

Assurance Contracts in Threshold Public Goods Provision with Incomplete Information

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Abstract

This paper studies private provision of a discrete public good using provision point mechanism by assuming the social planner has the ability to set up an “assurance contract” in case of provision failure. Our research is motivated by the positive prospects of dominant assurance contract (Tabarrok, 1998) to address the increasing popularity of using voluntary contribution mechanisms to fund public projects. We modify the existing assurance contract format with the inclusion of a minimum price (MP) and an assurance payment (AP), where an individual will obtain a compensation which equals to the assurance payment if she is willing to contribute above the minimum price in case of a provision failure. We analyze the Bayesian Nash equilibrium for a two-player public goods provision game allowing continuous bids under the assurance contract as well as a N-player public goods game allowing each player to either accept or reject an assurance contract. We show that using an assurance contract, a threshold public good may be provided with an arbitrarily high *ex-ante* provision probability while the social planner still receive a positive expected profit.

Keywords: Public Good, Assurance Contract, Provision Point Mechanism, Private Provision, Bayesian Nash Equilibrium

Keywords: Q56, Q57, C72

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1 Introduction

In public good provision problems, individuals' free-riding incentives constantly challenge the social planner's ability to fund public projects using decentralized market mechanisms. Intuitively, the provision failure is a result of individuals' dependence on others to bear the cost while enjoying the benefit when the public good is provided by the group. This paper explores the possibility of using assurance contract to counter individuals' free riding behaviors. The assurance contract changes the contribution behaviors by compensating significant contributors upon a provision failure and creates incentives for individuals to raise contribution to be eligible for an assurance payment, and eventually leads to a higher probability of provision success.

Our framework is based on the threshold public good provision game (Rondeau et al., 1999, 2005) where a minimum total of contributions is needed to provide one unit of a public good. Compared to the linear public good game where always contributing zero is the equilibrium strategy (Andreoni, 1995; Bernheim, 1986; Isaac and Walker, 1988), in the threshold public good game, the minimum total contribution required to reach the provision cost introduces an incentive for individuals to contribute to the public good with the efficient outcome being a part of feasible equilibrium outcomes. Bagnoli and Lipman (1989) prove that the non-provision outcome cannot be eliminated unless a stronger refinement is used. Different rebate mechanisms are proposed to reduce the non-provision outcomes (Marks and Croson, 1998; Spencer et al., 2009; Liu et al., 2016; Li et al., 2016), however, most of the rebate rules, such as the proportional rebate rule where excess contributions returned to individuals in proportion of their contributions, do not alter the efficient equilibrium strategy set.

Our paper is motivated by the dominant assurance contract introduced by Tabarrok (1998), under which the non-provision equilibria are successfully eliminated in a complete information setting. When individuals are faced with a dichotomous choice, i.e., contributing versus not contributing, contributing to the public good becomes a dominant strategy if those who agreed to contribute can be compensated with a positive benefit in situations where the public good is not provided. With the dominant assurance contract, the pure free-riding equilibrium is eliminated due to the existence of assurance contract and not contributing is no longer a best response. The assurance contract encourages commitments to pay for public good provision by offering compensation to individuals who commit to making a donation, and paying that compensation when the group fails to provide the public good (and no donations are collected). Recently, Zubrickas (2014) proposes a "rebate bonus" mechanism where individuals will be compensated proportion-

ally to their contribution upon *provision failure*, and the rebate bonus mechanism achieves Lindahl allocation (Lindahl, 1919) with complete information.

In our paper, we generalize the dominant assurance contract to a threshold public good environment by investigating individuals' equilibrium strategies under incomplete information. In order to be eligible for the assurance payment upon provision failures, individuals need to contribute at least the minimum price (MP); we also alter the level of assurance payment (AP), the compensation that eligible individuals receive upon a provision failure. We allow continuous contributions and specify the equilibrium bidding strategy for a two-players game and allow dichotomous choice¹ (i.e., accept or reject the assurance contract) for a group of arbitrary size under incomplete information.

Different from experiments using mechanisms penalizing free or cheap riders (Masclot et al., 2003; Bracht et al., 2008; Denant-Boemont et al., 2007) the assurance contract mechanism encourages a higher contribution by rewarding those who are committed to share the public good cost, which is in line of the spirits reflected in Tabarrok (1998) and Zubrickas (2014). Also, we are exploring the idea of increasing the expected share of consumers' surplus (or the expected contributors' surplus) to overcome the free or cheap riding incentives. Under the assurance contract, a producer (the social planner or the market maker) compensates the contributors if 1) the contributor entered a binding agreement to pay a significant, specified amount (or more) in support of the public good provision and 2) the public good is, nevertheless, not provided. As a consequence, the individuals' expected share of consumer surplus is higher than the provision point mechanism where consumer-donors realize zero benefits if the public good is not provided. Note that different rebate mechanisms, such as the proportional rebate redistributes all the excess contribution back to the contributors and thus leave the producer exactly zero surplus. Our assurance contract may either result in a positive or negative (or zero) producer surplus depending on the model parameters.

The funding for public project has been an important topic in public finance as government usually rely on tax (e.g., proportional or stepwise wage-income tax) to fund public projects that usually benefit a subgroup of population. The recently emerged and fast-growing crowd funding industry seeks potential possibilities to cover or partially offset the R&D cost from the general public. For example, the Kickstarter² and gofundme³ crowd

¹It is analytical difficult to derive the individual equilibrium contribution strategy for a group size larger than two in a standard threshold public good contribution without the assurance contract. Our specifications further complicate the currently unsolved problem and thus we restrict our attention to the discrete equilibrium strategies for a group size of N .

²<https://www.kickstarter.com>.

³<http://www.gofundme.com>.

funding websites gathered \$470 million and \$444 million of private donation (or pledges) in the year of 2014. Particularly, the Kickstarter specifies the “all-or-nothing” rule where “no one will be charged for a pledge towards a project unless it reaches its funding goal,” which is effectively using the provision point mechanism that provides a threshold public good (i.e., the project) while most of the projects listed carry some public good properties (such as the development of a new computer game). People with a similar preference constitute the donation base and decide the group size for the provision of a particular project. Our results using the assurance contract can be helpful for crowdfunding companies to test new rules in order to support more publicly valued projects.

Our analyses on the two-player threshold public good game with assurance contract reveal a discontinuous equilibrium bidding strategy that contrasts the continuous Bayesian Nash equilibrium strategy without the assurance contract (Alboth et al., 2001; Laussel and Palfrey, 2003; Barbieri and Malweg, 2008). Particularly, we find that the maximum contribution never exceeds the minimum price (minimum contribution required to be eligible for assurance contract in case of provision failure) and the equilibrium bidding strategy is closely linked to the specified minimum price in relation to the provision cost. Our analyses on with group size N is related to the work by Tabarrok (1998) but distinguish from the early work by considering the assurance contract using provision point mechanism with a more flexible compensation scheme. We devote a significant portion of discussion on the difference of allocative efficiency change due to the presence of assurance contract. We also explore the implications when the social planner (which will be used as a general term for the market maker or entrepreneur in case of the crowdfunder industry) can gain a private benefit from the successful provision of public good, which is compared to the extreme, less realistic scenario where the social planner has zero benefits from a successful provision. Our results suggest that by using different combinations of minimum price and assurance payment, the social planner has a large capacity and flexibility to achieve a higher provision probability in even large groups, at the cost of a smaller, even negative, producer surplus in cases where the social planner has a small private value from the successful provision.

The rest of the paper is organized as follows. Section 2 describes the assurance contract and analyzes the symmetric Bayesian Nash equilibrium for a 2-player public good provision allowing continuous contributions. Section 3 applies the assurance contract to N -player and characterizes the equilibrium outcome allowing each player to either accept or reject an assurance contract. Section 4 discusses the results and concludes the paper.

2 The Model, Two Players

A provision point mechanism is employed by a social planner to provide a discrete public good with a predetermined cost c . There are $N \in \mathcal{N} = \{2, \dots, n\}$ individuals who potentially contribute to the public good. Individual private value v follows an i.i.d. draw from a commonly known distribution function $F(\cdot)$ on the support $[\underline{v}, \bar{v}]$. We consider a mechanism (a.k.a “assurance contract”) where the social planner is able to compensate individuals whose contributions surpass a certain amount in case of a provision failure. The assurance contract is characterized by a contingent payment scheme (α, β) such that an individual contributes at least the minimum price (MP) α . If the public good is not provided, contribution will be returned and the individual will be compensated by an assurance payment (AP) β . Note that we maintain the assumptions that individuals’ induced values are private while the value distribution $F(\cdot)$, the cost c , the number of potential contributors N as well as the payment scheme (α, β) are common knowledge. Additionally, we assume $0 < \alpha \leq \bar{v}$ and $\bar{v} < c < 2\bar{v}$ to rule out the possibility that the public good can be provided with only one individual.

Let b_i be the contribution of individual i whose induced value is v_i . Then her expected payoff is given by

$$(v_i - b_i) \Pr \left(\sum_{j=1}^N b_j \geq c \right) + \mathbf{1}(b_i \geq \alpha) \beta \Pr \left(\sum_{j=1}^N b_j < c \right). \quad (1)$$

Given the setting and information structure specified above, the public good provision constitutes a game of incomplete information. Individual i ’s contributing strategy, denoted by $s_i(\cdot)$, is a mapping from the private value to contribution.

Definition 1. *A profile of contributing strategies $s^* = (s_i^*(\cdot), s_{-i}^*(\cdot))$ is a pure strategy Bayesian-Nash Equilibrium if for all $i \in \mathcal{N}$ and for all $v_i \in [\underline{v}, \bar{v}]$, we have that*

$$s_i^*(v_i) \in \arg \max_{0 \leq s'_i \leq c} \left\{ (v_i - s'_i) \Pr (s'_i + s_{-i}^*(v_{-i}) \geq c) + \mathbf{1}(s'_i \geq \alpha) \beta \Pr (s'_i + s_{-i}^*(v_{-i}) < c) \right\}, \quad (2)$$

Without loss of generality, let $\underline{v} = 0, \bar{v} = 1$. To simplify our analysis but still keep the game interesting, we first consider the assurance contracts with two players, i.e, $N = 2$ where we provide a Bayesian-Nash equilibrium and then analyze the probability of a successful provision as well as the expected payoff of the social planner in the equilibrium. We focus our analysis on the symmetric equilibria $(s_i, s_{-i}) \rightarrow (s, s)$ and assume the assurance payment is set to equal the minimum price, $\alpha = \beta$ (we use α in the following exposition) for now.

2.1 Individuals' Contributing Strategy

Before analyzing individuals' equilibrium contributing strategies, we introduce several important properties of the strategies defined in (2). If individual i 's contribution is given to be s'_i then at equilibrium the probability of providing the public good successfully depends on the value distribution $f(\cdot)$. Explicitly, the probability is written as

$$\Pr(s'_i + s^*_{-i}(v_{-i}) \geq c) = \int_{v_{-i}: s'_i + s^*_{-i}(v_{-i}) \geq c} f(v_{-i}) dv_{-i}.$$

Lemma 1. *Assume s_1 and s_2 are the equilibrium contribution strategies, then $s_i(v) \leq \max\{\alpha, 1 - \alpha\}$, $\forall \alpha \in (0, 1)$ and $i = 1, 2$*

Proof. See Appendix. □

Lemma 2. *In the Bayesian-Nash Equilibria defined in (2), the equilibrium strategy $s_i(v)$ is generally a nondecreasing function of v_i if $N = 2$.*

Proof. See Appendix. □

Lemma 3. *In the equilibrium, when $\alpha < \frac{1}{2}$, no one contributes less than α .*

Proof. See Appendix. □

According to Lemma 2, the highest contribution is $s(\bar{v})$. For the convenience of exposition, we introduce a threshold v^* such that

$$v^* \equiv \min \{v \in [\underline{v}, \bar{v}] : s(v) + s(\bar{v}) \geq c\}.$$

If there does not exist a v^* satisfies the above definition, then the public good will not be provided at all when individuals follow the strategy $s(\cdot)$. Suppose v^* exists, then the public good is provided with positive probability if and only if $v_i > v^*$, whereas the provision probability is zero if $v_i \leq v^*$. Below we further assume the distribution of induced values $F(\cdot)$ is a standard uniform distribution.

When the assurance contract is not available, Laussel and Palfrey (2003) characterize the equilibrium conditions for a two-player threshold public good provision game under incomplete information using a set of differential equations; Barbieri and Malueg (2008) derive the linear equilibrium strategy under boarder circumstances. Particularly, under our setting (uniformly distributed induced values on $[0, 1]$ and $N = 2$) the symmetric Bayesian Nash equilibrium strategy is

$$s(v) = \begin{cases} v & \text{for } v \leq \frac{1}{3} \\ \frac{1}{6} + \frac{1}{2}v & \text{for } v > \frac{1}{3}. \end{cases} \quad (3)$$

In our analysis, we will compare the equilibrium strategy under assurance contract with the strategy above in terms of provision probability and surplus allocation between contributors and the social planner.

To solve the equilibrium strategy with the assurance contract, first note that if the public good is provided with zero probability at the equilibrium, then $s(v) = \alpha$ and individuals would just obtain the assurance payment. In fact, as we will prove later, an equilibrium strategy is $s(v) = \alpha, \forall v \in [0, 1]$ when $\alpha < \frac{1}{2}$, which is consistent with Lemma 3.

When $\alpha \geq \frac{1}{2}$, the public good maybe provided with some positive probability. If so, there exists a $v^* \in (0, 1)$ such that $s(v^*) + s(1) \geq 1$. In this case, note that $s(v^*) \leq \alpha$ according to Lemma 1. Assume otherwise $s(v^*) > \alpha$, consider the strategy for the player when $v = 1$, the expected profit of contributing $s(1) = \alpha$ dominates the strategy $s'(1) > \alpha$, because the provision probability is $1 - v^*$ under both strategies while the profit is strictly less when the public good is provided. Note that the player when $v = v^*$ will not change between $s(1)$ and $s'(1)$ since the probability of provision is unaffected. Therefore $s(1) = \alpha$, which violates the monotonicity result from Lemma 2.

