

# Capital Reallocation And Profit-And-Loss Sharing Monetary Policy

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## Abstract

We study a steady-state monetary economy in which preference shocks give rise to the presence of idle capital. We introduce and analyze a new supply-side monetary policy idea, where the central bank aims to enhance the efficiency in the economy by reallocating idle capital from unproductive to productive agents, through the concept of profit-and-loss sharing joint-venture as practiced in islamic banking. We construct a microfounded model of money and capital to examine how such a monetary policy improves welfare and output compared to "laissez-faire" and to "Friedman's rule" policy in an economy with lump-sum taxes.

**Keywords:** preference shocks, capital reallocation, profit-and-loss sharing joint-venture, social welfare, Friedman's rule.

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# 1 Introduction

The presence of idle capital, that is not used for production at all, is an extreme form of capital misallocation in the manufacturing sector. As argued by Cooley, Hansen and Prescott (1995), one commonly observes idle plants, vacant office buildings and unused equipment. To motivate the presence of idle capital, Cooley et al. construct a real business cycle economy in which production takes place at individual plants that differ according to idiosyncratic technology shocks. Thus, in equilibrium, some plants will operate, given its realized technology shock, and other not, leading to an unused fraction of capital. Our approach is little different from them. We rationalize the existence of idle capital by considering preference shocks to which agents are exposed instead of technology shocks. Preference shocks involves a misallocation of capital in the sense that some agents hold capital but are not expected to produce in the immediate future, whereas others are, and may need to expand their capital in the short run in order to produce more output. Such distortion that impedes the production activity (i.e. the supply side) can not be corrected by a monetary policy that deals only with the optimal quantity of money for transactional purposes (i.e. the demand side).

In most search models of money such as models à la Lagos and Wright (2005), the Friedman's rule has been shown to be optimal in a monetary economy with lump-sum taxes because it maximizes the real value of money. It makes the money no costly to hold and thereby eliminates trade distortions arising from the fact that buyers are liquidity constrained. Nevertheless, changing the real value of cash balances through the Friedman's rule, does not reduce the costs related to the business activity such as the risk of production failure or the depreciation cost resulting from the utilization of capital in the process of production. Given this, we wonder whether an expansionary monetary policy, focused on reallocating idle capital and sharing some risk of production failure, could be more efficient than the Friedman's rule or at least than the constant money supply policy, termed in what follows the *laissez-faire* policy. We take an interest in comparing this monetary policy to the *laissez-faire* one because it becomes the second-best optimal policy when the central bank does not have the power to levy lump-sum taxes or in other words when the Friedman's rule is not feasible. Of course, an expansionary or inflationary monetary policy will decrease the real value of money and affect negatively the consumption, all things being equal. But the bottom line here is whether the positive effect on the production activity could outperform the negative effect on the demand of goods such that the efficiency is enhanced in the end.

To address this question, we propose in the present paper a specific monetary policy based on the use of the central bank's power of creating money from nothing (*ex nihilo*), to rent idle capital from non-productive agents and reallocate it to productive agents. The mechanism whereby the central bank transfers capital to producers, while at the same time absorbing some risk of production failure, is assumed following the concept of profit-and-loss sharing joint-venture (PLS joint-venture) as practiced by islamic banks. In such contract, the loss is shared between partners according to their capital contribution ratio, while the revenue is shared on the basis of a free agreement between the two parties.

However, for silent partners (non-active partners, who only contribute capital, in cash or in kind, such as a central bank here), the profit ratio can not be any higher than the capital contribution ratio. Given this, the profit ratio of the central bank under a PLS joint-venture can be written as  $\alpha(1-\theta)$  where  $\alpha$  is its capital contribution ratio, and at the same time its risk-sharing ratio, while  $0 \leq \theta < 1$  is a rate measuring the deviation from the pro-rata sharing basis. As it will be shown below, governing the risk-sharing ratio  $\alpha$  and the pro-rata deviation rate  $\theta$  gives the central bank a monetary policy instrument to influence the interest of producers in expanding their business activity through PLS joint-ventures. This policy will be termed, PLS monetary policy.

We make the assumption that the central bank can act as a direct intermediary in the economy, able to rent and transfer capital across agents. We are not concerned here with how such a monetary policy can be applied in the real world where the central bank is nothing to do with individual agents and firms. But, to bring some light on the matter, although it is beyond our scope here, one can imagine a banking sector where banks exercise their usual role of intermediaries whereas the central bank holds investment accounts in each bank, in which it inject whatever money it creates and from which it withdraws whatever money it extracts from the economy. Such investment accounts can be restricted so that they can only be used by banks to finance the real economy and in particular to reallocate capital across agents. One can imagine other forms of intermediation to reallocate capital. For instance, instead of entering in a PLS joint-venture, the central bank can rent capital to producers in exchange of a fixed rental price, in which case producers bear entirely the risk of expanding their capital stock. In this paper, we are interested rather in the question of sharing production risk via the concept of PLS joint-venture which will be explained in more details further down.

We report here the main findings of our model: we show that the optimal PLS monetary policy consists in a positive inflation rate and a maximum possible value of pro-rata deviation rate so that producers derive the maximum of surplus from the PLS joint-venture. Moreover, we find that the maximization of welfare under the optimal PLS monetary policy does not coincide with the maximization of allocation. In other words, an optimal PLS monetary policy can well maximize the social welfare without necessarily maximizing the equilibrium output, or even by sacrificing some of the the equilibrium output. In terms of comparison across policies, we show that the optimal PLS monetary policy is welfare-and-output improving compared to *laissez-faire*. Regarding the Friedman's rule, neither the improvement of welfare nor that of allocation are systematic. In our calibrated model, however, the PLS monetary policy increases the social welfare, as long as the business failure risk is significant, and that by decreasing slightly the aggregate output. In general, the gain in welfare under the PLS monetary policy comes from two factors. First, from the saving in depreciation cost because agent employ less their own capital and tend to expand their business through PLS joint-ventures. Second, from the gain in production cost due to the reallocation of idle capital. The PLS monetary policy generates however two specific costs in welfare. First, it gives rise to a new depreciation cost associated with the mobilization of idle capital into the production process. Second, as the central bank resorts to issuing new money to pay the capital rental price in case of production failure,

inflation is increased on average which affects negatively the consumption, other things being equal. The gain in welfare in the end, is the result of counter-balancing of these four effects.

The paper proceeds as follows. Section 2 presents the environment, Section 3 the social planner's problem and Section 4 the monetary economy with lump-sum transfers. In section 5, we present the economy with PLS monetary policy, we describe the agents' decision problems in the different markets and we derive equilibrium and feasibility conditions. Then, in Section 6, we study analytically the optimal PLS monetary policy with Cobb-Douglas specification. For illustration purposes, we provide a quantitative analysis in section 7. The last section concludes.

## 2 Environment

The basic environment we use is the divisible money model developed by Lagos and Wright (2005) and Berentsen, Camera, and Waller (2007), consisting of periodic meetings of centralized and decentralized markets. There is a  $[0,1]$  continuum of infinitely-lived agents. Time is discrete and each period is divided into two sub-periods. In the first sub-period there is a decentralized market referred to as (DM), where double coincidence problem and anonymity of trades give rise to an essential role for money. In the second sub-period there is a frictionless or centralized Arrow-Debreu market referred to as (CM). Agents discount between the (CM) and (DM) at rate  $\beta = \frac{1}{1+r} < 1$ .

We extend the basic framework by introducing capital as a factor of production while fiat money remains the only medium of exchange in the (DM)<sup>1</sup>. We assume that capital is acquired during the (CM) and can be used as input of production only in the (DM). One can assume that capital is used as a factor of production in both centralized and decentralized markets, as in Aruoba, Waller and Wright (2005), but this breaks the dichotomy between the two markets and complicates the analytical resolution. We prefer a model that dichotomizes for a certain degree of tractability. We mean by dichotomy between the (CM) and the (DM) the ability to solve for the outcome in the centralized market and the outcome in the decentralized markets independently (see for details Aruoba and Wright (2003)). In terms of pricing, we focus on competitive price taking rather than bargaining. Agents are assumed to meet in large groups in the (DM), instead of bilaterally and take prices as parametric.

At the beginning of the first sub-period, agents get a preference shock, such that they can produce a special good in the (DM) with probability  $n$ . With probability  $1 - n$  the agent can not produce and can either consume with probability  $\sigma$  or be a simple observer with probability  $1 - \sigma$ . Thus, the fraction of producers is  $n$ , the fraction of active consumers is  $\sigma(1 - n)$  and the fraction of passive consumers is  $(1 - \sigma)(1 - n)$ . Agents get

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<sup>1</sup>We are not considering here the question of how money and capital can co-exist as a medium of exchange.

utility  $u(q)$  from  $q$  consumption, where  $u'(q) > 0$ ,  $u''(q) < 0$ ,  $u'(0) = +\infty$ ,  $u'(+\infty) = 0$ .

To add a risky character to producers' activity in the (DM), we assume that with a probability  $1 - \omega$  the production process stops in an earlier step. In that case, the generation of the special good is unsuccessful and the capital depreciates by a coefficient  $\delta_1$ . With a probability  $\omega$ , the process arrives at its end and the producer succeeds to produce a quantity  $q^s$  of the special good that he offers to sell in the competitive market. In this instance, the capital depreciates by a coefficient  $\delta_2$  and the producer incurs a disutility  $-l(e)$  where  $q^s = f(e, K)$ ,  $e$  the level of effort made to produce the  $q^s$  units of output given  $K$  units of capital, and  $f$  the production function. To simplify, we assume  $\delta_1 = \delta_2 = \delta$ . One can consider that capital depreciates more if production attains the final stage, which implies  $\delta_2 > \delta_1$ , but this has no great importance.

In the second sub-period, the (CM) opens and all agents can produce and trade a general good. Each unit of the general good is produced with one unit of labor and can be consumed or transformed one-for-one into capital. Agents get a disutility of  $-h$  for  $h$  hours worked and utility  $U(x)$  from  $x$  consumption, where  $U'(x) > 0$ ,  $U''(x) \leq 0$ ,  $U'(0) = +\infty$ ,  $U'(+\infty) = 0$ . We assume an interior solution for hours worked  $h < h_{max}$  (see Lagos and Wright (2005) for details).

To include central bank intervention in the model through the PLS concept, we add an other extension to the basic framework: we assume there is a lapse of time in the first sub-period between the occurrence of the preference shock and the beginning of the (DM). In this interval of time, an intermediation market (IM) takes place where consumers, who are not expected to use their capital in the (DM), can rent a part or the totality of their idle capital to the central bank whereas producers can expand their capital by entering in PLS joint-venture with the central bank. It's only after producers express their interest in the partnership and accept the sharing ratios  $\alpha$  and  $\theta$  applied by the central bank, that the latter proceeds to rent the necessary capital from the (IM) to achieve the PLS contract. At the end of the (DM), profits, if any, and capital net of depreciation are distributed between producers and central bank according to the agreed ratios. Thereafter, the central bank returns the rented capital net of depreciation to consumers and pays them the capital rental price in money that it creates ex nihilo in case of loss or insufficient earning from the PLS joint-venture. This will be detailed below. Because of anonymity and limited commitment frictions, consumers don't rent their capital directly to producers since the latter can renege on the promise to pay the rent or to return the capital at the end of the (DM). We assume, however, that the central bank has the power to force producers to respect their contractual obligations.

In the first part of the paper, we establish the equations that govern the equilibrium using a generic production function. Further, to make things as simple as possible, we consider a Cobb-Douglas production function  $f(e, K) = e^{1-\psi} K^\psi$  and a linear disutility  $l(e) = e$ , which ensures a Cobb-Douglas production cost  $c(q, K) = q^a K^{1-a}$ . In general, it's convenient to invert the production function to express the level of effort as  $e = \xi(q, K)$ , so the cost function  $c(q, K)$  is expressed as  $c(q, K) = l(e) = l(\xi(q, K))$ . As shown in Aruoba, Waller and Wright (2005), we have under the usual monotonicity and convexity

assumptions on  $f$  and  $l$ :  $c_q > 0$ ,  $c_k < 0$ ,  $c_q > 0$ ,  $c_{qq} > 0$ ,  $c_{kk} > 0$  and  $c_{qk} < 0$  if  $f_k f_{ee} < f_e f_{ek}$  which holds if  $K$  is a normal input ( $\frac{\partial K}{\partial q} > 0$ ). In addition to these properties, we prove that  $c$  is convex and thereby the determinant of its Hessian matrix is positive,  $c_{qq}c_{kk} - c_{qk}^2 \geq 0$  and finally that  $c_k c_{qk} \leq c_q c_{kk}$  (see appendix A).

For notational ease variables corresponding to the next period are indexed by  $+1$ , and variables corresponding to the previous period are indexed by  $-1$ . We denote by  $M$  the per capital money stock at the beginning of period, and  $M_{+1}$  the per capital money stock after the money supply adjustments have taken place. The gross growth rate of the money supply at the end of period is:  $\gamma = \frac{M_{+1}}{M}$ . We denote by  $\phi_{-1}$  and  $\phi$ , respectively, the value of the money in terms of the general good at the beginning of a period and after money supply adjustments, and we study a stationary monetary equilibrium where end-of-period real value of the money supply is constant over time,  $\phi M_{+1} = \phi_{-1} M$ , which implies:  $\gamma = \frac{\phi_{-1}}{\phi}$ .

Before moving to an economy with PLS monetary policy, we first examine the social planner's problem, and thereafter, the case of a monetary economy with lump-sum transfers, as in the literature. This will allow us to compare equilibrium solutions and especially, to show in which extent a PLS-based monetary policy could be more efficient.

### 3 Social planner

Consider a powerful and benevolent social planner who makes all the decisions in a non-monetary economy, by imposing his choices of employment, consumption and production upon all agents, regardless of their desires.

Let  $X$ ,  $H$  and  $K$  be respectively the aggregate consumption, the aggregate hours worked and the aggregate capital. The social planner's problem is to maximize the lifetime utility of the representative agent:

$$P(K) = \max_{X, H, q, K_+} \{U(X) - H + \sigma(1-n)u(q) - \omega n c(q^s, K) + \beta P(K_+)\}$$

subject to the budget constraint:

$$X + K_+ - K(1 - n\delta) = H$$

Goods market clearing implies total demand equals total supply:  $\sigma(1-n)q = \omega n q^s$

Notice that aggregate capital depreciates by  $n\delta$ , rather than by  $\delta$ , because the capital depreciates only when it is used by producers. Eliminating  $H$ , using the budget constraint, leads to:

$$P(K) = K(1 - n\delta) + \max_{X, q, K_+} \left\{ U(X) - X - K_+ + \sigma(1-n)u(q) - \omega n c\left(\frac{\sigma(1-n)}{\omega n} q, K\right) + \beta P(K_+) \right\}$$

The first-order conditions are:

$$X : U'(X) = 1$$

$$q : \frac{u'(q)}{c_q\left(\frac{\sigma(1-n)}{\omega n}q, K\right)} = 1$$

$$K : 1 = \beta P'(K)$$

where the first-order condition for capital has been lagged one period.

From the envelope condition:  $P'(K) = 1 - n\delta - \omega n c_k\left(\frac{\sigma(1-n)}{\omega n}q, K\right)$ .

Insert this into the FOC for  $K$ , to get the first-best allocation chosen by the social planner:

$$\begin{cases} \widehat{X} = U'^{-1}(1) \\ \frac{u'(\widehat{q})}{c_q\left(\frac{\sigma(1-n)}{\omega n}\widehat{q}, \widehat{K}\right)} = 1 \\ -c_k\left(\frac{\sigma(1-n)}{\omega n}\widehat{q}, \widehat{K}\right) = \frac{r + n\delta}{\omega n} \end{cases} \quad (1)$$

## 4 Monetary Economy with lump-sum transfers

In this section we consider a monetary economy where the central bank maintains a constant money supply gross growth rate  $\gamma$ . If  $\gamma > 1$  then the central bank transfers a quantity,  $T > 0$ , of money to agents under a lump-sum fashion. If  $\gamma < 1$ , the central bank collects lump-sum taxes from agents (negative transfers  $T < 0$ ) and extract the collected quantity of money from the economy.  $\gamma = 1$  corresponds to the case of constant money supply or "Laissez-faire" policy. Agents observe  $\phi_{-1}$  and take  $\gamma$  as given, so they predict perfectly the end of period value of the money,  $\phi = \frac{\phi_{-1}}{\gamma}$ .

The budget constraint of the central bank is given by:

$$T = M_{+1} - M = (\gamma - 1)M \quad (2)$$

### 4.1 The centralized market

Let  $V(m, K)$  (resp.  $W(m, K)$ ) denotes the expected value from entering the first sub-period (resp. the second sub-period) with  $m$  units of money and  $K$  units of capital. In

what follows, we look at a representative period and work backwards from the second sub-period to the first sub-period.

In the (CM), agents work a number of hours  $h$ , produce and trade good  $x$ , adjust their money and capital holdings for the next period. The representative agent's program is given by:

$$W(m, K) = \max_{x, h, m_+, K_+} E[U(x) - h + \beta V_+(m_+, K_+)]$$

subject to the budget constraint:

$$\begin{cases} x + \phi m_+ + (K_+ - K) = h + \phi m + \phi T \\ m_+ \geq 0; \quad K_+ \geq 0; \quad h \geq 0; \end{cases} \quad (3)$$

After eliminating  $h$  using the budget equation, we have:  $W(m, K) = \phi m + \phi T + K + W_0$  where  $W_0 = \max_{x, m_+, K_+} \{U(x) - x - \phi m_+ - K_+ + \beta V_+(m_+, K_+)\}$

We underline that  $W$  is linear in  $m$  and in  $K$ :  $W_m(m, K) = \phi$  and  $W_K(m, K) = 1$

The first-order conditions are:

$$x : U'(x) = 1 \quad (4)$$

$$m : \phi_{-1} = \beta V_m(m, K) + \mu_m \quad (5)$$

$$K : 1 = \beta V_K(m, K) + \mu_K \quad (6)$$

where:

- the first-order conditions for money and capital have been lagged one period
- $V_m(m, K)$  (resp.  $V_K(m, K)$ ) is the marginal value of an additional unit of money (resp. capital) taken into the (DM)
- $\mu_m$  (resp.  $\mu_K$ ) is the Lagrangian multiplier on the money (resp. capital) constraint,  $\mu_m m = 0$ ,  $\mu_K K = 0$

Notice, from (4), that the optimal choice of  $x$  is the same across time for all agents. Also,  $(m_-, K_-)$  does not appear in (5) and (6), for any distribution of  $(m, K)$  across agents entering the (CM). Therefore, all agents enter the following period with the same quantity of money  $m$  and capital  $K$  regardless of how much they bring into the (CM). Thus, the distribution of money and capital holdings are degenerate.



## 4.2 The decentralize market

At the beginning of the first sub-period, agents get the preference shock such that they can either be producer or consumer in the (DM). The expected value from entering the (DM) as producer or consumer are denoted, respectively,  $V^p(m, K)$  and  $V^c(m, K)$ , so:

$$V(m, K) = nV^p(m, K) + (1 - n)V^c(m, K) \quad (7)$$

### Producers' decisions:

As explained above, the representative producer succeeds to generate and sell the special good with probability  $\omega$  and fails with probability  $1 - \omega$ , his capital depreciates by  $\delta$  in both cases. Therefore, the producer's problem in the (DM) is as follows:

$$V^p(m, K) = \max_{q^s} \{ \omega[-c(q^s, K) + W(m + \tilde{p}q^s, K(1 - \delta))] + (1 - \omega)W(m, K(1 - \delta)) \}$$

where  $\tilde{p}$  is the price level taken parametrically.

This simplifies, using the linearity of  $W$  on  $m$  and  $K$ , to:

$$V^p(m, K) = \phi m + \phi T + K(1 - \delta) + W_0 + \omega \max_{q^s} \{ -c(q^s, K) + \phi \tilde{p} q^s \}$$

The first order condition on  $q^s$  implies:

$$c_q(q^s, K) = \phi \tilde{p} \quad (8)$$

We get:

$$V^p(m, K) = \phi m + \phi T + K(1 - \delta) + W_0 + \omega[-c(q^s, K) + \phi \tilde{p} q^s] \quad (9)$$

where  $q^s(m, K)$  is solution of (8).

