Necessary and Sufficient Condition for the Existence of Equilibrium in Finite Dimensional Asset Markets with Short-Selling and Preferences with Half-Lines

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Abstract

We consider a pure exchange asset model with a finite number of agents and a finite number of states of nature where short sells are allowed. We present the definition of weak no-arbitrage price, a weaker notion of no-arbitrage price than the one of Werner, and prove that if the utility functions satisfy the maximal and closed gradients conditions we propose in this paper, then there exists an equivalence between existence of a general equilibrium and existence of a price which is weak no-arbitrage price for all the agents.
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JEL Classification: C62, D50, D81,D84,G1.

1 Introduction

In economic models of financial markets, where short-sales are allowed, the case of a finite number of states is well-treated in a huge literature. Since the consumption set is not any more compact, the literature focus on conditions which ensure the compactness of allocation set or of the utility set. These conditions are known as *no-arbitrage* conditions. We can classify them in three categories:

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- Conditions on prices, like Green [16], Grandmont [14], [15], Hammond [12] and Werner [21],
- Conditions on net trade, like Hart [13], Page [18], Nielsen [17], Page and Wooders [19], Allouch [1], Page, Wooders and Monteiro [20],
- Conditions on utility set, like Brown and Werner [5], Dana, Le Van, Magnien [8].

A natural question arises. Under which conditions there is an equivalence between these conditions. In [2], Allouch, Le Van and Page prove the equivalence between Hart's condition and No Unbounded Arbitrage of Page with the assumption that the utility functions have no half-line, i.e. there exists no trading direction in which the agent's utility is constant. These conditions imply existence of a general equilibrium. But the converse is not always true, i.e., the existence of equilibrium does not ensure these no-arbitrage conditions are satisfied. We can find in Ha-Huy and Le Van [11] an example of economy where these conditions fail but an equilibrium exists.

In this paper, we extend the idea of Dana and Le Van in [7] by using weak no-arbitrage price, a no-arbitrage price weaker than the one in Werner [21], or in [2]. Following [7], we use the derivatives of utility functions as weak no-arbitrage prices. Under the conditions of closed gradient (condition C), which is similar of the one of Chichilnisky [6], or maximal condition (condition **M**) for the utility functions, we can establish the equivalence between existence of a price which is weak no-arbitrage for every agent and existence of general equilibrium. Moreover, these conditions are equivalent to the compactness of the individually rational utility set. We emphasize that in our paper existence of half-lines i.e. trading directions in which the agent's utility is constant is not excluded.

The paper is organized as follows. In Section 2, we present the model with the definitions of equilibrium, individually rational attainable allocations set, individually rational utility set, useful vectors and useless vectors. In Section 3, we review some no-arbitrage conditions in the literature. In particular we define weak no-arbitrage prices. Section 4 links usual no-arbitrage conditions and existence of equilibrium. In Section 5, we introduce **C**, **M** conditions. In particular we show that a separable utility satisfies these conditions. In Section 6, we introduce the assumption that the utility functions, if they do not satisfy No Half-line condition then they satisfy either **C** or **M**. We then prove equivalence between the existence of a price which is weak no-arbitrage for every agent and existence of a general equilibrium. Moreover, these conditions are equivalent ! to the compactness of the individually rational utility set. Section 7 is the appendix where we put most of the proofs.

2 The model

We have an exchange economy $\mathcal{E}$ with m agents. Each agent is characterized by a consumption set $X^i = \mathbb{R}^S$, an endowment e^i and a utility function $U^i : \mathbb{R}^S \to \mathbb{R}^S$. We suppose that $\sup_{x\in\mathbb{R}^S} U(x) = +\infty$.

For the sake of simplicity, we suppose that utility functions are concave, strictly increasing and differentiable. In the general case of concave functions, the sub differential of U^i exists. The results and economic intuitions do not change, but the computations become tedious.

We first define an equilibrium of this economy.

Definition 1 *An equilibrium is a list* $\big((x^{*i})_{i=1,\dots,m}, p^*)\big)$ *such that* $x^{*i} \in X^i$ *for every* i *and* $p^* \in \mathbb{R}^S_+ \setminus \{0\}$ *and*
(a) For any i, $U^i(x) > U^i(x^{*i}) \Rightarrow p^* \cdot x > p^* \cdot e^i$
(b) $\sum_{i=1}^m x^{*i} = \sum_{i=1}^m e^i$.

Definition 2 *A quasi-equilibrium is is a list* $\big((x^{*i})_{i=1,\dots,m}, p^*)\big)$ *such that* $x^{*i} \in X^i$ *for every* i *and* $p^* \in \mathbb{R}^S_+ \setminus \{0\}$ *and*
(a) For any i, $U^i(x) > U^i(x^{*i}) \Rightarrow p^* \cdot x \geq p^* \cdot e^i$
(b) $\sum_{i=1}^m x^{*i} = \sum_{i=1}^m e^i$.

Since short-sales are allowed, from Geistdorfer-Florenzano [10], actually any quasi-equilibrium is an equilibrium.

Definition 3 *1. The individually rational attainable allocations set* $\mathcal{A}$ *is defined by*

$$\mathcal{A} = \{(x^i) \in (\mathbb{R}^S)^m | \sum_{i=1}^m x^i = \sum_{i=1}^m e^i \text{ and } U^i(x^i) \geq U^i(e^i) \text{ for all } i\}.$$

2. The individually rational utility set $\mathcal{U}$ *is defined by*

$$\mathcal{U} = \{(v^1, v^2, ..., v^m) \in \mathbb{R}^m \mid \exists\, x \in \mathcal{A} \text{ such that } U^i(e^i) \leq v^i \leq U^i(x^i) \text{ for all } i\}.$$

Definition 4 *i) The vector* w *is called* useful vector *of agent* i *if for any* $x \in \mathbb{R}^S$, *for any* $\lambda \geq 0$ *we have* $U^i(x + \lambda w) \geq U^i(x)$.

ii) The vector w *is called* useless vector *of agent* i *if for any* $x \in \mathbb{R}$, *for any* $\lambda \in \mathbb{R}$ *we have* $U^i(x + \lambda w) = U^i(x)$.

iii) We say that $w \in \mathbb{R}^S$ *is a* half-line direction *for agent* i *if there exists* $x \in \mathbb{R}^S$ *such that* $U^i(x + \lambda w) = U^i(x)$, $\forall\ \lambda \geq 0$.