Similarly, we can also demonstrate that $s(1) = \alpha$ whenever the public good is provided with some positive probability (Lemma 1). Intuitively, this is because when $s(1)$ increases beyond α will only adversely affect one's profit upon provision but the probability of provision is unchanged in the equilibrium. Based on the above characterizations of the equilibrium behaviors, we propose the equilibrium strategy below and prove that our strategy constitute the Bayesian Nash equilibrium.

Proposition 1. (i). When $\alpha < \frac{1}{2}$, an equilibrium bidding strategy is $s(v) = \alpha$. (ii). When $\alpha = \frac{1}{2}$, an equilibrium bidding strategy is

$$s(v) = \begin{cases} v & \text{for } v \leq 1 - \frac{\sqrt{2}}{2}; \\ \frac{1}{2} & \text{for } v > 1 - \frac{\sqrt{2}}{2}. \end{cases} \quad (4)$$

(iii) When $\alpha \in (\frac{1}{2}, \frac{2}{3})$, an equilibrium bidding strategy is

$$s(v) = \begin{cases} v & \text{for } v \leq \frac{1}{2}(2\alpha + 1 - \sqrt{4\alpha^2 + 1}) \\ \alpha & \text{for } v > \frac{1}{2}(2\alpha + 1 - \sqrt{4\alpha^2 + 1}), \end{cases} \quad (5)$$

(iv) When $\alpha \geq \frac{2}{3}$, the equilibrium bidding strategy is

$$s(v) = \begin{cases} v & \text{for } v < 1 - \alpha \\ 1 - \alpha & \text{for } v \in [1 - \alpha, 1 - \alpha + \sqrt{3\alpha^2 - 2\alpha}] \\ \alpha & \text{for } v > 1 - \alpha + \sqrt{3\alpha^2 - 2\alpha} \end{cases} \quad (6)$$

Note that among the equilibrium strategies represented in equations (4), (5) and (6), the strategy $s(v) = v$ in the low value range can be generalized to $s(v) = b, b \in [0, 1 - \alpha)$ and $s(v)$ satisfies the monotonic constraint. In the low value range $v < v^*$, the probability of provision is zero in equilibrium, thus, the bidding choice does not influence the equilibrium outcome as long as $b < 1 - \alpha$.

Figure 1 illustrates the equilibrium bidding strategy when the minimum price MP belongs to different ranges, as well as the liner equilibrium bidding strategy without the assurance contract. The equilibrium strategies suggest three possible equilibrium outcomes if we merge $MP = \frac{1}{2}$ and $MP \in (\frac{1}{2}, \frac{2}{3})$. When $MP < \frac{1}{2}$, both players will bid exactly α and the public good will be provided with zero probability. When $MP \in [\frac{1}{2}, \frac{2}{3})$, players with relative low values ($v_i \in [0, \frac{1}{2}(2\alpha + 1 - \sqrt{4\alpha^2 + 1})$) will bid their value and the expected provision probability and equilibrium profit are zero, while players with relative higher values ($v_i \in (\frac{1}{2}(2\alpha + 1 - \sqrt{4\alpha^2 + 1}), 1]$) will bid α and claim the assurance payment upon provision failure, thus receive a positive expected profit. When $MP > \frac{2}{3}$, a new bidding strategy emerges as players with value in the middle range bid $1 - \alpha$. If the player's value is higher than $1 - \alpha$, bidding $1 - \alpha$ is strictly better than bidding below $1 - \alpha$ since the provision probability is now positive and one can gain some surplus from provision, however, unless one's value reaches a threshold, in our case $v^{**} = 1 - \alpha + \sqrt{3\alpha^2 - 2\alpha}$, the expected benefit of bidding α and be eligible for the assurance payment upon provision failure is higher than the expected loss of paying an amount higher than one's value upon provision success.

The threshold value $v^* = \frac{1}{2}(2\alpha + 1 - \sqrt{4\alpha^2 + 1})$ in equilibrium strategy represented in equation (4) and (5) is calculated by setting the player v^* receives the same expected profit from bidding α and bidding v ; the threshold values $v^* = 1 - \alpha$ and $v^{**} = 1 - \alpha + \sqrt{3\alpha^2 - 2\alpha}$ in equilibrium strategy represented in equation (6) is calculated by setting the player v^* receives the same expected profit from bidding $1 - \alpha$ and bidding v , while at the same time, the player with v^{**} receives the same expected profit from bidding $1 - \alpha$ and bidding α . Based on the equilibrium bidding strategy, we investigate the provision probability and the surplus allocation between the social planner and the players under assurance contract.

2.2 The Allocation of Realized Social Surplus

A new feature of a threshold public good provision with assurance contract is that the social planner may need to secure fund for the potential assurance payment. Thus it is important to understand the determinants of the social planner's expected payoff from the provision. Individual equilibrium bidding strategy outlined in Proposition 1 enables

us to analyze the provision probability the expected payoff of the individuals as well as the social planner. The results are summarized as follows.

Proposition 2. *When $\alpha < \frac{1}{2}$, the provision probability is zero. The expected profit of the individuals and the social planner are α and -2α , respectively.*

When $MP = \frac{1}{2}$, the provision probability $P = \frac{1}{2}$. The player's expected profit

$$\pi(v) = \begin{cases} 0 & \text{for } v \leq 1 - \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2}(v - 1) + \frac{1}{2} & \text{for } v > 1 - \frac{\sqrt{2}}{2}, \end{cases} \quad (7)$$

the social planner's expected profit is $\pi_s = \frac{2-\sqrt{2}}{2} > 0$.

When $MP \in (\frac{1}{2}, \frac{2}{3})$, the provision probability $P \in (\frac{4}{9}, \frac{1}{2})$, which decreases in α . The player's expected profit is

$$\pi(v) = \begin{cases} 0 & \text{for } v \leq v^* \\ (1 - v^*)v + (2v^* - 1)\alpha & \text{for } v > v^*, \end{cases} \quad (8)$$

the social planner's expected profit is $\pi_s = 2\alpha(1 - v^)(1 - 2v^*) \in (\frac{2-\sqrt{2}}{2}, \frac{8}{27})$, which is increasing as MP increases, where $v^* = \frac{1}{2}(2\alpha + 1 - \sqrt{4\alpha^2 + 1})$.*

*When $MP \geq \frac{2}{3}$, the provision probability $P = (1 - v^{**})^2 + 2(1 - v^{**})(v^{**} - v^*) \in [0, \frac{4}{9}]$, which decreases as MP increases. The player's expected profit is*

$$\pi(v) = \begin{cases} 0 & \text{for } v < v^* \\ (1 - v^{**})(v - v^*) & \text{for } v \in [v^*, v^{**}] \\ (1 - v^*)v + (2v^* - 1)\alpha & \text{for } v > v^{**} \end{cases} \quad (9)$$

*the social planner's expected profit is $\pi_s = (1 - v^{**})(2\alpha^2 - v^* + (1 - 2\alpha)v^{**}) \in [0, \frac{8}{27}]$, which is decreasing as MP increases, where $v^* = 1 - \alpha$ and $v^{**} = 1 - \alpha + \sqrt{3\alpha^2 - 2\alpha}$.*

The player's expected profit is expected profit upon provision plus the expected profit upon provision failure, which is always positive if the assurance contract is applicable. The social planner's expected profit is calculated as

$$\pi_s = \sum_i b_i P - \alpha \sum_i p'_i, \quad (10)$$

where P is probability of success, p'_i is the probability that player i is eligible to receive the assurance payment and the expected profit equals the expected revenue ($\sum_i b_i P$) minus the expected assurance payment.

The above profit function does not subtract the cost for providing the public good if we are willing to assume that the social planner can also benefit from the provision

of the public good through decentralized contributions instead of supporting the public good from other sources, such as distortionary tax, or in case of crowdfunding where the entrepreneur holds a positive value from the successful provision of the project. Note that we can also specify the social planner receives a fixed profit (S) from provision of the public good, in this case, the social planner's expected profit becomes:

$$\pi_s = \left(\sum_i b_i + S - c \right) P - \alpha P', \quad (11)$$

thus, equation (10) is a special case of equation (11) where $S = c$. Proposition 2 uses equation (10) to calculate the social planner's expected profit. Proposition 2 also suggests that the public good is provided with the highest probability $\frac{1}{2}$ when $MP = \frac{1}{2}$ while the social planner receives the highest expected profit when $MP = \frac{2}{3}$, which can vary depends on the assumption we impose on the social planner's benefit from providing the public good.

When the assurance contract is not available, according to the linear equilibrium bidding strategy, the public good is provided when $s(v_1) + s(v_2) = \frac{1}{2}v_1 + \frac{1}{6} + \frac{1}{2}v_2 + \frac{1}{6} \geq 1$, $v_i \geq \frac{1}{3}$. Note that v_i follows a uniform distribution on $[0, 1]$. Therefore, $P(v_1 + v_2 \geq \frac{4}{3}, v_1 \geq \frac{1}{3}, v_2 \geq \frac{1}{3}) = P(v_1 + v_2 \geq \frac{4}{3})$. Let $v_s = v_1 + v_2$, then the probability distribution $f(v_s) = v_s$ if $v_s \in [0, 1]$ and $f(v_s) = 2 - v_s$ if $v_s \in [1, 2]$, which leads to the provision probability $P(v_s \geq \frac{4}{3}) = \frac{2}{9}$.

When the assurance payment is available, and particularly, when $MP = \frac{1}{2}$, the provision probability is $\frac{1}{2}$, which increases by 125% compared when there is no assurance payment. Furthermore, we find that as long as $MP \in [\frac{1}{2}, \frac{1}{2} + \frac{\sqrt{5}}{6})$, the assurance payment will increase the provision probability compared to the equilibrium outcome where such assurance payment scheme is not available.

The social planner's expected profit without assurance contract is $\pi_s = \int_{v_s \in [\frac{2}{3}, 1]} v_s f(v_s) = \frac{28}{81}$. When $MP = \frac{1}{2}$, the social planner's expected profit about 15% smaller (approximate 0.05 in absolute term) compared to the situation without assurance contract. In this case, the assurance contract increases the provision probability significantly while only moderately reduces the social planner's expected profit. If the social planner's value from providing the public good S is higher than c , it is possible that the assurance contract increases the both the provision probability and the social planner's expected profit. To see this, when $MP = \frac{1}{2}$, $\pi_s = (\sum_i b_i + S - c)P - \alpha P' = \frac{1}{2}S + \frac{1}{2} - \frac{\sqrt{2}}{2}$, while without assurance contract, $\pi_s = \int_{v_s \in [\frac{2}{3}, 1]} (v_s + S - 1)f(v_s) = \frac{28}{81} + \frac{2}{9}(S - 1)$. Thus, when the social planner's value for the public good $S \geq \frac{9\sqrt{2}}{5} - \frac{61}{45} \approx 1.19$, the assurance contract increases both expected provision probability and the social planner's expected profit. Note that when $S > c = 1$, the social planner has the incentive to provide the public good all by

herself. However, it is possible that a social planner or an entrepreneur has a budget or liquidity constraint so that the project can only be provided using private contributions.

In terms of player's expected profit, the assurance contract always yields a higher expected profit at $MP = \frac{1}{2}$. Specifically, when $v \in [1 - \frac{\sqrt{2}}{2}, \frac{1}{3})$, the difference in expected profit is $\Delta\pi(v) = \frac{\sqrt{2}}{2}(v-1) + \frac{1}{2}$ and when $v \in [\frac{1}{3}, 1]$, $\Delta\pi(v) = \frac{\sqrt{2}}{2}(v-1) + \frac{1}{2} - (\frac{1}{2}v - \frac{1}{6})(v - \frac{1}{3})$. Thus, the assurance contract weakly increases the expected profit for players of all values.

Efficiency The social efficiency is evaluated from an ex-post perspective and a socially efficient mechanism provides the public good whenever the sum of realized values are higher than (or equal to) the cost. When assurance contract is not available⁴ and according to the strategy in equation (3), if in the equilibrium, the players can provide the public good, it must be socially desirable to provide the public good since each player contributes lower than their values. Thus, the realized social surplus from the equilibrium when $S = c$ is,

$$R_s = \int_{v_1 \in [\frac{1}{3}, 1]} \int_{v_2 \in [s^{-1}(c-s(v_1)), 1]} (v_1 + v_2 + S - 1) dv_2 dv_1,$$

or

$$R_s = \int_{\frac{1}{3}}^1 \int_{\frac{4}{3}-v_1}^1 (v_1 + v_2 + S - 1) dv_2 dv_1 = \frac{28}{81}.$$

The potential maximum social surplus is

$$M_s = \int_0^1 \int_{1-v_1}^1 (v_1 + v_2 + S - 1) dv_2 dv_1 = \frac{2}{3}.$$

Therefore, the equilibrium strategy can realize about 24% potential social surplus without the assurance contract.

When the assurance contract is available, it is possible the public good will be provided when the sum of players' value is smaller than the cost, however, after considering the benefit to the social planners, the benefit from providing the public good still outweigh the provision cost. Thus,

$$R_s = \int_{1-\frac{\sqrt{2}}{2}}^1 \int_{1-\frac{\sqrt{2}}{2}}^1 (v_1 + v_2 + S - 1) dv_1 dv_2 = 1 - \frac{1}{2\sqrt{2}},$$

which is approximately 97% of the total social surplus. When $MP \in (\frac{1}{2}, \frac{2}{3})$, the social surplus can be calculated by

$$R_s = \int_{\frac{1}{2}(2\alpha+1-\sqrt{4\alpha^2+1})}^1 \int_{\frac{1}{2}(2\alpha+1-\sqrt{4\alpha^2+1})}^1 (v_1 + v_2 + S - 1) dv_1 dv_2 \in (\frac{16}{27}, 1 - \frac{1}{2\sqrt{2}}),$$

⁴For comparison purpose, we assume players will bid v when they are indifferent between a contribution in $[0, v]$ with and without assurance contract.

which ranges from 89% to 97% of the total social surplus and decreases as α increases. When $MP \in [\frac{2}{3}, 1]$, the realized social surplus can be calculated by

$$2 \int_{1-\alpha+\sqrt{3\alpha^2-2\alpha}}^1 \int_{1-\alpha}^{1-\alpha+\sqrt{3\alpha^2-2\alpha}} (v_1 + v_2 + S - 1) dv_1 dv_2 \\ + \int_{1-\alpha+\sqrt{3\alpha^2-2\alpha}}^1 \int_{1-\alpha+\sqrt{3\alpha^2-2\alpha}}^1 (v_1 + v_2 + S - 1) dv_1 dv_2,$$

which belongs to $[0, \frac{16}{27}]$ and decreases as the α increases. Thus, the social efficiency ranges from 0% to 89% in this situation. In summary, the above results show that the realized social surplus (or the social efficiency) is highest when $MP = \alpha = \frac{1}{2}$, while the social planner's expected profit is highest when $MP = \alpha = \frac{2}{3}$.