### Consumers' decisions:

Once entered in the (DM), a consumer is active with probability  $\sigma$ , in that case he buys and consumes  $q^b$  units of the special good, and is passive (simple observer) with probability  $1 - \sigma$ . Consumer's problem in the (DM) is then:

$$V^c(m, K) = \max_{q^b} \{ \sigma[u(q^b) + W(m - \tilde{p}q^b, K)] + (1 - \sigma)W(m, K) \} \quad \text{s.t.} \quad \tilde{p}q^b \leq m$$

This reduces, using the linearity of  $W$ , to:

$$V^c(m, K) = \phi m + \phi T + K + W_0 + \sigma \max_{q^b} \{ u(q^b) - \phi \tilde{p} q^b \} \quad \text{s.t.} \quad \tilde{p}q^b \leq m$$

The first order condition on  $q^b$  implies:

$$u'(q^b) = \phi\tilde{p} + \lambda_m \quad (10)$$

where  $\lambda_m \geq 0$ , is the Lagrangian multiplier on the budget constraint:  $\lambda_m(\tilde{p}q^b - m) = 0$

If  $q^b(m, K)$  is the solution of (10) then:

$$V^c(m, K) = \phi m + \phi T + K + W_0 + \sigma[u(q^b) - \phi\tilde{p}q^b] \quad (11)$$

The equilibrium competitive price  $\tilde{p}$  is the one that makes demand and supply quantities equal to each other:  $\sigma(1-n)q^b = \omega n q^s$ . Moreover, money market clearing implies:

$$M = m = \tilde{p}q^b \quad (12)$$

Let  $q \equiv q^b = \frac{\omega n}{\sigma(1-n)}q^s$  be the equilibrium consumption in the (DM). As  $q = \frac{m}{\tilde{p}}$  we get:

$$\frac{\partial q}{\partial m} = \frac{1}{\tilde{p}} \quad ; \quad \frac{\partial q^s}{\partial m} = \frac{\sigma(1-n)}{\omega n} \frac{1}{\tilde{p}} \quad \text{and} \quad \frac{\partial q}{\partial K} = \frac{\partial q^s}{\partial K} = 0 \quad (13)$$

Note that the constraint  $\tilde{p}q^b \leq m$  is binding so  $\lambda_m > 0$ , and (10) and (8) yields:

$$u'(q) > c_q(q^s, K) \quad (14)$$

implying trades are inefficient compared to the social planner's solution.

Differentiating (9) and (11), using (8) and (13), gives the marginal value of money and capital for producers and consumers:

$$V_m^p(m, K) = \phi \quad (15)$$

$$V_m^c(m, K) = \phi \left[ 1 + \sigma \left( \frac{u'(q)}{c_q(q^s, K)} - 1 \right) \right] \quad (16)$$

$$V_K^p(m, K) = 1 - \delta - \omega c_K(q^s, K) \quad (17)$$

$$V_K^c(m, K) = 1 \quad (18)$$

Differentiating (7) with respect to  $m$  and  $K$ , and inserting the derivatives above, leads to:

**Marginal value of money:**

$$V_m(m, K) = \phi \left[ 1 + \sigma(1-n) \left( \frac{u'(q)}{c_q(q^s, K)} - 1 \right) \right] \quad (19)$$

**Marginal value of capital:**

$$V_k(m, K) = 1 - n\delta - \omega n c_k(q^s, K) \quad (20)$$

### 4.3 Equilibrium

We now derive the equilibrium of the lump-sum transfers economy.

Substituting (19), into the FOC for  $m$ , (5), implies:

$$\phi_{-1} = \beta\phi \left[ 1 + \sigma(1-n) \left( \frac{u'(q)}{c_q(q^s, K)} - 1 \right) \right] \quad (21)$$

Notice, as we focus on equilibrium with  $q > 0$ , that  $m = \tilde{p}q > 0$ , so  $\mu_m = 0$  in (5).

(21) can be written, using  $\gamma = \frac{\phi_{-1}}{\phi}$ :

$$\frac{u'(q)}{c_q\left(\frac{\sigma(1-n)}{\omega n}q, K\right)} = 1 + \frac{(\gamma/\beta) - 1}{\sigma(1-n)} \quad (22)$$

From (14) and (22),  $\gamma > \beta$  is a necessary condition for equilibrium existence.

According to (21), the value of money in a period is equal to its discounted value at the next period plus a liquidity premium  $L = \sigma(1-n) \left( \frac{u'(q)}{c_q(q^s, K)} - 1 \right) = \frac{\gamma}{\beta} - 1$ , giving the marginal benefit of holding money in the (DM).

Substituting (20), into the FOC for  $K$ , (6), yields<sup>2</sup>:

$$1 = \beta [1 - n\delta - \omega n c_k(q^s, K)]$$

or equivalently, using  $\beta = \frac{1}{1+r}$ :

$$-c_k\left(\frac{\sigma(1-n)}{\omega n}q, K\right) = \frac{r + n\delta}{\omega n} \quad (23)$$

**Definition 1.** *A stationary equilibrium in lump-sum transfers economy is given by  $(x, q, K)$  satisfying (4), (22) and (23).*

**Definition 2.** *We define social welfare as the expected lifetime utility of the representative agent:*

$$\mathcal{W} = U(x) - E[h] + \sigma(1-n) u(q) - \omega n c(q^s, K) + \beta\mathcal{W} \quad (24)$$

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<sup>2</sup> Notice, since we are interested in equilibrium with  $K > 0$ , that  $\mu_K = 0$  in (6).

The first term  $U(x) - E[h]$  corresponds to the (CM) consumption utility net of the disutility of work while the other term corresponds to the (DM) consumers' consumption utility net of producers' disutility of production. In the appendix B, we show that at equilibrium  $E[h] = x + n\delta K$ . Therefore, we can rewrite (24) as:

$$\mathcal{W} = \frac{1}{1 - \beta} \{U(x) - x - n\delta K + \sigma(1 - n) u(q) - \omega n c(q^s, K)\} \quad (25)$$

**Proposition 1.** *If  $\gamma > \beta$ , it exists a unique stationary equilibrium in the lump-sum transfers economy. The social welfare and the (DM) output are both decreasing in  $\gamma$ .*

$$\frac{\partial q}{\partial \gamma} = \frac{c_q^2}{\beta\sigma(1 - n)} \left[ u'' c_q - \frac{\sigma(1-n)}{\omega n} u' \left( c_{qq} - \frac{c_{qk}^2}{c_{kk}} \right) \right]^{-1} < 0$$

$$\frac{\partial \mathcal{W}}{\partial \gamma} = \frac{\sigma(1 - n)}{1 - \beta} \left[ u' - c_q - \frac{r}{\omega n} \frac{c_{qk}}{c_{kk}} \right] \frac{\partial q}{\partial \gamma} < 0$$

**Corollary 1.** *If the central bank has the power to levy lump-sum taxes then the Friedman's rule,  $\gamma \rightarrow \beta$ , is welfare-maximizing and implements the first-best optimal allocation (1).*

*If the central bank does not have this power then "Laissez-faire",  $\gamma = 1$ , is welfare-maximizing and implements the second-best optimal allocation  $(x^*, q^*, K^*)$  given by:*

$$\begin{cases} x^* = U'^{-1}(1) \\ \frac{u'(q^*)}{c_q(\frac{\sigma(1-n)}{\omega n} q^*, K^*)} = 1 + \frac{r}{\sigma(1 - n)} \\ -c_k(\frac{\sigma(1-n)}{\omega n} q^*, K^*) = \frac{r + n\delta}{\omega n} \end{cases} \quad (26)$$

## 5 Economy with PLS monetary policy

Now, let's describe more the economy with PLS monetary policy. As mentioned in the description of the extended framework, we suppose that agents' preference shock occurs at the beginning of the first sub-period, such that an intermediation market (IM) takes place before the decentralized market starts. The PLS monetary policy is based on two principles. First, reallocate idle capital to where it's effectively used. Second, support the production activity by sharing some of the business risk with producers through profit-and-loss sharing joint-venture. To this aim, the central bank acts an intermediary between consumers, who are not willing to produce in the immediate future (next DM), and producers who could be interested in an additional capital stock before starting the

production of the special good in the (DM). The central bank determines its optimal capital contribution ratio in PLS joint-ventures  $0 \leq \tilde{\alpha} < 1$  (which represents also its optimal risk-sharing ratio) as well as the optimal pro-rata deviation rate  $0 \leq \theta < 1$  and makes a take-it-or-leave-it offer for producers. Recall that unlike the sharing of loss, the sharing of profit in a PLS contract may differ from the pro-rata share in capital,  $\tilde{\alpha}$ , in favor of the active partner. So the distribution of revenue could be:  $\tilde{\alpha}(1 - \theta)$  for the central bank and  $1 - \tilde{\alpha}(1 - \theta)$  for producers with some  $\theta > 0$ . The producers observe the parameters  $\tilde{\alpha}$  and  $\theta$  in the intermediate market, and ask the central bank for a PLS joint-venture if this increases their expected utility in the (DM). In that case, the central bank enters in the (IM), where consumers supply their idle capital, rents the aggregate capital required by all producers and enters concretely in a PLS joint-venture with each producer. The capital rental price is assumed to be determined competitively in the (IM).

In the (DM), agents trade the special good against money by taking price parametrically. Then, sales revenue, if any, and capital net of depreciation are shared between producers and the central bank. Thereafter, the central bank returns the capital rented to consumers and pays them the rent with the money earned eventually from the PLS joint-venture and with new money it creates in case of insufficient earning or no earning at all. Otherwise, if the central bank earning from the PLS joint-venture exceeds the capital rental price then the excess of money is destroyed. In such a way, the PLS monetary policy consists, on the one hand, on creating the right quantity of new money to absorb some of the loss due to production failure, on the other hand, on destroying any excess cash that is not immediately needed to finance production. By destroying the excess cash instead of transferring it to agents in the (CM), the central bank reduces the expected cost of holding money and hence promotes the consumption in the (DM). The sequence of events in a typical period and the PLS joint-venture mechanism are respectively illustrated in figure (1) and (2).

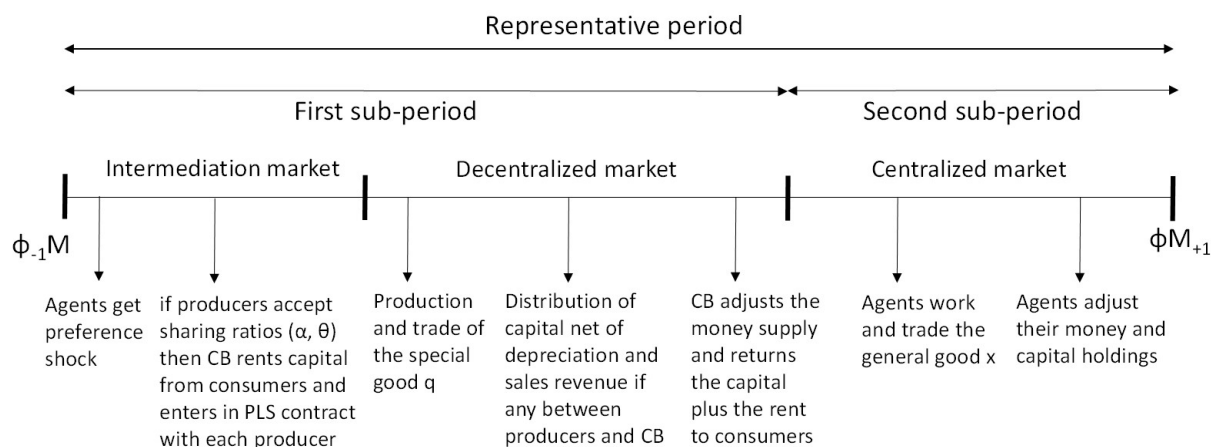


Figure 1: Sequence of events

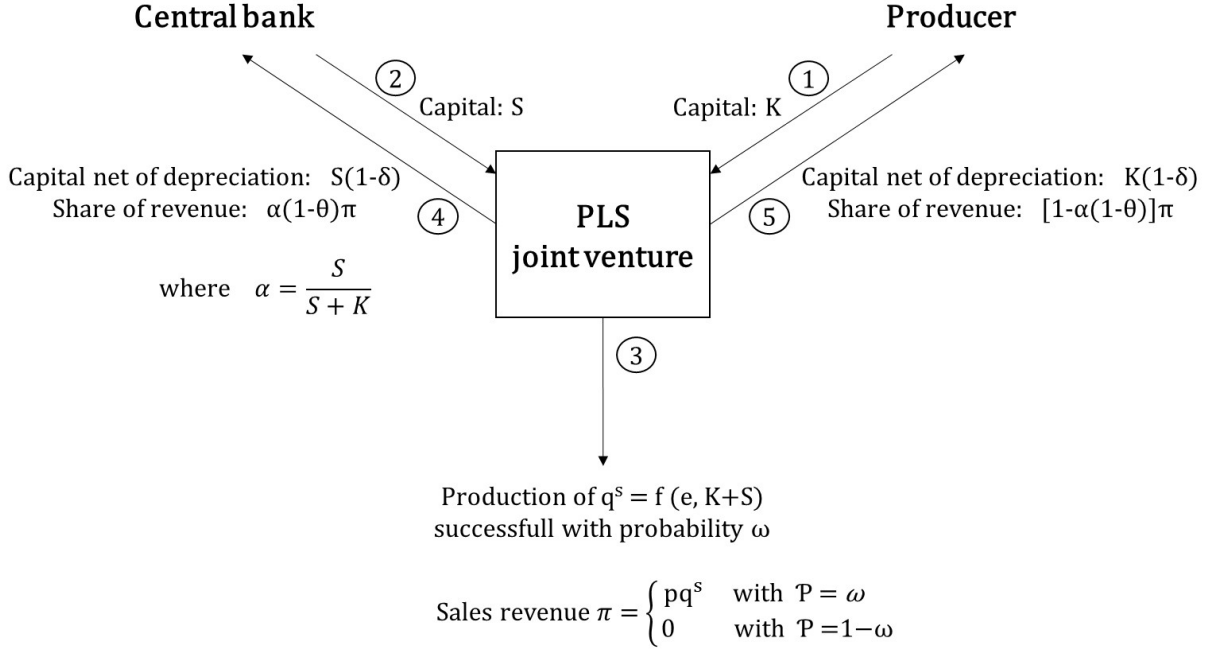


Figure 2: PLS joint-venture diagram

Let  $S$  denotes the central bank contribution in capital under the PLS joint-venture undertaken with a representative producer. When the producer is self-financed, meaning he does not request any PLS joint-venture from the central bank, we set  $S = 0$ . We denote by  $\alpha \equiv \frac{S}{S+K}$  the central bank's share in the total capital brought by the producer into the (DM). If the PLS joint-venture is formed then  $\alpha = \tilde{\alpha}$  where  $\tilde{\alpha}$  is the sharing ratio offered by the central bank, else  $\alpha$  is zero.

From the definition of  $\alpha$ ,  $S$  can be expressed as  $S = \frac{\alpha}{1-\alpha}K$ , and the total capital used for production as  $K + S = \frac{K}{1-\alpha}$ .

Furthermore, let  $\pi$  denotes the sales revenue of the business.  $\pi$  is random and can take one of two value,  $pq^s$  if the producer succeeds to generate the special good, or 0 if not. Finally, let  $\tau$  be the capital rental price, in nominal terms.

The budget constraint of the central bank is given by:

$$\tau(nS) = n[\alpha(1-\theta)\pi] + (M_+ - M)$$

By inserting the expression of  $S$  into the equation above, we obtain the money supply dynamic:

$$M_+ = M + n \left[ \tau \frac{\alpha}{1-\alpha} K - \alpha(1-\theta)\pi \right] \quad (27)$$

Since the sales revenue  $\pi$  is random, it's clear from (27) that the money supply dynamic is stochastic as long as  $\alpha > 0$ . In the case of self-financed producers,  $\alpha = 0$ , the money supply remains constant,  $M_+ = M$  as in laissez-faire policy.

At the beginning of a period, all agents know the current stock of money that is available for trade,  $M$  and the value of money that prevailed in the previous period,  $\phi_{-1}$ . But, as long as  $\alpha > 0$  and so the money supply dynamic is stochastic, they don't know neither the end-of-period money stock  $M_{+1}$ , nor the value of money that will prevail,  $\phi$ . Hence, agents will determine terms of trade in the (DM) based on the expected value of money  $\phi^e$  where:  $\phi^e = E[\phi] = E\left[\phi_{-1}\frac{M}{M_{+1}}\right] = \phi_{-1}E\left[\frac{M}{M_{+1}}\right]$ . However, once in the (CM), the end-of-period stock of money  $M_+$  and the real value of the money  $\phi$  are no longer stochastic,  $E[\phi] = \phi = \phi_{-1}M/M_{+1}$ . This is because the central bank adjusts the money supply at the end of the (DM) before the second sub-period starts.

### Feasibility condition on $\tilde{\alpha}$ :

To implement the PLS joint-venture if requested by producers ( $\alpha = \tilde{\alpha}$ ), the central bank needs to rent  $nS$  units of capital from the (IM), while the supply of capital for rent can not exceed the capital held by consumers, namely  $(1 - n)K$ . Therefore  $nS \leq (1 - n)K$ . This leads, using  $S = \frac{\tilde{\alpha}}{1 - \tilde{\alpha}}K$ , to a feasibility condition on  $\tilde{\alpha}$ :  $\frac{\tilde{\alpha}}{1 - \tilde{\alpha}} \leq \frac{1 - n}{n}$

$$\text{or equivalently} \quad \tilde{\alpha} \leq 1 - n \quad (28)$$

### Feasibility condition on $\theta$ :

The higher the pro-rata deviation rate  $\theta$  is, the more producers are advantaged in terms of distribution of revenue; and the more the central bank PLS joint-venture offer is attractive. Nevertheless, the deviation from the pro-rata sharing basis is constrained by the fact that the central bank still wants to avoid a capital loss when the business is making profit. In other words, the value of  $\theta$  is capped so that in case of a positive nominal profit of the PLS joint-venture,  $\Pi_{PLS} = pq^s - \frac{1}{\phi^e}\delta(K + S) > 0$ , the nominal profit of the central bank is at least zero,  $\Pi_{CB} = \tilde{\alpha}(1 - \theta)pq^s - \frac{1}{\phi^e}\delta S \geq 0$ . This reduces using  $S = \frac{\tilde{\alpha}}{1 - \tilde{\alpha}}K$  to a feasibility condition on  $\theta$ :

$$0 \leq \theta \leq 1 - \frac{\delta K}{(1 - \tilde{\alpha})\phi^e pq^s} \equiv \theta_{max} \quad (29)$$

where  $\theta_{max}$  defines the upper bound of the pro-rata deviation rate. Note that:

- $\theta_{max} < 1$
- $\theta_{max} > 0$  as long as  $\Pi_{PLS} > 0$  since  $\theta_{max} = \frac{pq^s - \frac{1}{\phi^e}\delta(K + S)}{pq^s} = \frac{\Pi_{PLS}}{pq^s}$

## 5.1 The second sub-period

The centralized market in the second sub-period remains the same as in the monetary economy with lump-sum transfers with the exception that there is no transfers,  $T = 0$ . The expression of  $W(m, K)$  simplifies to:

$$W(m, K) = \phi m + K + W_0 \quad (30)$$

The first order conditions are still given by (4), (5) and (6), and the distribution of money and capital holdings are degenerate.

## 5.2 The first sub-period

The expected value from entering the first sub-period is given by:

$$V(m, K) = nV^p(m, K + S) + (1 - n)V^c(m, K)$$

It yields using  $K + S = \frac{K}{1 - \alpha}$ :

$$V(m, K) = nV^p(m, \frac{K}{1 - \alpha}) + (1 - n)V^c(m, K) \quad (31)$$

**Producers' decisions:** the producer decides to enter, in the (IM), in a PLS joint-venture with the central bank (so  $\alpha = \tilde{\alpha}$ ), if this increases his expected value function.

$$\alpha = \begin{cases} \tilde{\alpha} & \text{if } V^p(m, \frac{K}{1 - \tilde{\alpha}}) > V^p(m, K) \\ 0 & \text{otherwise} \end{cases} \quad (32)$$

Let  $q_\alpha^s(m, K)$  be the production in the (DM) for a given value of  $\alpha$  among  $\{0, \tilde{\alpha}\}$ .

The value function of a producer in the first sub-period is:

$$V^p(m, \frac{K}{1 - \alpha}) = \max_{q_\alpha^s} \{ \omega E [-c(q_\alpha^s, \frac{K}{1 - \alpha}) + W(m + \tilde{p}q_\alpha^s[1 - \alpha(1 - \theta)], K(1 - \delta))] + (1 - \omega)E[W(m, K(1 - \delta))] \}$$

This reduces using (30) and rearranging terms to:

$$V^p(m, \frac{K}{1 - \alpha}) = \phi^e m + K(1 - \delta) + W_0 + \omega \max_{q_\alpha^s} \{ -c(q_\alpha^s, \frac{K}{1 - \alpha}) + \phi^e \tilde{p}q_\alpha^s[1 - \alpha(1 - \theta)] \}$$



The first order condition on  $q_\alpha^s$  yields:

$$c_q(q_\alpha^s, \frac{K}{1-\alpha}) = \phi^e \tilde{p} [1 - \alpha(1 - \theta)] \quad (33)$$

We arrive at:

$$V^p(m, \frac{K}{1-\alpha}) = \phi^e m + K(1 - \delta) + W_0 + \omega \left\{ -c(q_\alpha^s, \frac{K}{1-\alpha}) + \phi^e \tilde{p} q_\alpha^s [1 - \alpha(1 - \theta)] \right\} \quad (34)$$

where  $q_\alpha^s(m, K)$  is solution of (33) for a given value of  $\alpha$  in  $\{0, \tilde{\alpha}\}$ .

