every firm F_i is able to compute his n local market optimum price $p_i(n)$

$$p_i(n) = \sum_{m=0}^{n} 2^{-(m+1)} \sum_{v \in V} k_{i,v}^m \left(c_v + t\, L_v\right).$$

Let N_V denote the number of nodes in the network.

Theorem 7.2. *A Hotelling town satisfying the WB condition has a local market optimum price strategy $\boldsymbol{P}^L$ that is well approximated by the n local market optimum price $\boldsymbol{P}(n)$ with the following 2^{-n} bound*

$$0 \leq p_i^L - p_i(n) \leq 2^{-(n+1)} N_V (c_M + t\, l_M).$$

Furthermore, $\boldsymbol{P}(n+1)$ is the best response to $\boldsymbol{P}(n)$ for n sufficiently high.

Hence, by theorem 7.2, the profit $\pi_i(\mathbf{P}^L)$ is well approximated by $\pi_i(\mathbf{P}(n))$ with the following bound

$$|\pi_i(\mathbf{P}^L) - \pi_i(\mathbf{P}(n))| \leq 2^{-(n+2)}\, k_i\, N_V\, t^{-1} \left(c_M + t\, l_M\right) \left(\Delta(c) + 3\, t\, l_M\right).$$

8. **Profit degree growth.** We say that a price strategy has the *profit degree growth* property if the profits of the firms increase with the degree of the nodes in the neighborhoods in which they are located. We give an example where the local market optimum price strategy does not have the profit degree growth property. Hence, we introduce the degree-bound condition, that is similar to the weak bounded condition, and we prove, under this condition, that the local market optimum price strategy has the profit degree growth property. Hence, we introduce the *degree-bound DB* condition that gives a new bound for $\Delta(c)$ and $\Delta(l)$. We prove that for a Hotelling town network satisfying the DB condition the local market optimum price strategy $\mathbf{P}^L$ has the profit degree growth property.

We say that a price strategy $\mathbf{P}$ has the *profit degree growth* property if

$$k_i > k_j \Rightarrow \pi_i(\mathbf{P}, \mathbf{C}) > \pi_j(\mathbf{P}, \mathbf{C})$$

for every $i, j \in V$.

Lemma 8.1. *Let F_i be a firm located in a node of degree k_i and F_j a firm located in a node of degree k_j. Then, $\pi_i^L > \pi_j^L$ if, and only if,*

$$\frac{k_i - k_j}{k_j} > \frac{(p_j^L - c_j)^2 - (p_i^L - c_i)^2}{(p_i^L - c_i)^2}.$$

Recall from subsection 2.5 the definition of heterogeneous network measures.

Definition 8.2. A Hotelling town network satisfies the *degree-bound lengths and costs (DB)* condition if

$$\Delta(c) + t\,\Delta(l) < t\,l_m \min\left\{1, \left(\sqrt{1 + 1/k_M} - 1\right)\left(1 - \frac{\Delta(c)}{2\,t\,l_m}\right)\right\}. \tag{8}$$

Theorem 8.3. *A Hotelling town network satisfying the DB condition has the profit degree growth property.*

9. **Examples: Homogeneous town models.** We show that a homogeneous Hotelling town satisfies the SB and DB conditions. We present an example of a network with a local market optimum price strategy that is not a Nash equilibrium price.

A Hotelling town has *homogeneous costs* if $c_m = c_M$. Hence, for a Hotelling town with homogeneous costs, the WB condition is reduced to

$$2\,k_M\,l_M\Delta(l) \le l_m^2;$$

the SB condition is reduced to

$$\Delta(l) \le \frac{l_m^2}{2\,k_M\,l_M};$$

and the DB condition is reduced to

$$\Delta(l) < l_m\left(\sqrt{1 + 1/k_M} - 1\right).$$

A Hotelling town has *homogeneous lengths* if $l_m = l_M$. Hence, for a Hotelling town with homogeneous lengths, the WB condition is reduced to

$$8\,k_M\Delta(c) \le t\,l_M;$$

and the SB condition is reduced to

$$4\,t\,l\,(2\,k_M + 1)\Delta(c) \le 4t^2\,l^2 + \Delta(c)^2;$$

Therefore, the condition

$$(2\,k_M + 1)\Delta(c) \leq t\,l.$$

implies the SB condition. The DB condition is reduced to

$$\left(\sqrt{1 + 1/k_M} + 1\right)\,\Delta(c) < 2\,\left(\sqrt{1 + 1/k_M} - 1\right)\,t\,l.$$

A Hotelling town is *homogeneous* if $c_m = c_M$ and $l_m = l_M$. For a homogeneous Hotelling town, the WB, SB and DB conditions are satisfied independently of the degree of the nodes. Hence, there is a Nash equilibrium price. Furthermore, the Nash equilibrium price satisfies the profit degree growth property.

We are going to present an example satisfying the WB condition but not the SB condition. Furthermore, we will show that in this example the local market optimum price strategy do not form a Nash equilibrium price. Consider the Hotelling town network presented in figure 1. The parameter values are $c_i = 0$, $l_m = 4$, $l_M = 7$,

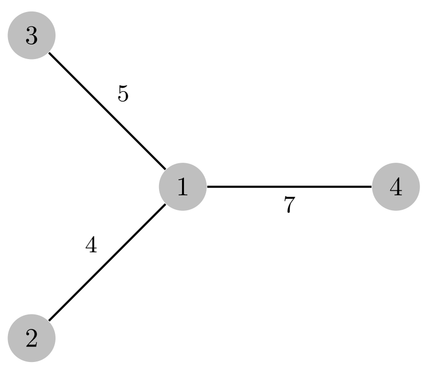

FIGURE 1. Star Network

$\Delta(l) = 3$ and $k_M = 3$. Hence, Network 1 satisfies the WB condition. By Theorem 4.3, the local market optimum prices and the correspondent profits are

$$\mathbf{P}^L = t\,\left(\frac{16}{3}, \frac{14}{3}, \frac{31}{6}, \frac{37}{6}\right);\ \pi^L = t\,\left(\frac{128}{3}, \frac{98}{9}, \frac{961}{72}, \frac{1369}{72}\right).$$

We will show that the local market optimum price strategy is not a Nash equilibrium. The profits of the firms are given by $\pi_i^L = p_i\,S(i, \mathbf{P}^L)$, and the local market sizes $S(i, \mathbf{P}^L)$ are

$$S(i, \mathbf{P}^L) = \frac{\pi_i^L}{p_i^L} = \frac{k_i\,p_i^L}{2\,t}$$

Hence, the local market sizes are

$$S(1, \mathbf{P}^L) = 8;\ S(2, \mathbf{P}^L) = \frac{14}{6};\ S(3, \mathbf{P}^L) = \frac{31}{12};\ S(4, \mathbf{P}^L) = \frac{37}{12}.$$

Suppose that firm F_2 decides to lower its price in order to capture the market of firm F_1. The firm F_2 captures the market of F_1, excluding F_1 from the game, if the firm F_2 charges a price p_2 such that $p_2 + 4\,t < p_1^L$ or, equivalently $p_2 < 4/3\,t$. Let us consider $p_2 = 4/3\,t - \epsilon$, where ϵ is sufficiently small. Hence, for this new price, firm F_2 keeps the market $M(2, \mathbf{P}^L)$ and, since the price of F_2 at location of F_1 is less that p_1^L, firm F_2 gains at least the market of firm F_1. Thus, the new market $M(2,\mathbf{P})$ of firm F_2 is such that $S(2,\mathbf{P}) > S(1,\mathbf{P}^L) + S(2,\mathbf{P}^L)$. Therefore, $S(2,\mathbf{P}) > 31/3$ and so

$$\pi_2 > p_2\, S(2,\mathbf{P}) = \left(\frac{4}{3}\,t - \epsilon\right)\frac{31}{3} = \frac{124}{9}\,t - \frac{31}{3}\,\epsilon.$$

Thus $\pi_2 > 98\,t/9 = \pi_2^L$, and so firm F_2 prefers to alter its price p_2^L. Therefore, $\mathbf{P}^L$ is not a Nash equilibrium price.