The extreme case when $S = 0$ It is obvious that a benevolent social planner prefers to provide the public good as it constitutes a Pareto improvement as long as the sum of individual values are higher than the provision cost, and prefers to provide the public good through cost-sharing among individuals who benefit from the public good rather than through transfer payment such as tax dollars raised from other sectors. While it is largely an empirical question what's the true magnitude of S , we analyze the extreme case where the social planner derives zero utility from providing the public good by setting $S = 0$. Compared with Proposition 2, when $\alpha < \frac{1}{2}$, the social planner's expected profit is still -2α ; when $MP = \frac{1}{2}$, the social planner's expected profit is $\pi_s = \frac{1-\sqrt{2}}{2} < 0$; when $MP \in (\frac{1}{2}, \frac{2}{3})$, the social planner's expected profit is $\pi_s = -2\alpha v^*(1 - v^*) < 0$, where $v^* = \frac{1}{2}(2\alpha+1-\sqrt{4\alpha^2+1})$; when $MP \geq \frac{2}{3}$, the provision probability $P = 2\alpha(1-\alpha) \in [0, \frac{4}{9}]$, which decreases as MP increases. The player's expected profit is $(\alpha - \sqrt{3\alpha^2 - 2\alpha})(4\alpha^2 - 3\alpha - 2\alpha\sqrt{3\alpha^2 - 2\alpha})$, which is always smaller than 0 when $MP \in [\frac{2}{3}, 1]$. The consideration of the extreme case indicates the use of assurance payment may always lead to a negative expected profit for the social planner if the social planner cannot benefit from the provision of the public good, which is very unlikely as the social planner is assumed to care about the overall social welfare and distortionary tax will surely introduce some inefficiency. This is less a problem for crowd fundraising companies since the public contribution to jump start a project has the potential to bring in a huge future profit, i.e., S is large, where the assurance payment may increase both the success probability and the social planner's expected profit, at least in the restricted circumstance we considered above.

The realized social surplus from the equilibrium when $S = 0$ is,

$$R_s = \int_{\frac{1}{3}}^1 \int_{\frac{4}{3}-v_1}^1 (v_1 + v_2 - 1) dv_2 dv_1 = \frac{181}{1296}.$$

The potential maximum social surplus is

$$M_s = \int_0^1 \int_{1-v_1}^1 (v_1 + v_2 - 1) dv_2 dv_1 = \frac{1}{6}.$$

Therefore, the equilibrium strategy can realize about 83.80% potential social surplus without the assurance contract.

When the assurance contract is available, it is possible the public good will be provided when the sum of players' value is smaller than the cost, which introduces inefficiency to the economy. In this case,

$$R_s = \int_{1-\frac{\sqrt{2}}{2}}^1 \int_{1-\frac{\sqrt{2}}{2}}^1 (v_1 + v_2 - 1) dv_1 dv_2 = \frac{1}{4}(2 - \sqrt{2}),$$

which is approximately 87.87% of the total social surplus. The assurance contract increases the social efficiency even in the extreme case and such efficiency gain becomes larger when the social planner has a higher value S from public good provision, as the provision probability is increased substantially with assurance contract.

When $MP \in (\frac{1}{2}, \frac{2}{3})$, the social surplus can be calculated by

$$R_s = \int_{\frac{1}{2}(2\alpha+1-\sqrt{4\alpha^2+1})}^1 \int_{\frac{1}{2}(2\alpha+1-\sqrt{4\alpha^2+1})}^1 (v_1 + v_2 - 1) dv_1 dv_2 \in (\frac{1}{4}(2 - \sqrt{2}), \frac{4}{27}),$$

which ranges from 87.87% to 88.89% of the total social surplus and increases as α increases.

When $MP \in [\frac{2}{3}, 1]$, the social surplus can be calculated by

$$2 \int_{1-\alpha+\sqrt{3\alpha^2-2\alpha}}^1 \int_{1-\alpha}^{1-\alpha+\sqrt{3\alpha^2-2\alpha}} (v_1 + v_2 - 1) dv_1 dv_2 + \int_{1-\alpha+\sqrt{3\alpha^2-2\alpha}}^1 \int_{1-\alpha+\sqrt{3\alpha^2-2\alpha}}^1 (v_1 + v_2 - 1) dv_1 dv_2,$$

which belongs to $[0, \frac{4}{27}]$ and decreases as the α increases. Thus, the social efficiency ranges from 0% to 88.89% in this situation. Different from above when $S = c$, the realized social surplus (or the social efficiency) is highest when $MP = \alpha = \frac{2}{3}$.

3 Model Extensions

Section 2 analyzes the assurance contract in a two-player setting in great details. One interesting observation is that compared with no assurance contract (cf. Alboth et al., 2001; Laussel and Palfrey, 2003; Barbieri and Malueg, 2008) where the bidding function depends on one's realized value, our equilibrium strategy depends on the level of the minimum price and the assurance payment and is discontinuous at certain value thresholds.

When we generalize the assurance contract to N players, it is almost impossible to derive the equilibrium bidding strategy allowing for continuous bids. Therefore, we restrict each player's choice to $s(v) = 0$, which means reject the assurance contract and $s(v) = \alpha$, which means accept the assurance contract. We first set $MP = \frac{c}{N}$ and analyze provision probability of the public good with N players and then generalize the results from a more flexibility assurance contract.

3.1 N Players

In the case of N players, let the set $\mathcal{N} = \{1, 2, \dots, N\}$ denote the individuals who may contribute to the public good, with the same value support $[\underline{v}, \bar{v}]$, following a commonly known distribution $F(\cdot)$. We consider an assurance contract where the social planner can compensate individuals who accept to contribute the minimum price ($MP = \alpha$) case of provision failure. Each individual's strategy space is limited to either accept ($s(v) = MP$) or reject ($s(v) = 0$). Thus, the assurance contract is a contingent payment scheme (MP, AP) such that an individual either accept the contract and pay the minimum price (MP) and if the public good is not provided, then her contribution will be returned and she will be compensated by an amount, assurance payment ($AP = \beta$). Let s_i denote the decision of individual i whose induced value is v_i . Then her expected payoff is given by

$$(v_i - s_i) \Pr \left(\sum_{j=1}^N s_j \geq c \right) + \mathbf{1}(s_i = \alpha) \beta \Pr \left(\sum_{j=1}^N s_j < c \right). \quad (12)$$

Individual i 's strategy is a mapping from her private value to her contribution given all other individuals' strategies, denoted by $s_i(\cdot)$.

Definition 2. *A profile of strategies $s^* = (s_1^*(\cdot), s_2^*(\cdot), \dots, s_N^*(\cdot))$ is a pure strategy Bayesian-Nash Equilibrium if for all $i \in \mathcal{N}$ and for all $v_i \in [\underline{v}, \bar{v}]$, we have that*

$$s_i^*(v_i) \in \arg \max_{s'_i \in (0, \alpha)} \left\{ (v_i - s'_i) \Pr (s'_i + s_{-i}^*(v_{-i}) \geq c) + \mathbf{1}(s'_i = \alpha) \beta \Pr (s'_i + s_{-i}^*(v_{-i}) < c) \right\}, \quad (13)$$

Similar to the 2-player case, given a fixed bidding strategy $s_{-i}^*(\cdot)$, the probability of providing the public good depends on the value distribution $f(v_{-i})$ for player $-i$ and thus can be calculated by:

$$\Pr (s'_i + s_{-i}^*(v_{-i}) \geq c) = \int_{v_{-i}: s'_i + s_{-i}^*(v_{-i}) \geq c} f(v_{-i}) dv_{-i}.$$

We still focus on the symmetric equilibria $(s_i, s_1, \dots, s_{i-1}, s_{i+1}, \dots) \rightarrow (s, \dots, s)$. Consider the case where $\underline{v} = 0, \bar{v} = 1, v$ follows a uniform distribution on $[0, 1]$ and $c = \alpha N$ with $N \geq 2$ players, $\alpha \in (\frac{1}{N}, 1)$. Each player's choice is to either accept the contract or reject the contract.

Proposition 3. *When $MP = \alpha$,*

$$s(v) = \begin{cases} 0 & \text{for } v < \tilde{v} \\ \alpha & \text{for } v \geq \tilde{v}, \end{cases}$$

where $(\tilde{v} - 2\alpha)(1 - \tilde{v})^{N-1} + \alpha = 0$. In addition, there exists one and only one $\tilde{v} \in (0, a)$.

Proposition 3 implies that players with value above the threshold \tilde{v} will accept the assurance contract and agrees to contribute α while players with value below \tilde{v} will reject the assurance contract. Realistically, the cost c for the public good can either be constant or changes as N changes. Denote the sequence of \tilde{v}_n as $\{\tilde{v}_n\}_{n=1}^{\infty}$ and the sequence of corresponding provision probability as $\{P_n\}_{n=1}^{\infty}$ where $P_n = (1 - \tilde{v}_n)^n$. We show that when 1) the cost increases proportionally as N changes (α is constant) and 2) the cost is fixed, the provision probability approaches to $\frac{1}{2}$ as $N \rightarrow \infty$ in both cases.

Proposition 4. *When α is fixed and $MP = \alpha$, $\lim_{n \rightarrow \infty} P_n = \frac{1}{2}$; when the provision cost c is fixed and $MP = \frac{c}{N}$, $\lim_{n \rightarrow \infty} P_n = \frac{1}{2}$.*

Intuitively, as the group size N groups large, if each individual above the same threshold value \tilde{v} still contributing according to MP , then the probability of provision is smaller, which will in turn make people with value smaller \tilde{v} want to contribute MP , which counters the influence of group size on decreasing the provision probability. Proposition 4 suggests the public good will be provided half of the time when N is sufficiently large. We conduct a series of numerical simulation⁵ given a fixed value of α ranges from 0 to 1 and the group size N ranges from 2 to 101. Figure 2 the threshold (smallest) value of accepting the contract and the expected provision probability, under different combination of α and N . Our simulation results show that when $\alpha > 0.3$ with $N > 10$ or $\alpha > 0.1$ with $N > 20$, the provision probability is very close to 0.5.

3.2 $MP \neq AP$

When $MP \neq AP$, if one contributes higher than α but the group fail to provide the public good, the received assurance payment is β instead of α when $MP = AP$.

⁵Note that in the assumption α is constraint to $(\frac{1}{N}, 1)$ so that $\bar{v} < c$, thus, the meaningful region in the figures are $\alpha > \frac{1}{N}$; points close to the axes are not attainable.

Proposition 5. Consider an assurance contract (MP, AP) where MP does not equal to AP , when $MP = \alpha$ and let $AP = \beta$, the equilibrium strategy

$$s(v) = \begin{cases} b \in [0, \alpha) & \text{for } v < \tilde{v} \\ \alpha & \text{for } v \geq \tilde{v}, \end{cases} \quad (14)$$

where $(\tilde{v} - \alpha - \beta)(1 - \tilde{v})^{N-1} + \beta = 0$. In addition, there is exists one and only one $\tilde{v} \in (0, a)$.

Rewrite $(\tilde{v} - \alpha - \beta)(1 - \tilde{v})^{N-1} + \beta = 0$,

$$P_N = (1 - \alpha + \beta) \frac{P_N}{1 - \tilde{v}_N} + \beta.$$

Thus,

$$P_N = \frac{\beta(1 - \tilde{v})}{\alpha + \beta - \tilde{v}}.$$

Proposition 6. When α is fixed and $MP = \alpha, AP = \beta$, $\lim_{n \rightarrow \infty} P_n = \frac{\beta}{\alpha + \beta}$; when the provision cost c is fixed and $MP = \frac{c}{N}$, $\lim_{n \rightarrow \infty} P_n = 1$.

The above equation shows that when α is fixed, if $N \rightarrow \infty$ and $\beta < \alpha$, the expected provision probability is smaller than 0.5 while $\beta > \alpha$, the expected provision probability is higher than 0.5. The above equation also implies that the when α is fixed, social planner can always choose a higher β so that the public good will be provided with a higher probability, while if the public good is more difficult to provide from the start (a high α), the social planner need to set a higher assurance payment β relative to α to achieve a higher provision probability. When c is fixed, the expected provision probability approaches 1. Figure 3 shows the threshold (smallest) value of accepting the assurance contract and the expected provision probability for $N = 10$ and $N = 100$ respectively under different combination of α and β . The expected provision probability sub-figures in Figure 3 show that when N is very large, $\alpha = \frac{c}{N}$ can be very small and the points close to x axis (assurance payment level, β) can be very close to 1.

3.3 The General Case

More generally, consider the case where $\underline{v} = 0, \bar{v} = 1$, v follows a uniform distribution on $[0, 1]$ and $c = \alpha N$ with $N \geq 2$ players, $\alpha \in (\frac{1}{N}, 1)$ with $MP = \frac{c}{N-k}, k = 0, 1, 2, \dots, N-c$, where α is now the cost sharing ratio and $AP = \beta$. We restrict $MP < 1$ so that the minimum price is strictly lower than the upper bound of the value distribution. Each player's choice is to either accept the contract or reject the contract.