It's straightforward from (33) that  $q_{\tilde{\alpha}}^s(m, K)$  and  $q_0^s(m, K)$  are related by:

$$\frac{c_q(q_{\tilde{\alpha}}^s(m, K), \frac{K}{1-\tilde{\alpha}})}{1 - \tilde{\alpha}(1 - \theta)} = c_q(q_0^s(m, K), K) \quad (35)$$

By combining (33) and (34), we can rewrite, the condition of PLS joint-venture implementation (32), as follows:

$$\alpha = \begin{cases} \tilde{\alpha} & \text{if } \Delta(q_{\tilde{\alpha}}^s(m, K), \frac{K}{1-\tilde{\alpha}}) > \Delta(q_0^s(m, K), K) \\ 0 & \text{otherwise} \end{cases}$$

where  $\Delta$  is the operator defined by:

$$\Delta(q, K) = -c(q, K) + q c_q(q, K) \quad (36)$$

Or again using the indicator function:

$$\alpha = \tilde{\alpha} \mathbb{1} \left\{ \Delta(q_{\tilde{\alpha}}^s(m, K), \frac{K}{1-\tilde{\alpha}}) > \Delta(q_0^s(m, K), K) \right\} \quad (37)$$

**Consumers' decisions:** a consumer who holds  $m$  units of money and  $K$  units of capital in the first sub-period, decides how much capital  $Z$  to rent to the central bank in the (IM) and how much special good  $q^b$  to consume in the (DM). The capital rented, since it is intended to be used in production in the (DM), depreciates by  $\delta$ . At the end of the (DM), the consumer receives  $\tau Z$  in cash for the rental of capital. His function value in the first sub-period satisfies:

$$V^c(m, K) = \max_{q^b, Z} \left\{ \sigma E[u(q^b) + W(m - pq^b + \tau Z, K - \delta Z)] + (1 - \sigma) E[W(m + \tau Z, K - \delta Z)] \right\}$$

subject to  $pq^b \leq m$  and  $0 \leq Z \leq K$

This boils down using (30) and rearranging terms to:

$$V^c(m, K) = \phi^e m + K + W_0 + \max_{q^b, Z} \left\{ (\phi^e \tau - \delta) Z + \sigma [u(q^b) - \phi^e p q^b] \right\} \quad (38)$$

The first order conditions on  $q^b$  and  $Z$  imply respectively:

$$u'(q^b) = \phi\tilde{p} + \lambda_m \quad (39)$$

$$\phi^e\tau - \delta = \lambda_K - \lambda_0 \quad (40)$$

where:

- $\lambda_m \geq 0$ , is the Lagrangian multiplier on the budget constraint:  $\lambda_m(\tilde{p}q^b - m) = 0$
- $\lambda_K \geq 0$ , is the Lagrangian multiplier on the budget constraint:  $\lambda_K(Z - K) = 0$
- $\lambda_0 \geq 0$ , is the Lagrangian multiplier on the budget constraint:  $\lambda_0 Z = 0$

At equilibrium, goods market clearing implies  $q_\alpha \equiv q^b = \frac{\omega n}{\sigma(1-n)} q_\alpha^s$ .

Money market clears:  $M = m = \tilde{p}q_\alpha$ . Therefore:

$$\frac{\partial q_\alpha}{\partial m} = \frac{1}{\tilde{p}} \quad \text{and} \quad \frac{\partial q_\alpha}{\partial K} = 0 \quad (41)$$

The constraint  $\tilde{p}q^b \leq m$  is binding so  $\lambda_m > 0$ . According to (39) and (33) we get:

$$u'(q_\alpha) > \frac{1}{1 - \alpha(1 - \theta)} c_q\left(\frac{\sigma(1-n)}{\omega n} q_\alpha, \frac{K}{1-\alpha}\right) \quad (42)$$

### Equilibrium rental price of capital:

if  $\phi^e\tau < \delta$  then (40) yields:  $\lambda_0 > \lambda_K \geq 0$ . Therefore,  $\lambda_0 > 0$  and  $Z = 0$ . In words, if the rental price expressed in expected real balances does not cover the depreciation of capital then the consumer is not interested in renting his capital.

On the contrary, if  $\phi^e\tau > \delta$  then (40) yields:  $\lambda_K > \lambda_0 \geq 0$ . Thus,  $\lambda_K > 0$  and  $Z = K$ . This means that the consumer is willing to rent all his capital as long as the expected real value of the rental price exceeds the capital depreciation factor.

Finally, if  $\phi^e\tau = \delta$ , (40) implies  $\lambda_0 = \lambda_K$ . The case  $\lambda_0 = \lambda_K > 0$  leads to  $Z = 0$  and  $K = 0$  which is impossible as long as  $K > 0$ . Thus  $\lambda_0 = \lambda_K = 0$  and  $0 < Z < K$ . This reflects that the consumer is indifferent to how much capital to rent between 0 and  $K$  if the expected real value of the rental price is equal to the capital depreciation factor.

At equilibrium, the competitive price of rent  $\tau$  makes supply and demand equal:

$$(1 - n)Z = nS$$

We obtain:

$$Z = \frac{n}{1-n} \frac{\alpha}{1-\alpha} K \quad (43)$$

Notice, if no PLS joint-venture has been requested by producers,  $\alpha = 0$ , then  $Z = 0$  according to (43). Moreover, if producers take the PLS joint-venture offered by the central bank ( $\alpha = \tilde{\alpha}$ ) with some sharing ratio,  $0 < \tilde{\alpha} \leq 1 - n$  then  $0 < Z \leq K$ . Nevertheless, if  $\tilde{\alpha} = 1 - n$ , then (43) implies  $Z = K$ , which leads, as discussed above, to an undetermined equilibrium rental price  $\tau$  in the region  $] \frac{\delta}{\phi^e}, +\infty [$ . We avoid this case by assuming that the feasibility condition on  $\tilde{\alpha}$  is not binding:  $\tilde{\alpha} < 1 - n$ . Then (43) implies  $0 < Z < K$ , which is possible, as explained above, only if the equilibrium rental price of capital is given by:

$$\tau = \frac{\delta}{\phi^e} \quad (44)$$

Taking this into account, (38) simplifies to:

$$V^c(m, K) = \phi^e m + K + W_0 + \sigma [u(q_\alpha) - \phi^e p q_\alpha] \quad (45)$$

Note that consumers get no surplus, on average, from capital intermediation because the money they receive at the end of the (DM) in exchange of renting their capital is barely enough to offset, on average, the depreciation of their capital. This surplus can be written as:  $(\phi\tau Z - \delta Z) = \left(\frac{\phi}{\phi^e} - 1\right) \delta Z$  (since  $\tau = \frac{\delta}{\phi^e}$ ). So, it can be positive or negative at each end of the (DM), depending on whether the value of money is rising or falling compared to its expected value, but in the average case, it is zero.

Now, differentiate (34) and (45), using (33) and (41), to obtain the marginal value of money and capital for producers and consumers:

$$V_m^p(m, \frac{K}{1-\alpha}) = \phi^e \quad (46)$$

$$V_m^c(m, K) = \phi^e \left\{ 1 + \sigma \left( [1 - \alpha(1 - \theta)] \frac{u'(q_\alpha)}{c_q(q_\alpha^s, \frac{K}{1-\alpha})} - 1 \right) \right\} \quad (47)$$

$$V_k^p(m, \frac{K}{1-\alpha}) = 1 - \delta - \frac{\omega}{1-\alpha} c_k(q_\alpha^s, \frac{K}{1-\alpha}) \quad (48)$$

$$V_k^c(m, K) = 1 \quad (49)$$

To get the marginal value of money and capital, differentiate (31) with respect to  $m$  and  $K$ , and insert the derivatives above:

**Marginal value of money:**

$$V_m(m, K) = \phi^e \left\{ 1 + \sigma(1 - n) \left( [1 - \alpha(1 - \theta)] \frac{u'(q_\alpha)}{c_q(q_\alpha^s, \frac{K}{1-\alpha})} - 1 \right) \right\} \quad (50)$$

**Marginal value of capital:**

$$V_k(m, K) = 1 - n\delta - \frac{\omega n}{1-\alpha} c_k(q_\alpha^s, \frac{K}{1-\alpha}) \quad (51)$$

### 5.3 Equilibrium

Now, we derive the equilibrium of the economy with PLS monetary policy.

As in the lump-sum transfers economy, the equilibrium consumption in the (CM) is given by (4) and is the same across time for all agents.

Substituting (50), into the FOC for  $m$ , (5), yields<sup>3</sup>:

$$\phi_{-1} = \beta\phi^e \left\{ 1 + \sigma(1-n) \left( [1 - \alpha(1-\theta)] \frac{u'(q_\alpha)}{c_q(q_\alpha^s, \frac{K}{1-\alpha})} - 1 \right) \right\} \quad (52)$$

The term  $\frac{\phi_{-1}}{\beta\phi^e} - 1$ , denoted in what follows by  $i$ , measures the expected cost of holding money for the next period. From the equation (52), the value of money in a period is equal to its discounted expected value at the next period plus a liquidity premium  $L(\alpha, \theta)$ , which gives the marginal benefit of holding money in the (DM), and counterbalances on the other hand the expected cost of holding money:

$$L(\alpha, \theta) = \sigma(1-n) \left( [1 - \alpha(1-\theta)] \frac{u'(q_\alpha)}{c_q(q_\alpha^s, \frac{K}{1-\alpha})} - 1 \right) = i$$

(52) can be written as:

$$\frac{u'(q_\alpha)}{c_q(\frac{\sigma(1-n)}{\omega n} q_\alpha, \frac{K}{1-\alpha})} = \frac{1}{1 - \alpha(1-\theta)} \left[ 1 + \frac{i}{\sigma(1-n)} \right] \quad (53)$$

From (42) and (53),  $i > 0$  is a necessary condition for equilibrium existence.

Substitute now (51), into the FOC for  $K$ , (6), to get<sup>4</sup>:

$$1 = \beta \left[ 1 - n\delta - \frac{\omega n}{1-\alpha} c_k(q_\alpha^s, \frac{K}{1-\alpha}) \right]$$

This reduces, using  $\beta = \frac{1}{1+r}$  to:

$$-c_k(\frac{\sigma(1-n)}{\omega n} q_\alpha, \frac{K}{1-\alpha}) = (1-\alpha) \frac{r+n\delta}{\omega n} \quad (54)$$

It should be pointed out that, when no PLS joint-venture is undertaken ( $\alpha = 0$ ), the money supply is constant  $M_+ = M$  and  $\phi^e = \phi_{-1}$ . Therefore,  $i_{(\alpha=0)} = \frac{1}{\beta} - 1 = r$  and

<sup>3</sup>Note, we take  $\mu_m = 0$  in (5) as we focus on equilibrium with  $m = \tilde{p}q_\alpha > 0$ .

<sup>4</sup> $\mu_K$  has been taken equal to 0 in (6), as we are interested in equilibrium with  $K > 0$ .

(53) becomes similar to (22) with  $\gamma = 1$ , while (54) boils down to (23). The equilibrium hence is the same as the one of the laissez-faire policy.

**Definition 3.** A stationary equilibrium in economy with PLS monetary policy, governed by  $(\tilde{\alpha}, \theta)$ , is quantities  $(x, \alpha, q_\alpha, K)$  satisfying (4), (37), (53) and (54):

$$\left\{ \begin{array}{l} x = U'^{-1}(1) \\ \frac{u'(q_\alpha)}{c_q(\frac{\sigma(1-n)}{\omega n} q_\alpha, \frac{K}{1-\alpha})} = \frac{1}{1-\alpha(1-\theta)} \left[ 1 + \frac{i}{\sigma(1-n)} \right] \\ -c_K(\frac{\sigma(1-n)}{\omega n} q_\alpha, \frac{K}{1-\alpha}) = (1-\alpha) \frac{r+n\delta}{\omega n} \\ \alpha = \tilde{\alpha} \mathbb{1}_{\{\Delta(q_{\tilde{\alpha}}^s(m, K), \frac{K}{1-\tilde{\alpha}}) > \Delta(q_0^s(m, K), K)\}} \end{array} \right. \quad (55)$$

The equilibrium is termed PLS-equilibrium if the PLS joint-venture is formed,  $\alpha = \tilde{\alpha}$ .<sup>5</sup>

**Lemma 1.** A PLS-equilibrium is characterized by an expected cost of holding money,  $i$ , satisfying:

$$i = r - \tilde{\alpha}\sigma(1-n)(1+r) \left\{ 1 - \theta - \left( \frac{n\delta}{r+n\delta} \right) \left[ \frac{1 - \tilde{\alpha}(1-\theta)}{1 - \tilde{\alpha}} \right] \left[ -\frac{\frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}} c_k(q_{\tilde{\alpha}}^s, \frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}})}{q_{\tilde{\alpha}}^s c_q(q_{\tilde{\alpha}}^s, \frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}})} \right] \right\}$$

**Proof:** See Appendix C. The demonstration follows from the dynamic of the money supply, (27) and the fact that  $\phi_{-1}M = \phi M_+$  in stationary equilibrium.

From the derivatives of the cost function explained in the appendix A, we have  $-\frac{c_k}{c_q} = f_k$ . Also,  $q_{\tilde{\alpha}}^s = f(e, \frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}})$ . Therefore, the bracketed term on the far right of the expression of  $i$  above can be viewed as:  $-\frac{\frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}} c_k(q_{\tilde{\alpha}}^s, \frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}})}{q_{\tilde{\alpha}}^s c_q(q_{\tilde{\alpha}}^s, \frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}})} = \frac{\frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}} f_k(e, \frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}})}{f(e, \frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}})}$

which represents the elasticity of production with respect to capital.

**Proposition 2.** If the PLS joint-venture is feasible,  $\alpha = \tilde{\alpha}$  in (55), then a PLS-equilibrium  $(x^*, q_{\tilde{\alpha}}, K_{\tilde{\alpha}})$  exists and tends to the laissez-faire equilibrium  $(x^*, q^*, K^*)$  when  $\tilde{\alpha}$  tends to 0. Furthermore, if the elasticity of production with respect to capital is constant or a strictly increasing function of production then the PLS-equilibrium is unique.

<sup>5</sup>Note, when no PLS joint-venture is implemented ( $\alpha = 0$ ), the equilibrium is equivalent to the laissez-faire equilibrium (26).

**Proof:** We prove in the Appendix C, that:

- \* The total capital held by a producer is an increasing function of  $q_{\tilde{\alpha}}^s$ ,  $\frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}} = g_{\tilde{\alpha}}(q_{\tilde{\alpha}}^s)$
- \* The inverse supply curve,  $\phi^e \tilde{p} = \frac{c_q(q_{\tilde{\alpha}}^s, \frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}})}{1 - \tilde{\alpha}(1 - \theta)} \equiv S(q_{\tilde{\alpha}}^s)$  is increasing in  $q_{\tilde{\alpha}}^s$ .
- \* The inverse demand curve,  $\phi^e \tilde{p} = \frac{u'(q_{\tilde{\alpha}}^b)}{1 + \frac{i}{\sigma(1-n)}} \equiv D(q_{\tilde{\alpha}}^b)$  satisfies  $\lim_{q_{\tilde{\alpha}}^b \rightarrow 0} D(q_{\tilde{\alpha}}^b) = +\infty$  and  $\lim_{q_{\tilde{\alpha}}^b \rightarrow +\infty} D(q_{\tilde{\alpha}}^b) = 0$
- \* Given this,  $\lim_{q_{\tilde{\alpha}} \rightarrow 0} D(q_{\tilde{\alpha}}) - S(\frac{\sigma(1-n)}{\omega n} q_{\tilde{\alpha}}) = +\infty$  and  $\lim_{q_{\tilde{\alpha}} \rightarrow +\infty} D(q_{\tilde{\alpha}}) - S(\frac{\sigma(1-n)}{\omega n} q_{\tilde{\alpha}}) < 0$

Consequently, it exists  $q_{\tilde{\alpha}} > 0$  such that  $D(q_{\tilde{\alpha}}) = S(\frac{\sigma(1-n)}{\omega n} q_{\tilde{\alpha}})$ . Hence, the inverse demand and supply curves intersect at least once, which gives the equilibrium consumption  $q_{\tilde{\alpha}} = q_{\tilde{\alpha}}^b = \frac{\omega n}{\sigma(1-n)} q_{\tilde{\alpha}}^s$ . The equilibrium capital holding is given by  $K_{\tilde{\alpha}} = (1 - \tilde{\alpha}) g_{\alpha}(\frac{\sigma(1-n)}{\omega n} q_{\tilde{\alpha}})$ . Thus, a PLS-equilibrium  $(x^*, q_{\tilde{\alpha}}, K_{\tilde{\alpha}})$  exists.

- \* It's straightforward that the second and the third equations of (55) tend to the equivalent ones of the laissez-faire equilibrium (26) when  $\tilde{\alpha}$  tends to 0, while the first equation on  $x$  is the same. Hence, a solution  $(x^*, q_{\tilde{\alpha}}, K_{\tilde{\alpha}})$  of the PLS-equilibrium tend to  $(x^*, q^*, K^*)$  of the laissez-faire equilibrium when  $\tilde{\alpha}$  tends to 0.
- \* It's straightforward, from lemma (1) that if the elasticity of production with respect to capital is constant or a strictly increasing function of production then the expected cost of holding money,  $i$ , is also constant or a strictly increasing in  $q_{\tilde{\alpha}}^b$ . In that case, the inverse demand curve  $D(q_{\tilde{\alpha}}^b) = \frac{u'(q_{\tilde{\alpha}}^b)}{1 + \frac{i}{\sigma(1-n)}}$  is strictly decreasing (as  $u'$  is strictly decreasing). Therefore, the inverse demand and supply curves, intersect exactly once and thereby the PLS-equilibrium is unique (see figure 3).

In the rest of the paper we consider, for tractability purpose, a linear disutility  $l(e) = e$  and a Cobb-Douglas production function  $f(e, K) = e^{1-\psi} K^{\psi}$  where  $0 < \psi < 1$ . This leads to a Cobb-Douglas cost function  $c(q, K) = q^{\frac{1}{1-\psi}} K^{1-\frac{1}{1-\psi}}$  which has the following characteristics:

- \* The Hessian matrix is equal to zero,  $c_{qq}(q, K) - \frac{c_{qk}^2(q, K)}{c_{kk}(q, K)} = 0$
- \* The operator  $\Delta$  simplifies to:  $\Delta(q, K) \equiv qc_q(q, K) - c_k(q, K) = \frac{\psi}{1-\psi} c(q, K)$
- \* The elasticity of production with respect to capital is constant,  $\frac{K f_k(e, K)}{f(e, K)} = \psi$

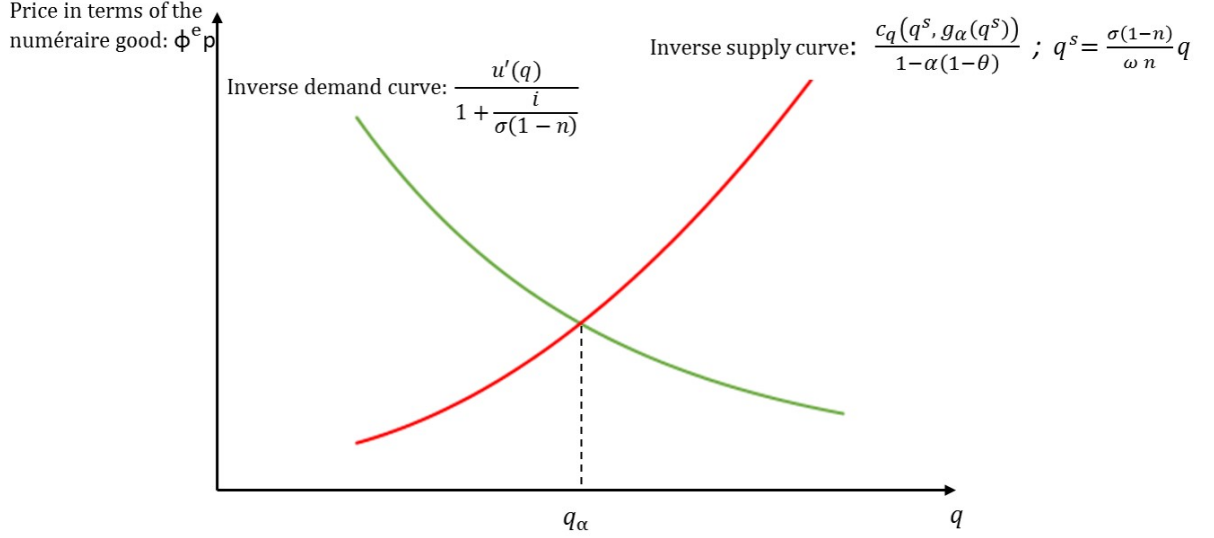


Figure 3: Inverse demand and supply curves in the (DM)

The constancy of the elasticity of production with respect to capital implies that of the expected cost of holding money in a PLS-equilibrium. From lemma (1), the expression of the latter simplifies indeed to:

$$i(\tilde{\alpha}, \theta) = r - \tilde{\alpha}\sigma(1-n)(1+r) \left\{ 1 - \theta - \psi \left( \frac{n\delta}{r+n\delta} \right) \left[ \frac{1 - \tilde{\alpha}(1-\theta)}{1 - \tilde{\alpha}} \right] \right\} \quad (56)$$

This ensures, according to proposition 2, the strict decrease of the inverse demand curve and thereby the uniqueness of the PLS-equilibrium, as long as it is feasible.

