The welfare comparison of ad-valorem tax and specific tax with quality choice of a consumer*

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Abstract

This paper compares ad-valorem and specific taxes in terms of welfare when consumers have both a quality and a quantity choice under perfect competition. In the setting, while ad-valorem tax causes only income effect, specific tax causes not only income effect but also substitution effect. This implies that if substitution effect exists, specific tax generates welfare losses. Using a constant elasticity of substitution (CES) utility function and a linear price function, we show that ad-valorem tax is superior to specific tax except for Leontief preference under which the two forms of commodity taxes are equivalent.

JEL Classification: H21, H22, H31
Keywords: quality choice of a consumer, specific tax, substitution effect, ad-valorem tax

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1. Introduction

In the field of public finance, there are many existing literatures of comparing social welfare under an ad-valorem tax and a specific tax. The seminal contribution of **?** examines the impact of the tax structure on the welfare in keeping the tax revenue under monopoly market and shows that an ad-valorem tax is superior to a specific tax. Some works dealing with the comparison under imperfect competition support the conclusion (see **?**, **?**, **?**, **?**, and **?**).

However, these studies ignore the impact of the tax structure on product quality. If both forms of taxation affect the quality, specific taxation can be optimal under perfect competition (see **??** and **?**).

The objective of the paper is to compare the two forms of commodity taxation in terms of welfare when consumers have both a quantity and a quality choice under perfect competition. The crucial difference is that aforementioned literatures assume that firms, not consumers, determine product quality and then consumers can only choose a quantity given homogeneous product.

We find that introducing a specific tax distorts consumers' choice between a quantity and a quality whereas introducing an ad-valorem tax does not affect the choice. In other words, while the marginal rate of substitution between a quantity and a quality includes tax rate under a specific tax, the corresponding marginal rate of substitution does not include tax rate under an ad-valorem tax. This implies that if substitution effect exists, specific taxation deters their consumption behavior from that without any taxes, and it generates welfare losses to some extent. Assuming that a price function is linear and individual preference is expressed by the constant elasticity of substitution (CES) utility function, we show that while an ad-valorem tax is superior to a specific tax in the presence of substitution effect, the two forms of taxation are equivalent when consumers have Leontief preference without substitution effect. The findings implies that substitution effect plays an important role in ad-valorem tax dominating specific tax in terms of consumer welfare. Moreover, we investigate the effects of changes in ad-valorem tax and specific tax to the choice between quality and quantity. Comparative statics results show that an increase in the ad-valorem tax decrease both quality and quantity owing to income effect, and an increase in the specific tax decrease quantity but the change in quality is unclear because of substitution effect. If we assume the CES utility function and a linear price function, the direction is determined by the degree of the elasticity of substitution.

The basic structure of our model follows **?**. They study the optimal ad-valorem and specific taxation as well as nonlinear income taxation, but our setting and concern differ from **?**. They allow the government to employ both an ad-valorem and a specific tax, whereas we allow either one of the two taxes. Furthermore, they investigate the two forms of commodity taxation under nonlinear labor income taxes when individuals have their different productivity and there is the asymmetric information between the policymaker and taxpayers with respect to individuals 'productivity. Both consumption taxes play a crucial role in relaxing incentive constraint, so such taxes are necessary to implement the second best allocation, which contradicts the canonical results provided by ?. Contrary to their paper, we allow the government to levy taxes only on such consumption and assume homogeneous individuals, that is, there is no asymmetric information. Instead of studying the linkage between commodity taxes and relaxing incentive constraints on income taxes, we present the welfare comparison between an ad-valorem and a specific tax in the sense of taxpayer's utility.

2. The model

In this model, there is a single representative consumer with initial wealth I. Taxpayers consume only one kind of good and they can choose both a quantity and a quality. Let $y \in \mathbb{R}_{++}$ be a quantity and $\theta \in \mathbb{R}_{++}$ be a quality. Both y and θ affect their utility, and the price for unit consumption is determined by θ. The price function is denoted by p and we assume that p is differentiable in θ. Here, it's natural to assume that the unit price increases in θ ($p'(\cdot) > 0$). It is assumed that all the taxpayers have the same utility on a quantity and a quality given by $v(y, \theta)$. To guarantee interior solutions with respect to a quantity and a quality, we assume that v is twice differentiable, strictly increasing and strictly concave in y and in θ.