Proposition 7. When $MP = \frac{c}{N-k}$, $k = 1, 2, \dots, N - 2$ and $AP = \beta$,

$$s(v) = \begin{cases} 0 & \text{for } v < \tilde{v} \\ MP & \text{for } v \geq \tilde{v}, \end{cases}$$

constitutes an equilibrium strategy where

$$\tilde{v} \binom{N-1}{k} \tilde{v}^k (1-\tilde{v})^{N-k-1} + \beta = \left(\frac{c}{N-k} + \beta \right) \mathcal{P}_1.$$

with

$$\mathcal{P}_1(\tilde{v}, k) = (1-\tilde{v})^{N-1} + \binom{N-1}{1} \tilde{v} (1-\tilde{v})^{N-2} + \dots + \binom{N-1}{k-1} \tilde{v}^{k-1} (1-\tilde{v})^{N-k} + \binom{N-1}{k} \tilde{v}^k (1-\tilde{v})^{N-k-1}.$$

The social planner's expected profit when $MP = \frac{c}{N-k}$ is

$$\begin{aligned} \pi_s &= (1-\tilde{v})^N \left(S + \frac{Nc}{N-k} - c \right) + \binom{N}{1} \tilde{v} (1-\tilde{v})^{N-1} \left(S + \frac{(N-1)c}{N-k} - c \right) + \dots \\ &+ \binom{N}{k} \tilde{v}^k (1-\tilde{v})^{N-k} \left(S + \frac{(N-k)c}{N-k} - c \right) \\ &- \beta \left((N-k-1) \binom{N}{k+1} \tilde{v}^{k+1} (1-\tilde{v})^{N-k-1} + \dots + \binom{N}{N-1} \tilde{v}^{N-1} (1-\tilde{v})^1 \right) \end{aligned} \quad (15)$$

Since players can always choose to reject the assurance contract which yield an expected utility of zero, thus, each player's expected profit in equilibrium is at least weakly positive.

Numerical Analyses An analytical solution to the threshold value \tilde{v} is almost impossible since there is no closed form solution to the partial sum of binomial coefficient. As a result, we are able to derive an closed form solution to the threshold value nor calculate the social planner's expected profit in the general case. However, we are able construct an computer algorithm to numerically analyze the influence of different assurance contracts on the threshold value, provision probability and the social planner's expected surplus.

To proceed, we choose the group size $N = \{10, 100\}$, the provision cost $c = \{0.4N, 0.8N\}$, the social planner's benefit $S = \{0, c\}$, the assurance payment $AP = \{0, 0.5MP, MP, 2MP\}$ with each players' value independently and identically drawn from $U[0, 1]$. For example, if $N = 10$, $c = 0.4N$, which indicate that there will be 10 players and the public good provision cost is 4. Thus, we can set $k = \{0, 1, 2, 3, 4, 5\}$ and the corresponding minimum price $MP = \{\frac{10}{10}, \frac{10}{9}, \frac{10}{8}, \frac{10}{7}, \frac{10}{6}, \frac{10}{5}\}$. We choose four different assurance payment levels: 1) $AP = 0$ with no assurance payment available, players are only asked whether or not they want to contribute the minimum price; 2) $AP = 0.5MP$, the assurance payment is half

of the minimum price, e.g., if $MP = 2$ and if a player decide to accept the assurance contract and commit to pay \$2, she will receive \$1 upon provision failure; 3) $AP = MP$, the assurance payment equals the minimum price and 3) $AP = 2MP$, the assurance payment is twice as much as the minimum price.

In addition, we assume the social planner can either benefit $S = c$ or $S = 0$ from the provision of the public good, which we think put a boundary on the social planner's actual benefit since if $S > c$, the social planner would just provide the public good herself and if $S = 0$, the social planner has no incentive to provide the public good. Note that the social planner's value S plays no role in determining the threshold values nor the provision probability; it only influences the social planner's expected profit as well as the social efficiency results due to a change in the social planner's value. threshold value

Figure 4 depict the social planner's expected profit when $S = c$, the social planner's expected profit when $S = 0$, the threshold value and the expected provision probability when the group size $N = 10$; Figure 5 show the corresponding scenarios when the group size $N = 100$. In each graph, we consider a range of minimum prices (with different k), two provision cost ($cost = 40, cost = 80$), as well as differential ratio of assurance payment over minimum price ($\beta = 0, 0.5MP, MP, 2MP$). Our numerical results reveal several systematic features using the assurance contract that we summarize below.

From Figure 4 and Figure 5, our numerical results indicate that when $S = c$, the assurance contract will result positive expected profit for the social planner; such profits generally first increase and then stabilize or decrease as the minimum price increases (a higher k). With a large group ($N = 100$), neither the provision cost nor the assurance payment has a huge impact on the social planner's expected profit, and social planner's expected profit is more sensitive to the change minimum price. When assurance contract is not available, the social planner's expected profit is positive and very close to zero. When the social planner's benefit $S = 0$, our results show that the expected profit is strictly negative and such loss will increase with a larger group size N , in exchange to a higher provision probability. With a relative small group size ($N = 5$), the expected loss increases with the provision cost and level of assurance payment; however, with a relative large group size ($N = 100$), the expected loss only varies with respect to the assurance payment and changes little for different provision cost.

Our results also show the change in the threshold value (\tilde{v}) of accepting the assurance contract in different scenarios. We find that when the assurance contract is not available ($\beta = 0$), the threshold value basically equals to the minimum price; that is, players whose values are higher than the minimum price will accept the assurance contract and contribute the minimum price, otherwise contribute zero, as each player only faces a binary

choice. We also find that the provision cost has little influence on the threshold value \tilde{v} . When the group size is small ($N = 10$), the threshold value decreases as the assurance payment level increases; when the group size becomes large ($N = 100$), the assurance payment level has little influence on the threshold value. As expected, the threshold value increases significantly as the required minimum price increases; the threshold value is strictly lower than the minimum price since players with values (slightly) less than the minimum price may have incentive to contribute the minimum price due to a positive probability of provision failure and the ability to claim the assurance payment.

In terms of the expected provision probability, we find that it remains relative stable when the minimum price changes (k changes), more so when the group size is large. We also find that the provision cost has little influence on the expected provision probability. The expected provision probability can be reasonably approximated by the ratio $\frac{\beta}{\alpha+\beta}$, which suggest that the expected provision probability is mostly determined by the cost sharing ratio ($\frac{c}{N}$) and assurance payment level; the minimum price has little impact on the expected provision probability, especially when the group size is large.

Efficiency With a group size of N and a provision of c , the potential maximum surplus is

$$\int \cdots \int_{\sum v_i \geq c} (v_1 + v_2 + \cdots + v_N) dv_1 \dots dv_N.$$

Since $v_i \in U[0, 1]$, then $\sum_1^N v_i \in I(N)$, where $I(\cdot)$ is the Irwin-Hall distribution with the probability density function,

$$f(x; N) = \frac{1}{2(N-1)!} \sum_{l=0}^N (-1)^l \binom{N}{l} (x-l)^{N-1} \text{sgn}(x-l).$$

The potential maximum surplus can be rewritten as

$$MSurplus = \int_c^{[c]} x f(x; N) dx + \sum_{l=[c]}^N \int_l^{l+1} x f(x; N) dx, l \in \mathcal{N}^+.$$

When $S = c$, $MP = \frac{c}{N-k}$, $k = 0, 1, 2, \dots, N - c$, the realized surplus can be rewritten as (assuming the equilibrium strategies)

$$\begin{aligned}
RSurplus &= \sum_{l=0}^k \binom{N}{l} \underbrace{\int_{\tilde{v}}^1 \cdots \int_{\tilde{v}}^1}_{N-l} \underbrace{\int_0^{\tilde{v}} \cdots \int_0^{\tilde{v}}}_l (v_1 + v_2 + \cdots + v_N) dv_1 \dots dv_N, \\
&= \sum_{l=0}^k \binom{N}{l} \frac{1}{2} \tilde{v}^l (1 - \tilde{v})^{N-l} (\tilde{v} * N + N - l).
\end{aligned}$$

If $S = 0$,

$$\begin{aligned}
RSurplus &= \sum_{l=0}^k \binom{N}{l} \underbrace{\int_{\tilde{v}}^1 \cdots \int_{\tilde{v}}^1}_{N-l} \underbrace{\int_0^{\tilde{v}} \cdots \int_0^{\tilde{v}}}_l (v_1 + v_2 + \cdots + v_N - c) dv_1 \dots dv_N, \\
&= \sum_{l=0}^k \binom{N}{l} \frac{1}{2} \tilde{v}^l (1 - \tilde{v})^{N-l} (\tilde{v} * N + N - l - 2c),
\end{aligned}$$

where \tilde{v} is the threshold value above which players will accept the assurance contract.

Figure 5 shows the realized social surplus in different scenarios for selected parameters. The numerical results suggest that when the social planner's value for the public good $S = c$, in all cases, the presence of assurance payment significantly increases the realized social surplus. This result greatly contrasts the situations when no assurance payment are available, which is likely to lead to close zero social surplus (due to a close to zero provision probability). We also find that when $S = c$, the provision cost makes a small difference in terms of realized social surplus, which is consistent with the results from the expected provision probability. When the minimum price is fixed, an increase in the assurance payment is likely to increase the social surplus due to an increased probability of provision success, and also due to assumption the social planner's value alone is sufficient to justify the cost of provision, which makes it always a Pareto improvement to provide the public good.

Figure 6 also includes the situations when the social planner's value $S = 0$. We find that in these situations, the cost of the public good is critical to assess the ranking realized social surplus with different specifications of assurance contracts. When the provision cost is chosen at 40% of the sum of the expected individual values, the assurance contract still increase the realized social surplus compared to a close to zero social surplus without the assurance contract. When the cost is relative high, i.e., in our situation, when the cost is set at 80% of the sum of the expected individual values, the assurance contract is most likely to lead to negative realized social surplus. We find the threshold values \tilde{v} are largely invariant with respect to the change of the provision cost, which indicate that when there cost is higher, it is more likely that the sum of realized values will be smaller than the cost even though individuals with the same induced values are both equally likely to accept

the assurance contract regardless the provision cost. As a result, in the high provision cost case, there is higher likelihood that sum of realized individuals values are smaller than the provision the cost but the public good still gets provided, which brings in social inefficiency as the social planner has zero benefits from provision by assumption.

4 Conclusions

In this paper, we explore the use of assurance contract to provide threshold public good for both the 2-player and the N -player environment. We derive the Bayesian Nash equilibrium strategy allowing continuous contribution for the 2-player and allowing discrete actions for the N -players. Our results illustrate the potential improvement over provision probability, and possibly the social planner's expected profit comparing to the situation where the assurance contract is not available. We find potential efficiency improvement from apply the assurance contract under certain circumstances, especially with a relative low provision cost.

Our framework can be easily generalized to any other distribution $F(v)$ with an arbitrary value boundary though we choose a uniform distribution for exposition purpose. When we replace the uniform distribution with other distribution, e.g., a normal distribution, the provision probability may still converge to the same value but the rate of convergence might change. Future research can explore the convergence rate of provision probability using different distributions. Also, in our framework, the group size N is a common information known by both the social planner and the contributors. Empirically, the number of individuals who will contribute to a project is hard to know; e.g., it is very difficult to predict how many backups a particular project can receive from the crowdfunding website, which is often crucial to the outcome. Thus, it is interesting to extend the model by introducing some uncertainties to the group size N and explore the implication of using assurance contract. We speculate that assurance contract may be able to overcome disadvantages due to uncertain group size compared to straight up donation approach.

Our results show important implications for real world fundraising practices, either for the social planner to implement a public project or for the entrepreneurs to offset the research cost using crowdfunding approach to collect contributions from individuals who can benefit from the research products. However, behavioral economics may better predict individual actual decisions compared to the equilibrium assumption as incentive system may crowd out some intrinsic motivations and achieve a less desirable outcome; thus, lab and field experiment results are useful. In other separate ongoing researches, Li

et al. (2014) and Liu and Swallow (2015) find supporting evidences from both lab and field experiments that the assurance contract will increase the social efficiency based on the student subjects playing public good provision game and the assurance contract is likely to increase the participation rate for residents to support a local environmental project.

Acknowledgement

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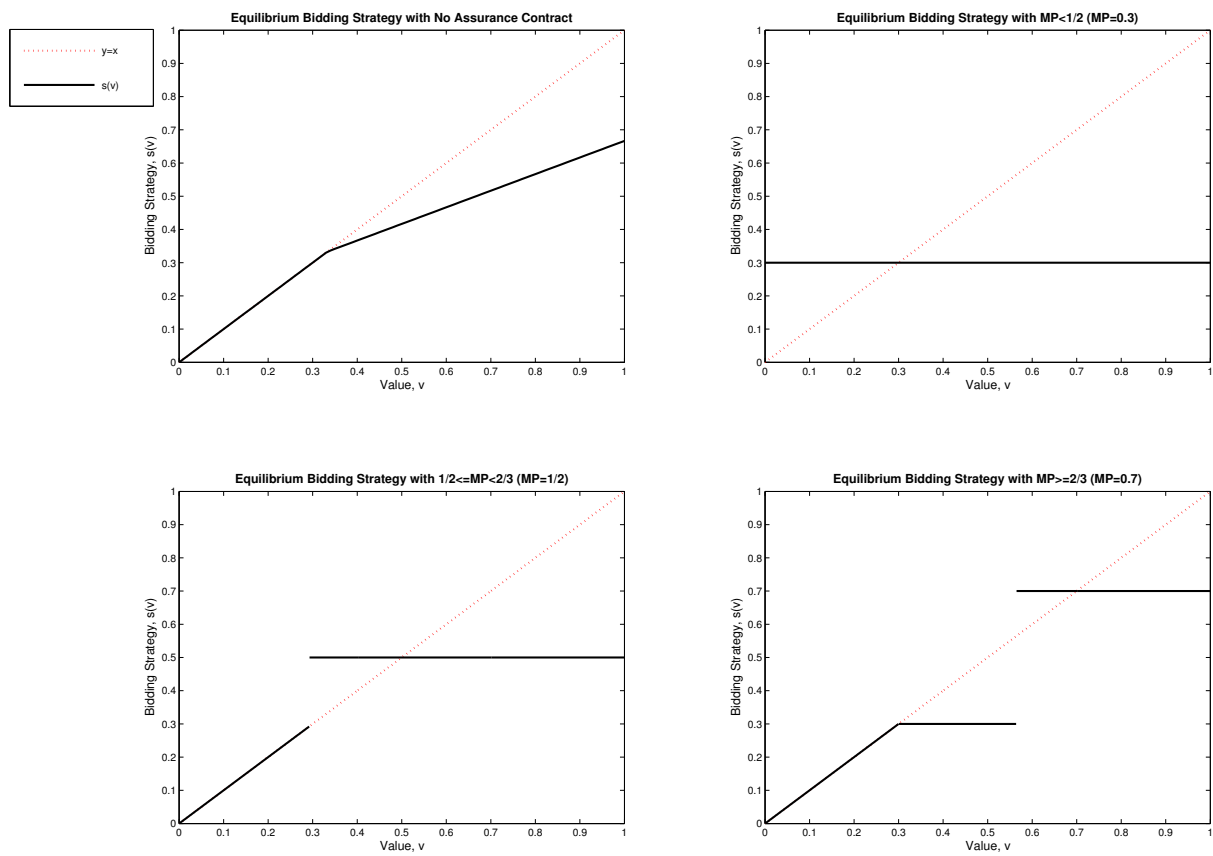
Figure 1: Equilibrium Bidding Strategies when MP belongs to Different Ranges