Denote by R^i the set of useful vectors. L_i the set of useless vectors. By the very definition, the set of useless vectors of agent i is the biggest linear subspace included in R^i:

$$L_i = R^i \cap (-R^i).$$

We have that R^i is the positive dual cone of P^i, i.e. $R^i = -(P^i)^o$. Observe that R^i has no empty interior since $\mathbb{R}^S_+ \subseteq R^i$.

3 No-arbitrage conditions in the literature

We will review some well-known no-abitrage conditions in the literature.
1. Hart [13] proposed the Weak No Market Arbitrage (*WNMA*) condition:

Definition 5 *The economy satisfies* WNMA *if* $(w^1, w^2, \ldots, w^m) \in R^1 \times R^2 \times \ldots \times R^m$ *satisfies* $\sum_{i=1}^m w^i = 0$ *then* $w^i \in L_i$ *for every* i.

2. Page [18] proposed the No Unbounded Arbitrage (*NUBA*) condition:

Definition 6 *The economy satisfies* NUBA *if* $(w^1, w^2, \ldots, w^m) \in R^1 \times R^2 \times \ldots \times R^m$ *satisfies* $\sum_{i=1}^m w^i = 0$ *then* $w^i = 0$ *for every* i.

3. We present the definition of *no-arbitrage prices* proposed by Werner [21].

Definition 7 *The vector* $p \in \mathbb{R}^S$ *is a* no-arbitrage price *for agent* i *if for any* $w \in R^i \setminus L_i$ *we have* $p \cdot w > 0$, *and for* $w \in L_i$, $p \cdot w = 0$.

Denote by S^i the set of no-arbitrage prices of agent i. It is a cone. The *no-arbitrage condition* is $\bigcap_i^m S^i \neq \emptyset$.
We have identity between no-arbitrage condition of Werner and WNMA of Hart:

Proposition 1 $\bigcap_i^m S^i \neq \emptyset \Leftrightarrow$ *WNMA*

For a proof see e.g. Allouch, Le Van and Page [2].

4. In [20], Page, Wooders and Monteiro introduced the notion of *Inconsequential arbitrage.*

Definition 8 *The economy satisfies* Inconsequential arbitrage condition *if for any* $(w_1, w_2, \ldots, w_m)$ *with* $w_i \in R_i$ *for all* i *and* $\sum_{i=1}^m w^i = 0$ *and* $(w^1, w^2, \ldots, w^m)$ *is the limit of* $\lambda_n(x^1(n), x^2(n), \ldots, x^m(n))$ *with* $(x^1(n), x^2(n), \ldots, x^m(n)) \in \mathcal{A}$ *and* λ_n *converges to zero when* n *tends to infinity, there exists* $\epsilon > 0$ *such that for* n *sufficiently big we have* $U^i(x^i(n) - \epsilon w^i) \geq U^i(x^i(n))$.

In [7], Dana and Le Van propose to use the derivative of utility function as no-arbitrage price. They introduce *weak no-arbitrage prices.*

Definition 9 *A vector p is a* weak no-arbitrage price *for agent i if their exists $\lambda > 0$ and $x^i \in \mathbb{R}^S$ such that $p = \lambda U^{i\prime}(x^i)$.*

Let P^i denote the set of weak no-arbitrage prices for the agent i. Their no-arbitrage condition is $\bigcap_i^m P^i \neq \emptyset$
We have $S^i \subset \text{conv}P^i \subset \overline{S}^i$ where $\text{conv}P^i$ denotes the convex hull of P^i and $\overline{S}^i$ is the closure of S^i. Actually we have $S^i = \text{ri}(\text{clconvex}P^i)$ (the relative interior of the closure of convex hull of P^i).
The next Lemma gives a property of useful vectors.

Lemma 1 *The vector w is useful a vector for agent i if, and only if, for all $p \in P^i$, we have: $p \cdot w \geq 0$.*

Proof: ($\Rightarrow$): Let $p = \mu U^{i\prime}(x)$ for some $\mu > 0$ some $x \in \mathbb{R}^S$. From the concavity of U we have:

$$0 \geq \mu U^i(x) - \mu U^i(x + \lambda w) \geq \mu U^{i\prime}(x) \cdot (-\lambda w) = p \cdot (-\lambda w) \text{ for all } \lambda \geq 0.$$

Hence $\mu U^{i\prime}(x) \cdot w \geq 0$.
($\Leftarrow$): Observe that for all $\lambda \geq 0$, all $\mu > 0$:

$$\mu U^i(x + \lambda w) - \mu U^i(x) \geq \mu U^{i\prime}(x + \lambda w) \cdot (\lambda w) \geq 0.$$

Hence $U^i(x^i + \lambda w) \geq U^i(x)$, $\forall\, x$. ■

4 No-arbitrage condition and existence of equilibrium

Existence of equilibrium can be derived from the following result.

Theorem 1 *The compactness of $\mathcal{U}$ implies the existence of equilibrium.*

Proof: For the existence of a quasi-equilibrium, see e.g. Brown and Werner [5] or Dana, Le Van and Magnien [9]. Since short-sales are allowed, in our model quasi-equilibrium is also equilibrium. See Geistdorfer-Florenzano [10]. ■

Theorem 2 *NUBA $\Rightarrow$ WNMA $\Rightarrow$ Inconsequential Arbitrage $\Rightarrow$ Compactness of $\mathcal{U}$ $\Rightarrow$ Existence of equilibirum.*

For a proof see e.g. Allouch, Le Van and Page [2].
It is well known that, with the assumptions we state until now, if an equilibrium exists, we are not sure either that the equilibrium price belongs to the set $S = \bigcap_i S^i$, or the set $\mathcal{U}$ is compact.
However we have with the weak no-arbitrage prices the following result.