The government can impose on the unit price $p(\theta)$ or the quantity y. If she adopts an ad-valorem tax denoted by t^{adv}, the individual's budget constraint must be:

$$(1 + t^{adv})p(\theta)y \leq I, \tag{1}$$

while if she chooses a specific tax denoted by t^s, the budget constraint should become

$$(p(\theta) + t^s)y \leq I. \tag{2}$$

The taxpayers maximize their own utility with respect to a quantity y and a quality θ given the budget constraint. From the first order conditions, we can derive the following ad-valorem and specific tax wedge:

$$\frac{v_y(y^{adv}, \theta^{adv})}{v_\theta(y^{adv}, \theta^{adv})} = \frac{p(\theta^{adv})}{p'(\theta^{adv})y^{adv}} \tag{3}$$

$$\frac{v_y(y^s, \theta^s)}{v_\theta(y^s, \theta^s)} = \frac{p(\theta^s) + t^s}{p'(\theta^s)y^s} \tag{4}$$

where v_k denotes the derivative of v with respect to $k = y, \theta$, the subscript adv the choice of an ad-valorem tax, and the subscript s the choice of a specific tax. Given initial wealth I, tax rate t and tax scheme $i = s, adv$, their quantity choice function, quality choice function and indirect utility function are defined as follows: $y(t^i, I)$, $\theta(t^i, I)$ and $V(t^i, I)$.

When the government employs an ad-valorem tax, the budget constraint faced by the government is:

$$t^{adv}p(\theta^{adv})y^{adv} \geq R \tag{5}$$

where R is an exogenous amount of public expenditure. On the other hand, if it imposes a specific tax, the budget constraint is:

$$t^{s}y^{s} \geq R \tag{6}$$

3. Comparative statics of (y, θ) on tax rate t under ad-valorem tax and specific tax

How do an ad-valorem tax and a specific tax affect taxpayers' choice of quantity y and quality θ? Hereafter, we study the sensitivity of their choice (y, θ) in terms of an increase of each tax rate.

Let $|K|$ and $|M|$ be the determinants of border Hessian matrix on their optimization problem under ad-valorem tax and specific tax respectively. By the comparative statics of y and θ under ad-valorem tax, these derivatives are:

$$\frac{\partial \theta}{\partial t^{adv}} = \frac{1}{|K|}\frac{v_\theta v_y p(\theta)}{\gamma^{adv}}(1 - \rho - \varepsilon_y^{v_\theta}) \tag{7}$$

$$\frac{\partial y}{\partial t^{adv}} = \frac{1}{|K|}\frac{v_\theta p(\theta) y}{\gamma^{adv}}\left(\frac{v_\theta}{y}(1 - \varepsilon_y^{v_\theta}) + v_y\{\frac{v_{\theta\theta}}{v_\theta} - \frac{p''(\theta)}{p'(\theta)}\}\right) \tag{8}$$

where γ^{adv} is the Lagrangian multiplier under ad-valorem tax, $\rho \equiv -\frac{v_{yy}y}{v_y}$ is the curvature of marginal utility on quantity y, $\varepsilon_y^{v_\theta} \equiv \frac{v_{\theta y}}{v_\theta}y$ is the demand elasticity of v_θ, and $v_{\theta y}$ is the cross-derivative of v. We assume that the second-order condition for utility maximization under ad-valorem tax holds, which means that $\varepsilon_y^{v_\theta} \geq 1$ and $p''(\theta) \geq 0$ (see Appendix A).

From equation (7) and (8), both quality θ and quantity y decrease in response to an increase of an ad-valorem tax, that is, $\frac{\partial \theta}{\partial t^{adv}} < 0$ and $\frac{\partial y}{\partial t^{adv}} < 0$, if the second-order condition is satisfied. The intuition is that, as the ad-valorem tax increases, their disposable income decreases by $\frac{I}{1+t^{adv}}$; on the other hand, the marginal rate of substitution between quantity and quality is the same as that before imposing the tax. Therefore, since only income effect occurs, decreasing their disposable income makes individuals to decrease both quantity and quality.