We are going to present an example satisfying the WB condition but not the DB condition. Furthermore, we will show that this example does not have the profit degree growth property. Consider the Hotelling town network presented in Figure 2.

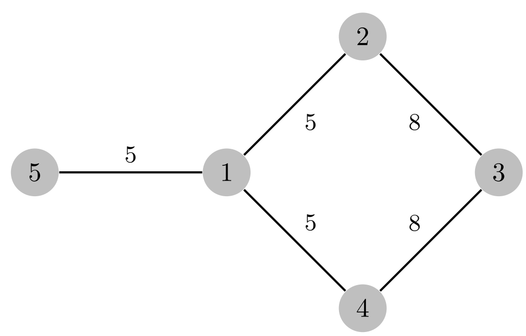

FIGURE 2. Network not satisfying the DB condition

The parameter values are $\epsilon_i = 0$, $c_i = 0$, $l_m = 5$, $l_M = 8$, $\Delta(l) = 3$ and $k_M = 3$.

Hence, the network 2 satisfies the WB condition. Thus, by Proposition 10.3, there is a local optimum price strategy P^L. The profits valued at the local optimal prices are given by

$$\pi^L = t\left(\frac{48387}{1058}, \frac{21904}{529}, \frac{27556}{529}, \frac{21904}{529}, \frac{14641}{1058}\right).$$

We observe that $k_1 > k_3$ and $\pi_3^L > \pi_1^L$. Hence, the profit degree growth property is not satisfied and so the DB condition does not hold.

10. **Firms at nodes neighborhoods.** In this section, we extend all the results to the general case where the firms are not necessarily located at the nodes, but can choose their locations in the neighbourhoods of the nodes. Furthermore, we give the proofs of all the results presented through the paper.

For every $v \in V$, let $\epsilon_v = d(v, y_v)$ and $j(v)$ be the node with the property that y_v is at the road $R_{v,j(v)}$. The *shift location matrix* $\mathbf{S}(v)$ associated to node v is defined by

$$s_{i,j}(v) = \begin{cases} \epsilon_v & \text{if } i = v \text{ and } j \in N_v \setminus \{j(v)\}\ ; \\ -\epsilon_v & \text{if } i = v \text{ and } j = j(v)\ ; \\ \epsilon_v & \text{if } j = v \text{ and } i \in N_v \setminus \{j(v)\}\ ; \\ -\epsilon_v & \text{if } j = v \text{ and } i = j(v)\ ; \\ 0 & \text{otherwise.} \end{cases}$$

The distance $\tilde{l}_{i,j} = d(y_i, y_j)$ between the location of firms F_i and F_j is given by

$$\tilde{l}_{i,j} = l_{i,j} + \sum_{v \in \{i,j\}} s_{i,j}(v). \tag{9}$$

Let $\epsilon = \max_{v \in V} \epsilon_v$. Hence, for every $i, j \in V$ we have

$$l_{i,j} - 2\,\epsilon \leq \tilde{l}_{i,j} \leq l_{i,j} + 2\,\epsilon.$$

10.1. **Local best response price strategy.** Under a local market structure, for every road $R_{i,j}$ there is an *indifferent buyer* located at a distance

$$0 < x_{i,j} = (2\,t)^{-1}(p_j - p_i + t\,\tilde{l}_{i,j}) < \tilde{l}_{i,j} \tag{10}$$

of firm F_i. Thus, a price strategy $\mathbf{P}$ determines a local market structure if, and only if, $|p_i - p_j| < t\,\tilde{l}_{i,j}$ for every road $R_{i,j}$. Hence, if

$$|p_i - p_j| < t\,l_{i,j} - 2\,t\,\epsilon \tag{11}$$

then condition (10) is satisfied. Therefore, if condition (11) holds then the price strategy $\mathbf{P}$ determines a local market structure.

If the price strategy determines a local market structure then

$$S(i, \mathbf{P}) = \sum_{j \in N_i} x_{i,j} - (k_i - 2)\,\epsilon_i$$

and

$$\begin{aligned} \pi_i(\mathbf{P}, \mathbf{C}) &= (p_i - c_i)\,S(i, \mathbf{P}) \\ &= (2\,t)^{-1}(p_i - c_i)\left(\sum_{j \in N_i}(p_j - p_i + t\,\tilde{l}_{i,j}) - 2\,t\,(k_i - 2)\,\epsilon_i\right). \end{aligned} \tag{12}$$

The Hotelling town *firm deviation* is the vector $\mathbf{Y}$ whose coordinates are

$$Y_i = k_i^{-1}\left(\sum_{j \in N_i} s_{i,j}(j) - \epsilon_i\,(k_i - 2)\right).$$

Lemma 10.1. *Let $\boldsymbol{P}$ and $\boldsymbol{P}^*$ be price strategies that determine local market structures. The price strategy $\boldsymbol{P}^*$ is the local best response to price strategy $\boldsymbol{P}$ if, and only if,*

$$\boldsymbol{P}^* = \frac{1}{2}\,(\boldsymbol{C} + t\,(\boldsymbol{L} + \boldsymbol{Y})) + \frac{1}{2}\,\boldsymbol{K}\,\boldsymbol{P} \tag{13}$$

and the price strategies $\tilde{\boldsymbol{P}}(i, \boldsymbol{P}, \boldsymbol{P}^*)$ determine local market structures for all $i \in V$.

Lemma 10.1 implies Lemma 3.1.