Proposition 2

If an equilibrium exists then the equilibrium price is a weak no-arbitrage price for any agent.

$$\textit{Hence } \bigcap_{i=1}^{m} P^i \neq \emptyset.$$

Proof: Denote by $(p^*, (x^{*i})_i)$ the equilibrium. The equilibrium allocation (x^{*i}) solves the problems:

$$\max U^i(x^i)$$
$$\text{s.t. } p^* \cdot x^i = p^* \cdot e^i.$$

From Theorem V.3.1, page 91, in Arrow-Hurwicz-Uzawa (1958) [3], for any i, there exists $\zeta_i > 0$ s.t.

$$U^i(x^{*i}) - \zeta_i p^* \cdot x^{*i} \geq U^i(x^i) - \zeta_i p^* \cdot x^i$$

for any $x^i \in \mathbb{R}^S$. Hence $U^{i\prime}(x^{*i}) = \zeta_i p^*, \forall\ i$. Let $\lambda_i = \frac{1}{\zeta_i}$, we have $p^* = \lambda_i U^{i\prime}(x^{*i}) \in P^i$ for all i. Hence $\bigcap_i P^i \neq \emptyset$. ■

When any agent has no half-line direction, we have

Theorem 3 *Assume no agent has half-line direction. Then the following claims are equivalent*

- $\bigcap_i S^i \neq \emptyset$
- *NUBA holds*
- $\mathcal{A}$ *is compact*
- *WNMA holds*
- $\mathcal{U}$ *is compact*
- *Inconsequential Arbitrage holds*
- *Existence of equilibrium*

For a proof see Allouch, Le Van and Page [2].

The interesting case is when at least one agent has half line directions. Our aim is to prove now

$$\bigcap_{i=1}^{m} P^i \neq \emptyset \text{ implies an equilibrium exists,}$$

even one agent has half line directions. In this case, we will have the equivalence

$$\bigcap_{i=1}^{m} P^i \neq \emptyset \Leftrightarrow \text{ Existence of an equilibrium.}$$

To obtain this result we require the utility functions satisfy some more properties (*Closed Gradient Condition* or *Maximality Condition*).
In [11], we give an example of economy where the no-arbitrage condition, NUBA, WNMA are not satisfied. However this economy has an equilibrium because there exists a common weak no-arbitrage price.

5 The closed gradient condition and the maximality condition

In this section, if there is no confusion, we will use $U(x)$ instead of $U^i(x)$, the set R will denote the set of useful vectors associated with U. Firstly, we present some preliminary results about the limits of a utility function along a useful direction.

Definition 10 *Given a concave function U, and $w \in R$. Define $V[U] : \mathbb{R}^S \times R \to \mathbb{R}$:*

$$V[U](x,w) = \sup_{\lambda \in \mathbb{R}} U(x+\lambda w) = \lim_{\lambda \to +\infty} U(x+\lambda w).$$

We will present here some properties of the function $V[U]$. From now on, when there exists no confusion, we will write $V(x,w)$ instead of $V[U](x,w)$. The proofs of lemmas 3, 4, 5 and 6 are given in Appendix.

Lemma 2 *The vector w is a useless vector of $V(\cdot,w)$.*

Proof: Let $\mu \in \mathbb{R}$. We have

$$\begin{aligned} V(x+\mu w, w) &= \lim_{\lambda \to +\infty} U(x+\mu w+\lambda w) \\ &= \lim_{\lambda \to +\infty} U(x+(\mu+\lambda) w) \\ &= \lim_{\zeta \to +\infty} U(x+\zeta w) \\ &= V(x,w). \end{aligned}$$

■

Lemma 3

- *(a) V is concave.*
- *(b) If there exists $\overline{x}$ such that $V(\overline{x},w) = +\infty$ then $V(x,w) = +\infty$, $\forall x \in \mathbb{R}^S$.*

- *(c) If $V(x,w) < +\infty \ \forall x \in \mathbb{R}^S$, $V(\cdot,w)$ is continuous on $\mathbb{R}^S$.*

Lemma 4 *If $w \in intR$ then $V(x,w) = +\infty$ for all x.*

Remark 1 *Let ∂R denote the boundary of R. From Lemma 4, the necessary condition to have $V(x,w) < +\infty$ is $w \in \partial R$.*

The set of useful vector of U is included in the set of useful vectors of V.

Lemma 5 *Given $w \in \partial R$ such that $V(x,w) \in \mathbb{R}$, a useful vector of U is a useful vector of $V(\cdot,w)$.*

Lemma 6 gives us the derivative of the function V at the points x satisfying $V(x,w) = \max_\lambda U(x+\lambda w)$.

Lemma 6 *Given $w \in \partial R$, $x \in \mathbb{R}^S$. Suppose that there exists $\overline{\lambda} \geq 0$ such that $U(x+\lambda w) = U(x+\overline{\lambda}w)$ for all $\lambda \geq \overline{\lambda}$. Then V is differentiable at x and $V'_x(x,w) = U'(x+\overline{\lambda}w)$.*

We recall that $w \in \mathbb{R}^S$ is a *half-line direction* if there exists $x \in \mathbb{R}^S$ such that $U(x+\lambda w) = U(x)$, $\forall\ \lambda \geq 0$. Let **HL** denote the set of half-line directions.