On the other hand, the comparative statics results under specific tax are:

$$\frac{\partial \theta}{\partial t^{s}} = \frac{1}{|M|}\frac{v_\theta v_y}{\gamma^{s}}(2 - \rho - \varepsilon_y^{v_\theta}) \tag{9}$$

$$\frac{\partial y}{\partial t^{s}} = \frac{1}{|M|}\left(-\frac{v_\theta v_{\theta y} y}{\gamma^{s}} + \frac{v_y v_\theta y}{\gamma^{s}}(\frac{v_{\theta\theta}}{v_\theta} - \frac{p''(\theta)}{p'(\theta)})\right) \tag{10}$$

where γ^s is the Lagrangian multiplier under specific tax. Similar to ad-valorem tax cases, we assume that the second-order condition under specific tax holds, which means that $\varepsilon_y^{v_\theta} \geq 1$ and $p''(\theta) \geq 0$ (see Appendix A).

From equation (9) and (10), while $\frac{\partial y}{\partial t^s}$ is negative, the sign of $\frac{\partial \theta}{\partial t^s}$ is ambiguous. Put it differently, employing a specific tax induces individuals to buy lower quantity but they may improve quality instead of decreasing quantity in contrast with the choice between quantity and quality under ad-valorem tax. This is because not only income effect but also substitution effect occurs as shown in equation (4). From equation (9), whether the quality is improved also depends on ρ, which means that how much the consumers do not want to dwindle the quantity. If ρ is sufficiently large (small), the bracket in (9) may be negative (positive), which makes $\frac{\partial \theta}{\partial t^s}$ negative (positive). The intuition is that when consumers are (not) sensitive to the decrease of quantity, they choose lower (greater) quality, that is, the income effect (does not) dominates substitution effect. We summarize the results as follows.

Proposition 1. *Assume that sufficient conditions for taxpayers' optimization problems are satisfied. Then,*

1. *Under an ad-valorem tax, both $\frac{\partial \theta}{\partial t^{adv}}$ and $\frac{\partial y}{\partial t^{adv}}$ are negative. In other words, both quality θ and quantity y decrease in response to a tax increase.*

2. *Under a specific tax, while $\frac{\partial y}{\partial t^s}$ is negative, which means that the quantity decreases due to a tax increase, the sign of $\frac{\partial \theta}{\partial t^s}$ is ambiguous and also depends on ρ.*

Here, we present a special case in which the second-order condition is satisfied. For example, we assume that the price function is linear such as $p(\theta) = a\theta$ where $a > 0$ and individuals' preference is the constant elasticity of substitution (CES) utility function which is expressed by $v(y, \theta) = (\alpha y^{-\sigma} + \beta\theta^{-\sigma})^{-\frac{1}{\sigma}}$, where $\alpha > 0$, $\beta > 0$, $\alpha + \beta = 1$, and σ is a measure of substitutability, assuming that $\sigma \geq 1$. In this case, it is obvious that $p''(\theta) \geq 0$ because $p(\theta)$ is linear. In addition, if $\sigma \geq 1$, $\varepsilon_y^{v_\theta} \geq 1$ under both ad-valorem tax and specific tax. Therefore, the results of proposition 1 hold in the environment. Moreover, we demonstrate that consumers remain quality or select lower one under specific tax, that is, $\frac{\partial \theta}{\partial t^s} \leq 0$. These results are shown in Appendix B.

4. Welfare Comparison

This section examines which the government should adopt an ad-valorem or a specific tax under the same tax revenue. From now on, we stick to the following environment: the price function is $p(\theta) = a\theta$ where $a > 0$, and individuals' preference is the CES utility function which is given by $v(y, \theta) = (\alpha y^{-\sigma} + \beta\theta^{-\sigma})^{-\frac{1}{\sigma}}$ where $\alpha > 0$, $\beta > 0$, $\alpha + \beta = 1$, and $\sigma \geq 1$. As mentioned in the above section, the assumptions on the price function and the utility function ensure the second-order conditions for maximization problem. In the setting, we can state the following.