Figure 2: Threshold Value of Accepting the Assurance Contract and Expected Provision Probability when $MP = AP$

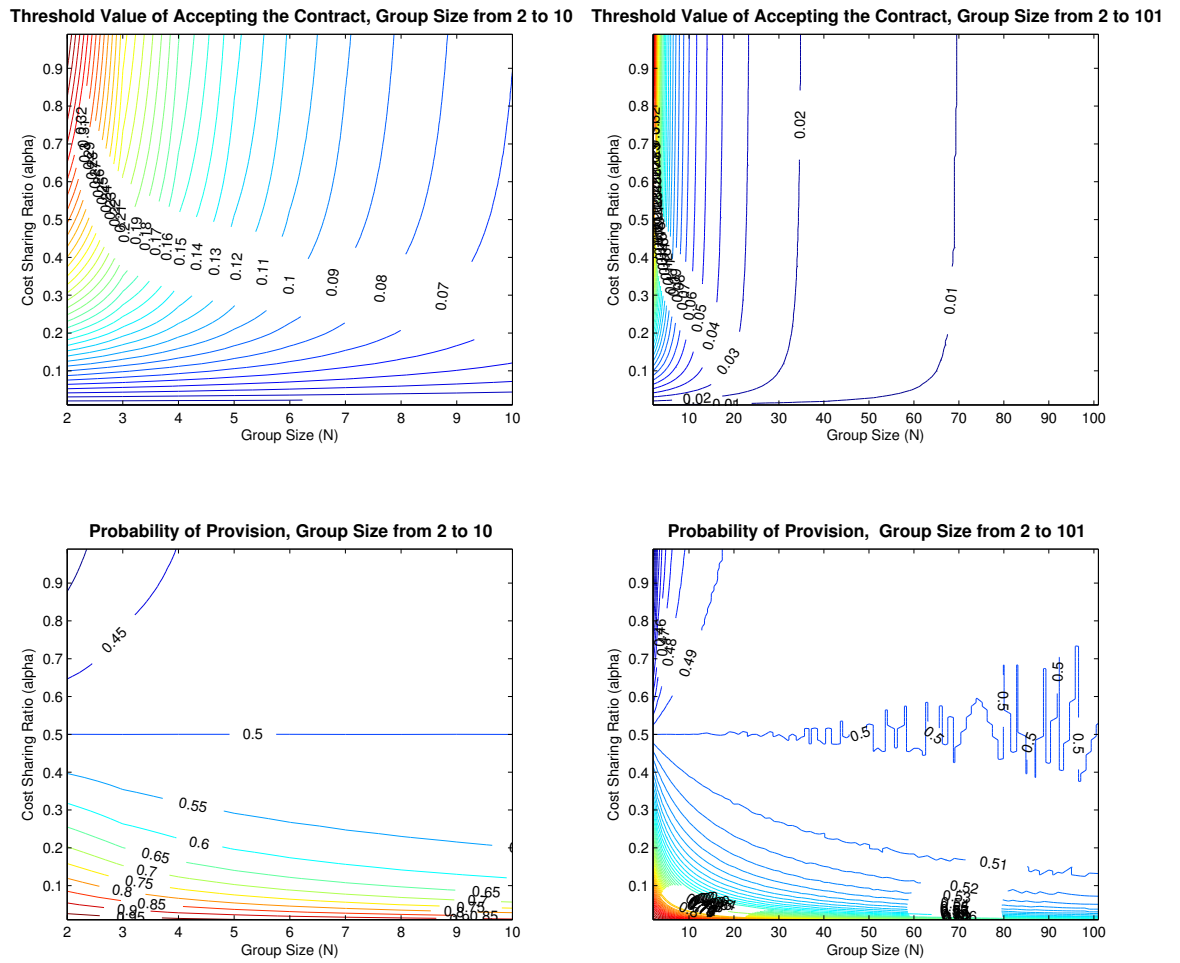


Figure 3: Threshold Value of Accepting the Assurance Contract and Expected Provision Probability when $MP \neq AP$

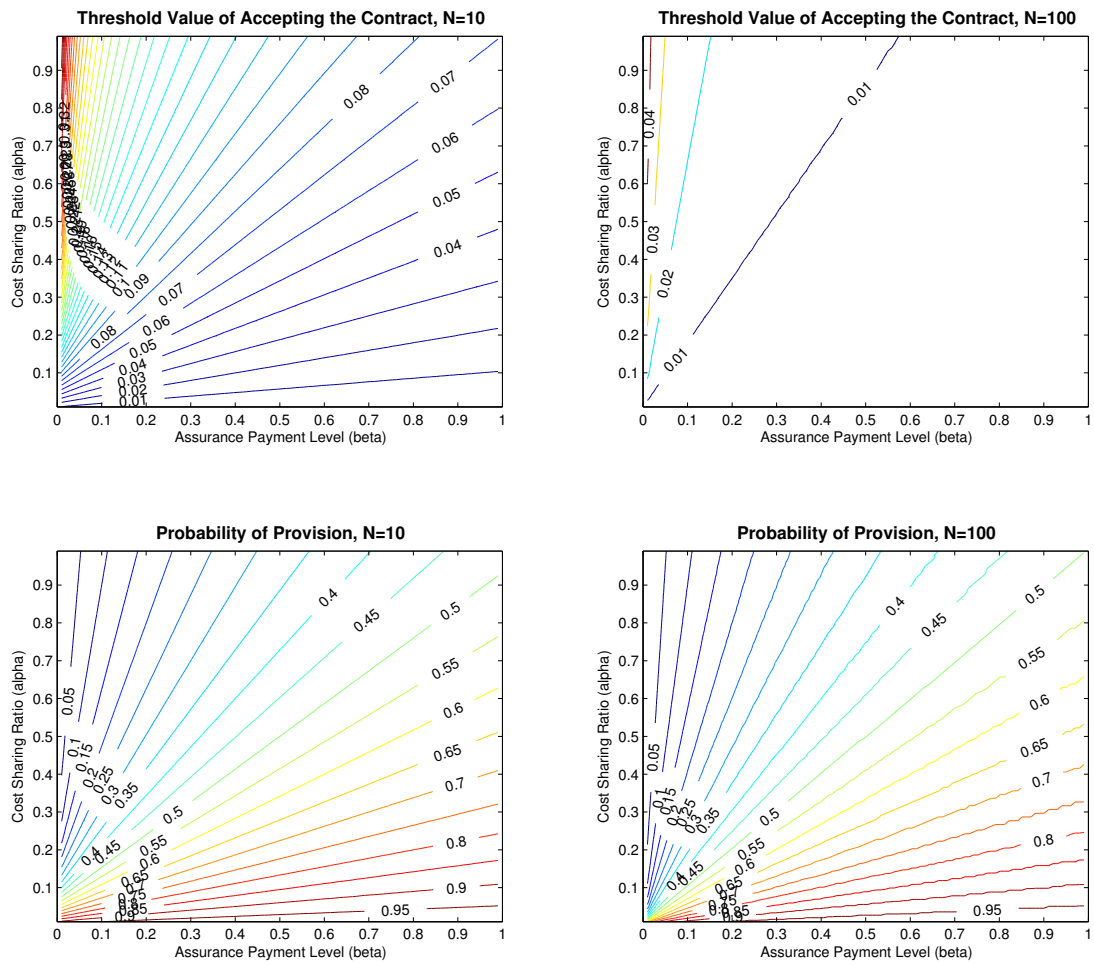


Figure 4: The Social Planner's Expected Profit, Threshold Value of Accepting the Assurance Contract and Expected Provision Probability when $N = 10$

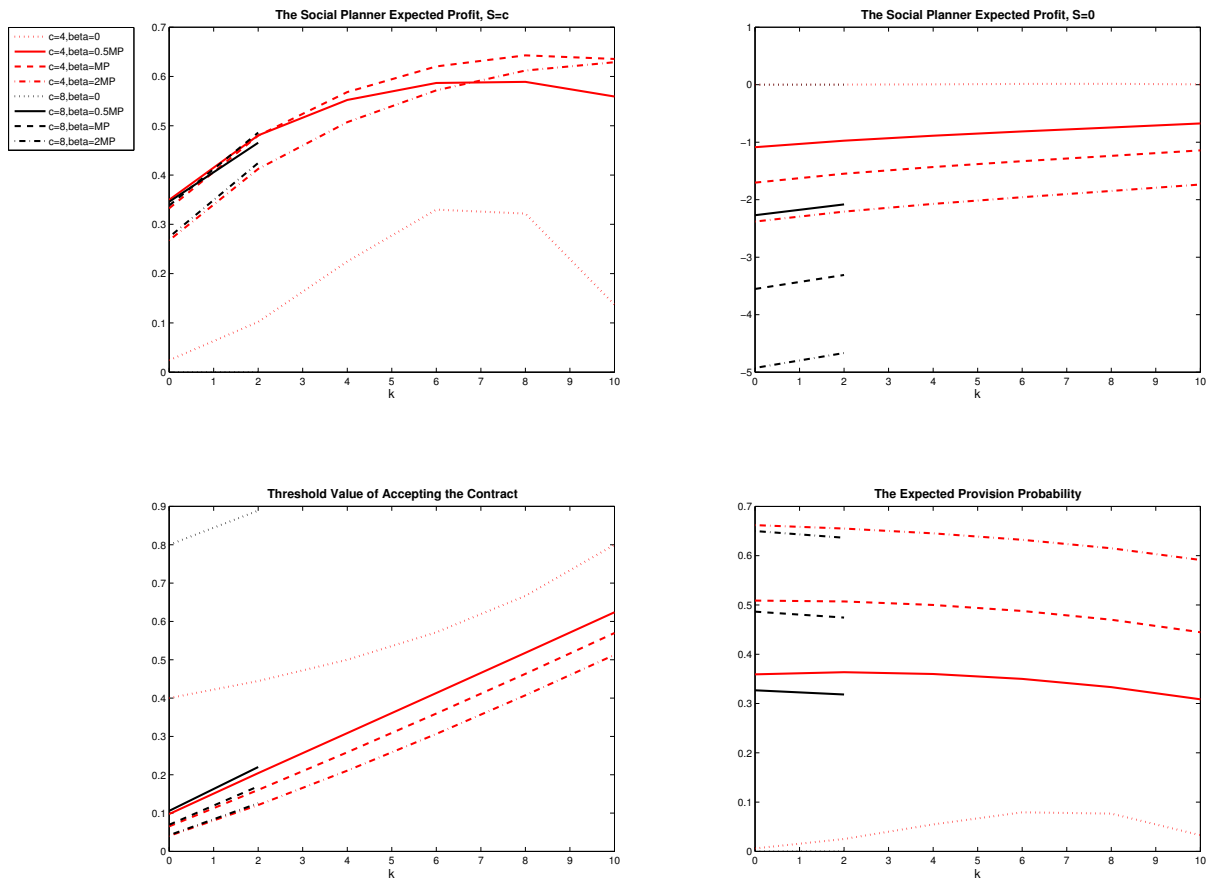


Figure 5: The Social Planner's Expected Profit, Threshold Value of Accepting the Assurance Contract and Expected Provision Probability when $N = 100$