Proposition 3 *(a) **HL** $\subset \partial R$.*

*(b) Assume that for any $\alpha \in \mathbb{R}$, the set $\{\frac{U'(x)}{\|U'(x)\|}\}_{x\in\sigma_\alpha}$ is closed, then **HL** $= \partial R$.*

Proof: See Appendix. ■

Given an affine space H of $\mathbb{R}^S$ such that $\sup_{x\in H} U(x) = +\infty$, define the function $U_H : H \to \mathbb{R}$ as the restriction of U on H. Define U'_H the projection of gradient of U on H. Let σ_α denote an indifference surface, $\sigma_\alpha = \{x \mid U(x) = \alpha\}$ with $\alpha \in \mathbb{R}$. We set the following assumptions.

C (*Closed gradient condition*): For any $\alpha \in \mathbb{R}$, any H affine space such that $\sup_H U(x) = +\infty$, define σ_α^H as the indifference surface of U on H. Then the map $x \in \sigma_\alpha^H \to U'_H(x)/\|U'_H(x)\|$ is closed.

M (*Maximality*) condition: For all affine space H of $\mathbb{R}^S$, only one of the two following cases holds:

(i) $\sup_H U(x) = +\infty$.

(ii) $\max_H U(x)$ exists.

Remark 2 ***C** condition is borrowed from Chichilnisky [6].*

Proposition 4 *Let U satisfy **C** condition. Let $w \in \partial R$.Then for all x, the function $\lambda \mapsto U(x + \lambda w)$ has a maximum.*

Proof: See Appendix. ■

Proposition 5 *Suppose $w \in R$ is such that for all x, $\max_{\lambda \geq 0} U(x+\lambda w)$ exists. Define $V(x, w) = \max_{\lambda \geq 0} U(x + \lambda w)$. If U satisfies the **C** condition, then so does $V(\cdot, w)$.*

Proof: See Appendix. ■

Proposition 6 *Suppose that U satisfies **M** condition. Given a vector $w \in R$ such that there exists $\tilde{x}$ satisfying $\sup_{\lambda \geq 0} U(\tilde{x} + \lambda w) < +\infty$, then the function $V(x, w) = \sup_{\lambda \geq 0} U(x + \lambda w)$ also satisfies **M**.*

Proof: See Appendix. ■

The next Proposition 7 gives us an example of utility function which satisfies **C** and **M** conditions.

Proposition 7 *We consider the case where the utility function is defined as $U(x) = \sum_{s=1}^{m} \pi_s u(x_s)$, with $u : \mathbb{R} \to \mathbb{R}$ is a concave, strictly increasing, differentiable function, π is a probability measure on $\{1, \dots, S\}$, with $\pi_s > 0$, $\forall$ s. Define $a = u'(+\infty)$, $b = u'(-\infty)$. Suppose that $0 < a \leq b < +\infty$ and there exists $z > 0$ such that $u'(x) = a$, $\forall$ $x > z$ and $u'(x) = b$, $\forall$ $x < -z$. Then we have:*

*(a) U satisfies **C** condition.*
*(b) U satisfies **M** condition.*

Proof: See Appendix. ■

6 Equivalence between the existence of general equilibrium and the existence of common weak no-arbitrage price under C and M

Let $\mathcal{E} = \{(U^i, e^i, X^i)_i\}$ be the initial economy, with $X^i = \mathbb{R}^S$ for any i.

Lemma 7 *Suppose that $\bigcap_i P^i \neq \emptyset$. Then $\mathcal{U}$ is bounded.*

Proof: Take any $\overline{p} \in \bigcap_i P^i$. For each i, there exists $\lambda_i > 0$, $\overline{x}^i \in \mathbb{R}^S$ such that $\overline{p} = \lambda_i U^{i\prime}(\overline{x}^i)$. For $(x^1, x^2, \dots, x^m) \in \mathcal{A}$, we have:

$$\begin{aligned}\sum_{i=1}^m \lambda_i [U^i(x^i) - U^i(\overline{x}^i)] &\leq \sum_{i=1}^m \overline{p} \cdot (\overline{x}^i - x^i) \\ &= \overline{p} \cdot (\sum_{i=1}^m \overline{x}^i - \overline{e})\end{aligned}$$

then for all i:

$$\lambda_i U^i(e^i) \leq \lambda_i U^i(x^i) \leq \overline{p} \cdot (\sum_{i=1}^m \overline{x}^i - \overline{e}) - \sum_{j \neq i} \lambda_j U^j(e^j) + \sum_{j \neq i} \lambda_j U^j(\overline{x}^j).$$

■

Denote

$$W = \{(w^1, w^2, \dots, w^m) \in R^1 \times R^2 \times \dots \times R^m \text{ such that } \sum_{i=1}^m w^i = 0\}.$$

The set W is a cone and will be called the cone of mutual opportunities of arbitrage. Fix a vector $w = (w^1, w^2, \dots w^m) \in W$, as in Definition 10, define $V^i(x^i, w^i) = V[U^i](x^i, w^i)$. We define $\mathcal{E}^V$ the economy with m agents where each agent i has a utility function defined by V^i, endowment e^i and consumption set $X^i = \mathbb{R}^S$. The attainable allocation set $\mathcal{A}^V$ is defined as:

$$\mathcal{A}^V = \{(x^1, x^2, \dots, x^m) \in (\mathbb{R}^S)^m \mid \sum_{i=1}^m x^i = \sum_{i=1}^m e^i \text{ and } V^i(x^i, w^i) \geq U^i(e^i)\}.$$

The attainable utility set $\mathcal{U}^V$ is defined as:

$$\mathcal{U}^V = \{(v^1, v^2, \dots, v^m) \in \mathbb{R}^m \mid \exists x \in \mathcal{A}^V : U^i(e^i) \leq v^i \leq V^i(x^i, w^i), \forall i\}.$$

We have the result:

Proposition 8 *Suppose that for all $(x^1, x^2, \dots, x^m) \in \mathcal{A}$, there exists $\max_{\lambda \geq 0} U^i(x^i + \lambda w^i)$, $\forall$ i. Then $\mathcal{U}^V = \mathcal{U}$.*

Proof: Evidently, since $U^i(x^i) \leq V^i(x^i, w^i)$, we have $\mathcal{U} \subset \mathcal{U}^V$.