Moreover, (54) leads to a linear relationship between the capital stock held by a representative producer,  $\frac{K}{1-\alpha}$ , and the quantity of special good he produces,  $q_\alpha^s$ :

$$\frac{K}{1-\alpha} = \frac{\chi}{(1-\alpha)^{1-\psi}} q_\alpha^s \quad (57)$$

$$\text{where } \chi = \left[ \frac{\psi}{1-\psi} \left( \frac{\omega n}{r+n\delta} \right) \right]^{1-\psi}$$

It follows that the function  $q_\alpha^s \rightarrow c_q(q_\alpha^s, \frac{K}{1-\alpha})$  reduces to a constant:

$$c_q(q_\alpha^s, \frac{K}{1-\alpha}) = \frac{1}{1-\psi} \left[ \frac{q_\alpha^s}{\frac{K}{1-\alpha}} \right]^{\frac{\psi}{1-\psi}} = \xi(1-\alpha)^\psi \quad (58)$$

$$\text{where } \xi = \frac{1}{1-\psi} \left[ \frac{\psi}{1-\psi} \left( \frac{\omega n}{r+n\delta} \right) \right]^{-\psi} = \frac{\chi}{\psi} \left( \frac{r+n\delta}{\omega n} \right)$$

As result, the inverse supply curve in figure (3) is an horizontal line,  $S(q_{\tilde{\alpha}}^s) = \frac{\xi(1 - \tilde{\alpha})^\psi}{1 - \tilde{\alpha}(1 - \theta)}$ , that intersects the strictly decreasing inverse demand curve at exactly one point.

Given that, the equilibrium consumption,  $q_\alpha$ , derives from:

$$u'(q_\alpha) = \xi \frac{(1 - \alpha)^\psi}{1 - \alpha(1 - \theta)} \left[ 1 + \frac{i}{\sigma(1 - n)} \right] \quad (59)$$

In particular, when  $\alpha = 0$ , the equilibrium consumption (equivalent to the one under laissez-faire policy) is given by:  $u'(q^*) = \xi \left[ 1 + \frac{r}{\sigma(1 - n)} \right]$

We return to the comparison of these two allocations later. Let's examine now the feasibility conditions on the parameters  $\tilde{\alpha}$  and  $\theta$  that the central bank should offer to make the PLS joint-venture attractive to producers.

## 5.4 PLS joint-venture feasibility conditions

According to (37) and using the simplified expression of the operator  $\Delta$  we get:

$$\alpha = \tilde{\alpha} \mathbb{1} \left\{ c(q_{\tilde{\alpha}}^s(m, K), \frac{K}{1 - \tilde{\alpha}}) > c(q_0^s(m, K), K) \right\}$$

Or equivalently, using the expression of  $c$ :

$$\alpha = \tilde{\alpha} \mathbb{1} \left\{ q_{\tilde{\alpha}}^s(m, K) > \frac{q_0^s(m, K)}{(1 - \tilde{\alpha})^\psi} \right\} \quad (60)$$

This reflects that a producer, who leaves the centralized market with  $m$  quantity of money and  $K$  quantity of capital, is interested in entering in a PLS joint-venture in the intermediation market if this will rise enough his production level in the decentralized market,  $q_{\tilde{\alpha}}^s(m, K)$ , compared with that of self-financing,  $q_0^s(m, K)$ .

The relation (35) between  $q_{\tilde{\alpha}}^s(m, K)$  and  $q_0^s(m, K)$  boils down, in the case of Cobb-Douglas cost function, to:

$$q_{\tilde{\alpha}}^s(m, K) = \frac{[1 - \tilde{\alpha}(1 - \theta)]^{\frac{1-\psi}{\psi}}}{1 - \tilde{\alpha}} q_0^s(m, K) \quad (61)$$

By combining (60) and (61) we arrived at:  $\alpha = \tilde{\alpha} \mathbb{1} \left\{ [1 - \tilde{\alpha}(1 - \theta)]^{\frac{1-\psi}{\psi}} > (1 - \tilde{\alpha})^{1-\psi} \right\}$

which reduces after rearrangement to:

$$\alpha = \tilde{\alpha} \mathbb{1} \left\{ \theta > \frac{(1 - \tilde{\alpha})^\psi - (1 - \tilde{\alpha})}{\tilde{\alpha}} \right\} \quad (62)$$



We can then state the following:

**Proposition 3.** *A PLS joint-venture with a risk-sharing ratio  $\tilde{\alpha}$  is requested by the producer if the pro-rata deviation rate  $\theta$ , favoring him in the distribution of revenue, is higher than a minimum value  $\theta_{min}(\tilde{\alpha}) = \frac{(1 - \tilde{\alpha})^\psi - (1 - \tilde{\alpha})}{\tilde{\alpha}} > 0$ .*

It should be highlighted that  $\theta_{min}(\tilde{\alpha}) \rightarrow 0$  when  $\psi \rightarrow 1$ . This means that the producer accepts to share the revenue on pro-rata basis under the PLS joint-venture if the share of capital in output is 1. Furthermore, the minimum feasible pro-rata deviation rate  $\theta_{min}(\tilde{\alpha})$  is decreasing with the Cobb-Douglas capital-share  $\psi$ , for a given  $\tilde{\alpha}$ :  $\frac{\partial \theta_{min}(\tilde{\alpha})}{\partial \psi} < 0$ . In words, the lower the share of capital in output is (i.e.. the higher the share of effort in output is) the more the producer becomes demanding in terms of distribution of revenue.

Taking into account the lower bound  $\theta_{min}(\tilde{\alpha})$  and the equation (33), the feasibility condition on  $\theta$ , (29), becomes:

$$\theta_{min}(\tilde{\alpha}) < \theta \leq 1 - \frac{\delta [1 - \tilde{\alpha}(1 - \theta)] \frac{K}{1-\tilde{\alpha}}}{q_{\tilde{\alpha}}^s c_q(q_{\tilde{\alpha}}^s, \frac{K}{1-\tilde{\alpha}})}$$

Using (57) and (58) we get: 
$$\frac{\frac{K}{1-\tilde{\alpha}}}{q_{\tilde{\alpha}}^s c_q(q_{\tilde{\alpha}}^s, \frac{K}{1-\tilde{\alpha}})} = \frac{\psi}{1 - \tilde{\alpha}} \frac{\omega n}{r + n\delta}$$

Therefore, the condition on  $\theta$  can be expressed as:  $\theta_{min}(\tilde{\alpha}) < \theta \leq 1 - \psi\omega \left(\frac{n\delta}{r+n\delta}\right) \left[\frac{1-\tilde{\alpha}(1-\theta)}{1-\tilde{\alpha}}\right]$

Which leads after rearrangement of terms to:

**Feasibility condition on  $\theta$ :**

$$\theta_{min}(\tilde{\alpha}) < \theta \leq \theta_{max}(\tilde{\alpha}) \equiv \frac{1 - \psi\omega \left(\frac{n\delta}{r+n\delta}\right)}{1 + \psi\omega \left(\frac{n\delta}{r+n\delta}\right) \left(\frac{\tilde{\alpha}}{1-\tilde{\alpha}}\right)} \quad (63)$$

In addition to the feasibility constraint  $\tilde{\alpha} < 1 - n$ , the inequality above, implies a second feasibility constraint on  $\tilde{\alpha}$ :  $\theta_{min}(\tilde{\alpha}) < \theta_{max}(\tilde{\alpha})$ .

Let  $\Omega$  be the function defined on  $[0,1]$  by  $\Omega(\alpha) = (1 - \alpha)^{1-\psi} + \alpha \left[1 - \psi\omega \left(\frac{n\delta}{r+n\delta}\right)\right] - 1$ .

It is simple to prove that  $\theta_{min}(\tilde{\alpha}) < \theta_{max}(\tilde{\alpha})$  is equivalent to  $\Omega(\tilde{\alpha}) > 0$  (see demonstration in appendix D). Besides, it is straightforward that,  $\Omega(0) = 0$ ,  $\Omega(1) = -\psi\omega \left(\frac{n\delta}{r+n\delta}\right) < 0$ ,

$\Omega'(0) = \psi \left(1 - \omega \frac{n\delta}{r+n\delta}\right) > 0$ ,  $\Omega$  is concave since  $\Omega''(\alpha) = -\psi(1 - \psi)(1 - \alpha)^{-1-\psi} < 0$ .

By consequence (see figure (4) and appendix D for rigorous demonstration),  $\Omega$  admits a unique root  $\alpha_f$  on  $]0, 1[$ , such that for all  $\tilde{\alpha}$  in the region  $]0, \alpha_f[$ :  $\Omega(\tilde{\alpha}) > 0$ .

We can then express the upper bound constraint on the central bank risk-sharing ratio:

**Feasibility condition on  $\tilde{\alpha}$ :**

$$0 < \tilde{\alpha} < \text{Min}(1 - n, \alpha_f) \quad (64)$$

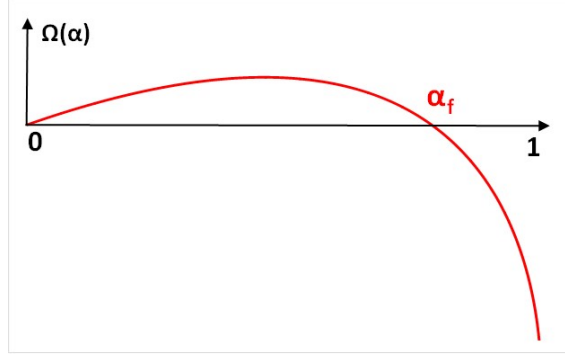


Figure 4

## 6 Optimal monetary policy

In this part, we analyze the optimal PLS monetary policy defined as the policy that maximizes the steady state social welfare. We define social welfare, as in (24), as the expected lifetime utility of the representative agent:

$$\mathcal{W} = U(x) - E[h] + \sigma(1 - n) u(q_\alpha) - \omega n c(q_\alpha^s, \frac{K}{1-\alpha}) + \beta \mathcal{W}$$

In the economy with PLS monetary policy, the appendix D verifies that

$$E[h] = x + n\delta K + \omega n \alpha(1 - \theta)\phi^e \tilde{p}q_\alpha^s$$

The third term, at the right hand side of the equation above, represents the expected additional hours an agent having been a producer in the (DM) and having succeeded to produce the special good (which occurs with probability  $\omega n$ ), has to work in the (CM) to offset the quantity of money  $\alpha(1 - \theta)\tilde{p}q_\alpha^s$  he gave to the central bank in the (DM) under profit-sharing. Substituting  $E[h]$  in the expression of welfare, yields:<sup>6</sup>

$$\mathcal{W}(\alpha, \theta) = \frac{1}{1 - \beta} \left\{ U(x) - x - n\delta K - \omega n \alpha(1 - \theta)\phi^e \tilde{p}q_\alpha^s + \sigma(1 - n)u(q_\alpha) - \omega n c(q_\alpha^s, \frac{K}{1-\alpha}) \right\}$$

<sup>6</sup>Notice, when no PLS joint-venture is implemented ( $\alpha = 0$ ,  $q_\alpha = q^*$  and  $K = K^*$ ), the social welfare reduces to the one of the laissez-faire policy.

By rearranging terms using (33), (57) and (58), we get the social welfare expression under the Cobb-Douglas specification:

$$\begin{aligned} \mathcal{W}(\alpha, \theta) = & \frac{1}{1-\beta} \left\{ U(x) - x - \chi \frac{\sigma(1-n)}{\omega} \delta (1-\alpha)^\psi q_\alpha - \sigma(1-n) \frac{\alpha(1-\theta)}{1-\alpha(1-\theta)} \xi (1-\alpha)^\psi q_\alpha \right. \\ & \left. + \sigma(1-n) [u(q_\alpha) - (1-\psi)\xi(1-\alpha)^\psi q_\alpha] \right\} \quad (65) \end{aligned}$$

The central bank's problem is to reach the maximum level of social welfare through the implementation of a PLS joint-venture. The social welfare maximization is subject to the feasibility conditions on the risk-sharing ratio and the pro-rata deviation rate seen in (63) and (64), so that producers accept the PLS joint-venture and the central bank does not incur any loss in capital when the business makes profit. The other constraint is the non-negativity of the expected cost of holding money which is required for the equilibrium existence.

**Central bank's problem:**  $\max_{\tilde{\alpha}, \theta} \mathcal{W}(\tilde{\alpha}, \theta)$

subject to the feasibility constraints:

$$\begin{aligned} 0 &< \tilde{\alpha} < \text{Min}(1-n, \alpha_f) \\ \theta_{\min}(\tilde{\alpha}) &< \theta \leq \theta_{\max}(\tilde{\alpha}) \\ 0 &< i(\tilde{\alpha}, \theta) \end{aligned}$$

**Lemma 2.** *At PLS-equilibrium, whatever the value of  $\tilde{\alpha}$ , the consumption,  $q_{\tilde{\alpha}}$  and the social welfare,  $\mathcal{W}(\tilde{\alpha}, \theta)$  are increasing with respect to the pro-rata deviation rate,  $\theta$ :*

$$\begin{aligned} \frac{\partial q_{\tilde{\alpha}}}{\partial \theta} &= -\frac{1}{u''(q_{\tilde{\alpha}})} \frac{\xi \tilde{\alpha} (1-\tilde{\alpha})^\psi}{[1-\tilde{\alpha}(1-\theta)]^2} \left[ \frac{r}{\sigma(1-n)} + r\psi \left( \frac{n\delta}{r+n\delta} \right) \left( \frac{\tilde{\alpha}}{1-\tilde{\alpha}} \right) \right] > 0 \\ (1-\beta) \frac{\partial \mathcal{W}}{\partial \theta} &= \sigma(1-n)\xi(1-\tilde{\alpha})^\psi \left\{ \frac{\tilde{\alpha} q_{\tilde{\alpha}}}{[1-\tilde{\alpha}(1-\theta)]^2} + \frac{\partial q_{\tilde{\alpha}}}{\partial \theta} \left( \frac{\psi r}{r+n\delta} + \frac{\frac{i}{\sigma(1-n)}}{1-\tilde{\alpha}(1-\theta)} \right) \right\} > 0 \end{aligned}$$

**Proof:** see Appendix D.

As direct corollary of lemma 2, the social welfare is maximized when the pro-rata deviation rate is at its maximum possible value,  $\theta = \theta_{\max}(\tilde{\alpha})$ . This means that it is optimal that the central bank get the right revenue, if any, that offsets the depreciation of its capital invested, while the producer derives the maximum possible profit from the PLS joint-venture.

**Lemma 3.** For  $\theta = \theta_{max}(\tilde{\alpha})$ , the expected cost of holding money, at PLS-equilibrium, is strictly positive whatever be the value of  $\tilde{\alpha}$ :

$$\tilde{i} = r + (1+r) \left( \frac{1}{\omega} - 1 \right) \sigma(1-n) \left[ 1 - \frac{1}{1 + \psi\omega \left( \frac{n\delta}{r+n\delta} \right) \left( \frac{\tilde{\alpha}}{1-\tilde{\alpha}} \right)} \right] > 0 \quad (66)$$

**Proof:** see Appendix D.

From lemma (3), the constraint of non-negativity of  $i$  is always met when  $\theta = \theta_{max}(\tilde{\alpha})$ . Thus, the central bank's optimization problem reduces to a single variable problem in  $\tilde{\alpha}$ . We can then state the following:

**Proposition 4** The optimal PLS monetary policy is a couple of positive numbers  $(\tilde{\alpha}_{opt}, \theta_{opt})$  satisfying:

$$\begin{cases} \tilde{\alpha}_{opt} = \arg \max_{\tilde{\alpha}} \tilde{W}(\tilde{\alpha}) & \text{s.t. } 0 < \tilde{\alpha} < \text{Min}(1-n, \alpha_f) \\ \theta_{opt} = \theta_{max}(\tilde{\alpha}_{opt}) \end{cases} \quad (67)$$

Where:

$$\begin{aligned} \tilde{W}(\tilde{\alpha}) &\equiv \mathcal{W}(\tilde{\alpha}, \theta_{max}(\tilde{\alpha})) \\ &= \frac{1-n}{1-\beta} \left\{ \frac{U(x) - x}{1-n} - \chi\delta \frac{\sigma}{\omega} \left[ \frac{1-\tilde{\alpha}(1-\omega)}{(1-\tilde{\alpha})^{1-\psi}} \right] q_{\tilde{\alpha}} + \sigma [u(q_{\tilde{\alpha}}) - (1-\psi)\xi(1-\tilde{\alpha})^\psi q_{\tilde{\alpha}}] \right\} \end{aligned} \quad (68)$$

**Proof:** For the expression of  $\mathcal{W}(\tilde{\alpha}, \theta_{max}(\tilde{\alpha}))$  see Appendix D.

In the last part of this paper, we will determine, through a quantitative example, the analytical solution to the optimization problem and discuss the results obtained. But before, let us examine the improvement of allocations under the specification  $\theta = \theta_{max}(\tilde{\alpha})$ .

### Improvement of the equilibrium allocation:

From (59), we have under  $\theta = \theta_{max}(\tilde{\alpha})$ :  $u'(q_{\tilde{\alpha}}) = \xi \Gamma(\tilde{\alpha})$  where:

$$\Gamma(\tilde{\alpha}) = \frac{(1-\tilde{\alpha})^\psi}{1-\tilde{\alpha}(1-\theta_{max}(\tilde{\alpha}))} \left[ 1 + \frac{\tilde{i}}{\sigma(1-n)} \right] \quad (69)$$

To see how  $q_{\tilde{\alpha}}$  responds to an increase in the risk-sharing ratio  $\tilde{\alpha}$ , we differentiate the previous equation with respect to  $\tilde{\alpha}$ :  $\frac{\partial q_{\tilde{\alpha}}}{\partial \tilde{\alpha}} u''(q_{\tilde{\alpha}}) = \xi \Gamma'(\tilde{\alpha})$

We substitute  $\xi$  by  $\frac{u'(q_{\tilde{\alpha}})}{\Gamma(\tilde{\alpha})}$  to get:  $\frac{\partial q_{\tilde{\alpha}}}{\partial \tilde{\alpha}} = \frac{u'(q_{\tilde{\alpha}})}{u''(q_{\tilde{\alpha}})} \frac{\Gamma'(\tilde{\alpha})}{\Gamma(\tilde{\alpha})}$

To economize in notation, we introduce the following constants:

$$\begin{aligned}\lambda &= 1 - \left( \frac{n\delta}{r+n\delta} \right) \left[ \omega + \frac{1+r}{1+\frac{r}{\sigma(1-n)}}(1-\omega) \right] \quad (\in ]0, 1[ ) \\ \mu &= 1 - \psi(1-\lambda) \quad (\in ]\lambda, 1[ ) \\ \tilde{\alpha}_m &= \frac{\lambda}{\mu} \quad (\in ]0, 1[ )\end{aligned}$$

Appendix D shows that:  $\frac{\Gamma'(\tilde{\alpha})}{\Gamma(\tilde{\alpha})} = \frac{\psi\mu}{(1-\tilde{\alpha})(1-\mu\tilde{\alpha})}(\tilde{\alpha} - \tilde{\alpha}_m)$ . Therefore:

$$\frac{\partial q_{\tilde{\alpha}}}{\partial \tilde{\alpha}} = \frac{u'(q_{\tilde{\alpha}})}{u''(q_{\tilde{\alpha}})} \frac{\psi\mu}{(1-\tilde{\alpha})(1-\mu\tilde{\alpha})}(\tilde{\alpha} - \tilde{\alpha}_m) \quad (70)$$

The term multiplied by  $\tilde{\alpha} - \tilde{\alpha}_m$ , at the right hand side of (70) is strictly negative. Thus, when  $\tilde{\alpha} < \tilde{\alpha}_m$ , the PLS-equilibrium consumption is increasing in  $\tilde{\alpha}$  and when  $\tilde{\alpha} > \tilde{\alpha}_m$ , the PLS-equilibrium consumption is decreasing in  $\tilde{\alpha}$ . This means that the PLS monetary policy improves the equilibrium allocation, compared to the laissez-faire policy, at least in the region  $]0, \tilde{\alpha}_m]$ . We recall however that  $\tilde{\alpha}$  is constrained by the feasibility upper bound  $Min(1-n, \alpha_f)$ , so the risk-sharing ratio that maximizes the PLS-equilibrium allocation is given by:

$$\tilde{\alpha}_q = Min(1-n, \alpha_f, \tilde{\alpha}_m) \quad (71)$$

Furthermore, it should be pointed that the optimal risk-sharing ratio  $\tilde{\alpha}_{opt}$ , that maximizes the social welfare, is not necessarily the one that maximizes the equilibrium allocation. We show in the quantitative analysis that  $\tilde{\alpha}_{opt}$  may be different from  $\tilde{\alpha}_q$ .

## 7 Quantitative analysis

We consider a logarithmic utility function in the (CM),  $U(x) = B \log(x)$  which implies  $x = B$  at equilibrium; and a CRRA utility function in the (DM):  $u(q) = D \frac{q^{1-\eta}}{1-\eta}$  where  $0 < \eta < 1$  is the relative risk aversion. Given that  $\eta = -\frac{q u''(q)}{u'(q)}$ , the equation (70)

simplifies to:  $\frac{\partial q_{\tilde{\alpha}}}{\partial \tilde{\alpha}} = \frac{1}{\eta} \frac{\psi\mu}{(1-\tilde{\alpha})(1-\mu\tilde{\alpha})}(\tilde{\alpha}_m - \tilde{\alpha}) q_{\tilde{\alpha}}$

If we differentiate (68) with respect to  $\tilde{\alpha}$  and insert the expression above, we find, after some calculation, the expression of the derivative of welfare (see proof in Appendix D):

$$\tilde{W}'(\tilde{\alpha}) = \frac{1-n}{1-\beta} \left[ \frac{\xi\psi\sigma q_{\tilde{\alpha}}}{(1-\mu\tilde{\alpha})(1-\tilde{\alpha})^{2-\psi}} \right] Q(\tilde{\alpha}) \quad (72)$$

where  $Q$  is a polynomial of degree 2 characterized by the following properties:

$$\begin{aligned} Q(y) &= ay^2 + by + d \\ Q(0) &= d > 0 \\ Q(1) &< 0 \end{aligned}$$

It is straightforward from (72) that  $\tilde{\mathcal{W}}'(0^+) > 0$  and  $\tilde{\mathcal{W}}'(1^-) = -\infty$ . Therefore, it exists a value  $\alpha_w$  in the region  $]0, 1[$  such that:  $\tilde{\mathcal{W}}'(\alpha_w) = 0$ . Furthermore, it is clear from (72), that  $\alpha_w$  is a root of the polynomial  $Q$ .