Proof of Lemma 10.1. By (12), the *profit function* $\pi_i(\mathbf{P}, \mathbf{C})$ of firm F_i, in a local market structure, is given by

$$\pi_i(\mathbf{P}, \mathbf{C}) = (2\,t)^{-1}(p_i - c_i)\left(\sum_{j \in N_i}(p_j - p_i + t\,\tilde{l}_{i,j}) - 2\,t\,(k_i - 2)\,\epsilon_i\right).$$

Let $\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*)$ be the price vector whose coordinates are $\tilde{p}_i = p_i^*$ and $\tilde{p}_j = p_j$, for every $j \in V \backslash \{i\}$. Since $\mathbf{P}$ and $\mathbf{P}^*$ are local price strategies, the local best response of firm F_i to the price strategy $\mathbf{P}$, is given by computing $\partial \pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*), \mathbf{C})/\partial \tilde{p}_i = 0$. Hence,

$$p_i^* = \frac{1}{2}\left(c_i - \frac{2\,t\,(k_i-2)}{k_i}\,\epsilon_i + \frac{1}{k_i}\sum_{j\in N_i} t\,\tilde{l}_{i,j} + p_j\right). \tag{14}$$

By (9), we obtain

$$p_i^* = \frac{1}{2}\left(c_i - \frac{2\,t\,(k_i-2)}{k_i}\,\epsilon_i + \frac{t}{k_i}\sum_{j\in N_i}\sum_{v\in\{i,j\}} s_{i,j}(v) + \frac{1}{k_i}\sum_{j\in N_i} t\,l_{i,j} + p_j\right).$$

We note that

$$\sum_{j\in N_i}\sum_{v\in\{i,j\}} s_{i,j}(v) = \sum_{j\in N_i} s_{i,j}(i) + \sum_{j\in N_i} s_{i,j}(j) = (k_i-2)\,\epsilon_i + \sum_{j\in N_i} s_{i,j}(j).$$

Hence,

$$p_i^* = \frac{1}{2}\left(c_i + \frac{t}{k_i}\left(\sum_{j\in N_i} s_{i,j}(j) - \epsilon_i(k_i-2)\right) + \frac{1}{k_i}\sum_{j\in N_i} t\,l_{i,j} + p_j\right).$$

Therefore, since $\partial^2 \pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}, \mathbf{P}^*), \mathbf{C})/\partial \tilde{p}_i^2 = -k_i/t < 0$, the local best response strategy prices $\mathbf{P}^*$ is given by

$$\mathbf{P}^* = \frac{1}{2}\,(\mathbf{C} + t\,(\mathbf{Y}+\mathbf{L}) + \mathbf{K}\,\mathbf{P})\,.$$

□

10.2. **Local market optimum price strategy.**

Definition 10.2. A Hotelling town satisfies the *weak bounded length and costs (WB)* condition, if

$$\Delta(c) + t\Delta(l) < t\,l_m - 6\,t\,\epsilon. \tag{15}$$

Hence, the WB condition implies $\epsilon < l_m/6$.

Let

$$\begin{aligned}\mathbf{P}^L &= \frac{1}{2}\left(\mathbf{1} - \frac{1}{2}\mathbf{K}\right)^{-1}(\mathbf{C} + t\,(\mathbf{L}+\mathbf{Y}))\\ &= \sum_{m=0}^{\infty} 2^{-(m+1)}\mathbf{K}^m\,(\mathbf{C} + t\,(\mathbf{L}+\mathbf{Y}))\,. \end{aligned} \tag{16}$$

Theorem 10.3. *If there is a local market optimum price strategy then it is the the candidate price strategy $\boldsymbol{P}^L$. Furthermore, there is a local market optimum price strategy if, and only if,*

$$|p_i^L - p_j^L| < t\,\tilde{l}_{i,j}$$

for all firms F_i and F_j that are neighbors. If the Hotelling town satisfies the WB condition, then the candidate price strategy is the unique local market optimum price strategy. Furthermore, the local market optimum prices p_i^L are uniformly bounded

$$t\,l_m + \frac{1}{2}\,(c_i + c_m) - 2\,t\,\epsilon \le p_i^L \le t\,l_M + \frac{1}{2}\,(c_i + c_M) + 2\,t\,\epsilon. \tag{17}$$

The local market optimum profit $\pi_i^L(\boldsymbol{P}, \boldsymbol{C})$ of firm F_i is given by

$$\pi_i^L(\boldsymbol{P}, \boldsymbol{C}) = (2t)^{-1}\,k_i\,(p_i^L - c_i)^2$$

and it is uniformly bounded

$$(8t)^{-1}\, k_i\, (2t\, l_m - \Delta(c) - 4\, t\, \epsilon)^2 \leq \pi_i^L(\boldsymbol{P}, \boldsymbol{C}) \leq (8t)^{-1}\, k_i\, (2\, t\, l_M + \Delta(c) + 4\, t\, \epsilon)^2.$$

Theorem 10.3 implies Theorem 4.1 and Theorem 4.3.

Proof of Theorem 10.3. The matrix $\mathbf{K}$ is a stochastic matrix (i.e., $\sum_{j\in V} k_{i,j} = 1$, for every $i \in V$) we have $\|\mathbf{K}\| = 1$. Hence, the matrix Q is well-defined by

$$\mathbf{Q} = \frac{1}{2}\left(\mathbf{1} - \frac{1}{2}\mathbf{K}\right)^{-1} = \sum_{m=0}^{\infty} 2^{-(m+1)}\, \mathbf{K}^m$$

and Q is also a non-negative and stochastic matrix. By Lemma 10.1, a local market optimum price strategy satisfies equality (13). Therefore,

$$\begin{aligned} \mathbf{P}^L &= \frac{1}{2}\left(\mathbf{1} - \frac{1}{2}\mathbf{K}\right)^{-1} (\mathbf{C} + t\,(\mathbf{L} + \mathbf{Y})) \\ &= \sum_{m=0}^{\infty} 2^{-(m+1)}\, \mathbf{K}^m\, (\mathbf{C} + t\,(\mathbf{L} + \mathbf{Y}))\,, \end{aligned} \tag{18}$$

and so $\mathbf{P}^L$ satisfies (16). By construction,

$$p_i^L = \sum_{v\in V} Q_{i,v}(c_v + t\,(L_v + Y_v)). \tag{19}$$

Let us prove that the price strategy $\mathbf{P}^L$ is local, i.e., the indifferent consumer $x_{i,j}$ satisfies $0 < x_{i,j} < \tilde{l}_{i,j}$ for every $R_{i,j} \in E$. We note that

$$l_m \leq L_v = k_v^{-1} \sum_{j\in N_v} l_{v,j} \leq l_M. \tag{20}$$

We note that

$$-k_v\, \epsilon \leq \sum_{j\in N_v} s_{v,j}(j) \leq k_v\, \epsilon$$

Hence, if $k_v = 1$ then

$$-\,\epsilon \leq -\epsilon + \epsilon_v \leq Y_v = k_v^{-1}\left(\sum_{j\in N_v} s_{v,j}(j) + \epsilon_v\right) \leq \epsilon + \epsilon_v \leq 2\,\epsilon; \tag{21}$$

if $k_v = 2$ then

$$-\,\epsilon \leq Y_v = k_v^{-1} \sum_{j\in N_v} s_{v,j}(j) \leq \epsilon; \tag{22}$$

and if $k_v \geq 3$ then

$$-\epsilon - \epsilon_v\, \frac{k_v - 2}{k_v} \leq Y_v = k_v^{-1}\left(\sum_{j\in N_v} s_{v,j}(j) - \epsilon_v(k_v - 2)\right) \leq \frac{1}{k_v}\,(k_v\,\epsilon - \epsilon_v(k_v - 2))\,.$$