Take $(v^1, v^2, \dots, v^m) \in \mathcal{U}^V$. There exists $(x^1, x^2, \dots, x^m) \in \mathcal{U}^V$ such that $U^i(e^i) \leq v^i \leq V^i(x^i, w^i)$, $\forall i$. By assumption, there exists $\lambda \geq 0$ big enough such that $V^i(x^i, w^i) = U^i(x^i + \lambda w^i)$. Observe that $(x^1 + \lambda w^1, x^2 + \lambda w^2, \dots, x^m + \lambda w^m) \in \mathcal{A}$. This implies $(v^1, v^2, \dots, v^m) \in \mathcal{U}$. ■

Theorem 4 *Consider economy $\mathcal{E}$. Suppose that for all i, if U^i does not satisfy No Half-line condition then it satisfies* **C** *condition. In this case we have*

$$\bigcap_{i=1}^{m} P^i \neq \emptyset \Leftrightarrow \mathcal{U} \text{ is compact} \Leftrightarrow \text{ there exists an equilibrium.}$$

Proof: Suppose there exists an equilibrium. By Proposition 2, we have $\bigcap_i P^i \neq \emptyset$.

We prove the converse. Suppose that $\bigcap_i P^i \neq \emptyset$. We will prove that $\mathcal{U}$ is compact, and this implies the existence of equilibrium. Take any $\overline{p} \in \bigcap_i P^i$. There exists $\lambda_i > 0$, $\overline{x}_i \in \mathbb{R}^S$ such that $\overline{p} = \lambda_i U^{i\prime}(\overline{x}_i)$, $\forall\ i$.

Suppose that $(w^1, w^2, \ldots, w^m) \in W$. From the Lemma 1, we know that for all i, $\overline{p} \cdot w \geq 0$, so we have:

$$0 = \overline{p} \cdot \sum_{i=1}^{m} w^i \geq 0.$$

This implies $\overline{p} \cdot w^i = 0$ for all i.

Define $L(W) := W \bigcap (-W)$, the biggest sublinear space included in W. If W is a linear subspace, then the *WNMA* condition is satisfied. By Theorem 2, $\mathcal{U}$ is compact and there exists an equilibrium.

Suppose that $W \neq L(W)$, or there exists $w = (w^1, w^2, \ldots, w^m) \in W$ such that $-w \notin W$, i.e. $w \notin L(W)$. Denote $U^i_1(x^i) = V[U^i](x^i, w^i)$ as in Definition 10.

Suppose U^i satisfies No Half-line condition. In this case, $w^i = 0$. Indeed, we have $p \cdot w^i = 0$. This implies $U^{i\prime}(\overline{x}^i) \cdot w^i = 0$, so $U^i(\overline{x}^i + \lambda w^i) = U^i(\overline{x}^i), \forall \lambda \geq 0$. If $w^i \neq 0$, this direction will be a half-line of i which is a contradiction. Hence if U^i has no half-line, $w^i = 0$ and this implies $U^i_1 = U^i$. Evidently, U^i_1 satisfies no half-line property.

Now suppose U^i does not satisfy No Half-line condition. In this case, U^i satisfies **C** condition. By Proposition 5, U^i_1 also satisfies **C** condition.

Now we define $\mathcal{E}_1$ the economy which is characterized by $\{(U^i_1, e^i, X^i)\}$. As above, denote by W_1 the cone of mutual opportunities of arbitrage.

Observe that any useful vector of U^i is also a useful vector of U^i_1. So, $W \subset W_1$. From the very definition of $V[U^i]$, we have that, for all i, w^i is a useless vector of U^i_1. Thus, $(w^1, w^2, \ldots, w^m) \in L(W_1)$. Hence we have $\dim L(W_1) \geq \dim L(W) + 1$.

We will prove that the utility set $\mathcal{U}_1$ equals $\mathcal{U}$. Recall that w^i is a half-line. If U^i satisfies no half-line condition, then $w^i = 0$, and evidently $\max_\lambda U^i(x^i + \lambda w^i)$ exists. If U^i does not satisfy no half-line condition, then U^i satisfies **C** condition. By Proposition 4, $\max_\lambda U^i(x^i + \lambda w^i)$ exists. By applying Proposition 8, we have $\mathcal{U}_1 = \mathcal{U}$.

If we denote by P_1^i, the set of weak no-arbitrage prices of agent i, we have $\overline{p} \in P_1^i$ for all i. Indeed, $\overline{p} \cdot w^i = 0$, this implies $U^i(\overline{x}^i) = V^i(\overline{x}^i, w^i)$. Hence by Lemma 6, $\overline{p} = \lambda_i V_x^{i\prime}(\overline{x}^i, w^i)$.

The economy $\mathcal{E}_1$ satisfies $\bigcap_i P_1^i \neq \emptyset$. Its utility functions satisfy either no half-line condition, or **C** condition. If W_1 is linear, using the *WNMA* condition, we have the compactness of $\mathcal{U}_1$. In the contrary case, take $\tilde{w} = (\tilde{w}^1, \tilde{w}^2, \ldots, \tilde{w}^m) \in W_1$ such that $-\tilde{w} \notin W_1$. We define $U^2(x^i) = V[U_1^i](x^i, \tilde{w}^i)$.

By induction and the same arguments, suppose that we have constructed the economy $\mathcal{E}_t$, with $t \geq 1$. If W_t is not a linear space, we can construct a new economy $\mathcal{E}_{t+1}$, with $\dim L(W_{t+1}) \geq \dim L(W_t) + 1$. Our economies have the same utility sets $\mathcal{U}_t = \mathcal{U}$, $\forall\ t$.