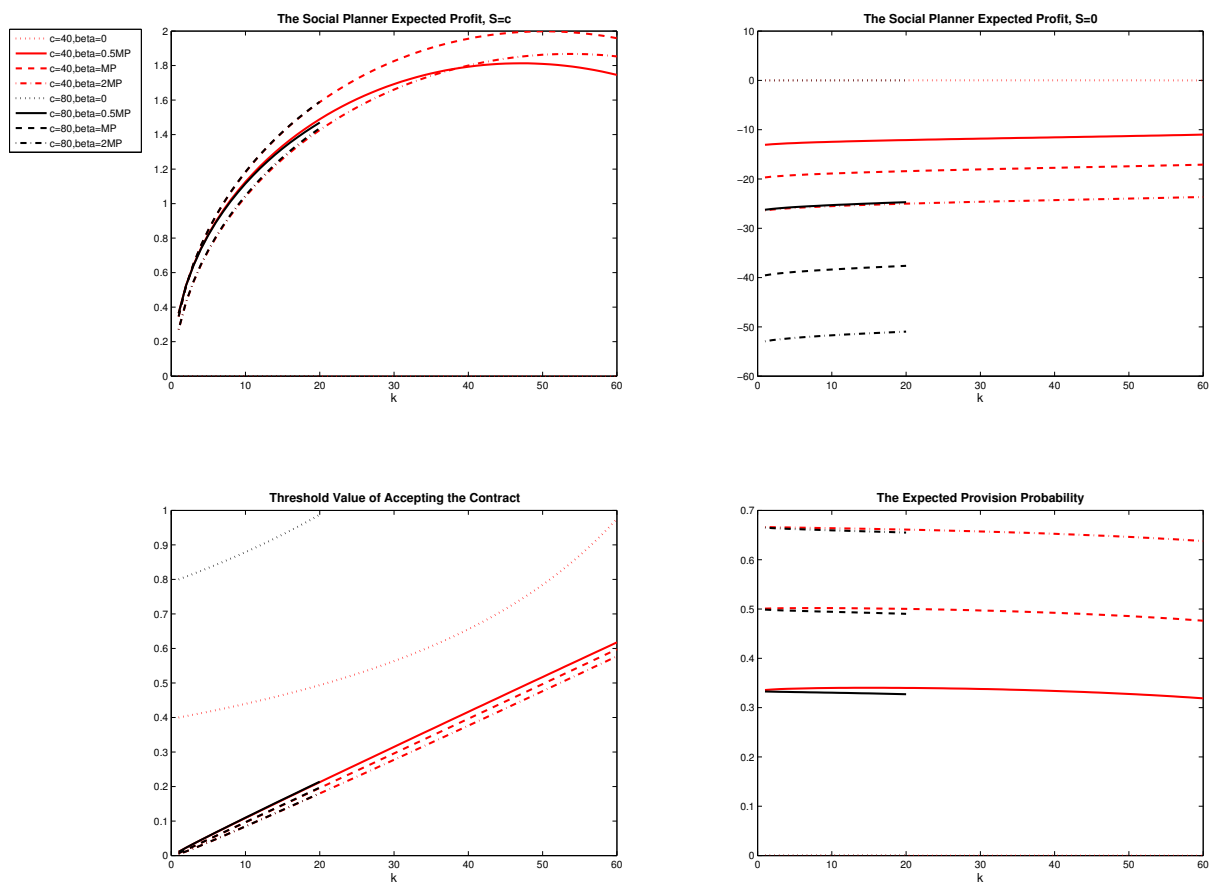
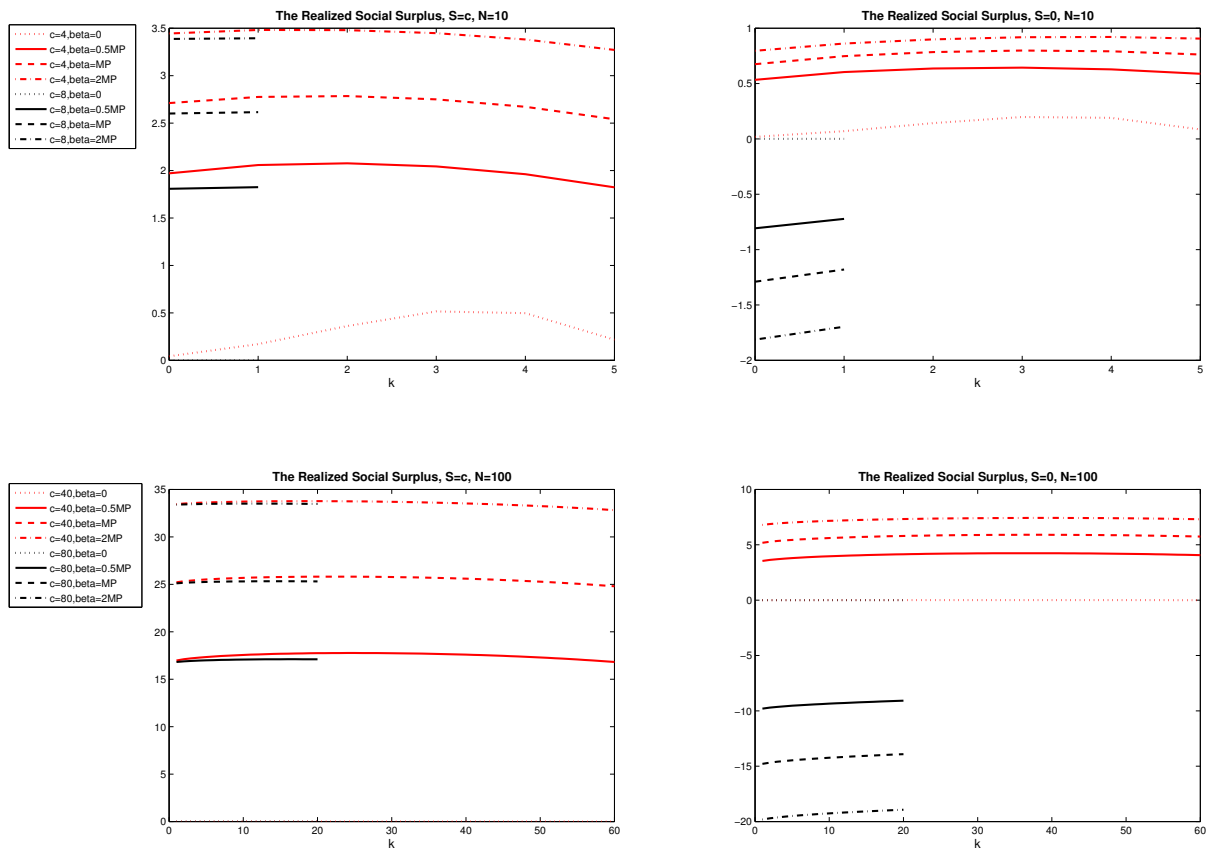


Figure 6: The Realized Social Surplus



Appendix

Proof for Lemma 1 Lemma 1 can be alternatively stated as

$$\begin{cases} s(v) \leq \alpha & \text{if } \alpha \geq \frac{1}{2}, \quad \forall v \in [0, 1] \\ s(v) \leq 1 - \alpha & \text{if } \alpha < \frac{1}{2}, \quad \forall v \in [0, 1]. \end{cases} \quad (16)$$

First, note the expected profit of contribution α is

$$E\pi(s_1(v) = \alpha) = (v - \alpha)Pr(s_2 \geq 1 - \alpha) + \alpha(1 - Pr(s_2 \geq 1 - \alpha)), \quad (17)$$

where $Pr(\cdot)$ is the probability that the public good is provided. For any $\epsilon > 0$,

$$E\pi(s_1(v) = \alpha + \epsilon) = (v - (\alpha + \epsilon))(Pr(s_2 \geq 1 - \alpha) + p(\epsilon)) + \alpha(1 - (Pr(s_2 \geq 1 - \alpha) + p(\epsilon))), \quad (18)$$

where $p(\epsilon)$ denotes the incremental probability due to the increase of ϵ from s_1 and $p(\epsilon) \geq 0$. Thus, $Pr(s_2 \geq 1 - \alpha) \leq Pr(s_2 \geq 1 - \alpha - \epsilon)$. Therefore,

$$\begin{aligned} & E\pi(s_1(v) = \alpha) - E\pi(s_1(v) = \alpha + \epsilon) \\ &= (v - \alpha)Pr + \alpha(1 - Pr) - ((1 - \alpha)Pr + (v - \alpha)p(\epsilon) - \epsilon - \epsilon p(\epsilon) + \alpha(1 - Pr) - \alpha p(\epsilon)) \\ &= -(v - 2\alpha)p(\epsilon) + \epsilon(Pr + p(\epsilon)) \\ &= \epsilon(Pr + p(\epsilon)) - (v - 2\alpha)p(\epsilon). \end{aligned} \quad (19)$$

When $\alpha \geq \frac{1}{2}$, $2\alpha \geq 1 \geq v$ for all $v \in [0, 1]$. As a result

$$E\pi(s_1(v) = \alpha) - E\pi(s_1(v) = \alpha + \epsilon) \geq 0. \quad (20)$$

When $\alpha < \frac{1}{2}$,

$$E\pi(s_1(v) = \alpha) - E\pi(s_1(v) = \alpha + \epsilon) = \epsilon Pr + (\epsilon + v - 2\alpha)p(\epsilon). \quad (21)$$

If $\epsilon + 2\alpha \geq 1$, i.e., $\epsilon \geq 1 - 2\alpha > 0$, which implies $\epsilon + \alpha \geq 1 - \alpha > 0$. We also have $E\pi(s_1(v) = \alpha) - E\pi(s_1(v) = \alpha + \epsilon) \geq 0$. Individual will always choose α instead of $1 - \alpha$.

Proof for Lemma 2 Assume $s_1(\cdot)$ and $s_2(\cdot)$ are equilibrium contribution functions. For any $v_h > v_l$, we have

$$\begin{cases} Pr(s_1(v_h) + s_2(\cdot) \geq c) > 0 \\ Pr(s_1(v_l) + s_2(\cdot) \geq c) > 0. \end{cases} \quad (22)$$

By the definition of equilibrium, we have

$$\begin{aligned} & (v_h - s_h)Pr(s_h + s_l \geq c) + 1(s_h \geq \alpha)\alpha(1 - Pr(s_h + s_2 \geq c)) \\ & \geq (v_h - s_l)Pr(s_l + s_2 \geq c) + 1(s_l \geq \alpha)\alpha(1 - Pr(s_l + s_2 \geq c)) \end{aligned} \quad (23)$$

and

$$\begin{aligned} & (v_l - s_l)Pr(s_l + s_2 \geq c) + 1(s_l \geq \alpha)\alpha(1 - Pr(s_l + s_2 \geq c)) \\ & \geq (v_l - s_h)Pr(s_h + s_2 \geq c) + 1(s_h \geq \alpha)\alpha(1 - Pr(s_h + s_2 \geq c)). \end{aligned} \quad (24)$$

Combing the above two equations, we have

$$(v_h - v_l)(Pr(s_h + s_2 \geq c) - Pr(s_l + s_2 \geq c)) \geq 0. \quad (25)$$

When $Pr(s_h + s_2 \geq c) > Pr(s_l + s_2 \geq c)$, $Pr(s_2 \geq c - s_h) > Pr(s_2 \geq c - s_l)$. As a result, $s_h > s_l$.

When $Pr(s_h + s_2 \geq c) = Pr(s_l + s_2 \geq c) > 0$, we have

$$(s_l - s_h)Pr \geq [1(s_l \geq \alpha) - 1(s_h \geq \alpha)]\alpha(1 - Pr) \quad (26)$$

and

$$(s_h - s_l)Pr \geq [1(s_h \geq \alpha) - 1(s_l \geq \alpha)]\alpha(1 - Pr). \quad (27)$$

If $s_l > s_h$, in order to be consistent with the above two inequalities, we need $s_h < \alpha \leq s_l$ since $(s_l - s_h)Pr > 0$ and $(s_h - s_l)Pr < 0$.

According to Lemma 1, $s_i(\cdot) \leq \max\{\alpha, 1 - \alpha\}$ for all $\alpha \in (0, 1)$ and $v \in [0, 1]$, $i = 1, 2$. When $\alpha \geq \frac{1}{2}$, $s(\cdot) \leq \alpha$, therefore, $s_h < \alpha = s_l$. Since $Pr > 0$ and $s_2 \leq \alpha$, we can also infer $s_h \geq 1 - \alpha$. Therefore, we can conclude when $\alpha \geq \frac{1}{2}$,

$$1 - \alpha \leq s_h < \alpha = s_l. \quad (28)$$

When $\alpha = \frac{1}{2}$, s_h does not exist. In this case, $s_l < s_h$.

When $\alpha > \frac{1}{2}$, since $Pr(s_h + s_2 \geq c) = Pr(\alpha + s_2 \geq c) > 0$ ($s_h < \alpha$), we can infer $s_2(\cdot)$ is discontinuous at $s_2(\cdot)$ at $s_2(\cdot) = c - s_h$, in particular, there is a jump between $[c - \alpha, c - s_h]$ for $s_2(\cdot)$ because $Pr(\alpha + s_2 \geq c) > Pr(s_h + s_2 \geq c)$. Since $Pr > 0$, $s_2 \in [1 - s_h, \alpha]$, then we have

$$(v_h - s_h)Pr \geq (v_h - \alpha)Pr + \alpha(1 - Pr), \quad (29)$$

or $(2\alpha - s_h)Pr \geq \alpha$, and

$$(v_l - \alpha)Pr + \alpha(1 - Pr) \geq (v_l - s_h)Pr \quad (30)$$

or $\alpha \geq (2\alpha - s_h)Pr$. The above two inequalities imply $\alpha = (2\alpha - s_h)Pr$, or

$$Pr = \frac{\alpha}{2\alpha - s_h} \in [0, 1]. \quad (31)$$

Note that, for $s_l > s_h$ to hold, equation (31) is required. In other words, $s_h \geq s_l$ whenever equation (31) does not hold.

When $\alpha \leq \frac{1}{2}$, $s(\cdot) \leq 1 - \alpha$ for all $v \in [0, 1]$, then $s_2 \leq 1 - \alpha$ and $s_h < s_l$ together imply that $s_h \geq 1 - (1 - \alpha) = \alpha = s_l$.

In summary, we find that the monotonicity condition is satisfied unless $Pr = \frac{\alpha}{2\alpha - s_h}$ when $\alpha > \frac{1}{2}$ which is quite arbitrary.

Proof for Lemma 3 First, we show that in equilibrium, when $\alpha < \frac{1}{2}$, no one contributes less than α . If not, the probability of provision is zero, then any agents would be strictly better off by contribution α . Assume $s(v) = \alpha$, when $E\pi(\alpha) \geq E\pi(1 - \alpha + \epsilon)$, $\epsilon > 0$, then $s_2 \in [0, 1 - \alpha)$, which implies $E\pi(s_1(v) < \alpha) = 0 < E\pi(\alpha) = \alpha$. Therefore, $s_i \in [\alpha, 1 - \alpha]$.

When $\alpha < \frac{1}{2}$, $s(v) \geq \alpha$ in equilibrium.

If $s(v) \geq \alpha$ in equilibrium, for any $v \in [0, 1]$, $s(v) < 1 - \alpha$ since

$$E\pi(1 - \alpha) = (v - (1 - \alpha))Pr(s_2(\cdot) \geq 1 - (1 - \alpha)) = v - 1 + \alpha < \alpha = E\pi. \quad (32)$$

Note that $Pr(s(v) = \alpha) = 0$ as $s(v) < 1 - \alpha$ for all $v \in [0, 1]$. Thus,

$$E\pi(\alpha + \epsilon) = (v - (\alpha + \epsilon))p(\epsilon) + \alpha(1 - p(\epsilon)) = \alpha + (v - 2\alpha - \epsilon)p(\epsilon). \quad (33)$$

When $v \leq 2\alpha$, $E\pi(\alpha) \geq E\pi(\alpha + \epsilon)$, combined with $s(v) \in [\alpha, 1 - \alpha]$, we can show that if $v \leq \alpha$, $s(v) = \alpha$, if $v > 2\alpha$, $s(v) \in [\alpha, 1 - \alpha]$.