Hence,

$$-\,2\,\epsilon \leq -\epsilon - \epsilon_v \leq Y_v = k_v^{-1}\left(\sum_{j\in N_v} s_{v,j}(j) - \epsilon_v(k_v - 2)\right) \leq \epsilon - \epsilon_v\, \frac{k_v - 2}{k_v} \leq \epsilon. \tag{23}$$

Therefore, from (21), (22) and (23), we have

$$-2\,\epsilon \le Y_v = k_v^{-1}\left(\sum_{j\in N_v} s_{v,j}(j) - \epsilon_v(k_v-2)\right) \le 2\,\epsilon. \tag{24}$$

Since $\mathbf{Q}$ is a nonnegative and stochastic matrix, we obtain

$$\sum_{v\in V} Q_{i,v}(c_m + t\,l_m - 2\,t\,\epsilon) = c_m + t\,l_m - 2\,t\,\epsilon$$

and

$$\sum_{v\in V} Q_{i,v}(c_M + t\,l_M + 2\,t\,\epsilon) = c_M + t\,l_M + 2\,t\,\epsilon.$$

Hence, putting (19), (20) and (24) together we obtain that

$$c_m + t\,l_m - 2\,t\,\epsilon \le p_i^L \le c_M + t\,l_M + 2\,t\,\epsilon.$$

Since the last relation is satisfied for every firm, we obtain

$$-\,(c_M - c_m + t(l_M - l_m) + 4\,t\,\epsilon) \le p_i^L - p_j^L \le c_M - c_m + t(l_M - l_m) + 4\,t\,\epsilon.$$

Therefore,

$$|p_i^L - p_j^L| \le \Delta(c) + t\,\Delta(l) + 4\,t\,\epsilon.$$

Hence, by the WB condition, we conclude that

$$|p_i^L - p_j^L| < t\,l_m - 2\,t\,\epsilon.$$

Thus, by equation (11), we obtain that the indifferent consumer is located at $0 < x_{i,j} < \tilde{l}_{i,j}$ for every road $R_{i,j} \in E$. Hence, the price strategy $\mathbf{P}^L$ is local and is the unique local market optimum price strategy.

From (19), we have that

$$p_i^L = \sum_{v\in V} Q_{i,v}\,(c_v + t\,(L_v + Y_v)).$$

From (20) and (24), we obtain

$$p_i^L \ge \sum_{v\in V} Q_{i,v}(t\,l_m - 2\,t\,\epsilon) + \sum_{v\in V\setminus\{i\}} Q_{i,v}\,c_m + Q_{i,i}\,c_i.$$

By construction of matrix $\mathbf{Q}$, we have $Q_{i,i} > 1/2$. Furthermore, since $\mathbf{Q}$ is stochastic,

$$\sum_{v\in V\setminus\{i\}} Q_{i,v} < 1/2,$$

$\sum_{v\in V} Q_{i,v} t\,l_m = t\,l_m$ and $\sum_{v\in V} Q_{i,v} 2\,t\,\epsilon = 2\,t\,\epsilon$. Hence,

$$p_i^L \ge t\,l_m - 2\,t\,\epsilon + \frac{1}{2}\,(c_i + c_m).$$

Similarly, we obtain

$$p_i^L \le t\,l_M + 2\,t\,\epsilon + \frac{1}{2}\,(c_i + c_M),$$

and so the local local market optimum prices p_i^L are uniformly bounded and satisfy (17).

We can write the the profit function (12) of firm F_i for the price strategy P^L as

$$\pi_i^L = \pi_i(\mathbf{P}^L, \mathbf{C}) = (2t)^{-1}(p_i^L - c_i)\left(-k_i\, p_i^L - 2\,t\,(k_i-2)\,\epsilon_i + \sum_{j\in N_i}(p_j^L + t\,\tilde{l}_{i,j})\right) \tag{25}$$

Since $\mathbf{P}^L$ satisfies the best response function (14), we have

$$2\,p_i^L = c_i - \frac{2\,t\,(k_i-2)}{k_i}\,\epsilon_i + \frac{1}{k_i}\sum_{j\in N_i}\left(t\,\tilde{l}_{i,j} + p_j^L\right).$$

Therefore, $\sum_{j\in N_i}\left(t\,\tilde{l}_{i,j} + p_j^L\right) = 2\,k_i\,p_i^L - k_i\,c_i + 2\,t\,(k_i-2)\,\epsilon_i$, and replacing this sum in the profit function (25), we obtain

$$\pi_i^L = (2t)^{-1}(p_i^L - c_i)\left(-k_i\,p_i^L + 2\,k_i\,p_i^L - k_i\,c_i\right) = (2t)^{-1}\,k_i\,(p_i^L - c_i)^2.$$

Hence, using the price bounds (17), we conclude

$$(2t)^{-1}\,k_i\,(t\,l_m - \Delta(c)/2 - 2\,t\,\epsilon)^2 \le \pi_i^L \le (2t)^{-1}\,k_i\,(t\,l_M + \Delta(c)/2 + 2\,t\,\epsilon)^2.$$

□

10.3. **Nash equilibrium price strategy.**

Lemma 10.4. *In a Hotelling town satisfying the WB condition, if there is a Nash price* $\boldsymbol{P}^*$ *then* $\boldsymbol{P}^*$ *is unique and* $\boldsymbol{P}^* = \boldsymbol{P}^L$.

Lemma 10.4 implies Lemma 5.1.

Proof of Lemma 10.4. Suppose that P^* is a Nash price strategy and that $\mathbf{P}^* \neq \mathbf{P}^L$. Hence, $\mathbf{P}^*$ does not determine a local market structure, i.e., there exists $i \in V$ such that

$$M(i, \mathbf{P}^*) \not\subset \cup_{j\in N_i} R_{i,j}.$$

Hence, there exists $j \in N_i$ such that $M(j, \mathbf{P}^*) = 0$ and, therefore, $\pi_j^* = 0$. Moreover, in this case, we have that

$$p_j^* > p_i^* + t\,\tilde{l}_{i,j}.$$

Consider, now, that F_j changes his price to $p_j = c_j + t\,\Delta(l) + 4\,t\,\epsilon$. Since $p_i^* > c_i$ and $c_j - c_i \le \Delta(c)$ we have that

$$p_j - p_i^* = c_j + t\,\Delta(l) + 4\,t\,\epsilon - p_i^* < c_j + t\,\Delta(l) + 4\,t\,\epsilon - c_i \le \Delta(c) + t\,\Delta(l) + 4\,t\,\epsilon$$

Since the Hotelling town satisfies the WB condition, $\Delta(c)+t\,\Delta(l)+4\,t\,\epsilon < t\,l_m - 2\,t\,\epsilon$, we have

$$p_j - p_i^* < t\,l_m - 2\,t\,\epsilon \le t\,l_{i,j} - 2\,t\,\epsilon \le t\,\tilde{l}_{i,j}.$$

Hence, $M(j, \tilde{\mathbf{P}}(j, \mathbf{P}^*, \mathbf{P})) > 0$ and $\pi_j = (c_j + t\,\Delta(l) + 4\,t\,\epsilon)\,S(j, \tilde{\mathbf{P}}(j, \mathbf{P}^*, \mathbf{P})) > 0$. Therefore, F_j will change its price and so $\mathbf{P}^*$ is not a Nash equilibrium price strategy. Hence, if there is a Nash price $\mathbf{P}^*$ then $\mathbf{P}^* = \mathbf{P}^L$. □

Lemma 10.5. *In a Hotelling town satisfying the WB condition,*

$$M(i, \tilde{\boldsymbol{P}}(i, \boldsymbol{P}^L, \boldsymbol{P}^{L,*})) \subset \mathcal{N}(i, 2)$$

for every $i \in V$.