For all t, $W_t \subset \mathbb{R}^{S\times m}$. So, we have to stop at some step T. In economy $\mathcal{E}_T$, W_T is a linear space, and hence satisfies *WNMA* condition. By Theorem 2, we have $\mathcal{U}_T$ is compact. This implies $\mathcal{U}$ is compact and our initial economy $\mathcal{E}$ has an equilibrium. ■

Theorem 5 *Suppose that for any agent i, if U^i does not satisfy No Half-line condition then it satisfies* **M** *condition. In this case we have*

$$\bigcap_{i=1}^{m} P^i \neq \emptyset \Leftrightarrow \mathcal{U} \text{ is compact } \Leftrightarrow \text{ there exists an equilibrium.}$$

Proof: Suppose that there exists equilibrium. From Theorem 2 we have $\bigcap_i P^i \neq \emptyset$.

We now prove the converse. Suppose that there exists a common weak no-arbitrage price or equivalently, $\bigcap_i P^i \neq \emptyset$. Take $\overline{p} \in \bigcap_i P^i$. There exists $\lambda_i > 0$, $\overline{x}^i \in \mathbb{R}^S$ such that $\overline{p} = \lambda_i U^{i\prime}(\overline{x}^i)$. Suppose that $(w^1, w^2, \ldots, w^m) \in W$. Denote $V^i(x^i, w^i) = \sup_\lambda U^i(x^i + \lambda w^i)$. We have for all $\lambda > 0$, $U^i(\overline{x}^i + \lambda w^i) = U^i(\overline{x}^i)$ and hence, $V^i(\overline{x}^i, w^i) < +\infty$. By Lemma 3, we have $V^i(x^i, w^i) < +\infty$ for all $x^i \in \mathbb{R}^S$.

If U^i satisfies No-half-line condition, then evidently $w^i = 0$, and $V^i(x^i, w^i) = U^i(x^i)$.
If U^i does not satisfy No-half-line condition then U^i satisfies **M** condition. By Lemma 6, the function $x^i \to V^i(x^i, w^i) = \max_{\lambda \geq 0} U^i(x^i + \lambda w^i)$ also satisfies this condition. So $V^i(., w^i)$ satisfies either No Half-line condition or **M** condition.

Observe that, by Lemma 6, we have $V^i(\cdot, w^i)$ is differentiable, and if we denote

$$P_V^i = \{p \in \mathbb{R}^S \mid \exists\ \lambda > 0,\ \exists x \in \mathbb{R}^S \text{ such that } p = \lambda V^{i\prime}(x^i, w^i)\}$$

we have $\bigcap_i P_V^i \neq \emptyset$. Indeed, since $\overline{p} \cdot w^i = 0$, we have $U^i(\overline{x}^i) = V(\overline{x}^i, w^i)$. Using Lemma 6, we have $\overline{p} \in V^{i\prime}(\overline{x}^i, w^i)$, for any i.

We have constructed a new economy with m agents where the utility functions $V^i(\cdot, w^i)$ satisfy either No Half-line condition or **M** condition, and $\bigcap_i P^i_V \neq \emptyset$. By using the same arguments as in the proof of Theorem 4, we have $\mathcal{U}$ is compact and hence, there exists an equilibrium for the initial economy $\mathcal{E}$. ■

Remark 3 *The condition $\bigcap_i P^i \neq \emptyset$ is weaker than $\bigcap_i S^i \neq \emptyset$. In [11], we have an example in which $\bigcap_{i=1}^m S^i = \emptyset$, but $\bigcap_i P^i \neq \emptyset$. In this example, an equilibrium exists.*

Corollary 1 *Suppose the utility functions are separable, i.e for any agent i,*

$$U^i(x) = \sum_{s=1}^{S} \pi^i_s u^i(x_s)$$

with $\pi^i_s > 0$ for all i, for all s. We have

$$\bigcap_{i=1}^{m} P^i \neq \emptyset \Leftrightarrow \mathcal{U} \text{ is compact } \Leftrightarrow \text{ an equilibrium exists.}$$

Proof: For each agent i, if $u^{i\prime}(z) < u^{i\prime}(-\infty)$ for any z, or $u^{i\prime}(z) > u^{i\prime}(+\infty)$ for any z, then the function U^i has no half-line. For this result, see [7]. In the other case, U^i satisfies **C** and **M** conditions. The corollary is a direct consequence of theorems 4, 5 and 7.

Another proof of the corollary can be found in [11].

■

7 Appendix

7.1 Proof of Lemma 3

- (a) Consider two couples $(x, w), (x', w')$ with $w \in R, w' \in R$ and $\theta \in [0, 1]$. We have

  $$U\left(\theta x + (1-\theta)x' + \lambda(\theta w + (1-\theta)w')\right) \geq \theta U(x+\lambda w) + (1-\theta)U(x'+\lambda w')$$

  By taking the limits

  $$\begin{aligned} & \lim_{\lambda \to +\infty} U\left(\theta x + (1-\theta)x' + \lambda(\theta w + (1-\theta)w')\right) \\ \geq & \lim_{\lambda \to +\infty} [\theta U(x+\lambda w) + (1-\theta)U(x'+\lambda w')] \\ = & \theta \lim_{\lambda \to +\infty} U(x+\lambda w) + (1-\theta) \lim_{\lambda \to +\infty} U(x'+\lambda w'). \end{aligned}$$

  Equivalently

  $$V(\theta x + (1-\theta)x', \theta w + (1-\theta)w') \geq \theta V(x, w) + (1-\theta)V(x', w').$$

- (b) Let $x \in \mathbb{R}^S$. There exist $\theta \in (0,1)$ and $y \in \mathbb{R}^S$ such that $x = \theta\overline{x} + (1-\theta)y$. We use the concavity of U:

$$\begin{aligned} \text{Let } \lambda \in \mathbb{R}_+.\ U(x+\lambda w) &= U\big(\theta(\overline{x}+\lambda w) + (1-\theta)(y+\lambda w)\big) \\ &\geq \theta U(\overline{x}+\lambda w) + (1-\theta)U(y+\lambda w) \\ &\geq \theta U(\overline{x}+\lambda w) + (1-\theta)U(y). \end{aligned}$$

  Take the limits:

$$\begin{aligned} V(x,w) = \lim_{\lambda\to+\infty} U(x+\lambda w) &\geq \theta \lim_{\lambda\to+\infty} U(\overline{x}+\lambda w) + (1-\theta)U(y) \\ &= \theta V(\overline{x},w) + (1-\theta)U(y) = +\infty. \end{aligned}$$

- (c) The function $V(.,w)$ is concave, real valued over $\mathbb{R}^S$ is therefore continuous everywhere on $\mathbb{R}^S$.