Proof for Proposition 1 When $\alpha < \frac{1}{2}$, for any player who tries to get a higher profit by providing the public good, $s(v) \geq 1 - \alpha$, while the profit is $\pi(s(v)|v) = v - 1 + \alpha < \alpha = \pi(\alpha|v)$.

When $MP = \frac{1}{2}$, obviously, contribute $s(v) = c$ to provide the public good is not profitable for any $v \in [0, 1]$. Notice that for $v \leq 1 - \frac{\sqrt{2}}{2}$, in order to influence the provision outcome, the new contribution $s(v) \geq \frac{1}{2}$. Contributing $s(v) > \frac{1}{2}$ is dominated by $s(v) = \frac{1}{2}$ since the profit is less (more negative in this case) when the public is provided. When $s(v) = \frac{1}{2}$, the expected profit is $\pi(s = \frac{1}{2}|v \leq 1 - \frac{\sqrt{2}}{2}) = \frac{\sqrt{2}}{2}(v - \frac{1}{2}) + \frac{1}{2}(1 - \frac{\sqrt{2}}{2}) = \frac{1}{2}(\sqrt{2}v + 1 - \sqrt{2}) \leq 0$, while maintaining the original strategy yields an expected profit of 0.

For $v > 1 - \frac{\sqrt{2}}{2}$, deviate to $s(v) > \frac{1}{2}$ is less profitable when good is provided, equally profitable when the good is not provided and such a deviation is not consequential in terms of the provision outcome. Any deviation to $s(v) < \frac{1}{2}$ will yield exactly zero profit, while for any $v > 1 - \frac{\sqrt{2}}{2}$ has a strictly positive expected profit since $\pi(s = \frac{1}{2}|v > 1 - \frac{\sqrt{2}}{2}) =$

$\frac{\sqrt{2}}{2}(v - \frac{1}{2}) + \frac{1}{2}(1 - \frac{\sqrt{2}}{2}) = \frac{1}{2}(\sqrt{2}v + 1 - \sqrt{2}) > 0$. Note that when $v = 1 - \frac{\sqrt{2}}{2}$, contributing either α or v yields the same expected profit of 0.

When $MP \in (\frac{1}{2}, \frac{2}{3})$, When $v \leq \frac{1}{2}(2\alpha + 1 - \sqrt{4\alpha^2 + 1})$ and $s(v) = v < 1 - \alpha$, $\forall MP \in (\frac{1}{2}, \frac{2}{3})$, the expected profit is 0. To prove this, let $f(\alpha) = 1 - \alpha - v = 1 - \alpha - \frac{1}{2}(2\alpha + 1 - \sqrt{4\alpha^2 + 1})$, we can first show that $f(\alpha)$ is a decreasing function and $f(\frac{2}{3}) = 0$. If $s(v) = 1 - \alpha$, then the expected profit of contributing $1 - \alpha$ is

$$\begin{aligned}\pi(1 - \alpha|v) &= (v - 1 + \alpha)(1 - \frac{1}{2}(2\alpha + 1 - \sqrt{4\alpha^2 + 1})) \\ &\leq (\frac{1}{2}(2\alpha + 1 - \sqrt{4\alpha^2 + 1}) - 1 + \alpha)(1 - \frac{1}{2}(2\alpha + 1 - \sqrt{4\alpha^2 + 1})) \\ &= f(\alpha).\end{aligned}$$

We can show that the $f(\alpha)$ immediate above is an increasing function w.r.t. α and $f(\frac{2}{3}) = 0$. Thus, when $MP \in (\frac{1}{2}, \frac{2}{3})$, $\pi(1 - \alpha|v) \leq f(\alpha) < 0$. If $s(v) = \alpha$ when $v \leq \frac{1}{2}(2\alpha + 1 - \sqrt{4\alpha^2 + 1})$,

$$\begin{aligned}\pi(\alpha|v) &= (v - \alpha)(1 - \frac{1}{2}(2\alpha + 1 - \sqrt{4\alpha^2 + 1})) + \frac{1}{2}\alpha(2\alpha + 1 - \sqrt{4\alpha^2 + 1}) \\ &\leq (\frac{1}{2}(2\alpha + 1 - \sqrt{4\alpha^2 + 1}) - \alpha)(1 - \frac{1}{2}(2\alpha + 1 - \sqrt{4\alpha^2 + 1})) + \frac{1}{2}\alpha(2\alpha + 1 - \sqrt{4\alpha^2 + 1}) \\ &= 0.\end{aligned}$$

Clearly, when $v \leq \frac{1}{2}(2\alpha + 1 - \sqrt{4\alpha^2 + 1})$, $s(v) \in (1 - \alpha, \alpha)$ yields less profit than $s(v) = 1 - \alpha$ and $s(v) > \alpha$ yields less profit than $s(v) = \alpha$. Therefore, in this case, $s(v) = v$ and one receives an expected value of zero.

When $v > \frac{1}{2}(2\alpha + 1 - \sqrt{4\alpha^2 + 1})$ and $s(v) = \alpha$, the expected profit is

$$\begin{aligned}\pi(\alpha|v) &= (v - \alpha)(1 - \frac{1}{2}(2\alpha + 1 - \sqrt{4\alpha^2 + 1})) + \frac{1}{2}\alpha(2\alpha + 1 - \sqrt{4\alpha^2 + 1}) \\ &> (\frac{1}{2}(2\alpha + 1 - \sqrt{4\alpha^2 + 1}) - \alpha)(1 - \frac{1}{2}(2\alpha + 1 - \sqrt{4\alpha^2 + 1})) + \frac{1}{2}\alpha(2\alpha + 1 - \sqrt{4\alpha^2 + 1}) \\ &= 0.\end{aligned}$$

In this situation, if $s(v) > \alpha$, the probability of provision is unchanged but one's profit upon provision success is strictly less; if $s(v) < 1 - \alpha$, the expected profit is 0; if $s(v) = 1 - \alpha$, the expected profit compared when $s(v) = \alpha$ is,

$$\begin{aligned}\pi(1 - \alpha|v) - \pi(\alpha|v) &= (v - 1 + \alpha)(1 - \frac{1}{2}(2\alpha + 1 - \sqrt{4\alpha^2 + 1})) \\ &\quad - ((v - \alpha)(1 - \frac{1}{2}(2\alpha + 1 - \sqrt{4\alpha^2 + 1})) + \frac{1}{2}\alpha(2\alpha + 1 - \sqrt{4\alpha^2 + 1})) \\ &= (2\alpha - 1)(1 - \frac{1}{2}(2\alpha + 1 - \sqrt{4\alpha^2 + 1})) - \frac{1}{2}\alpha(2\alpha + 1 - \sqrt{4\alpha^2 + 1}) \\ &= f(\alpha).\end{aligned}$$

We can show that the $f(\alpha)$ immediate above is an increasing function w.r.t. α and $f(\frac{2}{3}) = 0$. Thus, when $MP \in (\frac{2}{3}, 1)$, $\pi(1 - \alpha|v) < f(\alpha|v)$. Finally, if $s(v) \in (1 - \alpha, \alpha)$, the

expected profit is smaller compared when $s(v) = 1 - \alpha$. Therefore, when $MP \in (\frac{1}{2}, \frac{2}{3})$, $s(v) = \alpha$ for $v > \frac{1}{2}(2\alpha + 1 - \sqrt{4\alpha^2 + 1})$.

When $MP \geq \frac{2}{3}$, first note that no one is willing to provide the good alone even when $MP = 1$, under which only the highest type with $v = 1$ is indifferent between provision and not provision. For players with $v \leq 1 - \alpha$, contribution higher than $1 - \alpha$ but lower than α will enable the good to be provided some with positive probability, however, since the profit is negative upon provision and the player is unable to receive the assurance payment when fails, the expected profit of contributing $s(v) \in [1 - \alpha, \alpha)$ is negative. If $s(v) = \alpha$ when $v \leq 1 - \alpha$, the probability of provision success is α while the probability of failure is $1 - \alpha$, thus, $(v - \alpha)\alpha + \alpha(1 - \alpha) = (v + 1 - 2\alpha)\alpha < (2 - 3\alpha)\alpha \leq 0$ since $\alpha = MP \geq \frac{2}{3}$.

When $v \in [1 - \alpha, 1 - \alpha + \sqrt{3\alpha^2 - 2\alpha}]$:

- If $s(v) < 1 - \alpha$, the probability of provision fails to zero and the expected profits is smaller compared when $s(v) = 1 - \alpha$, where $\pi(1 - \alpha|v) = (v - 1 + \alpha)(\alpha - \sqrt{3\alpha^2 - 2\alpha}) \geq \alpha - \sqrt{3\alpha^2 - 2\alpha} > 0$.
- If $s(v) \in (1 - \alpha, \alpha)$, the probability of provision is the same while the profit conditional on provision is smaller as the assurance contract does not apply, the expected profit is smaller compared when $s(v) = 1 - \alpha$.
- If $s(v) = \alpha$, $\pi(\alpha|v) = (v - \alpha)\alpha + \alpha(1 - \alpha)$, since $\pi(1 - \alpha|v) = (v - 1 + \alpha)(\alpha - \sqrt{3\alpha^2 - 2\alpha})$, then

$$\begin{aligned}
\pi(1 - \alpha|v) - \pi(\alpha|v) &= (v - 1 + \alpha)(\alpha - \sqrt{3\alpha^2 - 2\alpha}) - (v + 1 - 2\alpha)\alpha \\
&= \sqrt{3\alpha^2 - 2\alpha}(\sqrt{3\alpha^2 - 2\alpha} - (v - 1 + \alpha)) \\
&\geq \sqrt{3\alpha^2 - 2\alpha}(\sqrt{3\alpha^2 - 2\alpha} + 1 - \alpha - 1 + \alpha - \sqrt{3\alpha^2 - 2\alpha}) \\
&= 0
\end{aligned}$$

- If $s(v) > \alpha$, $\pi(s(v) > \alpha|v) < (v - \alpha)\alpha + \alpha(1 - \alpha) = \pi(s(v) = \alpha|v) < \pi(s(v) = 1 - \alpha|v)$.

Therefore, when $v \in [1 - \alpha, 1 - \alpha + \sqrt{3\alpha^2 - 2\alpha}]$, $s(v) = 1 - \alpha$.

When $v > 1 - \alpha + \sqrt{3\alpha^2 - 2\alpha}$, following similar steps as $v \in [1 - \alpha, 1 - \alpha + \sqrt{3\alpha^2 - 2\alpha}]$:

- If $s(v) < 1 - \alpha$, the probability of provision fails to zero and the expected profits is smaller compared when $s(v) = \alpha$, where $\pi(\alpha|v) = (v - \alpha)\alpha + \alpha(1 - \alpha) = (v + 1 - 2\alpha)\alpha > 0$ requires that $v > 2\alpha - 1$. Since $v > 1 - \alpha + \sqrt{3\alpha^2 - 2\alpha}$, and $1 - \alpha + \sqrt{3\alpha^2 - 2\alpha} - 2\alpha + 1 = 2 - 3\alpha + \sqrt{3\alpha^2 - 2\alpha} > 0$ when $\alpha \in (\frac{2}{3}, 1)$, then $\pi(\alpha|v) > 0 = \pi(s(v) < 1 - \alpha|v)$.

- If $s(v) = 1 - MP$, $\pi(1 - \alpha|v) = (v - 1 + \alpha)(\alpha - \sqrt{3\alpha^2 - 2\alpha})$ and since $\pi(\alpha|v) = (v - \alpha)\alpha + \alpha(1 - \alpha)$, compare the two expected profit,

$$\begin{aligned}
\pi(1 - \alpha|v) - \pi(\alpha|v) &= (v - 1 + \alpha)(\alpha - \sqrt{3\alpha^2 - 2\alpha}) - (v + 1 - 2\alpha)\alpha \\
&= \sqrt{3\alpha^2 - 2\alpha}(\sqrt{3\alpha^2 - 2\alpha} - (v - 1 + \alpha)) \\
&< \sqrt{3\alpha^2 - 2\alpha}(\sqrt{3\alpha^2 - 2\alpha} + 1 - \alpha - 1 + \alpha - \sqrt{3\alpha^2 - 2\alpha}) \\
&= 0.
\end{aligned}$$

- If $s(v) > \alpha$, $\pi(s(v) > \alpha|v) < (v - \alpha)\alpha + \alpha(1 - \alpha) = \pi(s(v) = \alpha|v)$.

Therefore, when $v > 1 - \alpha + \sqrt{3\alpha^2 - 2\alpha}$, $s(v) = \alpha$.

Proof for Proposition 3 When $MP = \alpha$, for $v < \tilde{v}$, if $s(v) = \alpha$, then

$$\pi(\alpha|v) = (v - \alpha)(1 - \tilde{v})^{N-1} + \alpha(1 - (1 - \tilde{v})^{N-1}) < (\tilde{v} - 2\alpha)(1 - \tilde{v})^{N-1} + \alpha = 0.$$

For $v \geq \tilde{v}$, if $s(v) = 0$, then

$$\pi(s(v) = 0|v) = 0 < \pi(\alpha|v).$$

To prove there is exists one and only one $\tilde{v} \in (0, \alpha)$, let

$$F(x) = (x - 2\alpha)(1 - x)^{N-1} + \alpha,$$

$F(0) = -2\alpha + \alpha = -\alpha < 0$, $F(\alpha) = -\alpha(1 - \alpha)^{N-1} + \alpha > 0$ and $F(\cdot)$ is a continuous function, then there exist $\tilde{v} \in (0, \alpha)$ such that $F(\tilde{v}) = 0$ (Intermediate Value Theorem).