To determine the expression of  $\alpha_w$ , we distinguish three cases:

- Case 1: if  $a > 0$  then  $\lim_{y \rightarrow +\infty} Q(y) = +\infty$ . Combining this by  $Q(1) < 0$  implies that  $Q$  admits a second root in  $]1, +\infty[$ . So,  $\alpha_w$  is the smaller root of  $Q$  given by:

$$\alpha_w = \frac{-b - \sqrt{b^2 - 4ad}}{2a}$$

- Case 2: if  $a < 0$  then  $\lim_{y \rightarrow -\infty} Q(y) = -\infty$ . Combining this by  $Q(0) > 0$  implies that  $Q$  admits a second root in  $] -\infty, 0[$ . So,  $\alpha_w$  is the bigger root of  $Q$  given again by:

$$\alpha_w = \frac{-b - \sqrt{b^2 - 4ad}}{2a}$$

- Case 3: if  $a = 0$  then  $Q(y) = by + d$ . So,  $\alpha_w$  is the unique root of  $Q$  given by:

$$\alpha_w = \frac{-d}{b}$$

In all three cases,  $\tilde{\mathcal{W}}'$  is zero in the region  $]0, 1[$  exactly at the unique point  $\alpha_w$ , positive when  $\alpha < \alpha_w$  and negative when  $\alpha > \alpha_w$ . Consequently, the welfare is enhanced at least in the region  $]0, \alpha_w[$  and is maximum at  $\alpha_w$ . Taking into account the fact that the risk-sharing ratio can not exceed the feasibility upper bound  $Min(1 - n, \alpha_f)$ , we have proven the following:

**Proposition 5** *Under a CRRA utility function in the (DM), the PLS optimal monetary policy is always welfare-improving compared to laissez-faire. It is given by the couple  $(\tilde{\alpha}_{opt}, \theta_{opt})$  satisfying:*

$$\begin{aligned} \tilde{\alpha}_{opt} &= Min(1 - n, \alpha_f, \alpha_w) \\ \theta_{opt} = \theta_{max}(\tilde{\alpha}_{opt}) &= \frac{1 - \psi\omega \left( \frac{n\delta}{r + n\delta} \right)}{1 + \psi\omega \left( \frac{n\delta}{r + n\delta} \right) \left( \frac{\tilde{\alpha}_{opt}}{1 - \tilde{\alpha}_{opt}} \right)} \end{aligned}$$

We turn now to the calibration of the model and discussion of its implications.

## 7.1 Model Calibration and Results

We calibrate the model at an annual frequency and hence set the time preference rate to  $r = 0.05$ . We set the relative risk aversion to be lower than and near to 1,  $\eta = 0.98$  and we take the elasticity of production with respect to capital (Cobb-Douglas capital-share) at  $\psi = 1/3$ , as usual in the literature. Next, we consider a relatively high depreciation rate for capital used in production,  $\delta = 0.2$  (we think here about machinery and equipment), while the depreciation of idle capital held by non-productive agents is neglected in the model. Our benchmark calibration is based on the laissez-faire policy (i.e.. an inflation rate of zero), and a success probability of  $\omega = 0.85$ . But, we also calibrate the model under different values of  $\omega$  in the region  $[0.1, 1]$ , in order to assess the sensitivity of the monetary policy to the business failure risk. The remaining four parameters, namely the utility functions parameters  $B$  and  $D$ , the probability of being a producer in the (DM),  $n$  and the probability to be an active consumer in the (DM),  $\sigma$  are calibrated to match the following four targets:

- First, an investment-capital ratio of  $I/K = 0.05$ . In the model, the investment is the quantity of the general good that offsets the depreciation of capital,  $I = n\delta K$ . This gives us directly the probability of being a producer in the (DM),  $n = (I/K)/\delta = 0.25$ .

- Second, a money demand elasticity of  $e_M = -0.226$ , following the estimation of Aruoba et al. (2009). By definition,  $e_M = \frac{\partial(\phi M)}{\partial i} \frac{i}{\phi M}$  which reduces under the benchmark calibration (i.e.  $i = r$ ) to  $e_M = \frac{r}{q^*} \frac{\partial q}{\partial i} \Big|_{i=r}$  since  $\phi M = \phi p q = \xi q$  at equilibrium. Using  $Dq^{-\eta} = \xi[1 + \frac{i}{\sigma(1-n)}]$ , it is straightforward that:  $e_M = \frac{-r/\eta}{r + \sigma(1-n)}$ . Therefore, the probability to consume in the (DM) is given by:  $\sigma = \frac{r}{1-n} \left[ \frac{1}{\eta(-e_M)} - 1 \right] = 0.2343$ .

- Third, we target an average hours worked of  $E[h^*] = 1/3$  where  $E[h^*] = B + n\delta K^*$ .

- Fourth, we target an average money velocity of 5.381, following Aruoba et al. (2011). Velocity  $v_M$  is computed by considering the price  $1/\phi$  of the general good as the unit of account. Expected real output is  $Y_C = E[h^*]$  in the (CM) and  $Y_D = \phi[\sigma(1-n)pq^*]$  in the (DM). Expected total real output is then  $Y = Y_C + Y_D$  and velocity is given by  $v_M = (1/\phi Y)/M$ . Using  $M = pq^*$  at laissez-faire equilibrium,  $\phi p = \xi$  and  $E[h^*] = 1/3$ , we get after some simple calculations:  $v_M = \frac{1/3 + \sigma(1-n)\xi q^*}{\xi q^*}$  or equivalently  $q^* = \frac{1/3}{\xi[v_M - \sigma(1-n)]}$ .

Then, using  $D(q^*)^{-\eta} = \xi[1 + r/\sigma(1-n)]$ , we obtain the value of the parameter  $D$ :  $D = \xi \left[ 1 + \frac{r}{\sigma(1-n)} \right] \left( \frac{1/3}{\xi[v_M - \sigma(1-n)]} \right)^\eta = 0.0875$ .

Finally,  $B$  is determined from  $E[h^*] = B + n\delta K^*$ :  $B = \frac{1}{3} - n\delta \chi \frac{\sigma(1-n)}{\omega n} q^* = 0.331$ .

The parameters of the baseline calibration and the results are recorded in Tables (1) and (2). In figures (5) and (6), we plot, respectively, the variation of the social welfare  $\tilde{W}$  and that of the (DM) consumption  $q$  in function of the risk-sharing ratio applied by the central bank,  $\tilde{\alpha}$ . We compare graphically the welfare and the (DM) consumption across the three regimes, namely, the laissez-faire policy, the Friedman's rule and the PLS monetary policy.

Table 1: Parameters and results of the benchmark calibration

Simple parameters	$r$	$\omega$	$\psi$	$\eta$	$\delta$
Values	5%	85%	33.33%	0.98	20%

Other parameters	$n$	$\sigma$	$D$	$B$
Targets	$I/K = 5\%$	$e_M = -0.226$	$v_M = 5.381$	$E[h] = 1/3$
Calibrated values	25%	23,43%	0.0875	0.331

PLS policy results	$\alpha_f$	$\tilde{\alpha}_q$	$a$	$\alpha_w$	$\tilde{\alpha}_{opt}$	$\theta_{opt}$	$i_{\tilde{\alpha}_{opt}}$	$\frac{\phi-1}{\phi^e} - 1$
values	87%	61.31%	0.9 > 0	59.84%	59.84%	70.87%	5.57%	0.54%

Table 2: Gain in allocations and welfare under the optimal PLS monetary policy

Compared to the Friedman's rule of the economy with lump-sum transfers					
$\frac{E[Y_{\tilde{\alpha}_{opt}}]}{E[\hat{Y}]}$	$\frac{E[h_{\tilde{\alpha}_{opt}}]}{E[\hat{h}]}$	$\frac{q_{\tilde{\alpha}_{opt}}}{\hat{q}}$	$\frac{K_{\tilde{\alpha}_{opt}}}{\hat{K}}$	reallocation: $\frac{\left(\frac{K_{\tilde{\alpha}_{opt}}}{1-\tilde{\alpha}_{opt}}\right)}{\hat{K}}$	welfare gain: $\frac{\tilde{W}(\tilde{\alpha}_{opt})}{\mathcal{W}_{\gamma=\beta}} - 1$
0.9985	1.003	0.8471	0.625	1.556	1.28%
Compared to the laissez-faire policy					
$\frac{E[Y_{\tilde{\alpha}_{opt}}]}{E[Y^*]}$	$\frac{E[h_{\tilde{\alpha}_{opt}}]}{E[h^*]}$	$\frac{q_{\tilde{\alpha}_{opt}}}{q^*}$	$\frac{K_{\tilde{\alpha}_{opt}}}{K^*}$	reallocation: $\frac{\left(\frac{K_{\tilde{\alpha}_{opt}}}{1-\tilde{\alpha}_{opt}}\right)}{K^*}$	welfare gain: $\frac{\tilde{W}(\tilde{\alpha}_{opt})}{\mathcal{W}_{\gamma=1}} - 1$
1.0067	1.0047	1.0937	0.807	2.009	7.44%

## 7.2 Discussion

In this section, we discuss the main results of the model. First, the mean inflation rate arising from the optimal monetary policy is always positive. With our benchmark calibration, we get an average inflation rate of  $\frac{\phi-1}{\phi^e} - 1 = 0.54\%$ . For an analytical proof that this is always the case whatever the parameters of the model, one can note from (66) that  $\tilde{i} > r$ . Therefore  $\frac{\phi-1}{\phi^e} - 1 = \beta(1 + \tilde{i}) - 1 = \frac{1 + \tilde{i}}{1 + r} - 1 > 0$ . Inflation is positive on average because the optimal PLS monetary policy consists in a positive growth rate of the money supply, in the average case. Indeed, when the representative producer succeeds to generate the special good in the (DM), the PLS joint-venture (with  $\theta = \theta_{max}$ )



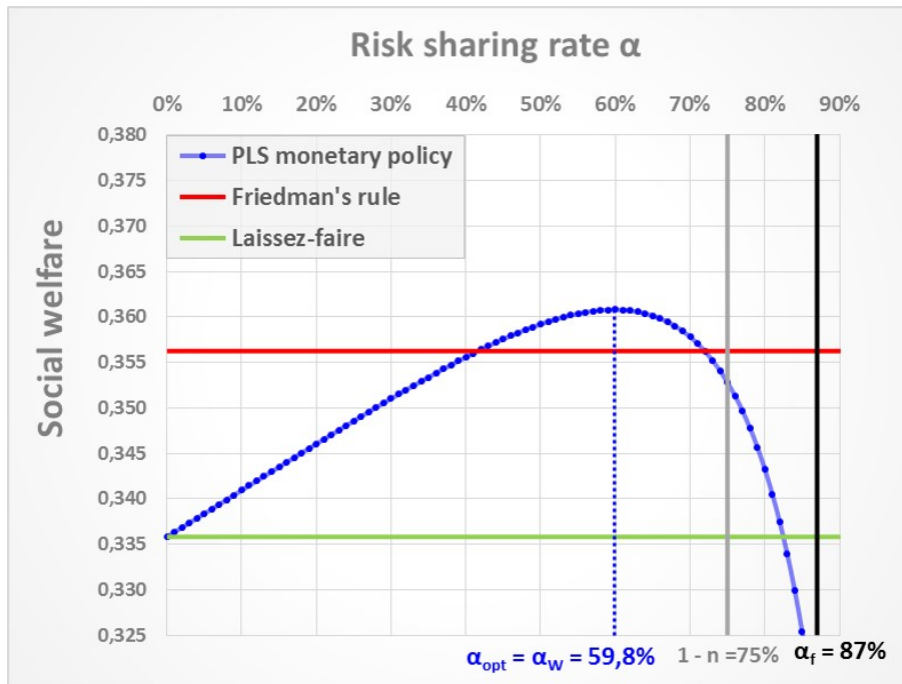


Figure 5

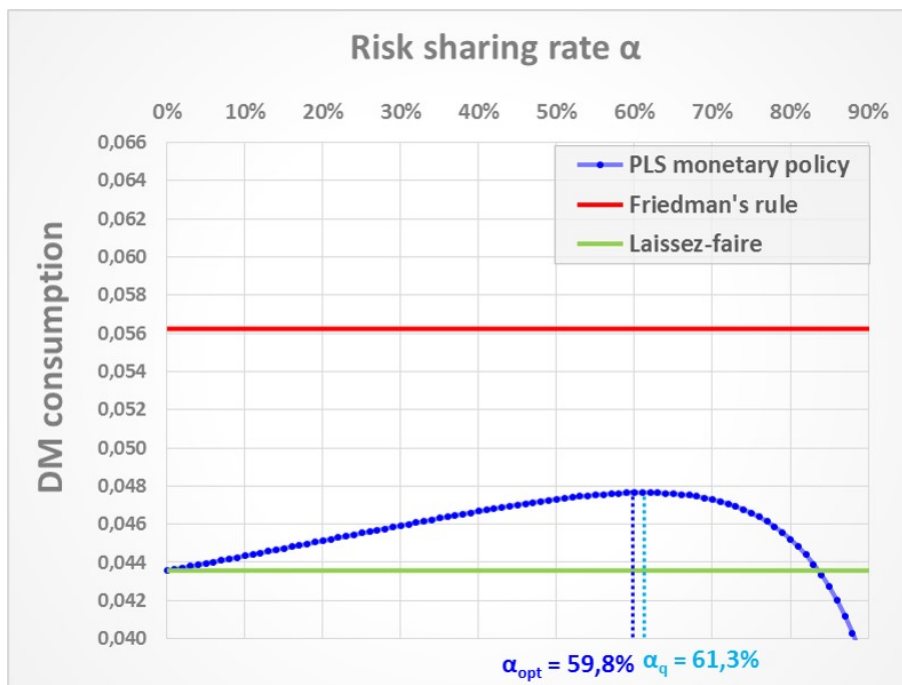


Figure 6

generates the right revenue for the central bank required to pay the capital rental price to consumers. So, the central bank has no need to expand the money supply in this case and the inflation rate is zero at the end of period. However, when the producer fails, the central bank gets no benefit from the PLS joint-venture and resorts to create, ex-nihilo, the quantity of money it needs to pay the renters. Thus, the inflation rate is positive in this case and, by consequence, in the average case. Second, the calibrated model suggests that it would be optimal for the central bank, in order to maximize welfare, to hold  $\tilde{\alpha}_{opt} = 60\%$  of shares in each PLS joint-venture whereby it transfers capital to producers, which means that it bears 60 % of production failure in the decentralized market. The optimal PLS monetary policy requires also a pro-rata deviation rate of  $\theta = 70\%$ . Hence, if the business is successful, the central bank gets 17.5 % of the total amount of revenue ( $\tilde{\alpha}_{opt}(1 - \tilde{\theta}_{opt}) \approx 17.5\%$ ), whereas the representative producer receives 82.5 %. In case of production failure, the central bank makes a loss of 7.45% of total capital in the economy, because of capital depreciation ( $\delta nS = \delta n \frac{\tilde{\alpha}_{opt}}{1 - \tilde{\alpha}_{opt}} K = 7.45\% K$ ) in which case, it creates, ex nihilo, the necessary quantity of money needed to pay the renters,  $M_+ - M = \tau(nS) = \delta nS / \phi^e = 7.49\% K / \phi_{-1}$ .

Third, as shown in Table (2) and figure (5), the PLS monetary policy arising from the benchmark calibration, increases the social welfare both compared to the laissez-faire policy (by 7.44%) as well as to the Friedman's rule policy (by 1.28%). We underline the fact that the gain in welfare always takes place with regards to laissez-faire, as it has been proven analytically in proposition (5). Although not consistently, the gain in welfare relative to the Friedman's rule policy is guaranteed for a large range of business failure risk, precisely, when the latter is high enough ( $\omega$  below 0.94). Figure (7) below, shows the gain in welfare for different values of the probability of success  $\omega$  set in the calibration.<sup>7</sup> Furthermore, we plot in figure (8) the optimal PLS monetary policy ( $\tilde{\alpha}_{opt}, \tilde{\theta}_{opt}$ ) and the inflation rate in function of  $\omega$ . It is interesting to note that the higher the business risk is, the greater the optimal  $\tilde{\alpha}_{opt}$  and  $\tilde{\theta}_{opt}$  are, reflecting the growing requirement for the sharing of risk between producers and the central bank within the PLS monetary policy.<sup>8</sup>

Finally, as shown in figure (9), the optimal PLS monetary policy increases the laissez-faire output in the (DM) by about 10 percent but it decreases, on the contrary, that of the Friedman's rule by about 15 percent. We emphasizes however, as in figure (10), that this positive (resp. negative) effect on the (DM) output compared to laissez-faire (resp. Friedman's rule), remains marginal on the total output because of the relatively small size of the (DM) in regard to the (CM).<sup>9</sup>

To conclude our analysis, let's discuss where the gain in social welfare could come from. The first reason is the saving in depreciation cost due to the fact that agents hold

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<sup>7</sup> Only  $B$  and  $D$  change with  $\omega$ . Other parameters remain equal to our benchmark calibration.

<sup>8</sup> Notice, for  $\omega$  low enough (below 42%), the optimal PLS monetary policy requires a high level of  $\alpha$ . But, the feasibility condition  $\alpha < 1 - n$  constraints  $\alpha$  to don't exceed this upper bound.

<sup>9</sup> The share of output produced in the (DM) is roughly 3%.

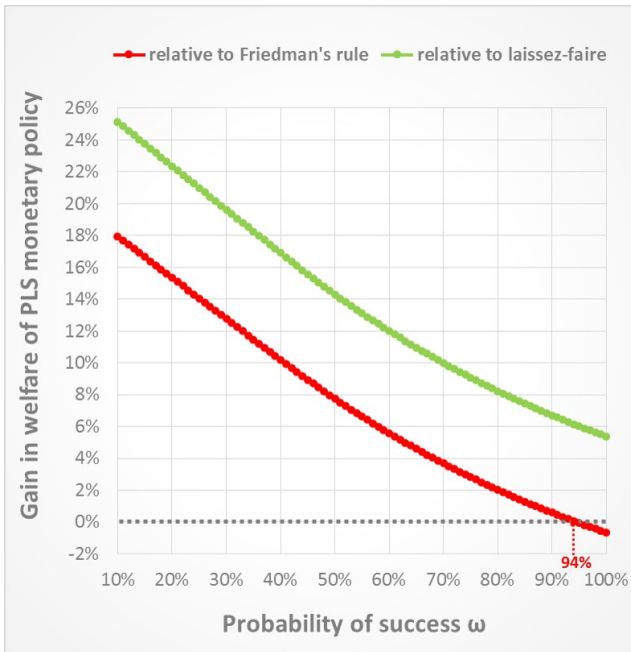


Figure 7

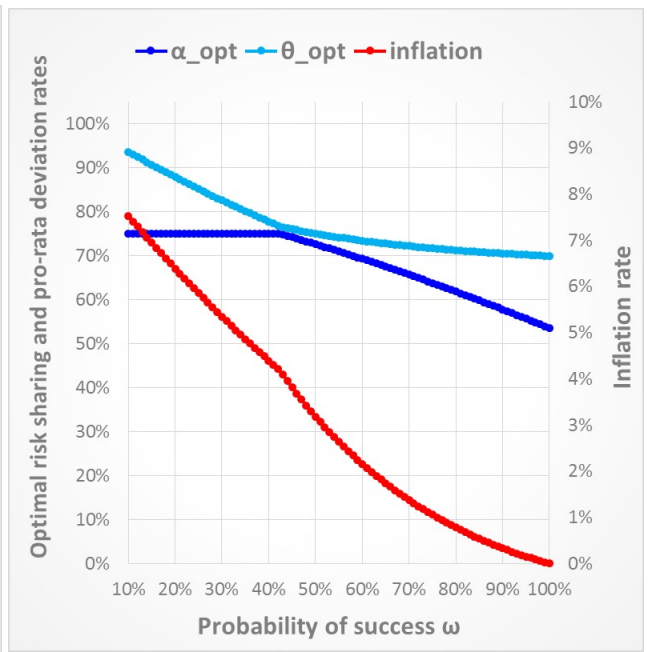


Figure 8

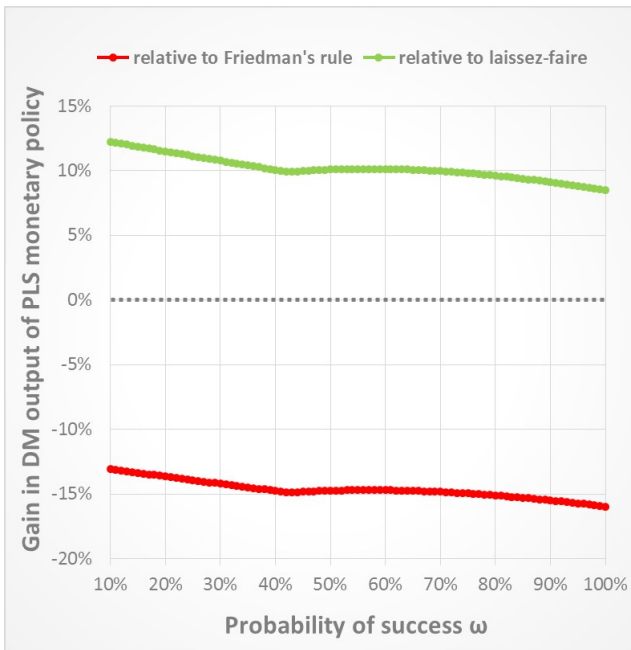


Figure 9

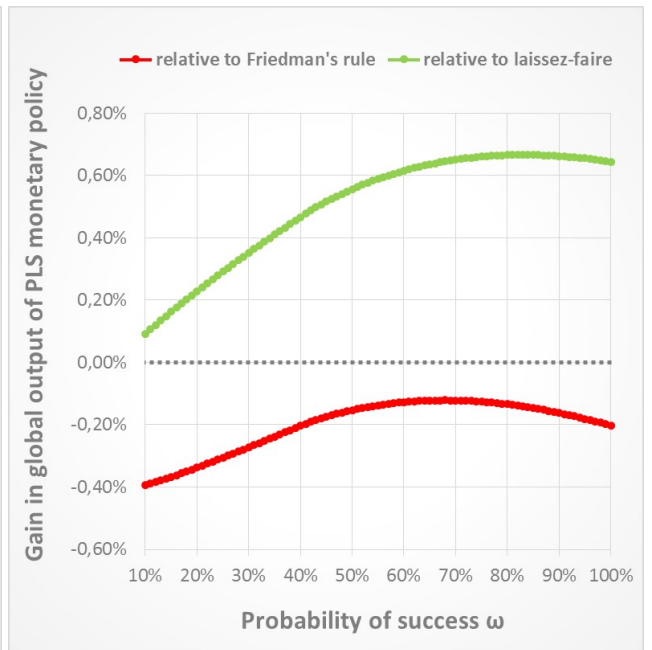


Figure 10

less capital and economize in work needed to offset its depreciation. More specifically, because of the presence of an idle capital that can be intermediated, agents tend to employ less their own capital and resort, once they are decided to produce, to PLS joint-ventures with the central bank whereby they expand the capital stock they put into the business. The saving in depreciation cost arising from the PLS monetary policy compared to the Friedman's rule policy can be written, for a representative period, as:  $n\delta(\widehat{K} - K_{\tilde{\alpha}_{opt}})$ . The second reason is the gain resulting from the reduction of capital misallocation, which reduces in turn the production cost ( $c$  is decreasing with respect to capital,  $c_k < 0$ ). This effect can be assessed, for each period, by the reduction of the disutility of production:  $c(\widehat{q}^s, \widehat{K}) - c(q_{\tilde{\alpha}_{opt}}^s, \frac{K_{\tilde{\alpha}_{opt}}}{1-\tilde{\alpha}_{opt}})$ .