Lemma 10.5 implies Lemma 5.2.

Proof of Lemma 10.5. By contradiction, let us consider a consumer $z \in M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}))$ and $z \notin \mathcal{N}(i,2)$. The price that consumer z pays to buy in firm F_i is given by

$$e = p_i + t\ (l_{i_1,i_2}(\epsilon) + l_{i_2,i_3}(\epsilon) + d\,(y_{i_3}, z)) \geq p_i + t\ (l_{i_1,i_2} + l_{i_2,i_3} - 2\,\epsilon + d\,(y_{i_3}, z))$$

where $p_i = p_i^{L,*}$ is the coordinate of the vector $\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})$ and for the 2-path $(R_{i_1,i_2}, R_{i_2,i_3})$ with $i_1 = i$. If the consumer z buys at firm F_{i_3}, then the price that has to pay is

$$\tilde{e} = p_{i_3}^L + t\,d\,(y_{i_3}, z).$$

Since, by hypothesis, $z \in M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}))$, we have $e < \tilde{e}$. Therefore

$$p_i < p_{i_3}^L - t\ (l_{i_1,i_2} + l_{i_2,i_3} - 2\,\epsilon)\,.$$

By (17), $p_i^L \leq t\,l_M + 2\,t\,\epsilon + \dfrac{1}{2}(c_i + c_M)$ for all $i \in V$. Since $l_{i,j} \geq l_m$ for all $R_{i,j} \in E$,

$$p_i < t\,l_M + \frac{1}{2}(c_M + c_{i_3}) - 2\,t\,l_m + 4\,t\,\epsilon \leq c_M + t\,\Delta(l) - t\,l_m + 4\,t\,\epsilon.$$

Furthermore,

$$p_i - c_i < \Delta(c) + t\,\Delta(l) - t\,l_m + 4\,t\,\epsilon.$$

By the WB condition, $p_i - c_i < 0$. Hence, $\pi_i^{L,*} < 0$ which contradicts the fact that p_i is the best response to $\mathbf{P}^L$ (since $\pi_i^L > 0$). Therefore, $z \in \mathcal{N}(i,2)$ and $M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})) \subset \mathcal{N}(i,2)$. □

For every firm F_i and every $0 < p < p_i^L$, let $\hat{N}_i(p) \subset N_i$ be the set of all $j \in N_i$ such that

$$|p - p_j| < t\,\tilde{l}_{i,j}.$$

Let

$$\hat{S}_i(p) = \sum_{j \in \hat{N}_i(p)} \frac{p_j - p + t\,\tilde{l}_{i,j}}{2\,t} + \sum_{j \in N \setminus \hat{N}_i(p)} \sum_{k \in N_j} \tilde{l}_{i,j} + \frac{p_k - p + t\,(\tilde{l}_{j,k} - \tilde{l}_{i,j})}{2\,t}.$$

Theorem 10.6. *The local market optimum price strategy $\boldsymbol{P}^L$ is a Nash equilibrium price if, and only if,*

$$\hat{S}_i(p)\,p \leq \pi_i^L$$

for every firm F_i and every $0 < p < p_i^L$.

Theorem 10.6 implies Theorem 5.3.

Proof. For every firm F_i and every $0 < p < p_i^L$, let

$$\hat{P}_i(p) = (p_1^L, \ldots, p_{i-1}^L, p, p_{i+1}^L, \ldots p_N^L).$$

Let $\overline{N}_i(p) \subset N_i$ be the set of all $j \in N_i$ such that

$$|p - p_j| = t\,\tilde{l}_{i,j}.$$

Let

$$\begin{aligned} \overline{S}_i(p) \quad &= \sum_{j \in \hat{N}_i(p)} \frac{p_j - p + t\,\tilde{l}_{i,j}}{2\,t} + \sum_{j \in \overline{N}_i(p)} \sum_{k \in N_j} \frac{\tilde{l}_{i,j}}{2} + \frac{p_k - p + t\,(\tilde{l}_{j,k} - \tilde{l}_{i,j})}{4\,t} \\ &+ \sum_{j \in N \setminus (\hat{N}_i(p) \cup \overline{N}_i(p))} \sum_{k \in N_j} \tilde{l}_{i,j} + \frac{p_k - p + t\,(\tilde{l}_{j,k} - \tilde{l}_{i,j})}{2\,t}. \end{aligned}$$

By Lemma 10.5 and by construction of $\hat{S}_i(p)$, we obtain

$$S(i, \hat{P}_i(p)) = \overline{S}_i(p) \leq \hat{S}_i(p).$$

Hence,

$$\pi_i(\hat{P}_i(p)) = p\, S(i, \hat{P}_i(p)) = p\, \overline{S}_i(p) \leq p\, \hat{S}_i(p) \leq \pi_i^L.$$

□

Definition 10.7. A Hotelling town satisfies the *strong bounded length and costs (SB)* condition, if

$$\Delta(c) + t\Delta(l) \leq \frac{(2\,t\,l_m - \Delta(c) - 4\,t\,\epsilon)^2}{8\,t\,k_M\,(l_M + \epsilon)} - 3\,t\,\epsilon. \tag{26}$$

Theorem 10.8. *If a Hotelling town satisfies the SB condition then there is a unique Hotelling town Nash equilibrium price strategy* $\boldsymbol{P}^* = \boldsymbol{P}^L$.

Theorem 10.8 implies Theorem 5.5.

Proof of Theorem 10.8. By Theorem 10.3 and Lemma 10.4, if there is a Nash equilibrium price strategy $\mathbf{P}^*$ then $\mathbf{P}^*$ is unique and $\mathbf{P}^* = \mathbf{P}^L$.

We note that if $M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})) \subset \mathcal{N}(i, 1)$ for every $i \in V$ then $\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}) = p_i^L$ and so $\mathbf{P}^L$ is a Nash equilibrium.

By Lemma 10.5, we have that $M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})) \subset \mathcal{N}(i, 2)$ for every $i \in V$. Now, we will prove that condition (26) implies that firm F_i earns more competing only in the 1-neighborhood than competing in a 2-neighborhood.