7.2 Proof of Lemma 4

Take $\overline{w} = (1,1,\ldots,)$. Since U is strictly increasing, $\mathbb{R}^S_+ \subset R$. Therefore, $\overline{w} \in \operatorname{int}R$ since $\overline{w} \in \operatorname{int}\mathbb{R}^S_+$. Let x be given and let $\{y^n\}$ be a sequence which satisfies $\lim_n U(y^n) = \sup_{z\in\mathbb{R}^S} U(z) = +\infty$. For any n there exists $\lambda_n > 0$ such that $x_s + \lambda_n\overline{w}_s > y^n_s$ for any $s = 1,\ldots,S$. That implies $U(x+\lambda_n\overline{w}) > U(y^n)$ since U is strictly increasing. Hence $V(x,\overline{w}) = \sup_{\lambda\in\mathbb{R}} U(x+\lambda\overline{w}) = +\infty$.
Now let $w \in \operatorname{int}R$. There exist $\theta \in (0,1)$, $r \in R$ such that $w = \theta\overline{w} + (1-\theta)r$. Let $\{\lambda_n\}_n$ be an increasing sequence of positive numbers which converge to $+\infty$. We have $x + \lambda_n w = \theta(x+\lambda_n\overline{w}) + (1-\theta)(x+\lambda_n r)$ and hence

$$U(x+\lambda_n w) \geq \theta U(x+\lambda\overline{w}) + (1-\theta)U(x+\lambda_n r) \geq \theta U(x+\lambda\overline{w}) + (1-\theta)U(x).$$

Since $\lim_n U(x+\lambda_n\overline{w}) = +\infty$ we have also $\lim_n U(x+\lambda_n w) = +\infty$ and $V(x,w) = +\infty$.

7.3 Proof of Lemma 5

Indeed, take $r \in R$. For all $\lambda, \mu \geq 0$ we have:

$$U(x+\mu r+\lambda w) \geq U(x+\lambda w).$$

This implies $\sup_\lambda U(x+\mu r+\lambda w) \geq \sup_\lambda U(x+\lambda w)$. ■

7.4 Proof of Lemma 6

Take $p \in \partial V(x,w)$. We have $p \cdot w = 0$. Indeed, observe that w is an useless vector of $V(\cdot,w)$, then for all $\lambda \in \mathbb{R}$, by the concavity of V we have:

$$0 = V(x+\lambda w) - V(x,w) \leq \lambda p \cdot w.$$

Hence $p \cdot w = 0$.

By assumption, $U(x+\lambda w) = U(x+\overline{\lambda}w)\ \forall\ \lambda \geq \overline{\lambda}$, so $V(x,w) = U(x+\overline{\lambda}w)$. For all $y \in \mathbb{R}^S$ we have:

$$U(y) - U(x+\overline{\lambda}w) \leq V(y,w) - V(x,w) \leq p \cdot (y-x) = p \cdot (y-(x+\overline{\lambda}w)).$$

Then $p = U'(x+\overline{\lambda}w)$.

Take $p = U'(x+\overline{\lambda}w)$. Observe that $p \cdot w = 0$. We have:

$$\begin{aligned} V(y,w) - V(x,w) &= \lim_{\lambda \to +\infty} [U(y+\lambda w) - U(x+\overline{\lambda}w)] \\ &\leq \lim_{\lambda \to +\infty} p \cdot (y - x + \lambda w - \overline{\lambda}w) \\ &= \lim_{\lambda \to +\infty} p \cdot (y-x) = p \cdot (y-x). \end{aligned}$$

Then $p \in \partial V(x,w)$. ■

7.5 Proof of Proposition 3

(a) Suppose that w is **HL**, by using Lemma 4, we have $w \in \partial R$.

(b) With $\alpha \in \mathbb{R}$, let $\sigma_\alpha = \{x \mid U(x) = \alpha\}$.

We will prove that if $w \in \partial R$, then there exists $\widehat{x} \in \sigma_\alpha$ such that $U(\widehat{x}+\lambda w) = U(\widehat{x}) = \alpha$ for all $\lambda \geq 0$. If $w = 0$, obviously, the claim is true. We assume now that $w \in \partial R \backslash \{0\}$.

Take some $x \in \sigma_\alpha$. If $U(x+\lambda w) = U(x)\ \forall\ \lambda \geq 0$, the claim is true. Assume that there exists $\mu > 0$ such that $U(x+\mu w) > U(x)$. If $\mu > 1$, the concavity of U implies that $U(x+w) > U(x)$. If $\mu \leq 1$, then $U(x+w) \geq U(x+\mu w) > U(x)$. This implies that $U(x+w) > U(x)$. From the continuity of U, there exists $\rho > 0$ such that for all $y \in B(x+w,\rho)$, a ball center $x+w$ and radius ρ, we have $U(y) > U(x)$. Since $w \in \partial R$, we can take a sequence w^n converging to w such that $x + w^n \in B(x+w,\rho)$ and $w^n \notin R$. We have $U(x+w^n) > U(x)$. Therefore, for any n, there exists $\lambda^n > 1$ such that $U(x+\lambda^n w^n) = U(x)$ and $U(x+\lambda w^n) < U(x)$ for all $\lambda > \lambda^n$.