Take the first order derivative,

$$F'(x) = (1 - x)^{N-1} - (x - 2\alpha)(N - 1)(1 - x)^{N-2} = (1 - x)^{N-2}(-Nx + 2\alpha(N - 1) + 1)$$

Since $F'(0) = 2\alpha(N - 1) + 1 > 0$ and $F'(\alpha) = (1 - \alpha)^{N-2}(\alpha(N - 2) + 1) > 0$,

$$F''(x) = -N(1 - x)^{N-2} - (N - 2)(1 - x)^{N-3}(-Nx + 2\alpha(N - 1) + 1) < 0,$$

therefore, $F(x)$ is strictly increasing on $(0, \alpha)$ and $F(x) = 0$ has one and only one solution, which is defined by $(x - 2\alpha)(1 - x)^{N-1} + \alpha = 0$

Proof for Proposition 4 When α is fixed, since $\tilde{v} < \alpha$ from Proposition 3, we have

$$\frac{\partial \tilde{v}(\alpha, N)}{\partial N} = \frac{(2\alpha - \tilde{v})(1 - \tilde{v}) \ln(1 - \tilde{v})}{1 - 2\alpha + (2\alpha - \tilde{v})N} < 0,$$

which implies that \tilde{v} is decreasing as N increases for a given α . Note that

$$(1 - \tilde{v}_N)^N = (1 - 2\alpha)(1 - \tilde{v}_N)^{N-1} + \alpha,$$

which implies

$$P_N = (1 - 2\alpha) \frac{P_N}{1 - \tilde{v}_N} + \alpha.$$

For any N ,

$$\frac{\alpha}{2\alpha - \tilde{v}_N} \leq P_N = \frac{\alpha(1 - \tilde{v}_N)}{2\alpha - \tilde{v}_N} \leq \frac{\alpha(1 - \tilde{v}_N)}{2\alpha}.$$

The above equation suggests when $\lim_{N \rightarrow \infty} \tilde{v}_N = 0$,

$$\lim_{N \rightarrow \infty} P_N = \lim_{N \rightarrow \infty} \frac{\alpha}{2\alpha - \tilde{v}_N} = \lim_{N \rightarrow \infty} \frac{\alpha(1 - \tilde{v}_N)}{2\alpha} = \frac{1}{2}.$$

Note that $\{\tilde{v}_n\}_{n=1}^{\infty}$ is a decreasing and bounded sequence, then it is convergent. Let

$$\lim_{N \rightarrow \infty} \tilde{v}_N = \lim_{N \rightarrow \infty} \tilde{v}_{N+1} = K,$$

thus $(\tilde{v}_N - 2\alpha)(1 - \tilde{v}_N)^{N-1} + \alpha = (\tilde{v}_{N+1} - 2\alpha)(1 - \tilde{v}_{N+1})^N + \alpha$, then

$$\frac{\tilde{v}_N - 2\alpha}{\tilde{v}_{N+1} - 2\alpha} = \left(\frac{1 - \tilde{v}_{N+1}}{1 - \tilde{v}_N} \right)^N (1 - \tilde{v}_N).$$

Take the limit on both side, we have,

$$1 = 1 \times (1 - K),$$

therefore,

$$\lim_{N \rightarrow \infty} \tilde{v}_N = 0 \text{ and } \lim_{N \rightarrow \infty} \tilde{P}_N = \frac{1}{2}.$$

When c is fixed, $\alpha = \frac{c}{N}$, still, denote the sequence of \tilde{v}_n as $\{\tilde{v}_n\}_{n=1}^{\infty}$ and the sequence of corresponding provision probability as $\{P_n\}_{n=1}^{\infty}$ where $P_n = (1 - \tilde{v}_n)^n$. According to $(\tilde{v} - 2\frac{c}{N})(1 - \tilde{v})^{N-1} + \frac{c}{N} = 0$, we have

$$\begin{aligned} \frac{\partial \tilde{v}(N)}{\partial N} &= \frac{(1-\tilde{v})[-c+c\tilde{v}+(1-\tilde{v})^N(2c-2cN \ln(1-\tilde{v})+\tilde{v}N^2 \ln(1-\tilde{v}))]}{(1-\tilde{v})^N N(2c-N-2cN+\tilde{v}N^2)} \\ &= \frac{(1-\tilde{v})^2[(2(1-\tilde{v})^{N-1}-1)+(1-\tilde{v})^{N-1}N \ln(1-\tilde{v})(\tilde{v}N-2c)]}{(1-\tilde{v})^N N(2c-\tilde{v}N)(1-N)} \\ &< 0 \end{aligned}$$

which implies that \tilde{v} is decreasing when c is fixed. Note that $\{\tilde{v}_n\}_{n=1}^{\infty}$ is a decreasing and bounded sequence, then it is convergent. Let

$$\lim_{N \rightarrow \infty} \tilde{v}_N = \lim_{N \rightarrow \infty} \tilde{v}_{N+1} = K.$$

Since $(\tilde{v}_N - 2\frac{c}{N})(1 - \tilde{v}_N)^{N-1} + \frac{c}{N} = (\tilde{v}_{N+1} - 2\frac{c}{N+1})(1 - \tilde{v}_{N+1})^N + \frac{c}{N+1}$, then take the limit on both side, we have,

$$K(1 - K)^{N-1} = K(1 - K)^{N-1}(1 - K).$$

Since $K \in [0, \frac{c}{N}]$, if $K > 0$, then

$$1 - K = 0,$$

which contradicts $\sup(K) = \frac{c}{N} < 1$. Therefore, $K = 0$ and

$$\lim_{N \rightarrow \infty} \tilde{v}_N = 0, \lim_{N \rightarrow \infty} \tilde{P}_N = \frac{1}{2}.$$

Proof for Proposition 5 This proof is similar to Proposition 3. When $MP = \alpha$, for $v < \tilde{v}$, if $s(v) = \alpha$, then

$$\pi(\alpha|v) = (v - \alpha)(1 - \tilde{v})^{N-1} + \beta(1 - (1 - \tilde{v})^{N-1}) < (\tilde{v} - \alpha - \beta)(1 - \tilde{v})^{N-1} + \beta = 0$$

For $v \geq \tilde{v}$, if $s(v) = 0$, then

$$\pi(s(v) = 0|v) = 0 < \pi(\alpha|v)$$

To prove there is exists one and only one $\tilde{v} \in (0, \alpha)$, let

$$F(x) = (x - \alpha - \beta)(1 - x)^{N-1} + \beta,$$

$F(0) = -\alpha - \beta + \beta = -\alpha < 0$, $F(\alpha) = -\beta(1 - \alpha)^{N-1} + \beta > 0$ and $F(\cdot)$ is a continuous function, then there exist $\tilde{v} \in (0, \alpha)$ such that $F(\tilde{v}) = 0$ (Intermediate Value Theorem).

Take the first order derivative,

$$F'(x) = (1 - x)^{N-1} - (x - \alpha - \beta)(N - 1)(1 - x)^{N-2} = (1 - x)^{N-1}(-Nx + (\alpha + \beta)(N - 1) + 1).$$

Since $F'(0) = (\alpha + \beta)(N - 1) + 1 > 0$ and $F'(\alpha) = (1 - \alpha)^{N-2}(\beta(N - 1) + 1) > 0$,

$$F''(x) = -N(1 - x)^{N-2} - (N - 2)(1 - x)^{N-3}(-Nx + (\alpha + \beta)(N - 1) + 1) < 0,$$

therefore, $F(x)$ is strictly increasing on $(0, \alpha)$ and $F(x) = 0$ has one and only one solution, which is defined by $(x - \alpha - \beta)(1 - x)^{N-1} + \beta = 0$

Proof for Proposition 6 This proof is similar to Proof for Proposition 4. **When** α is fixed, since $\tilde{v} < \alpha$ from Proposition 3, we have

$$\frac{\partial \tilde{v}(\alpha, N)}{\partial N} = \frac{(\alpha + \beta - \tilde{v})(1 - \tilde{v}) \ln(1 - \tilde{v})}{1 - \alpha - \beta + (\alpha + \beta - \tilde{v})N} < 0,$$

which implies that \tilde{v} is decreasing as N increases for a given α . Note that $\{\tilde{v}_n\}_{n=1}^{\infty}$ is a decreasing and bounded sequence, then it is convergent. Let

$$\lim_{N \rightarrow \infty} \tilde{v}_N = \lim_{N \rightarrow \infty} \tilde{v}_{N+1} = K,$$

thus $(\tilde{v}_N - \alpha - \beta)(1 - \tilde{v}_N)^{N-1} + \beta = (\tilde{v}_{N+1} - \alpha - \beta)(1 - \tilde{v}_{N+1})^N + \beta$, then

$$\frac{\tilde{v}_N - \alpha - \beta}{\tilde{v}_{N+1} - \alpha - \beta} = \left(\frac{1 - \tilde{v}_{N+1}}{1 - \tilde{v}_N} \right)^N (1 - \tilde{v}_N).$$

Take the limit on both side, we have,

$$1 = 1 \times (1 - K),$$

therefore,

$$\lim_{N \rightarrow \infty} \tilde{v}_N = 0 \text{ and } \lim_{N \rightarrow \infty} \tilde{P}_N = \frac{1}{2},$$

since $P_N = \frac{\beta(1-\tilde{v}_N)}{\alpha+\beta-\tilde{v}_N}$.

When c is fixed, $\alpha = \frac{c}{N}$, still, denote the sequence of \tilde{v}_n as $\{\tilde{v}_n\}_{n=1}^{\infty}$ and the sequence of corresponding provision probability as $\{P_n\}_{n=1}^{\infty}$ where $P_n = (1 - \tilde{v}_n)^n$. According to $(\tilde{v} - \frac{c}{N} - \beta)(1 - \tilde{v})^{N-1} + \beta = 0$, we have

$$\begin{aligned} \frac{\partial \tilde{v}(N)}{\partial N} &= -\frac{(\tilde{v} - \frac{c}{N} - \beta)(1 - \tilde{v}) \ln(1 - \tilde{v})}{1 + c - \beta + (\beta - \tilde{v})N - \frac{c}{N}} \\ &= -\frac{(\tilde{v} - \frac{c}{N} - \beta)(1 - \tilde{v}) \ln(1 - \tilde{v})}{c - \tilde{v}N + (N-1)\beta + 1 - \frac{c}{N}} \\ &< 0 \end{aligned}$$

which implies that \tilde{v} is decreasing when c is fixed. Note that $\{\tilde{v}_n\}_{n=1}^{\infty}$ is a decreasing and bounded sequence, then it is convergent. Let

$$\lim_{N \rightarrow \infty} \tilde{v}_N = \lim_{N \rightarrow \infty} \tilde{v}_{N+1} = K.$$

Since $(\tilde{v}_N - \frac{c}{N} - \beta)(1 - \tilde{v}_N)^{N-1} + \frac{c}{N} = (\tilde{v}_{N+1} - \frac{c}{N+1} - \beta)(1 - \tilde{v}_{N+1})^N + \frac{c}{N+1}$, then take the limit on both side, we have,

$$(K - \beta)(1 - K)^{N-1} = (K - \beta)(1 - K)^{N-1}(1 - K).$$

Since $K \in [0, \frac{c}{N}]$, if $K > 0$ and $K \neq \beta$, then

$$1 - K = 0,$$

which contradicts $\sup(K) = \frac{c}{N} < 1$. If $K = \beta$, then $\lim_{N \rightarrow \infty} \tilde{P}_N = \infty$, which is impossible. Therefore, $K = 0$ and

$$\lim_{N \rightarrow \infty} \tilde{v}_N = 0, \lim_{N \rightarrow \infty} \tilde{P}_N = 1.$$

Proof for Proposition 7 First, note that

$$\mathcal{P}_1(\tilde{v}, k) = (1 - \tilde{v})^{N-1} + \binom{N-1}{1} \tilde{v}(1 - \tilde{v})^{N-2} + \dots + \binom{N-1}{k-1} \tilde{v}^{k-1}(1 - \tilde{v})^{N-k} + \binom{N-1}{k} \tilde{v}^k(1 - \tilde{v})^{N-k-1}$$

is the probability that the public good will be provided if the player with value \tilde{v} accept the the contract, and

$$\mathcal{P}_0(\tilde{v}, k) = \mathcal{P}_1(\tilde{v}, k-1) = (1 - \tilde{v})^{N-1} + \binom{N-1}{1} \tilde{v}(1 - \tilde{v})^{N-2} + \dots + \binom{N-1}{k-1} \tilde{v}^{k-1}(1 - \tilde{v})^{N-k}$$

is the probability that the public good will be provided if the player with value \tilde{v} reject the the contract. The expected profit of accepting the contract for the player \tilde{v} is

$$\pi(MP|\tilde{v}) = (\tilde{v} - MP)\mathcal{P}_1 + \beta(1 - \mathcal{P}_1);$$

The expected profit of rejecting the contract for the player \tilde{v} is

$$\pi(0|\tilde{v}) = \tilde{v}\mathcal{P}_0$$

By equating the above two equations we can find the \tilde{v} above which players will accept the contract and vice versa. Thus,

$$\tilde{v}(\mathcal{P}_1 - \mathcal{P}_0) - \frac{c}{N-k}\mathcal{P}_1 + \beta(1 - \mathcal{P}_1) = 0,$$

or

$$\tilde{v}(\mathcal{P}_1 - \mathcal{P}_0) + \beta(1 - \mathcal{P}_1) = \frac{c}{N-k}\mathcal{P}_1.$$

The LHS of the above equation is the marginal benefit of accepting the contract, which equals to the RHS the marginal cost of accepting the contract.

$$\tilde{v} \binom{N-1}{k} \tilde{v}^k(1 - \tilde{v})^{N-k-1} = (\frac{c}{N-k} + \beta)\mathcal{P}_1 - \beta.$$

Thus,

$$\tilde{v} \binom{N-1}{k} \tilde{v}^k(1 - \tilde{v})^{N-k-1} = (\frac{c}{N-k} + \beta)\mathcal{P}_1 - \beta.$$

If $k = 0$, the above equation reduces to the condition described in Section 4.2.