On the other hand, the reallocation of capital via the PLS monetary policy is not without cost. The mobilization of idle capital involves a new depreciation cost, which we can call the capital reallocation cost. As explained above, the renters (consumers) are compensated by the right quantity of money that offsets on average the depreciation of their rented capital and the payment of this rent comes from one of two ways which each give rise to a specific loss in welfare. The first way is when the production is successful. In this case, the central bank receives a total revenue, from PLS joint-venture undertaken with producers, barely enough to pay the renters,  $n\tilde{\alpha}_{opt}(1 - \theta_{opt})\phi^e pq_{\tilde{\alpha}_{opt}}^s$ . This payment generates an equivalent disutility of work in the (CM) and thereby a loss in social welfare. Indeed, the producers should offset the portion of revenue they share with the central bank in order to accumulate the quantity of cash necessary to purchase (eventually) the special good in the next (DM). To this aim, they should work some supplement hours in the (CM) in addition to what it is needed to purchase the general good and repair the depreciated capital<sup>10</sup>. The second way to pay the capital rental price is when the production fails. In this case, the rent is paid using a new money created by the central bank ex nihilo which tends to raise inflation and tax consumption. The capital reallocation cost can be measured then by the eventual decline of the consumption utility  $u(q_{\tilde{\alpha}_{opt}}) - u(\widehat{q})$ . In the light of the above, and using the definition of the social welfare, we can decompose the gain in welfare under the PLS monetary policy as the sum of these four effects, (i) the gain in depreciation cost; (ii) the gain in production cost; and the two forms of capital reallocation cost, (iii) the additional disutility of work and (iv) the reduction of consumption utility:

$$\begin{aligned} \tilde{W} - \widehat{W} &= \frac{1}{1-\beta} n\delta(\widehat{K} - K_{\tilde{\alpha}_{opt}}) + \frac{1}{1-\beta} \omega n \left[ c(\widehat{q}^s, \widehat{K}) - c(q_{\tilde{\alpha}_{opt}}^s, \frac{K_{\tilde{\alpha}_{opt}}}{1-\tilde{\alpha}_{opt}}) \right] \\ &\quad - \frac{1}{1-\beta} \omega n \tilde{\alpha}_{opt}(1 - \theta_{opt})\phi^e pq_{\tilde{\alpha}_{opt}}^s - \frac{1}{1-\beta} \sigma(1 - n) [u(\widehat{q}) - u(q_{\tilde{\alpha}_{opt}})] \end{aligned}$$

The table (3), based on the benchmark calibration, shows the contribution of each effect in the welfare gain of the PLS monetary policy.

<sup>10</sup> It is easy to see it by looking at the expected hours worked in the (CM):  $E[h_{\tilde{\alpha}_{opt}}] = x + n\delta K_{\tilde{\alpha}_{opt}} + \omega n \tilde{\alpha}_{opt}(1 - \theta_{opt})\phi^e pq_{\tilde{\alpha}_{opt}}^s$ .

<b>Compared to the Friedman's rule welfare</b>	
gain in depreciation cost	5.35%
gain in production cost	21.42%
additional disutility of work	-11.30%
difference in consumption utility	-14.19%
<b>Total: welfare gain</b>	1.28%
<b>Compared to the laissez-faire welfare</b>	
gain in depreciation cost	2.27%
gain in production cost	9.06%
additional disutility of work	-11.99%
difference in consumption utility	8.10%
<b>Total: welfare gain</b>	7.44%

Table 3

## 8 Conclusion

In this paper, we have introduced and analyzed a new monetary policy idea, based on a specific profit-and-loss joint-venture between the central bank and productive agents. Through an optimal sharing of risk and profit, the central bank aims to correct capital misallocation in the economy and absorb some of the failure risk that hits the business and hence improve the social welfare. Our main finding is that the optimal PLS monetary policy, even if it generates some inflation, is welfare-and-output improving with regards to laissez-faire policy. Compared to the Friedman's rule in an economy with lump-sum taxes, the PLS monetary policy increases the social welfare, as long as the business failure risk is significant, and that by sacrificing some aggregate output. In a market with frictions and characterized by a substantial idle capital, the supply-side monetary policy presented in this paper, is a simplified but a relevant example of how it can be optimal for the central bank to use its money creation power to support the real economy and reduce the distortions that impede the production activity.

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## Appendix A, characteristics of the cost function $c$

- The cost function  $c(q, K) = l(e) = l(\xi(q, K))$  derives from a production function  $q = f(K, e)$  that is strictly increasing and concave, and a disutility of effort function  $l(e)$  that is strictly increasing and convex.

We have under the usual assumptions:  $f_e > 0$ ;  $f_k > 0$ ;  $f_{ee} < 0$ ;  $f_{kk} < 0$ ;  $f_{ek} = f_{ke} > 0$ . In Aruoba, Waller and Wright (2005), it's proven that:  $c_q > 0$ ,  $c_k < 0$ ,  $c_{qq} > 0$ ,  $c_{kk} > 0$  and  $c_{qk} < 0$  if  $f_k f_{ee} < f_e f_{ek}$  which holds if  $K$  is a normal input ( $\frac{\partial K}{\partial q} = f_e f_{ek} - f_k f_{ee} > 0$ ). We assume this in our model.

Inverting the production function yields:  $e = \xi(q, K)$ . Then:

$$\begin{aligned} \frac{\partial e}{\partial q} &= \xi_e = 1/f_e > 0 ; & \frac{\partial e}{\partial K} &= \xi_k = -f_k/f_e < 0 \\ \xi_{qq} &= \frac{\partial(1/f_e)}{\partial q} = -f_{ee}/f_e^3 > 0 ; & \xi_{qk} &= \frac{\partial(1/f_e)}{\partial K} = -(f_e f_{ek} - f_k f_{ee})/f_e^3 < 0 \\ \xi_{kk} &= \frac{\partial(-f_k/f_e)}{\partial K} = -(f_e^2 f_{kk} - 2f_k f_e f_{ek} + f_k^2 f_{ee})/f_e^3 > 0 \end{aligned}$$

- Convexity of  $\xi$ :

We can easily prove that  $\xi$  is convex by examining the sign of the determinant and the trace of the Hessian matrix:  $H_\xi = \begin{pmatrix} \xi_{qq} & \xi_{qk} \\ \xi_{qk} & \xi_{kk} \end{pmatrix}$

$$Tr(H_\xi) = \xi_{qq} + \xi_{kk} > 0$$

$Det(H_\xi) = \xi_{qq}\xi_{kk} - \xi_{qk}^2$ . After some calculation we find:

$$Det(H_\xi) = (f_{ee}f_{kk} - f_{ek}^2)/f_e^4 \geq 0 \text{ since } f \text{ is concave.}$$

Indeed,  $f$  is concave if and only if  $Tr(H_f) < 0$  and  $Det(H_f) = f_{ee}f_{kk} - f_{ek}^2 \geq 0$

We have then  $Tr(H_\xi) > 0$  and  $Det(H_\xi) \geq 0$ . Thus  $H_\xi$  is strictly definite positive and thereby  $\xi$  is convex.

- Convexity of the cost function  $c$ :

$\xi(q, k)$  convex,  $l(e)$  convex and increasing, implies  $c(q, k) = l(\xi(q, k))$  is convex.

To prove this, let  $\lambda \in [0, 1]$ ,  $e_1 = \xi(q_1, K_1)$  and  $e_2 = \xi(q_2, K_2)$

$\xi$  is convex, implies:  $\xi(\lambda (q_1, K_1) + (1 - \lambda) (q_2, K_2)) \leq \lambda e_1 + (1 - \lambda) e_2$

On the one hand  $l$  is increasing, therefore:

$$l(\xi(\lambda (q_1, K_1) + (1 - \lambda) (q_2, K_2))) \leq l(\lambda e_1 + (1 - \lambda) e_2)$$

On the other hand,  $l$  is convex, so:  $l(\lambda e_1 + (1 - \lambda) e_2) \leq \lambda l(e_1) + (1 - \lambda) l(e_2)$

Thus:  $l(\xi(\lambda (q_1, K_1) + (1 - \lambda) (q_2, K_2))) \leq \lambda l(e_1) + (1 - \lambda) l(e_2)$

or equivalently:  $c(\lambda (q_1, K_1) + (1 - \lambda) (q_2, K_2)) \leq \lambda c(q_1, K_1) + (1 - \lambda) c(q_2, K_2)$

We conclude that,  $c$  is convex and the determinant of its Hessian matrix is positive:  
 $c_{qq}c_{kk} - c_{qk}^2 \geq 0$

- Proof of  $c_k c_{qk} \leq c_q c_{kk}$ :

Following the appendix of "Aruoba, Waller and Wright (2005)", we have:

$$c_q = l'/f_e > 0 ; \quad c_k = -l' f_k/f_e < 0 ; \quad c_{qq} = [l''l'^2 f_e - l' f_{ee}] / f_e^3 > 0$$

$$c_{kk} = -[l''(f_e f_{kk} - 2f_e f_k f_{ek} + f_k^2 f_{ee}) - f_e f_k^2 l'''] / f_e^3 > 0$$

$$c_{qk} = -[l'' f_e f_k - l'(f_k f_{ee} - f_e f_{ek})] / f_e^3 < 0 \text{ under the assumption that } K \text{ is normal input}$$

After some calculations we arrive at:

$$c_k c_{qk} - c_q c_{kk} = \frac{l'^2}{f_e^4} [-f_e f_k f_{ek} + f_e f_{kk}] \leq 0$$

## Appendix B; Economy with lump-sum transfers

- **Expression of  $E[h]$ :**

Let  $h^p$  (resp.  $h^c$ ) denotes the number of hours worked in the (CM) by an agent who was a producer (resp. consumer) in the (DM) of the previous subperiod. We have:

$$E[h] = nE[h^p] + (1 - n)E[h^c] \tag{73}$$

If the producer succeeds to produce the special good in the (DM) then the budget constraint (3) at the steady state leads to:

$$x + \phi m_+ + (K - K(1 - \delta)) = h^p + \phi(m + pq^s) + \underbrace{\phi T}_{\phi(m_+ - m)}$$



which simplifies to  $h^p = x + \delta K - \phi p q^s$

If the producer fails to produce the special good, (3) gives :

$$x + \phi m_+ + (K - K(1 - \delta)) = h^p + \phi m + \underbrace{\phi T}_{\phi(m_+ - m)}$$

which reduces to  $h^p = x + \delta K$

Therefore:  $E[h^p] = \omega(x + \delta K - \phi p q^s) + (1 - \omega)(x + \delta K) = x + \delta K - \omega \phi p q^s$

For an agent who was active consumer in the (DM):

$$x + \phi m_+ + (K - K) = h^c + \phi(m - p q^b) + \underbrace{\phi T}_{\phi(m_+ - m)}$$

which simplifies to  $h^c = x + \phi p q^b$

For an agent who was a passive consumer in the (DM):

$$x + \phi m_+ + (K - K) = h^c + \phi m + \underbrace{\phi T}_{\phi(m_+ - m)}$$

which reduces to  $h^c = x$

Hence:  $E[h^c] = \sigma(x + \phi p q^b) + (1 - \sigma)x = x + \sigma \phi p q^b$

Substituting in (73) and using  $\omega n q^s = \sigma(1 - n)q^b$ , we obtain:

$$E[h] = x + n \delta K$$

## • Proof of proposition 1

### Existence and uniqueness of equilibrium:

Since  $K \rightarrow -c_k(\frac{\sigma(1-n)}{\omega n}q, K)$  is strictly decreasing for any given  $q$  ( $-c_{kk} < 0$ ), we can invert (23) to express  $K$  as function of  $q$ :  $K = g(q) > 0$  where:

$$-c_k(\frac{\sigma(1-n)}{\omega n}q, g(q)) = \frac{r + n\delta}{\omega n}$$

Differentiating this with respect to  $q$ , implies:

$$g'(q) = -\frac{\sigma(1-n)}{\omega n} \frac{c_{qk}(\frac{\sigma(1-n)}{\omega n}q, g(q))}{c_{kk}(\frac{\sigma(1-n)}{\omega n}q, g(q))} > 0$$

Let  $G$  be the function, strictly positive, defined by:  $G(q) = \frac{u'(q)}{c_q(\frac{\sigma(1-n)}{\omega n}q, g(q))}$

Differentiate the expression of  $G$ :

$$G'(q) = \frac{u'' c_q - u' \left[ \frac{\sigma(1-n)}{\omega n} c_{qq} + g' c_{qk} \right]}{c_q^2}$$

Using the expression of  $g'$  this reduces to:

$$G'(q) = \frac{u'' c_q - u' \left[ \frac{\sigma(1-n)}{\omega n} \left( c_{qq} - \frac{c_{qk}^2}{c_{kk}} \right) \right]}{c_q^2} < 0$$

Hence  $G$  is strictly decreasing.

Let's prove that  $\lim_{q \rightarrow 0} G(q) = +\infty$  and  $\lim_{q \rightarrow +\infty} G(q) = 0$

The function  $q \rightarrow c_q(\frac{\sigma(1-n)}{\omega n}q, g(q))$  is increasing since:

$$\left( c_q(\frac{\sigma(1-n)}{\omega n}q, g(q)) \right)' = \frac{\sigma(1-n)}{\omega n} \left( c_{qq} - \frac{c_{qk}^2}{c_{kk}} \right) \geq 0$$

So,  $\forall 0 < q \leq 1$   $0 < c_q(\frac{\sigma(1-n)}{\omega n}q, g(q)) \leq c_q(\frac{\sigma(1-n)}{\omega n}, g(1))$

It yields:  $\forall 0 < q \leq 1$   $G(q) \geq \frac{u'(q)}{c_q(\frac{\sigma(1-n)}{\omega n}, g(1))}$ . Therefore  $\lim_{q \rightarrow 0} G(q) = +\infty$  since  $\lim_{q \rightarrow 0} u'(q) = +\infty$

Also,  $\forall q \geq 1$   $c_q(\frac{\sigma(1-n)}{\omega n}q, g(q)) \geq c_q(\frac{\sigma(1-n)}{\omega n}, g(1))$

So,  $\forall q \geq 1$   $0 < G(q) \leq \frac{u'(q)}{c_q(\frac{\sigma(1-n)}{\omega n}, g(1))}$ . So,  $\lim_{q \rightarrow +\infty} G(q) = 0$  since  $\lim_{q \rightarrow +\infty} u'(q) = 0$

Finally, from (22), the equilibrium consumption satisfies:

$$G(q) = 1 + \frac{\gamma/\beta - 1}{\sigma(1-n)}$$

As  $1 + \frac{\gamma/\beta - 1}{\sigma(1-n)} > 0$  for  $\gamma > \beta$ , we can invert  $G$  in the equation above to obtain the unique solution  $(q, K)$ :

$$\begin{cases} q = G^{-1}\left(1 + \frac{\gamma/\beta - 1}{\sigma(1-n)}\right) \\ K = g(q) \end{cases}$$

**Expression of  $\frac{\partial q}{\partial \gamma}$ :**

Differentiating (22) with respect to  $\gamma$ , yields:

$$\frac{1}{c_q^2} \left\{ \frac{\partial q}{\partial \gamma} u'' c_q - u' \left[ \frac{\sigma(1-n)}{\omega n} \frac{\partial q}{\partial \gamma} c_{qq} + \frac{\partial K}{\partial \gamma} c_{qk} \right] \right\} = \frac{1}{\beta \sigma(1-n)}$$

Differentiating (23) with respect to  $\gamma$ , implies:

$$\frac{\sigma(1-n)}{\omega n} \frac{\partial q}{\partial \gamma} c_{qk} + \frac{\partial K}{\partial \gamma} c_{kk} = 0$$

We get:

$$\frac{\partial K}{\partial \gamma} = -\frac{\sigma(1-n)}{\omega n} \frac{c_{qk}}{c_{kk}} \frac{\partial q}{\partial \gamma} \quad (74)$$

We insert this in the equation above, and rearrange terms to arrive at:

$$\frac{\partial q}{\partial \gamma} = \frac{c_q^2}{\beta \sigma(1-n)} \left[ u'' c_q - \frac{\sigma(1-n)}{\omega n} u' \underbrace{\left[ c_{qq} - \frac{c_{qk}^2}{c_{kk}} \right]}_{\geq 0} \right]^{-1} < 0$$

**Expression of  $\frac{\partial \mathcal{W}}{\partial \gamma}$ :**

Differentiating (25) with respect to  $\gamma$ , implies:

$$\frac{\partial \mathcal{W}}{\partial \gamma} = \frac{1}{1-\beta} \left\{ -n\delta \frac{\partial K}{\partial \gamma} + \sigma(1-n) \frac{\partial q}{\partial \gamma} (u' - c_q) - \omega n \frac{\partial K}{\partial \gamma} c_k \right\}$$

From (23), we have:  $-n\delta - \omega n c_k = r$ . Therefore:

$$\frac{\partial \mathcal{W}}{\partial \gamma} = \frac{1}{1-\beta} \left\{ r \frac{\partial K}{\partial \gamma} + \sigma(1-n) \frac{\partial q}{\partial \gamma} (u' - c_q) \right\}$$

Inserting (74), leads to:

$$\frac{\partial \mathcal{W}}{\partial \gamma} = \frac{\sigma(1-n)}{1-\beta} \left[ \underbrace{u' - c_q}_{>0} - \underbrace{\frac{r}{\omega n} \frac{c_{qk}}{c_{kk}}}_{<0} \right] \underbrace{\frac{\partial q}{\partial \gamma}}_{<0} < 0$$

## Appendix C; Economy with PLS monetary policy

### Proof of lemma 1

$i_{\alpha=0} = r$  follows from the fact that money supply is constant if no PLS joint-venture is requested by producers. So,  $\frac{1}{\beta} \frac{\phi_{-1}}{\phi^e} - 1 = \frac{1}{\beta} - 1 = r$ . If the PLS joint-venture is formed then  $\alpha = \tilde{\alpha}$ .