Proposition 2. *Assume that the price function is linear and individual's preference is the CES utility function, that is, $p(\theta) = a\theta$ where $a > 0$ and $v(y, \theta) = (\alpha y^{-\sigma} + \beta\theta^{-\sigma})^{-\frac{1}{\sigma}}$ where $\alpha > 0$, $\beta > 0$, $\alpha + \beta = 1$, and $\sigma \geq 1$. If the tax revenue keeps the same under ad-valorem and specific tax,*

1. *ad-valorem tax is superior to specific tax in terms of the consumer's welfare except for $\sigma = \infty$;*
2. *the difference of the welfare, i.e., $v(y^{adv}, \theta^{adv}) - v(y^s, \theta^s)$ decreases as σ increases;*
3. *ad-valorem tax and specific tax are indifferent in terms of the welfare when $\sigma = \infty$, that is, the utility function is Leontief preference $v(y, \theta) = \min\{y, \theta\}$.*

It is shown in Appendix C. This result is very intuitive. As shown by equation (3) and (4), the marginal rate of substitution between a quantity and a quality includes tax rate under a specific tax, the corresponding marginal rate of substitution does not include tax rate under an ad-valorem tax. In other words, income effect only changes their consumption choice under an ad-valorem tax whereas substitution effect as well as income effect distorts their choice under a specific tax. This implies that if substitution effect exists, specific taxation generates welfare losses since it distorts individual's behavior relative to the case without any tax policy. Therefore, an ad-valorem tax is superior to a specific tax. In addition, the increase of σ implies that the substitution effect become smaller. In other words, welfare losses caused by specific tax diminishes. Consequently, the welfare difference decreases, and in particular, when the utility function is Leontief preference, it's possible for the policymaker to achieve the same utility level via a specific tax as an ad-valorem tax. This is because there is no substitution effect under Leontief preference, which leads that their consumption decision toward changing the tax rate is affected only by income effect, regardless of an ad-valorem tax and a specific tax.

4.1 Numerical Simulation

For the purpose of justifying that substitution effect plays an important role in the comparison between ad-valorem and specific taxes under the same tax revenue, we conduct numerical simulation. To assess the crucial role, we examine how the difference of the utility level under these two tax systems that get the same tax revenue changes as σ varies. We set the following environment: $R = 1$, $I = 6$, $a = 1$, $\alpha = \beta = 0.5$, and $\sigma \in [1, \infty)$. Figure 1 exhibits the difference of the utility level in changing the elasticity of substitution σ. The graph shows that the difference is positive and decreases as σ increases, and it converges to 0 as σ goes to infinity. Therefore, the result of proposition 2 is reasonable.

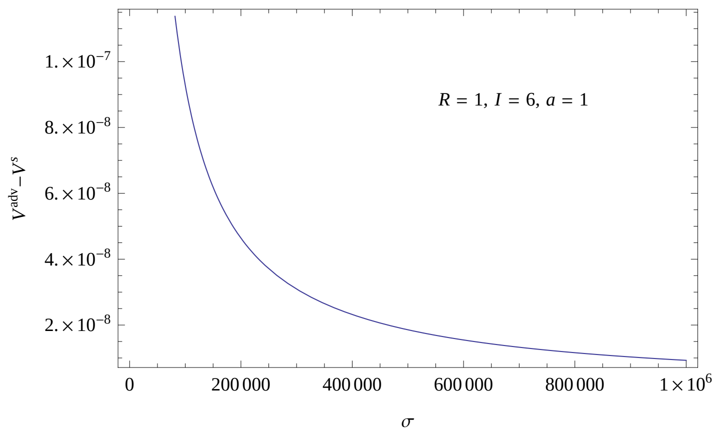

Figure 1: Simulations of $v(y^{adv}, \theta^{adv}) - v(y^s, \theta^s)$ with different elasticity of substitution

5. Conclusion

This paper develops the comparison of ad-valorem tax and specific tax from the point of representative consumer's welfare when consumers have both a quantity and a quality choice. Contrary to most of papers investigating such comparison of social welfare under imperfect competition, we allow for consumers' quality choice as well as competitive environment and consumers' welfare. Though our model has a limitation in linear price function on quality and CES utility function, we identify that the substitution effect distorts the consumer's optimal choice under specific tax, which leads to that ad-valorem tax dominates it in the sense of consumer's utility. On the other hand, since substitution effect vanishes under Leontief preferences, the two forms of taxes are equivalent. Our main result is different from that from previous researches which show that specific tax is better that ad-valorem tax for social welfare under quality choice by production sector as **??** and **?**.

Although our model is simple, its result can be applied to reality. For instance, in housing choice, we select its location and scale, and governments often collect taxes from such consumption. Our result suggests that they should levy taxes on (unit) land price if consumers have substitution between location and scale. Consequently, it has implications for the understanding of the effect of two forms of commodity tax to consumer's quality choice. However, we can't completely cover the situation (e.g. limited capacity for land use). In addition, it is possible to increase the number of goods, but we have priority over unveiling the interaction between preference and two tax schemes. It is possible to extend our model to these directions for future research.