By Lemma 10.5,

$$\begin{aligned}\pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})) &\leq (p_i - c_i) \sum_{j \in N_i} \left(\tilde{l}_{i,j} + \sum_{k \in N_j \setminus \{i\}} l_{j,k}(\epsilon) \right) \\ &\leq (p_i - c_i) \sum_{j \in N_i} \sum_{k \in N_j} l_{j,k}(\epsilon),\end{aligned}$$

where $p_i = p_i^{L,*}$ is the coordinate of the vector $\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})$. Hence,

$$\pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})) \leq (p_i - c_i) \sum_{j \in N_i} \sum_{k \in N_j} (l_{j,k} + \epsilon) \leq (p_i - c_i) k_i\, k_M\,(l_M + \epsilon). \tag{27}$$

By contradiction, let us consider a consumer $z \in M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}))$ and $z \notin \mathcal{N}(i, 1)$. Let $i_2 \in N_i$ be the vertex such that $z \in \mathcal{N}(i_2, i)$. The price that consumer z pays to buy in firm F_i is given by

$$e = p_i + t\,l_{i,i_2}(\epsilon) + t\,d\,(y_{i_2}, z) \geq p_i + t\,l_{i,i_2} + t\,d\,(y_{i_2}, z) - t\,\epsilon.$$

If the consumer y buys at firm F_{i_2}, then the price that has to pay is

$$\tilde{e} = p_{i_2}^L + t\,d\,(y_{i_2}, z).$$

Since, by hypothesis, $z \in M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}))$, we have $e < \tilde{e}$. Therefore

$$p_i < p_{i_2}^L - t\,l_{i,i_2} + t\,\epsilon.$$

By (17), $p_{i_2}^L \leq t\,l_M + 2\,t\,\epsilon + \dfrac{1}{2}(c_M + c_{i_2})$. Since $l_{i,i_2} \geq l_m$, we have

$$p_i < t\,l_M + \frac{1}{2}(c_M + c_{i_2}) + 2\,t\,\epsilon - t\,l_m + t\,\epsilon \leq c_M + t\,\Delta(l) + 3\,t\,\epsilon.$$

Thus,

$$p_i - c_i < \Delta(c) + t\,\Delta(l) + 3\,t\,\epsilon.$$

Hence, from (27) we obtain

$$\pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})) < k_i\, k_M\, (l_M + \epsilon)\, (\Delta(c) + t\,\Delta(l) + 3\,t\,\epsilon)\,.$$

By the SB condition,

$$\pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})) < (2\,t)^{-1}\, k_i\, (t\, l_m - \Delta(c)/2 - 2\,t\,\epsilon)^2. \tag{28}$$

By Theorem 10.3 and (28),

$$\pi_i^L \geq (2\,t)^{-1}\, k_i\, \left(t\, l_m - \Delta(c)/2 - 2\,t\,\epsilon\right)^2 > \pi_i(\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})),$$

which contradicts the fact that p_i is the best response to $\mathbf{P}^L$. Therefore, $z \in \mathcal{N}(i,1)$ and $M(i, \tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*})) \subset \mathcal{N}(i,1)$. Hence, $\tilde{\mathbf{P}}(i, \mathbf{P}^L, \mathbf{P}^{L,*}) = p_i^L$ and so $\mathbf{P}^L$ is a Nash equilibrium. □

10.4. **Firm position stability.** For every firm F_i, with $k_i = 2$, let $v(i) \in N_i \backslash \{j(i)\}$ be the neighboring vertex of i that is different from $j(i)$. Let us denote

$$U_i = \frac{Q_{i,v(i)}}{k_{v(i)}} - \frac{Q_{i,j(i)}}{k_{j(i)}}.$$

Theorem 10.9. *The marginal rate of the price of a firm F_i with respect to the deviation of the localization of the firm from the node is given by*

$$\begin{aligned} \partial p_i^L / \partial \epsilon_i &= t \left(Q_{i,i}\, \partial Y_i / \partial \epsilon_i + \sum_{j \in N_i} Q_{i,j}\, \partial Y_j / \partial \epsilon_i \right) \\ &= t \left(Q_{i,i}\, \frac{2 - k_i}{k_i} - \frac{2\, Q_{i,j(i)}}{k_{j(i)}} + \sum_{j \in N_i} \frac{Q_{i,j}}{k_j} \right). \end{aligned}$$

The marginal rate of the profit of a firm F_i with respect to the deviation of the localization of the firm from the node is given by

$$\partial \pi_i^L / \partial \epsilon_i = \frac{k_i\, (p_i^L - c_i)}{t} \cdot \partial p_i^L / \partial \epsilon_i.$$

Furthermore,

(i) Case $k_i \geq 1$. Then $\partial \pi_i^L / \partial \epsilon_i > 0$.
(ii) Case $k_i = 2$. Let $j = i(j)$ and v the other adjacent node of i.
If $U_i > 0$ then $\partial \pi_i^L / \partial \epsilon_i > 0$; if $U_i < 0$ then $\partial \pi_i^L / \partial \epsilon_i < 0$; and if $U_i = 0$ then $\partial \pi_i^L / \partial \epsilon_i = 0$.
(iii) Case $k_i \geq 3$ and $k_v \geq 3$, for every $v \in N_i$. Then $\partial \pi_i^L / \partial \epsilon_i < 0$.
(iv) Case $k_i \geq 4$ and $k_v \geq 2$, for every $v \in N_i$. Then $\partial \pi_i^L / \partial \epsilon_i < 0$.

Theorem 10.9 implies Theorem 6.2.

Proof of Theorem 10.9. From Theorem 10.8, we have

$$p_i^L = \sum_{v \in V} Q_{i,v} (c_v + t\, L_v + t\, Y_v), \tag{29}$$

and

$$\pi_i^L = (2\,t)^{-1}\, k_i\, (p_i^L - c_i)^2.$$

Hence,

$$\partial \pi_i^L / \partial \epsilon_i = \frac{k_i\,(p_i^L - c_i)}{t} \cdot \partial p_i^L / \partial \epsilon_i.$$

Hence, to study the influence of ϵ_i in the profit π_i^L, we only have to study the signal of $\partial p_i^L / \partial \epsilon_i$. By (29), we have

$$\partial p_i^L / \partial \epsilon_i = \sum_{v \in V} \partial p_i^L / \partial Y_v \cdot \partial Y_v / \partial \epsilon_i.$$

Since, for every $v \in V$, $\partial p_i^L / \partial Y_v = t\,Q_{i,v}$, we have

$$\partial p_i^L / \partial \epsilon_i = t \sum_{v \in V} Q_{i,v}\, \partial Y_v / \partial \epsilon_i.$$

Recall that

$$Y_v = \frac{1}{k_v} \left(\sum_{j \in N_v} s_{v,j}(j) - \epsilon_v\,(k_v - 2) \right)$$

Hence, for $v = i$, we have

$$\partial Y_i / \partial \epsilon_i = \frac{2 - k_i}{k_i};$$

for $v \in N_i$, we have

$$\partial Y_v / \partial \epsilon_i = \partial / \partial \epsilon_i \left(\frac{1}{k_v} s_{v,i}(i) \right) = \pm \frac{1}{k_v};$$

and for $v \notin N_i$, we have $\partial Y_i / \partial \epsilon_i = 0$. Therefore,

$$\partial p_i^L / \partial \epsilon_i = t \left(Q_{i,i}\, \frac{2 - k_i}{k_i} - \frac{2\, Q_{i,j(i)}}{k_{j(i)}} + \sum_{j \in N_i} \frac{Q_{i,j}}{k_j} \right)$$