According to the dynamic of the money supply, (27), we have:

$$\phi M_+ = \phi M + \phi n \left[ \tau \frac{\tilde{\alpha}}{1 - \tilde{\alpha}} K_{\tilde{\alpha}} - \tilde{\alpha}(1 - \theta)\pi \right]$$

Since  $\phi_{-1}M = \phi M_+$  in stationary equilibrium, it yields:

$$\phi_{-1}M = \phi M + n \left[ \phi \tau \frac{\tilde{\alpha}}{1 - \tilde{\alpha}} K_{\tilde{\alpha}} - \tilde{\alpha}(1 - \theta)\phi\pi \right]$$

Applying expectation in both sides, and using  $\phi^e \tau = \delta$ , implies:

$$\phi_{-1}M = \phi^e M + n\delta \left( \frac{\tilde{\alpha}}{1 - \tilde{\alpha}} \right) K_{\tilde{\alpha}} - n\tilde{\alpha}(1 - \theta)E[\phi\pi]$$

The term on the far right is calculated as follows:

$$\begin{aligned} E[\phi\pi] &= \omega E[\phi\pi / \text{production of } q^s] + (1 - \omega)E[\phi\pi / \text{production failure}] \\ &= \omega E[\phi \tilde{p} q_{\tilde{\alpha}}^s] + (1 - \omega)E[0] \\ &= \omega \phi^e \tilde{p} q_{\tilde{\alpha}}^s \end{aligned}$$

Insert this into the equation above and divide by  $\phi^e M = \phi^e \tilde{p} q_{\tilde{\alpha}}^s = \frac{\omega n}{\sigma(1-n)} \phi^e \tilde{p} q_{\tilde{\alpha}}^s$  to get:

$$\frac{\phi_{-1}}{\phi^e} = 1 + \frac{\sigma(1-n)}{\omega n} \frac{n\delta}{\phi^e \tilde{p} q_{\tilde{\alpha}}^s} \left( \frac{\tilde{\alpha}}{1 - \tilde{\alpha}} \right) K_{\tilde{\alpha}} - \sigma(1-n)\tilde{\alpha}(1 - \theta)$$

From (33) we have:  $\phi^e \tilde{p} = \frac{c_q(q_{\tilde{\alpha}}^s, \frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}})}{1 - \tilde{\alpha}(1 - \theta)}$ . Therefore:

$$\frac{\phi_{-1}}{\phi^e} = 1 - \sigma(1-n)\tilde{\alpha}(1 - \theta) + \frac{\sigma(1-n)}{\omega n} n\delta \tilde{\alpha} [1 - \tilde{\alpha}(1 - \theta)] \frac{\frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}}}{q_{\tilde{\alpha}}^s c_q(q_{\tilde{\alpha}}^s, \frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}})}$$

Rearrange this using (54) to obtain:

$$\frac{\phi_{-1}}{\phi^e} = 1 - \sigma(1-n)\tilde{\alpha}(1-\theta) + \sigma(1-n)\tilde{\alpha} \left( \frac{n\delta}{r+n\delta} \right) \left[ \frac{1-\tilde{\alpha}(1-\theta)}{1-\tilde{\alpha}} \right] \left[ -\frac{\frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}} c_k(q_{\tilde{\alpha}}^s, \frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}})}{q_{\tilde{\alpha}}^s c_q(q_{\tilde{\alpha}}^s, \frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}})} \right]$$

Or equivalently

$$\frac{\phi_{-1}}{\phi^e} = 1 - \tilde{\alpha}\sigma(1-n) \left\{ 1 - \theta - \left( \frac{n\delta}{r+n\delta} \right) \left[ \frac{1-\tilde{\alpha}(1-\theta)}{1-\tilde{\alpha}} \right] \left[ -\frac{\frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}} c_k(q_{\tilde{\alpha}}^s, \frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}})}{q_{\tilde{\alpha}}^s c_q(q_{\tilde{\alpha}}^s, \frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}})} \right] \right\}$$

Finally, divide by  $\beta = \frac{1}{1+r}$  and subtract 1 to arrive at the expression of the expected cost of holding money:

$$i = r - \tilde{\alpha}\sigma(1-n)(1+r) \left\{ 1 - \theta - \left( \frac{n\delta}{r+n\delta} \right) \left[ \frac{1-\tilde{\alpha}(1-\theta)}{1-\tilde{\alpha}} \right] \left[ -\frac{\frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}} c_k(q_{\tilde{\alpha}}^s, \frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}})}{q_{\tilde{\alpha}}^s c_q(q_{\tilde{\alpha}}^s, \frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}})} \right] \right\}$$

## Proof of proposition 2:

- **Proof of  $\frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}} = g_{\alpha}(q_{\tilde{\alpha}}^s)$  where  $g_{\alpha}$  is an increasing function:**

For given  $q$ , let  $h_q$  be the function defined on  $R_+$  by:  $h_q(K) = -c_k(q, K)$

We have  $h'_q(K) = -c_{kk}(q, K) < 0$ . Therefore  $h_q$  is strictly decreasing and admits an inverse function  $h_q^{-1}$ .

For given  $\tilde{\alpha}$ , let  $g_{\tilde{\alpha}}$  be the function defined on  $R_+$  by:  $g_{\tilde{\alpha}}(q) = h_q^{-1}(\frac{r+n\delta}{\omega n}(1-\tilde{\alpha}))$ . It's clear that  $g_{\tilde{\alpha}}$  is positive since  $h_q^{-1}$  takes its values in  $R_+$ .

Furthermore:  $h_q(g_{\tilde{\alpha}}(q)) = \frac{r+n\delta}{\omega n}(1-\tilde{\alpha})$ . Using the expression of  $h_q$ , the function  $g_{\tilde{\alpha}}$  satisfies the property:

$$-c_k(q, g_{\tilde{\alpha}}(q)) = \frac{r+n\delta}{\omega n}(1-\tilde{\alpha}) \quad \text{for all } q. \quad (75)$$

For  $q_{\tilde{\alpha}}^s$  particularly we have:  $-c_k(q_{\tilde{\alpha}}^s, g_{\tilde{\alpha}}(q_{\tilde{\alpha}}^s)) = \frac{r+n\delta}{\omega n}(1-\tilde{\alpha})$

According to (54),  $-c_k(q_{\tilde{\alpha}}^s, \frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}}) = \frac{r+n\delta}{\omega n}(1-\tilde{\alpha})$ .

It follows:  $c_k(q_{\tilde{\alpha}}^s, \frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}}) = c_k(q_{\tilde{\alpha}}^s, g_{\tilde{\alpha}}(q_{\tilde{\alpha}}^s))$ .

Hence  $\frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}} = g_{\tilde{\alpha}}(q_{\tilde{\alpha}}^s)$  since  $K \rightarrow c_k(\frac{1-n}{n}q_{\tilde{\alpha}}, K)$  is strictly increasing ( $c_{kk} > 0$ ).

To prove that  $g_{\tilde{\alpha}}$  is increasing, differentiate (75) with respect to  $q$ :

$$-c_{qk}(q, g_{\tilde{\alpha}}(q)) - g'_{\tilde{\alpha}}(q) c_{kk}(q, g_{\tilde{\alpha}}(q)) = 0$$

which leads to:

$$g'_{\tilde{\alpha}}(q) = -\frac{c_{qk}(q, g_{\tilde{\alpha}}(q))}{c_{kk}(q, g_{\tilde{\alpha}}(q))} > 0 \quad (76)$$

- **Proof of that the inverse supply curve,  $\phi^e \tilde{p} = \frac{c_q(q_{\tilde{\alpha}}^s, \frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}})}{1-\tilde{\alpha}(1-\theta)} \equiv S(q_{\tilde{\alpha}}^s)$  is an increasing function of  $q_{\tilde{\alpha}}^s$ :**

Using  $\frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}} = g_{\tilde{\alpha}}(q_{\tilde{\alpha}}^s)$ , we have:  $S(q_{\tilde{\alpha}}^s) = \frac{c_q(q_{\tilde{\alpha}}^s, g_{\tilde{\alpha}}(q_{\tilde{\alpha}}^s))}{1-\tilde{\alpha}(1-\theta)}$

We derive  $S$  with respect to  $q_{\tilde{\alpha}}^s$ :

$$S'(q_{\tilde{\alpha}}^s) = \frac{1}{1-\tilde{\alpha}(1-\theta)} [c_{qq}(q_{\tilde{\alpha}}^s, g_{\tilde{\alpha}}(q_{\tilde{\alpha}}^s)) + g'_{\tilde{\alpha}}(q_{\tilde{\alpha}}^s) c_{qk}(q_{\tilde{\alpha}}^s, g_{\tilde{\alpha}}(q_{\tilde{\alpha}}^s))]$$

This reduces using (76) to:

$$S'(q_{\tilde{\alpha}}^s) = \frac{1}{1-\tilde{\alpha}(1-\theta)} \left[ \underbrace{\left( c_{qq} - \frac{c_{qk}^2}{c_{kk}} \right)}_{\geq 0} (q_{\tilde{\alpha}}^s, g_{\tilde{\alpha}}(q_{\tilde{\alpha}}^s)) \right] \geq 0$$

- **Proof of  $\lim_{q_{\tilde{\alpha}}^b \rightarrow +\infty} D(q_{\tilde{\alpha}}^b) = 0$**

$i > 0$ , is a necessary condition for equilibrium existence.

Given this,  $0 \leq D(q_{\tilde{\alpha}}^b) = \frac{u'(q_{\tilde{\alpha}}^b)}{1 + \frac{i}{\sigma(1-n)}} \leq u'(q_{\tilde{\alpha}}^b)$

Therefore  $\lim_{q_{\tilde{\alpha}}^b \rightarrow +\infty} D(q_{\tilde{\alpha}}^b) = 0$  since  $\lim_{q_{\tilde{\alpha}}^b \rightarrow +\infty} u'(q_{\tilde{\alpha}}^b) = 0$ .

• **Proof of  $\lim_{q_{\tilde{\alpha}}^b \rightarrow 0} D(q_{\tilde{\alpha}}^b) = +\infty$**

As  $\lim_{q_{\tilde{\alpha}} \rightarrow 0} u'(q_{\tilde{\alpha}}) = +\infty$ , it's sufficient to prove that  $\lim_{q_{\tilde{\alpha}} \rightarrow 0} i$  is finite, to arrive at

$$\lim_{q_{\tilde{\alpha}} \rightarrow 0} \frac{u'(q_{\tilde{\alpha}})}{1 + \frac{i}{\sigma(1-n)}} = +\infty.$$

Let  $\zeta_{\tilde{\alpha}}$  be the function defined by:  $\zeta_{\tilde{\alpha}} : q \rightarrow -\frac{g_{\tilde{\alpha}}(q) c_k(q, g_{\tilde{\alpha}}(q))}{q c_q(q, g_{\tilde{\alpha}}(q))}$ .

From the expression of  $i$  in lemma 1 and  $\frac{K_{\tilde{\alpha}}}{1-\tilde{\alpha}} = g_{\alpha}(\frac{\sigma(1-n)}{\omega n} q_{\tilde{\alpha}})$ , it's clear that:

$$i = r - \tilde{\alpha}\sigma(1-n)(1+r) \left\{ 1 - \theta - \left( \frac{n\delta}{r+n\delta} \right) \left[ \frac{1 - \tilde{\alpha}(1-\theta)}{1 - \tilde{\alpha}} \right] \zeta_{\tilde{\alpha}}\left(\frac{\sigma(1-n)}{\omega n} q_{\tilde{\alpha}}\right) \right\}$$

It follows that a sufficient condition for  $\lim_{q_{\tilde{\alpha}} \rightarrow 0} i < +\infty$  is  $\lim_{q \rightarrow 0} \zeta_{\tilde{\alpha}}(q) < +\infty$ .

Let's prove this. Using the Hopital's rule we have:

$$\lim_{q \rightarrow 0} \zeta_{\tilde{\alpha}}(q) = \lim_{q \rightarrow 0} -\frac{[g_{\tilde{\alpha}}(q) c_k(q, g_{\tilde{\alpha}}(q))]' }{[q c_q(q, g_{\tilde{\alpha}}(q))]' } \quad (77)$$

On the one hand:

$$\begin{aligned} [g_{\tilde{\alpha}}(q) c_k(q, g_{\tilde{\alpha}}(q))]' &= g_{\tilde{\alpha}}'(q) c_k + g_{\tilde{\alpha}}(q) \underbrace{[c_{qk} + g_{\tilde{\alpha}}'(q) c_{kk}]}_{= 0 \text{ using (76)}} \\ &= -\frac{c_{qk} c_k}{c_{kk}} < 0 \end{aligned}$$

On the other hand:

$$\begin{aligned} [q c_q(q, g_{\tilde{\alpha}}(q))]' &= c_q + q [c_{qq} + g_{\tilde{\alpha}}'(q) c_{qk}] \\ &= c_q + q \underbrace{\left[ c_{qq} - \frac{c_{qk}^2}{c_{kk}} \right]}_{\text{Hessian positive}} \geq c_q \end{aligned}$$

As a result:

$$0 \leq -\frac{[g_{\tilde{\alpha}}(q) c_k(q, g_{\tilde{\alpha}}(q))]' }{[q c_q(q, g_{\tilde{\alpha}}(q))]' } \leq \frac{c_{qk} c_k}{c_{kk} c_q}$$

Recall the property of the cost function that we proved in Appendix A:

$c_{qk} c_k \leq c_{kk} c_q$  or equivalently  $\frac{c_{qk} c_k}{c_{kk} c_q} \leq 1$ . This implies:

$$0 \leq -\frac{[g_{\tilde{\alpha}}(q) c_k(q, g_{\tilde{\alpha}}(q))]'}{[q c_q(q, g_{\tilde{\alpha}}(q))]'} \leq 1$$

Given this, (77) yields:

$$0 \leq \lim_{q \rightarrow 0} \zeta_{\tilde{\alpha}}(q) \leq 1$$

We conclude that  $\lim_{q \rightarrow 0} \zeta_{\tilde{\alpha}}(q)$  is finite and  $\lim_{q_{\tilde{\alpha}}^b \rightarrow 0} D(q_{\tilde{\alpha}}^b) = +\infty$ .

## Appendix D; PLS monetary policy with Cobb-Douglas

**Proof of  $\theta_{min}(\tilde{\alpha}) < \theta_{max}(\tilde{\alpha}) \Leftrightarrow \Omega(\tilde{\alpha}) > 0$**

$$\begin{aligned} \theta_{min}(\tilde{\alpha}) < \theta_{max}(\tilde{\alpha}) &\Leftrightarrow \frac{(1-\tilde{\alpha})^\psi - (1-\tilde{\alpha})}{\tilde{\alpha}} < \frac{1-\psi\omega\left(\frac{n\delta}{r+n\delta}\right)}{1+\psi\omega\left(\frac{n\delta}{r+n\delta}\right)\left(\frac{\tilde{\alpha}}{1-\tilde{\alpha}}\right)} \\ &\Leftrightarrow [(1-\tilde{\alpha})^\psi - (1-\tilde{\alpha})] \left[1 + \psi\omega\left(\frac{n\delta}{r+n\delta}\right)\left(\frac{\tilde{\alpha}}{1-\tilde{\alpha}}\right)\right] < \tilde{\alpha} \left[1 - \psi\omega\left(\frac{n\delta}{r+n\delta}\right)\right] \\ &\Leftrightarrow [(1-\tilde{\alpha})^{\psi-1} - 1] \left[1 - \tilde{\alpha} + \tilde{\alpha}\psi\omega\left(\frac{n\delta}{r+n\delta}\right)\right] < \tilde{\alpha} \left[1 - \psi\omega\left(\frac{n\delta}{r+n\delta}\right)\right] \\ &\Leftrightarrow (1-\tilde{\alpha})^{\psi-1} - 1 - (1-\tilde{\alpha})^{\psi-1}\tilde{\alpha} \left[1 - \psi\omega\left(\frac{n\delta}{r+n\delta}\right)\right] < 0 \\ &\Leftrightarrow \Omega(\tilde{\alpha}) = (1-\tilde{\alpha})^{1-\psi} + \tilde{\alpha} \left[1 - \psi\omega\left(\frac{n\delta}{r+n\delta}\right)\right] - 1 > 0 \end{aligned}$$

**Proof of the existence and uniqueness of  $\alpha_f$**

As  $\Omega'(0) > 0$  and  $\Omega(0) = 0$ ,  $\Omega$  is strictly positive for  $\alpha$  close enough to zero. Therefore,  $\Omega(1) < 0$  and continuity of  $\Omega$  implies that  $\Omega$  admits at least one root root in  $]0, 1[$ . To prove that this root is unique, we denote by  $\alpha_f$  the smaller root of  $\Omega$  in  $]0, 1[$  such that for all  $0 < \alpha < \alpha_f$ :  $\Omega(\alpha) > 0$ . Given that  $\Omega(\alpha_f) = 0$ , we have for  $\alpha < \alpha_f$  and close enough to  $\alpha_f$ :

$$\Omega(\alpha) = \Omega'(\alpha_f)(\alpha - \alpha_f) + \Omega''(\alpha_f)(\alpha - \alpha_f)^2 + o((\alpha - \alpha_f)^2)$$



It follows:

$$\Omega'(\alpha_f) = \frac{\Omega(\alpha)}{\alpha - \alpha_f} - \Omega''(\alpha_f)(\alpha - \alpha_f) + o(\alpha - \alpha_f)$$

Consequently,  $\Omega'(\alpha_f) < 0$  since  $\alpha - \alpha_f < 0$ ,  $\Omega(\alpha) > 0$  and  $\Omega''(\alpha_f) < 0$  ( $\Omega$  is concave).

Next, concavity of  $\Omega$  implies: for all  $\alpha > \alpha_f$ ,  $\Omega'(\alpha) < \Omega'(\alpha_f) < 0$ . Thus,  $\Omega$  is strictly decreasing on  $]\alpha_f, 1]$ . So, for all  $\alpha > \alpha_f$ ,  $\Omega(\alpha) < \Omega(\alpha_f) = 0$ . Hence,  $\Omega$  is strictly negative on  $]\alpha_f, 1]$  and strictly positive on  $]0, \alpha_f[$  which ensures the uniqueness of the root  $\alpha_f$ .

### Expression of $E[\mathbf{h}]$

We have:  $E[h] = nE[h^p] + (1 - n)E[h^c]$  where  $h^p$  (resp.  $h^c$ ) denotes the number of hours worked in the (CM) by an agent who was a producer (resp. consumer) in the (DM) of the previous subperiod.

If the producer succeeds to produce the special good in the (DM) then the budget constraint (3) at the steady state leads to:

$$x + \phi m + (K - K(1 - \delta)) = h^p + \phi [m + \tilde{p}q^s (1 - \alpha(1 - \theta))]$$

which simplifies to  $h^p = x + \delta K - \phi \tilde{p}q^s (1 - \alpha(1 - \theta))$

If the producer fails to produce the special good, the budget constraint implies:

$$x + \phi m + (K - K(1 - \delta)) = h^p + \phi m$$

which reduces to  $h^p = x + \delta K$

Therefore:  $E[h^p] = \omega [x + \delta K - \phi^e \tilde{p}q^s (1 - \alpha(1 - \theta))] + (1 - \omega)(x + \delta K)$

which simplifies to:  $E[h^p] = x + \delta K - \omega \phi^e \tilde{p}q^s (1 - \alpha(1 - \theta))$

For an agent who was an active consumer in the (DM):

$$x + \phi m + K = h^c + (K - \delta Z) + \phi(\tau Z)$$

As  $m = \tilde{p}q^b$  at equilibrium, we get  $h^c = x + \phi \tilde{p}q^b + (\delta - \phi\tau)Z$

For an agent who was a passive consumer in the (DM):

$$x + \phi m + K = h^c + (K - \delta Z) + \phi(\tau Z) + \phi m$$

which reduces to  $h^c = x + (\delta - \phi\tau)Z$

Therefore:  $E[h^c] = \sigma [x + \phi^e \tilde{p}q^b + (\delta - \phi^e \tau)Z] + (1 - \sigma) [x + (\delta - \phi^e \tau)Z] = x + \sigma \phi^e \tilde{p}q^b$   
since  $\delta - \phi^e \tau = 0$  according to (44).