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Appendix A: Comparative statics under varying tax rates

Under the ad-valorem tax, individuals problem is formulated as follows.

$$\max_{y,\theta} \; v(y,\theta)$$
$$s.t. \; (1+t^{adv})p(\theta)y \leq I.$$

The corresponding Lagrangian is:

$$\mathcal{L} = v(y,\theta) + \gamma^{adv}[I - (1+t^{adv})p(\theta)y]$$

The first-order conditions are given by:

$$\frac{\partial \mathcal{L}}{\partial y} = v_y - \gamma^{adv}(1+t^{adv})p(\theta) = 0$$
$$\frac{\partial \mathcal{L}}{\partial \theta} = v_\theta - \gamma^{adv}(1+t^{adv})p'(\theta)y = 0$$
$$\frac{\partial \mathcal{L}}{\partial \gamma^{adv}} = I - (1+t^{adv})p(\theta)y = 0$$

The bordered Hessian matrix for this problem is as follows.

$$K = \begin{bmatrix} v_{yy} & -\gamma^{adv}(1+t^{adv})p'(\theta) + v_{y\theta} & -(1+t^{adv})p(\theta) \\ -\gamma^{adv}(1+t^{adv})p'(\theta) + v_{y\theta} & v_{\theta\theta} - \gamma^{adv}(1+t^{adv})p''(\theta)y & -(1+t^{adv})p'(\theta)y \\ -(1+t^{adv})p(\theta) & -(1+t^{adv})p'(\theta)y & 0 \end{bmatrix}$$

Its determinant $|K|$ is:

$$|K| = 2\frac{v_y(v_\theta)^2}{y(\gamma^{adv})^2}(\varepsilon_y^{v_\theta} - 1) - \left[(\frac{v_y}{\gamma^{adv}})^2 v_\theta \{ \frac{v_{\theta\theta}}{v_\theta} - \frac{p''(\theta)}{p'(\theta)} \} + (\frac{v_\theta}{\gamma^{adv}})^2 v_{yy}) \right]$$

Here, we assume that $\varepsilon_y^{v_\theta} \geq 1$ and $p''(\theta) \geq 0$ for utility maximization, that is, $|K|$ is positive. Therefore, it ensures the existence of the inverse matrix K^{-1} and then Cramer's rule is available. Then, the derivative of θ with respect to t^{adv} is:

$$\frac{\partial \theta}{\partial t^{adv}} = \frac{1}{|K|} \begin{vmatrix} v_{yy} & \gamma^{adv}p(\theta) & -(1+t^{adv})p(\theta) \\ -\gamma^{adv}(1+t^{adv})p'(\theta) + v_{y\theta} & \gamma^{adv}p'(\theta)y & -(1+t^{adv})p'(\theta)y \\ -(1+t^{adv})p(\theta) & p(\theta)y & 0 \end{vmatrix}$$
$$= \frac{1}{|K|}\frac{v_y v_\theta p(\theta)}{\gamma^{adv}}(1 - \rho - \varepsilon_y^{v_\theta})$$

Also, the derivative of y can be found as follows:

$$\frac{\partial y}{\partial t^{adv}} = \frac{1}{|K|} \begin{vmatrix} \gamma^{adv}p(\theta) & -\gamma^{adv}(1+t^{adv})p'(\theta) + v_{y\theta} & -(1+t^{adv})p(\theta) \\ \gamma^{adv}p'(\theta)y & v_{\theta\theta} - \gamma^{adv}(1+t^{adv})p''(\theta)y & -(1+t^{adv})p'(\theta)y \\ p(\theta)y & -(1+t^{adv})p'(\theta)y & 0 \end{vmatrix}$$

$$= \frac{1}{|K|}\frac{v_\theta p(\theta) y}{\gamma^{adv}}\left(\frac{v_\theta}{y}(1-\varepsilon_y^{v_\theta}) + v_y\{\frac{v_{\theta\theta}}{v_\theta} - \frac{p''(\theta)}{p'(\theta)}\}\right)$$