If $k_i = 1$, then

$$\partial p_i^L / \partial \epsilon_i = t\, Q_{i,i} > 0.$$

If $k_i = 2$, then

$$\partial p_i^L / \partial \epsilon_i = t \left(\frac{Q_{i,v(i)}}{k_{v(i)}} - \frac{Q_{i,j(i)}}{k_{j(i)}} \right) = t\, U_i.$$

If $k_i \geq 3$, then

$$\partial p_i^L / \partial \epsilon_i \leq t \left(Q_{i,i}\, \frac{2 - k_i}{k_i} + \sum_{j \in N_i} Q_{i,j}\, \frac{1}{k_j} \right)$$

By construction, $Q_{i,i} > 1/2$ and $\sum_{j \in N_i} Q_{i,j} < 1/2$. Hence, if $k_v \geq 3$, for every $v \in N_i$, then

$$\partial p_i^L / \partial \epsilon_i < t \left(\frac{-1}{6} + \frac{1}{6} \right) = 0.$$

Furthermore, if $k_i \geq 4$ and $k_v \geq 2$, for every $v \in N_i$, then

$$\partial p_i^L / \partial \epsilon_i < t \left(\frac{-1}{4} + \frac{1}{4} \right) = 0.$$

□

10.5. **Space bounded information.**

Definition 10.10. A Hotelling town has *n-space bounded information (n-I)* if for every $1 \le m \le n$, for every firm F_i and for every non-empty set $\mathcal{R}(i,j;m)$: (i) firm F_i knows the cost c_j and the average length road L_j and the firm deviation Y_j of firm F_j; (ii) for every m path $R \in \mathcal{R}(i,j;m)$, firm F_i knows the corresponding weight $k(R)$.

The *n-local market optimum price vector* is

$$\mathbf{P}(n) = \sum_{m=0}^{n} 2^{-(m+1)} \mathbf{K}^m \left(\mathbf{C} + t\left(\mathbf{L} + \mathbf{Y} \right) \right).$$

We observe that in a n-I Hotelling town, the firms might not be able to compute $\mathbf{K}$, $\mathbf{C}$, $\mathbf{L}$ or $\mathbf{Y}$. However, every firm F_i is able to compute his n local market optimum price $p_i(n)$

$$p_i(n) = \sum_{m=0}^{n} 2^{-(m+1)} \sum_{v \in V} k_{i,v}^m \left(c_v + t\left(L_v + Y_v \right) \right).$$

Theorem 10.11. *A Hotelling town satisfying the WB condition has a local market optimum price strategy $\boldsymbol{P}^L$ that is well approximated by the n local market optimum price $\boldsymbol{P}(n)$ with the following 2^{-n} bound*

$$0 \le p_i^L - p_i(n) \le 2^{-(n+1)} N_V (c_M + t\,(l_M + 2\,\epsilon)).$$

Furthermore, $\boldsymbol{P}(n+1)$ is the best response to $\boldsymbol{P}(n)$ for n sufficiently high.

Theorem 10.11 implies Theorem 7.2.

Proof of Theorem 10.11. By Theorem 10.3, if a Hotelling town satisfies the WB condition then there is local market optimum price strategy $\mathbf{P}^L$ given by

$$\mathbf{P}^L = \sum_{m=0}^{\infty} 2^{-(m+1)} \mathbf{K}^m \left(\mathbf{C} + t\left(\mathbf{L} + \mathbf{Y} \right) \right).$$

Considering $\mathbf{Q} = \sum_{m=0}^{\infty} 2^{-(m+1)} \mathbf{K}^m$, we can write the equilibrium prices as

$$p_i^L = \sum_{v \in V} Q_{i,v} \left(c_v + t\left(L_v + Y_v \right) \right), \text{ where } Q_{i,v} = \sum_{m=0}^{\infty} 2^{-(m+1)} k_{i,v}^m.$$

For the space bounded information Hotelling town, the n local market optimum price $\mathbf{P}(n)$ is given by

$$\mathbf{P}(n) = \sum_{m=0}^{n} 2^{-(m+1)} \mathbf{K}^m \left(\mathbf{C} + t\left(\mathbf{L} + \mathbf{Y} \right) \right)$$

and

$$p_i(n) = \sum_{v \in V} Q_{i,v}(n) \left(c_v + t\left(L_v + Y_v \right) \right), \text{ where } Q_{i,v}(n) = \sum_{m=0}^{n} 2^{-(m+1)} k_{i,v}^m.$$

The difference $R_i(n)$ between p_i^L and $p_i(n)$ is positive and is given by

$$R_i(n) = \sum_{v \in V} \left(Q_{i,v} - Q_{i,v}(n) \right) \left(c_v + t\left(L_v + Y_v \right) \right).$$

We note that

$$Q_{i,v} - Q_{i,v}(n) = \sum_{m=n+1}^{\infty} 2^{-(m+1)} k_{i,v}^m.$$

Since $0 \le k_{i,v}^m \le 1$, for all $m \in \mathbb{N}$ and all $i, v \in V$ and $\sum_{m=n+1}^{\infty} 2^{-(m+1)} = 2^{-(n+1)}$, we have that

$$Q_{i,v} - Q_{i,v}(n) \le 2^{-(n+1)}.$$

Hence,

$$R_i(n) \le \sum_{v \in V} 2^{-(n+1)} \left(c_v + t\,(L_v + Y_v)\right).$$

Since $L_v \le l_M$, $Y_v \le 2\,\epsilon$ and $c_v \le c_M$, we have that

$$R_i(n) \le 2^{-(n+1)}\, N_V\, \left(c_M + t\,(l_M + 2\,\epsilon)\right). \tag{30}$$

Therefore,

$$0 \le p_i^L - p_i(n) \le 2^{-(n+1)}\, N_V\, \left(c_M + t\,(l_M + 2\,\epsilon)\right).$$

By (13), the best response $\mathbf{P}'$ to $\mathbf{P}(n)$ is given by

$$\begin{aligned} \mathbf{P}' &= \frac{1}{2}\,(\mathbf{C} + t\,(\mathbf{L} + \mathbf{Y})) + \frac{1}{2}\,\mathbf{K}\,\mathbf{P}(n) \\ &= \frac{1}{2}\,(\mathbf{C} + t\,(\mathbf{L} + \mathbf{Y})) + \sum_{m=0}^{n} 2^{-(m+2)} \mathbf{K}^{m+1}\,(\mathbf{C} + t\,(\mathbf{L} + \mathbf{Y})) \\ &= \sum_{m=0}^{n+1} 2^{-(m+1)} \mathbf{K}^m\,(\mathbf{C} + t\,(\mathbf{L} + \mathbf{Y})) = \mathbf{P}(n+1). \end{aligned}$$

□

10.6. **Profit degree growth.** Let F_i be a firm located in a node of degree k_i and F_j a firm located in a node of degree k_j. Let $\bar{p}_i = p_i^L - c_i$ and $\bar{p}_j = p_j^L - c_j$ represent the unit profit of firm F_i and F_j, respectively.