Lastly, we insert  $E[h^c]$  and  $E[h^p]$  into the expression  $E[h]$  and use  $\omega nq^s = \sigma(1 - n)q^b$  to arrive at:

$$E[h] = x + n\delta K + \omega n\alpha(1 - \theta)\phi^e \tilde{p}q^s$$

## Proof of Lemma 2

- **Expression of  $\frac{\partial q_{\tilde{\alpha}}}{\partial \theta}$**

The PLS-equilibrium consumption is given by (59) with  $\alpha = \tilde{\alpha}$ :

$$u'(q_{\tilde{\alpha}}) = \xi \frac{(1 - \tilde{\alpha})^\psi}{1 - \tilde{\alpha}(1 - \theta)} \left[ 1 + \frac{i}{\sigma(1 - n)} \right]$$

Differentiating this with respect to  $\theta$  yields:

$$\frac{\partial q_{\tilde{\alpha}}}{\partial \theta} u''(q_{\tilde{\alpha}}) = \xi \frac{(1 - \tilde{\alpha})^\psi}{[1 - \tilde{\alpha}(1 - \theta)]^2} \left\{ \frac{\frac{\partial i}{\partial \theta}}{\sigma(1 - n)} [1 - \tilde{\alpha}(1 - \theta)] - \tilde{\alpha} \left[ 1 + \frac{i}{\sigma(1 - n)} \right] \right\}$$

The differentiation of (56) implies:

$$\frac{\frac{\partial i}{\partial \theta}}{\sigma(1 - n)} = \tilde{\alpha}(1 + r) \left[ 1 + \psi \left( \frac{n\delta}{r + n\delta} \right) \left( \frac{\tilde{\alpha}}{1 - \tilde{\alpha}} \right) \right]$$

if we combine this by the expression of  $i$  given by (56), we obtain after some calculation:

$$\frac{\frac{\partial i}{\partial \theta}}{\sigma(1 - n)} [1 - \tilde{\alpha}(1 - \theta)] - \tilde{\alpha} \left[ 1 + \frac{i}{\sigma(1 - n)} \right] = -\tilde{\alpha} \left[ \frac{r}{\sigma(1 - n)} + r\psi \left( \frac{n\delta}{r + n\delta} \right) \left( \frac{\tilde{\alpha}}{1 - \tilde{\alpha}} \right) \right]$$

As result:

$$\frac{\partial q_{\tilde{\alpha}}}{\partial \theta} = -\frac{1}{u''(q_{\tilde{\alpha}})} \frac{\xi \tilde{\alpha}(1 - \tilde{\alpha})^\psi}{[1 - \tilde{\alpha}(1 - \theta)]^2} \left[ \frac{r}{\sigma(1 - n)} + r\psi \left( \frac{n\delta}{r + n\delta} \right) \left( \frac{\tilde{\alpha}}{1 - \tilde{\alpha}} \right) \right] > 0$$

- **Expression of  $\frac{\partial \mathcal{W}}{\partial \theta}$**

Differentiating (65) gives:

$$(1-\beta)\frac{\partial\mathcal{W}}{\partial\theta} = -\chi\frac{\sigma(1-n)}{\omega}\delta(1-\tilde{\alpha})^\psi\frac{\partial q_{\tilde{\alpha}}}{\partial\theta} + \sigma(1-n)\xi\tilde{\alpha}(1-\tilde{\alpha})^\psi\left\{\frac{q_{\tilde{\alpha}}}{[1-\tilde{\alpha}(1-\theta)]^2} - \frac{\partial q_{\tilde{\alpha}}}{\partial\theta}\frac{1-\theta}{1-\tilde{\alpha}(1-\theta)}\right\} \\ + \sigma(1-n)\left[\frac{\partial q_{\tilde{\alpha}}}{\partial\theta}u'(q_{\tilde{\alpha}}) - (1-\psi)\xi(1-\tilde{\alpha})^\psi\frac{\partial q_{\tilde{\alpha}}}{\partial\theta}\right]$$

We rearrange terms:

$$(1-\beta)\frac{\partial\mathcal{W}}{\partial\theta} = \sigma(1-n)\xi(1-\tilde{\alpha})^\psi\left\{\frac{\tilde{\alpha}q_{\tilde{\alpha}}}{[1-\tilde{\alpha}(1-\theta)]^2} + \frac{\partial q_{\tilde{\alpha}}}{\partial\theta}\left[\frac{-\chi\delta}{\omega\xi} - \frac{\tilde{\alpha}(1-\theta)}{1-\tilde{\alpha}(1-\theta)} + \frac{u'(q_{\tilde{\alpha}})}{\xi(1-\tilde{\alpha})^\psi} - 1 + \psi\right]\right\}$$

From (59), we get:

$$-\frac{\tilde{\alpha}(1-\theta)}{1-\tilde{\alpha}(1-\theta)} + \frac{u'(q_{\tilde{\alpha}})}{\xi(1-\tilde{\alpha})^\psi} - 1 = \frac{\frac{i}{\sigma(1-n)}}{1-\tilde{\alpha}(1-\theta)}$$

and from (58):

$$\frac{\chi\delta}{\omega\xi} = \psi\frac{n\delta}{r+n\delta} \quad (78)$$

Substitute this into the equation above to find:

$$(1-\beta)\frac{\partial\mathcal{W}}{\partial\theta} = \sigma(1-n)\xi(1-\tilde{\alpha})^\psi\left\{\frac{\tilde{\alpha}q_{\tilde{\alpha}}}{[1-\tilde{\alpha}(1-\theta)]^2} + \underbrace{\frac{\partial q_{\tilde{\alpha}}}{\partial\theta}}_{>0}\left(\frac{\psi r}{r+n\delta} + \frac{\frac{i}{\sigma(1-n)}}{1-\tilde{\alpha}(1-\theta)}\right)\right\} > 0$$

### Proof of lemma 3: expression and non-negativity of $i(\tilde{\alpha}, \theta_{max}(\tilde{\alpha}))$

Insert  $\theta = \theta_{max}(\tilde{\alpha})$  into (56) yields:

$$\tilde{i} \equiv i(\tilde{\alpha}, \theta_{max}(\tilde{\alpha})) = r - \tilde{\alpha}\sigma(1-n)(1+r)\left\{1 - \theta_{max}(\tilde{\alpha}) - \psi\left(\frac{n\delta}{r+n\delta}\right)\left[\frac{1-\tilde{\alpha}(1-\theta_{max}(\tilde{\alpha}))}{1-\tilde{\alpha}}\right]\right\}$$

From the definition of  $\theta_{max}$  in (63), it is straightforward that:

$$1 - \theta_{max}(\tilde{\alpha}) = \frac{\psi\omega\left(\frac{n\delta}{r+n\delta}\right)\left(\frac{1}{1-\tilde{\alpha}}\right)}{1 + \psi\omega\left(\frac{n\delta}{r+n\delta}\right)\left(\frac{\tilde{\alpha}}{1-\tilde{\alpha}}\right)} \quad (79)$$

Therefore:

$$\psi \left( \frac{n\delta}{r+n\delta} \right) \left[ \frac{1 - \tilde{\alpha}(1 - \theta_{max}(\tilde{\alpha}))}{1 - \tilde{\alpha}} \right] = \frac{\psi \left( \frac{n\delta}{r+n\delta} \right) \left( \frac{1}{1 - \tilde{\alpha}} \right)}{1 + \psi\omega \left( \frac{n\delta}{r+n\delta} \right) \left( \frac{\tilde{\alpha}}{1 - \tilde{\alpha}} \right)} = \frac{1 - \theta_{max}(\tilde{\alpha})}{\omega}$$

Insert this into the expression of  $\tilde{i}$  leads to:

$$\tilde{i} = r + \tilde{\alpha}\sigma(1 - n)(1 + r) [1 - \theta_{max}(\tilde{\alpha})] \left( \frac{1}{\omega} - 1 \right)$$

Finally, substitute  $1 - \theta_{max}(\tilde{\alpha})$  in the previous equation by (79):

$$\tilde{i} = r + (1 + r) \left( \frac{1}{\omega} - 1 \right) \sigma(1 - n) \left[ 1 - \frac{1}{1 + \psi\omega \left( \frac{n\delta}{r+n\delta} \right) \left( \frac{\tilde{\alpha}}{1 - \tilde{\alpha}} \right)} \right] > 0$$

### Expression of $\mathcal{W}(\tilde{\alpha}, \theta_{max}(\tilde{\alpha}))$

From (65) we have:

$$\begin{aligned} \mathcal{W}(\tilde{\alpha}, \theta_{max}(\tilde{\alpha})) &= \frac{1 - n}{1 - \beta} \left\{ \frac{U(x) - x}{1 - n} - \chi\delta \frac{\sigma}{\omega} (1 - \tilde{\alpha})^\psi q_{\tilde{\alpha}} \right. \\ &\quad \left. - \sigma \frac{\tilde{\alpha}(1 - \theta_{max}(\tilde{\alpha}))}{1 - \tilde{\alpha}(1 - \theta_{max}(\tilde{\alpha}))} \xi(1 - \tilde{\alpha})^\psi q_{\tilde{\alpha}} + \sigma [u(q_{\tilde{\alpha}}) - (1 - \psi)\xi(1 - \tilde{\alpha})^\psi q_{\tilde{\alpha}}] \right\} \end{aligned}$$

Using (79), we can write:

$$\frac{\tilde{\alpha}(1 - \theta_{max}(\tilde{\alpha}))}{1 - \tilde{\alpha}(1 - \theta_{max}(\tilde{\alpha}))} = -1 + \frac{1}{1 - \tilde{\alpha}(1 - \theta_{max}(\tilde{\alpha}))} = \psi\omega \left( \frac{n\delta}{r+n\delta} \right) \left( \frac{\tilde{\alpha}}{1 - \tilde{\alpha}} \right)$$

Therefore:

$$\chi\delta \frac{\sigma}{\omega} (1 - \tilde{\alpha})^\psi q_{\tilde{\alpha}} + \sigma \frac{\tilde{\alpha}(1 - \theta_{max}(\tilde{\alpha}))}{1 - \tilde{\alpha}(1 - \theta_{max}(\tilde{\alpha}))} \xi(1 - \tilde{\alpha})^\psi q_{\tilde{\alpha}} = \chi\delta \frac{\sigma}{\omega} (1 - \tilde{\alpha})^\psi q_{\tilde{\alpha}} \left[ 1 + \frac{\omega\xi}{\chi\delta} \psi\omega \left( \frac{n\delta}{r+n\delta} \right) \left( \frac{\tilde{\alpha}}{1 - \tilde{\alpha}} \right) \right]$$

It's straightforward from (78) that  $\frac{\omega\xi}{\chi\delta} \psi\omega \left( \frac{n\delta}{r+n\delta} \right) = \omega$ . The equation above reduces to:

$$\chi\delta \frac{\sigma}{\omega} (1 - \tilde{\alpha})^\psi q_{\tilde{\alpha}} + \sigma \frac{\tilde{\alpha}(1 - \theta_{max}(\tilde{\alpha}))}{1 - \tilde{\alpha}(1 - \theta_{max}(\tilde{\alpha}))} \xi(1 - \tilde{\alpha})^\psi q_{\tilde{\alpha}} = \chi\delta \frac{\sigma}{\omega} \left[ \frac{1 - \tilde{\alpha}(1 - \omega)}{(1 - \tilde{\alpha})^{1-\psi}} \right] q_{\tilde{\alpha}}$$

Insert this into the expression of  $\mathcal{W}(\tilde{\alpha}, \theta_{max}(\tilde{\alpha}))$  to arrive at:

$$\begin{aligned}\tilde{\mathcal{W}}(\tilde{\alpha}) &\equiv \mathcal{W}(\tilde{\alpha}, \theta_{max}(\tilde{\alpha})) \\ &= \frac{1-n}{1-\beta} \left\{ \frac{U(x)-x}{1-n} - \chi \delta \frac{\sigma}{\omega} \left[ \frac{1-\tilde{\alpha}(1-\omega)}{(1-\tilde{\alpha})^{1-\psi}} \right] q_{\tilde{\alpha}} + \sigma [u(q_{\tilde{\alpha}}) - (1-\psi)\xi(1-\tilde{\alpha})^\psi q_{\tilde{\alpha}}] \right\}\end{aligned}$$

**Expression of  $\frac{\Gamma'(\tilde{\alpha})}{\Gamma(\tilde{\alpha})}$**

First, let's write  $\Gamma$  in terms of  $\tilde{\alpha}$ . From (79), it's straightforward that:

$$\frac{1}{1-\tilde{\alpha}(1-\theta_{max}(\tilde{\alpha}))} = 1 + \psi\omega \left( \frac{n\delta}{r+n\delta} \right) \left( \frac{\tilde{\alpha}}{1-\tilde{\alpha}} \right)$$

Insert this in (69) and using the expression of  $\tilde{i}$  given by lemma 3, leads to:

$$\Gamma(\tilde{\alpha}) = (1-\tilde{\alpha})^\psi \left[ 1 + \psi\omega \left( \frac{n\delta}{r+n\delta} \right) \left( \frac{\tilde{\alpha}}{1-\tilde{\alpha}} \right) \right] \left( 1 + \frac{r}{\sigma(1-n)} + (1+r)\left(\frac{1}{\omega} - 1\right) \left[ 1 - \frac{1}{1+\psi\omega \left( \frac{n\delta}{r+n\delta} \right) \left( \frac{\tilde{\alpha}}{1-\tilde{\alpha}} \right)} \right] \right)$$

By rearranging terms, we get:

$$\Gamma(\tilde{\alpha}) = (1-\tilde{\alpha})^\psi \left[ 1 + \frac{r}{\sigma(1-n)} \right] \left\{ 1 + \psi \left( \frac{\tilde{\alpha}}{1-\tilde{\alpha}} \right) \left( \frac{n\delta}{r+n\delta} \right) \underbrace{\left[ \omega + \frac{1+r}{1+\frac{r}{\sigma(1-n)}}(1-\omega) \right]}_{1-\lambda} \right\}$$

We simplify this by using the constants  $\lambda$  and  $\mu = 1 - \psi(1 - \lambda)$ :

$$\Gamma(\tilde{\alpha}) = \left[ 1 + \frac{r}{\sigma(1-n)} \right] \frac{1 - \mu\tilde{\alpha}}{(1-\tilde{\alpha})^{1-\psi}}$$

Now, differentiate  $\Gamma(\tilde{\alpha})$  with respect to  $\alpha$ :

$$\Gamma'(\tilde{\alpha}) = \left[ 1 + \frac{r}{\sigma(1-n)} \right] \frac{\psi}{(1-\tilde{\alpha})^{2-\psi}} (\mu\tilde{\alpha} - \lambda)$$

Therefore, using  $\tilde{\alpha}_m = \frac{\lambda}{\mu}$ :

$$\frac{\Gamma'(\tilde{\alpha})}{\Gamma(\tilde{\alpha})} = \frac{\psi\mu}{(1-\tilde{\alpha})(1-\mu\tilde{\alpha})} (\tilde{\alpha} - \tilde{\alpha}_m)$$

**Expression of  $\tilde{\mathcal{W}}'(\tilde{\alpha})$  under CRRA utility function:**

Differentiating (68) with respect to  $\tilde{\alpha}$  implies:

$$\begin{aligned} \frac{1-\beta}{1-n} \tilde{\mathcal{W}}'(\tilde{\alpha}) = & -\chi\delta\frac{\sigma}{\omega} \left\{ \left[ \frac{1-\tilde{\alpha}(1-\omega)}{(1-\tilde{\alpha})^{1-\psi}} \right] \frac{\partial q_{\tilde{\alpha}}}{\partial \tilde{\alpha}} + \left[ \frac{-(1-\omega)(1-\tilde{\alpha})+(1-\tilde{\alpha}(1-\omega))(1-\psi)}{(1-\tilde{\alpha})^{2-\psi}} \right] q_{\tilde{\alpha}} \right\} \\ & + \sigma \left\{ \frac{\partial q_{\tilde{\alpha}}}{\partial \tilde{\alpha}} u'(q_{\tilde{\alpha}}) - (1-\psi)\xi(1-\tilde{\alpha})^\psi \frac{\partial q_{\tilde{\alpha}}}{\partial \tilde{\alpha}} + \psi(1-\psi)\xi(1-\tilde{\alpha})^{\psi-1} q_{\tilde{\alpha}} \right\} \end{aligned}$$

Using  $\frac{\partial q_{\tilde{\alpha}}}{\partial \tilde{\alpha}} = \frac{1}{\eta} \frac{\psi}{(1-\tilde{\alpha})(1-\mu\tilde{\alpha})} (\lambda - \mu\tilde{\alpha}) q_{\tilde{\alpha}}$ ,  $u'(q_{\tilde{\alpha}}) = \xi \Gamma(\tilde{\alpha})$  and  $\frac{\chi\delta}{\omega} = \xi\psi \frac{\delta n}{r+n\delta}$  we get:

$$\begin{aligned} \frac{1-\beta}{1-n} \tilde{\mathcal{W}}'(\tilde{\alpha}) = & \frac{\xi\psi\sigma q_{\tilde{\alpha}}}{(1-\mu\tilde{\alpha})(1-\tilde{\alpha})^{2-\psi}} \left( -\frac{\delta n}{r+n\delta} \left\{ \left[ \frac{1-\tilde{\alpha}(1-\omega)}{\eta} \frac{\psi(\lambda-\mu\tilde{\alpha})}{\eta} \right] + (1-\mu\tilde{\alpha})[-(1-\omega)(1-\tilde{\alpha})+(1-\tilde{\alpha}(1-\omega))(1-\psi)] \right\} \right. \\ & \left. + \frac{(\lambda-\mu\tilde{\alpha})}{\eta} \Gamma(\tilde{\alpha})(1-\tilde{\alpha})^{1-\psi} - \frac{(1-\psi)(1-\tilde{\alpha})(\lambda-\mu\tilde{\alpha})}{\eta} + (1-\psi)(1-\tilde{\alpha})(1-\mu\tilde{\alpha}) \right) \end{aligned}$$

Let  $Q(\tilde{\alpha})$  denotes the term in parentheses at the right hand side of the equation above. We have then:

$$\tilde{\mathcal{W}}'(\tilde{\alpha}) = \frac{1-n}{1-\beta} \left[ \frac{\xi\psi\sigma q_{\tilde{\alpha}}}{(1-\mu\tilde{\alpha})(1-\tilde{\alpha})^{2-\psi}} \right] Q(\tilde{\alpha})$$

Using  $\Gamma(\tilde{\alpha}) = \left[ 1 + \frac{r}{\sigma(1-n)} \right] \frac{1-\mu\tilde{\alpha}}{(1-\tilde{\alpha})^{1-\psi}}$ , the expression of  $Q(\tilde{\alpha})$  simplifies to:

$$\begin{aligned} Q(\tilde{\alpha}) = & -\frac{\delta n}{r+n\delta} \left\{ [1-\tilde{\alpha}(1-\omega)] \left( \frac{\psi(\lambda-\mu\tilde{\alpha})}{\eta} + (1-\psi)(1-\mu\tilde{\alpha}) \right) - (1-\omega)(1-\tilde{\alpha})(1-\mu\tilde{\alpha}) \right\} \\ & + \left[ 1 + \frac{r}{\sigma(1-n)} \right] \frac{(\lambda-\mu\tilde{\alpha})(1-\mu\tilde{\alpha})}{\eta} - (1-\psi)(1-\tilde{\alpha}) \left[ \frac{\lambda-\mu\tilde{\alpha}}{\eta} - (1-\mu\tilde{\alpha}) \right] \quad (80) \end{aligned}$$

It's straightforward from (80) that  $Q(\tilde{\alpha})$  is a quadratic function of  $\tilde{\alpha}$ . After rearranging terms,  $Q(\tilde{\alpha})$  can be written as:  $Q(\tilde{\alpha}) = a\tilde{\alpha}^2 + b\tilde{\alpha} + d$  where:

$$\begin{aligned} a = & \left[ 1 + \frac{r}{\sigma(1-n)} \right] \frac{\mu^2}{\eta} - \mu(1-\psi) \left( \frac{1}{\eta} - 1 \right) - \frac{\delta n}{r+n\delta} \mu\psi(1-\omega) \left( \frac{1}{\eta} - 1 \right) \\ b = & - \left[ 1 + \frac{r}{\sigma(1-n)} \right] \frac{\mu}{\eta} (1+\lambda) + (1-\psi) \left[ \frac{\lambda}{\eta} - 1 + \mu \left( \frac{1}{\eta} - 1 \right) \right] + \frac{\delta n}{r+n\delta} \left\{ \psi \frac{\mu}{\eta} + \mu(1-\psi) + (1-\omega) \left[ \psi \left( \frac{\lambda}{\eta} - 1 \right) - \mu \right] \right\} \\ d = & \underbrace{\frac{\lambda}{\eta} \left[ \frac{r}{\sigma(1-n)} + \psi \left( 1 - \frac{\delta n}{r+n\delta} \right) \right]}_{>0} + \underbrace{1 - \psi \left( 1 - \frac{\delta n}{r+n\delta} \right) - \omega \left( \frac{\delta n}{r+n\delta} \right)}_{> (1-\psi) \left( 1 - \frac{\delta n}{r+n\delta} \right) > 0} > 0 \end{aligned}$$

Finally, let's prove that  $Q(1) < 0$ . From (80), we have:

$$\begin{aligned} Q(1) &= \left[1 + \frac{r}{\sigma(1-n)}\right] (1-\mu) \left(\frac{\lambda-\mu}{\eta}\right) - \frac{\delta n}{r+n\delta} \left\{ \frac{\omega\psi(\lambda-\mu)}{\eta} + (1-\psi)(1-\mu) \right\} \\ &= \left(\frac{\lambda-\mu}{\eta}\right) \left\{ \left(1 + \frac{r}{\sigma(1-n)}\right) (1-\mu) - \frac{\delta n}{r+n\delta} \omega\psi \right\} - \left(\frac{\delta n}{r+n\delta}\right) (1-\psi)(1-\mu) \end{aligned}$$

Using  $1-\mu = \psi \left(\frac{n\delta}{r+n\delta}\right) \left[\omega + \frac{1+r}{1+\frac{r}{\sigma(1-n)}}(1-\omega)\right]$  we arrive at:

$$Q(1) = \frac{\psi\delta n}{r+n\delta} \underbrace{\left(\frac{\lambda-\mu}{\eta}\right)}_{<0} \left\{ \underbrace{\left(1 + \frac{r}{\sigma(1-n)}\right) \left[\omega + \frac{1+r}{1+\frac{r}{\sigma(1-n)}}(1-\omega)\right]}_{>0} - \omega \right\} - \underbrace{\frac{\delta n(1-\psi)(1-\mu)}{r+n\delta}}_{>0} < 0$$