On the other hand, individuals' problem under the specific tax is:

$$\max_{y,\theta}\ v(y,\theta)$$
$$s.t.\ (p(\theta)+t^s)y \le I.$$

The corresponding Lagrangian is:

$$\mathcal{L} = v(y,\theta) + \gamma^s[I - (p(\theta)+t^s)y]$$

The first-order conditions are given by:

$$\frac{\partial \mathcal{L}}{\partial y} = v_y - \gamma^s(p(\theta)+t^s) = 0$$
$$\frac{\partial \mathcal{L}}{\partial \theta} = v_\theta - \gamma^s p'(\theta) y = 0$$
$$\frac{\partial \mathcal{L}}{\partial \gamma} = I - (p(\theta)+t^s)y = 0$$

The bordered Hessian matrix for this problem is:

$$M = \begin{bmatrix} v_{yy} & -\gamma^s p'(\theta) + v_{y\theta} & -(p(\theta)+t^s) \\ -\gamma^s p'(\theta) + v_{y\theta} & v_{\theta\theta} - \gamma^s p''(\theta) y & -p'(\theta)y \\ -(p(\theta)+t^s) & -p'(\theta)y & 0 \end{bmatrix}$$

Its determinant $|M|$ is:

$$|M| = 2\frac{v_y(v_\theta)^2}{y(\gamma^s)^2}(\varepsilon_y^{v_\theta}-1) - \left[(\frac{v_y}{\gamma^s})^2 v_\theta\{\frac{v_{\theta\theta}}{v_\theta} - \frac{p''(\theta)}{p'(\theta)}\} + (\frac{v_\theta}{\gamma^s})^2 v_{yy})\right]$$

Again, we assume that $\varepsilon_y^{v_\theta} \ge 1$ and $p''(\theta) \ge 0$ for utility maximization, that is, $|M|$ is positive. Using Cramer's rule,

$$\frac{\partial \theta}{\partial t^s} = \frac{1}{|M|}\begin{vmatrix} v_{yy} & \gamma^s & -(p(\theta)+t^s) \\ -\gamma^s p'(\theta) + v_{y\theta} & 0 & -p'(\theta)y \\ -(p(\theta)+t^s) & y & 0 \end{vmatrix}$$
$$= \frac{1}{|M|}\frac{v_\theta v_y}{\gamma^s}(2-\rho-\varepsilon_y^{v_\theta})$$

Similarly, we observe the derivative of y on t^s as follows.

$$\frac{\partial y}{\partial t^s} = \frac{1}{|M|}\begin{vmatrix} \gamma^s & -\gamma^s p'(\theta) + v_{y\theta} & -(p(\theta)+t^s) \\ 0 & v_{\theta\theta} - \gamma^s p''(\theta) y & -p'(\theta)y \\ y & -p'(\theta)y & 0 \end{vmatrix}$$
$$= \frac{1}{|M|}\left(-\frac{v_\theta v_{\theta y} y}{\gamma^s} + \frac{v_y v_\theta y}{\gamma^s}(\frac{v_{\theta\theta}}{v_\theta} - \frac{p''(\theta)}{p'(\theta)})\right)$$

□

Appendix B: A sufficient condition for the maximization problem

Assume $p(\theta) = a\theta$ and $v(y, \theta) = (\alpha y^{-\sigma} + \beta\theta^{-\sigma})^{-\frac{1}{\sigma}}$, where $a > 0$, $\alpha > 0$, $\beta > 0$, $\alpha + \beta = 1$, and $\sigma \geq 1$. In this case, $\varepsilon_y^{v_\theta}$ is given by:

$$\varepsilon_y^{v_\theta} = \alpha(\sigma + 1)(\alpha y^{-\sigma} + \beta\theta^{-\sigma})^{-1} y^{-\sigma} \tag{B.1}$$

Before examining the second-order conditions under ad-valorem tax and specific tax, we suggest the first-order conditions in the setting from equation (3) and (4):

$$\alpha y^{-\sigma} = \beta\theta^{-\sigma} \tag{B.2}$$

$$y^{\sigma} = \frac{\alpha}{\beta} \frac{a\theta^{\sigma+1}}{a\theta + t^s} \tag{B.3}$$

Equation (B.2) is the first-order condition under ad-valorem tax and equation (B.3) is one under specific tax.