Let $\theta(p) = p_i^L - p_j^L$, $\theta(c) = c_i - c_j$, $\theta(k) = k_i - k_j$ and $\theta(\bar{p}) = \bar{p}_i - \bar{p}_j = \theta(p) - \theta(c)$.

Lemma 10.12. *Given the local market optimum price strategy $\boldsymbol{P}^L$, $\pi_i^L > \pi_j^L$ if and only if*

$$\frac{k_i - k_j}{k_j} > \frac{\bar{p}_j^2 - \bar{p}_i^2}{\bar{p}_i^2}.$$

Lemma 10.12 implies Lemma 8.1.

Proof of Lemma 10.12. If F_j is a firm located in a node of degree k_j, then

$$\pi_j^L = (2t)^{-1}\, k_j\, (p_j^L - c_j)^2 = (2t)^{-1}\, k_j\, \bar{p}_j^2.$$

Similarly, if F_i is a firm located in a node of degree k_i, then

$$\pi_i^L = (2t)^{-1}\, k_i\, (p_i^L - c_i)^2 = (2t)^{-1}\, k_i\, \bar{p}_i^2 = (2t)^{-1}\, (k_j + \theta(k))\, \left(\bar{p}_j + \theta(\bar{p})\right)^2.$$

Hence,

$$\begin{aligned} 2\,t\,\pi_i^L &= k_j\,\bar{p}_j^2 + k_j\,\theta(\bar{p})\,(2\,\bar{p}_j + \theta(\bar{p})) + \theta(k)\,(\bar{p}_j + \theta(\bar{p}))^2 \\ &= 2\,t\,\pi_j^L + k_j\,\theta(\bar{p})\,(\bar{p}_j + \bar{p}_i) + \theta(k)\,\bar{p}_i^2, \end{aligned}$$

and so

$$2\,t\,(\pi_i^L - \pi_j^L) = k_j\,(\bar{p}_i - \bar{p}_j)\,(\bar{p}_j + \bar{p}_i) + \theta(k)\,\bar{p}_i^2 = k_j\,(\bar{p}_i^2 - \bar{p}_j^2) + (k_i - k_j)\,\bar{p}_i^2.$$

Therefore,

$$\pi_i^L > \pi_j^L \text{ if, and only if, } \frac{k_i - k_j}{k_j} > \frac{\bar{p}_j^2 - \bar{p}_i^2}{\bar{p}_i^2}.$$

□

Definition 10.13. A Hotelling town network satisfies the *degree-bound lengths and costs (DB)* condition if

$$\Delta(c) + t\,\Delta(l) < \left(\sqrt{1 + 1/k_M} - 1\right) (t\,l_m - \Delta(c)/2 - 2\,t\,\epsilon) - 4\,t\,\epsilon. \tag{31}$$

Theorem 10.14. *A Hotelling town network satisfying the WB and DB conditions has the profit degree growth property.*

Theorem 10.14 implies Theorem 8.3.

Proof of Theorem 10.14. Let F_i and F_j be firms in the Hotelling town network such that $k_i > k_j$. We need to prove that $\pi_i^L > \pi_j^L$. From Lemma 10.12 we say that $\pi_i^L > \pi_j^L$, if and only, if

$$k_j\,\theta(\bar{p})\,(\bar{p}_j + \bar{p}_i) + \theta(k)\,\bar{p}_i^2 > 0. \tag{32}$$

Since $k_i > k_j$, then $\theta(k) > 0$. Hence, if $\theta(\bar{p}) > 0$, i.e. $\bar{p}_i > \bar{p}_j$, then condition (32) is satisfied.

Let us now consider the case where $\theta(\bar{p}) < 0$. Condition (32) is equivalent to

$$k_j\,\theta(\bar{p})^2 - 2\,k_i\,\bar{p}_i\,\theta(\bar{p}) - \theta(k)\,\bar{p}_i^2 < 0. \tag{33}$$

Solving the second degree equation $k_j\,\theta(\bar{p})^2 - 2\,k_i\,\bar{p}_i\,\theta(\bar{p}) - \theta(k)\,\bar{p}_i^2 = 0$, we obtain

$$\theta(\bar{p})_\pm = \bar{p}_i \left(1 \pm \sqrt{1 + \theta(k)/k_j}\right).$$

Let $f(\theta(k), k_j)$ be the function given by

$$f(\theta(k), k_j) = \sqrt{1 + \theta(k)/k_j} - 1.$$

We note that $f(\theta(k), k_j) > 0$ and $\theta(\bar{p})_- = -f(\theta(k), k_j)\,\bar{p}_i$. If $\theta(\bar{p})_- < \theta(\bar{p}) < 0$ then condition (33) is satisfied. By hypothesis $\theta(\bar{p}) < 0$ and, so, if

$$f(\theta(k), k_j)\,\bar{p}_i > -\theta(\bar{p}) \tag{34}$$

then (33) is satisfied.

Since $\theta(\bar{p}) = \bar{p}_i - \bar{p}_j$, from (17) we have $|\theta(\bar{p})| < \Delta(c) + t\,\Delta(l) + 4\,t\,\epsilon$. Hence, if

$$f(\theta(k), k_j)\,\bar{p}_i > \Delta(c) + t\,\Delta(l) + 4\,t\,\epsilon \tag{35}$$

then (34) is satisfied. Noting that $f(\theta(k), k_j) > f(1, k_M) = \sqrt{1 + 1/k_M} - 1$, if

$$\Delta(c) + t\,\Delta(l) + 4\,t\,\epsilon < \left(\sqrt{1 + 1/k_M} - 1\right)\bar{p}_i \tag{36}$$

then (35) is satisfied. By (17), we have $\bar{p}_i \geq t\,l_m - \Delta(c)/2 - 2\,t\,\epsilon$. Hence, if

$$\Delta(c) + t\,\Delta(l) + 4\,t\,\epsilon < \left(\sqrt{1 + 1/k_M} - 1\right) (t\,l_m - \Delta(c)/2 - 2\,t\,\epsilon) \tag{37}$$

then (36) is satisfied. Hence, if condition (37) is satisfied, then (32) is satisfied, $\pi_j^L > \pi_i^L$ for every firms F_i and F_j such that $k_j > k_j$, and, so, the network has the profit degree growth property. □

11. **Conclusion.** Under the weak bounded costs condition, we proved that the price subgame has a unique local market optimum price strategy. We gave an explicit closed formula and an explicit series expansion formula for the local market optimum price strategy. We showed that the influence of a firm in the local market optimum price strategy of other firm decreases exponentially fast with the distance between the firms. We showed that each firm is able to compute an approximation of its own local market optimum price strategy that improves exponentially fast with the space bounded information knowledge of the firm. Under the strong bounded condition, we proved that the local market optimum price strategy is a Nash equilibrium price strategy. We proved that the firms prefer to be located at the crossroads of the network but they do not prefer to be located at the ends of no-exit roads.

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