We now turn to the analysis of the second-order conditions. First, we derive a sufficient condition under an ad-valorem tax. Substituting equation (B.2) into (B.1) yields:

$$\varepsilon_y^{v_\theta} = \frac{1}{2}(\sigma + 1) \tag{B.4}$$

Therefore, if $\sigma \geq 1$, $\varepsilon_y^{v_\theta} \geq 1$. On the other hand, substituting equation (B.3) into (B.1), $\varepsilon_y^{v_\theta}$ under specific tax can be rewritten as follows:

$$\varepsilon_y^{v_\theta} = (\sigma + 1)\frac{a\theta + t^s}{2a\theta + t^s} \tag{B.5}$$

This means that if $\sigma \geq \frac{a\theta}{a\theta + t^s}$, $\varepsilon_y^{v_\theta} \geq 1$. Note that $\frac{a\theta}{a\theta + t^s} < 1$ under $R > 0$.

To sum up, $\sigma \geq 1$ is a sufficient condition to yield a locally maximum solution under both ad-valorem and specific tax.

Next, we compute ρ under the setting. By the definition and equation (B.1), it can be rewritten as follows:

$$\rho = -(\sigma + 1)\left[\alpha(\alpha y^{-\sigma} + \beta\theta^{-\sigma})^{-1} y^{-\sigma} - 1\right] = 1 - \varepsilon_y^{v_\theta} + \sigma \tag{B.6}$$

Substituting equation (B.6) into equation (9), it yields:

$$\frac{\partial\theta}{\partial t^s} = \frac{1}{|M|}\frac{v_\theta v_y}{\gamma^s}(1 - \sigma) \tag{B.7}$$

As a result, the sign of $\frac{\partial\theta}{\partial t^s}$ is determined by σ. If σ is lower than 1, it is positive since the substitutability is high. However, σ is equal to or greater than 1 for utility maximization, which means that the complementarity is high. Therefore, $\frac{\partial\theta}{\partial t^s}$ is non-positive under $\sigma \geq 1$.

□

Appendix C: The welfare comparison

Assume that the price function is linear and individual's preference is the CES utility function. First, we compute the optimal indirect utility function under ad-valorem tax. Substituting (B.2) into equation (1) yields:

$$y^{adv} = \sqrt{\frac{I}{(1+t^{adv})a}\left(\frac{\alpha}{\beta}\right)^{\frac{1}{\sigma}}} \tag{C.1}$$

In addition, using (B.2) and (C.1), equation (5) is rewritten as follows:

$$t^{adv} = \frac{R}{I-R} \tag{C.2}$$

Combining (C.1) and (C.2) yields:

$$y^{adv} = \sqrt{\frac{I-R}{a}\left(\frac{\alpha}{\beta}\right)^{\frac{1}{\sigma}}} \tag{C.3}$$

Thus, we can derive the optimal indirect utility function, using (B.2) and then substituting (C.3) into the CES utility function as follows:

$$\begin{aligned} v(y^{adv},\theta^{adv}) &= (\alpha y^{-\sigma}+\beta\theta^{-\sigma})^{-\frac{1}{\sigma}} = (2\alpha)^{-\frac{1}{\sigma}} y^{adv} \\ &= 2^{-\frac{1}{\sigma}}(\alpha\beta)^{-\frac{1}{2\sigma}} R^{\frac{1}{2}}\left(\frac{\frac{I}{R}-1}{a}\right)^{\frac{1}{2}} \end{aligned} \tag{C.4}$$

Here, we assume that $\frac{I}{R} > 1$ to avoid a complex number, as seen in (C.4). Next, we compute the optimal indirect utility function under specific tax. From equation (6), we can get

$$y^{s} = \frac{R}{t^{s}} \tag{C.5}$$

Substituting (C.5) into equation (2) yields:

$$\theta^{s} = \frac{t^{s}(\frac{I}{R}-1)}{a} \tag{C.6}$$

Moreover, substituting (C.6) into (B.3) and using (C.5) yields:

$$y^{s} = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{2\sigma}}\left[\frac{(\frac{I}{R}-1)^{\sigma+1}}{a^{\sigma+1}(1+\frac{\frac{I}{R}-1}{a})}\right]^{\frac{1}{2\sigma}} R^{\frac{1}{2}} \tag{C.7}$$

On the other hand, substituting (C.5) into (C.6) yields:

$$\theta^{s} = \frac{I-R}{ay^{s}} \tag{C.8}$$

Substituting (C.7) into (C.8) yields:

$$\theta^s = \frac{\frac{I}{R}-1}{a}\left(\frac{\alpha}{\beta}\right)^{\frac{-1}{2\sigma}}\left[\frac{(\frac{I}{R}-1)^{\sigma+1}}{a^{\sigma+1}(1+\frac{\frac{I}{R}-1}{a})}\right]^{\frac{-1}{2\sigma}} R^{\frac{1}{2}} \tag{C.9}$$

Thus, we can derive the optimal indirect utility function, substituting (C.7) and (C.9) into the CES utility function as follows:

$$v(y^s,\theta^s) = (\alpha\beta)^{-\frac{1}{2\sigma}} R^{\frac{1}{2}}\left(\frac{\frac{I}{R}-1}{a}\right)^{\frac{\sigma+1}{2\sigma}}\left(1+\frac{\frac{I}{R}-1}{a}\right)^{\frac{1}{2\sigma}} a^{\frac{1}{\sigma}}\left(a+\frac{2I}{R}-2\right)^{\frac{-1}{\sigma}} \tag{C.10}$$

Now, we compare the utility level under ad-valorem tax with one under specific tax. Using (C.4) and (C.10), the difference is

$$\begin{aligned}
&v(y^{adv},\theta^{adv}) - v(y^s,\theta^s)\\
&= (\alpha\beta)^{-\frac{1}{2\sigma}} R^{\frac{1}{2}}\left[2^{-\frac{1}{\sigma}}\left(\frac{\frac{I}{R}-1}{a}\right)^{\frac{1}{2}} - \left(\frac{\frac{I}{R}-1}{a}\right)^{\frac{\sigma+1}{2\sigma}}\left(1+\frac{\frac{I}{R}-1}{a}\right)^{\frac{1}{2\sigma}} a^{\frac{1}{\sigma}}\left(a+\frac{2I}{R}-2\right)^{\frac{-1}{\sigma}}\right]\\
&= (\alpha\beta)^{-\frac{1}{2\sigma}} R^{\frac{1}{2}}\left(\frac{\frac{I}{R}-1}{a}\right)^{\frac{1}{2}}\left[\left(\frac{1}{2}\right)^{\frac{1}{\sigma}} - \left(\frac{\frac{aI}{R}-a+(\frac{I}{R}-1)^2}{(a+\frac{2I}{R}-2)^2}\right)^{\frac{1}{2\sigma}}\right]\\
&= (\alpha\beta)^{-\frac{1}{2\sigma}} R^{\frac{1}{2}}\left(\frac{\frac{I}{R}-1}{a}\right)^{\frac{1}{2}}\left[\left(\frac{1}{4}\right)^{\frac{1}{2\sigma}} - \left(\frac{\frac{aI}{R}-a+(\frac{I}{R}-1)^2}{(a+\frac{2I}{R}-2)^2}\right)^{\frac{1}{2\sigma}}\right]
\end{aligned} \tag{C.11}$$

Note that the second term in the bracket is smaller than $(\frac{1}{4})^{\frac{1}{2\sigma}}$. To show the fact, we define $f(\frac{I}{R})$ as follows:

$$\begin{aligned}
f(\frac{I}{R}) &\equiv (a+\frac{2I}{R}-2)^2 - 4\left(\frac{aI}{R}-a+(\frac{I}{R}-1)^2\right)\\
&= a^2
\end{aligned} \tag{C.12}$$

Therefore, $f(\frac{I}{R})$ is positive. This implies that

$$\left(\frac{1}{4}\right)^{\frac{1}{2\sigma}} > \left(\frac{\frac{aI}{R}-a+(\frac{I}{R}-1)^2}{(a+\frac{2I}{R}-2)^2}\right)^{\frac{1}{2\sigma}} \tag{C.13}$$

Therefore, $v(y^{adv},\theta^{adv}) - v(y^s,\theta^s)$ is positive except for $\sigma=\infty$. Moreover, $v(y^{adv},\theta^{adv}) - v(y^s,\theta^s)$ is close to zero as σ goes to ∞ from (C.11). In addition, $\frac{1}{\alpha\beta} > 1$ implies that $(\alpha\beta)^{-\frac{1}{2\sigma}}$ is decreasing in σ. With the bracket decreasing in σ, we can state that $v(y^{adv},\theta^{adv}) - v(y^s,\theta^s)$ is decreasing in σ.

□