Define $x^n := x + \lambda^n w^n$. We claim that $\lambda^n \to +\infty$. Indeed, in the contrary, $\lambda^n \to \overline{\lambda} \geq 1$. We have $U(x+\overline{\lambda}w) \geq U(x+w) > U(x)$: a contradiction because $U(x+\overline{\lambda}w) = \lim_n U(x^n) = U(x)$.

Since $\lambda^n \to \infty$, we have $\|x^n\| \to \infty$.

By the concavity of U we have:

$$0 = U(x^n) - U(x) \geq U'(x^n) \cdot (x^n - x).$$

This implies:

$$\frac{U'(x^n)}{\|U'(x^n)\|} \cdot \frac{x^n}{\|x^n\|} \leq \frac{U'(x^n)}{\|U'(x^n)\|} \cdot \frac{x}{\|x^n\|}.$$

Observe that when $n \to \infty$, $\|x^n\| \to +\infty$, and $\frac{x^n}{\|x^n\|} \to w$. From the closeness of $\{\frac{U'(x)}{\|U'(x)\|}\}_{x\in\sigma_\alpha}$ we have:

$$\frac{U'(x^n)}{\|U'(x^n)\|} \to \frac{U'(\hat{x})}{\|U'(\hat{x})\|}$$

where $\widehat{x} \in \sigma_\alpha$. We obtain that:

$$\frac{U'(\widehat{x})}{\|U'(\widehat{x})\|} \cdot w \leq 0.$$

So $U'(\widehat{x}) \cdot w = 0$ and that implies $U(\widehat{x} + \lambda w) = U(\widehat{x})\ \forall\ \lambda \geq 0$. ■

7.6 Proof of Proposition 4

From Proposition 3, there exist $\hat{x}$ such that $U(\hat{x} + \lambda w) = U(\hat{x}),\ \forall\ \lambda \geq 0$. Hence $V(\hat{x}, w) < +\infty$. By Lemma 3, we have $V(x, w) < +\infty\ \forall\ x \in \mathbb{R}^S$, or $\lim_{\lambda\to+\infty} U(x + \lambda w) < +\infty$.

Suppose that there exists x such that $U(x + \lambda w)$ has no maximum.

Take any $p \in \partial V(x, w)$. Take any z in interior of R. Let Π denote the plane spanned by (w, z) and $\Pi_x = \Pi + \{x\}$. We will prove that p is orthogonal to Π and then orthogonal to z.

By the way of choosing z and Lemma 3, we have $\sup_\lambda U(x + \lambda z) = +\infty$, so $\sup_{\Pi_x} U(x) = +\infty$. Let $\alpha := \lim_{\lambda\to\infty} U(x+\lambda w) = V(x, w)$. Since $U(x+\lambda w) < \alpha$ and $U(x + \lambda z) \to +\infty$ when $\lambda \to \infty$, there exists $y \in \Pi_x$ which satisfies $U(y) = \alpha$. In other words, $C_\alpha := \Pi_x \bigcap \sigma_\alpha \neq \emptyset$.

By using the same argument as in the proof of Proposition 3 and the closed gradient condition for the affine space Π_x, we can prove that there exists $\hat{x} \in C_\alpha$ such that

$$U(\hat{x}) = U(\hat{x} + \lambda w) = \alpha, \forall\ \lambda \geq 0.$$

We have $V(x, w) = V(\hat{x}, w) = \alpha$, so

$$0 = V(x, w) - V(\hat{x}, w) \geq p \cdot (x - \hat{x}).$$

We obtained $p \cdot (x - \hat{x}) \leq 0$.

By Lemma 6, we have $V'_x(\hat{x}, w) = U'(\hat{x})$, and then $V'_x(\hat{x}, w) \cdot w = 0$.

Denote $q = V'(\hat{x}, w)$. Let p_1, q_1 respectively the orthogonal projections of p and q on Π. Since $p \cdot w = q \cdot w = 0$, then p_1 and q_1 are orthogonal to w, and so $p_1 = \mu q_1$ with $\mu \in \mathbb{R}$. Since $z \in \text{int}(R)$, z is also a useful vector of $V(\cdot, w)$, and $V'(x, w) \cdot z \geq 0$ which implies $p_1 \cdot z \geq 0$. We have also $U'(\hat{x}) \cdot z > 0 \Rightarrow q_1 \cdot z > 0$. These inequalities imply $\mu \geq 0$.

Observe that:

$$U(\hat{x}) - U(x+\lambda w) \geq U'(\hat{x})\cdot(\hat{x}-x) - \lambda U'(\hat{x})\cdot w = U'(\hat{x})\cdot(\hat{x}-x) = q_1\cdot(\hat{x}-x).$$

Let $\lambda \to +\infty$. We obtain $q_1\cdot(\hat{x}-x)\leq 0$. And then $p_1\cdot(\hat{x}-x)\leq 0$. Remember that $p\cdot(x-\hat{x})\leq 0$ or $p_1\cdot(x-\hat{x})\leq 0$. Then, we have $p_1\cdot(x-\hat{x})=0$.
By the assumption of the non existence of a maximum of $U(x+\lambda w)$, $\hat{x}$ does not belong to the line $\{x+\lambda w\}$, so w and $x-\hat{x}$ are linearly independent and orthogonal to p_1, so p is orthogonal to Π, or $p\cdot z = 0$.

Since z is arbitrarily chosen, p is orthogonal to all vectors who belongs to $\text{int}R$, and hence to all vectors in $\mathbb{R}^S_+$. This implies $p=0$. Then $V(x,w)\geq V(y,w)\geq U(y)$ for all y, or $\sup_y U(y) < +\infty$: a contradiction. ■

7.7 Proof of Proposition 5

By Proposition 4, for all $x\in\mathbb{R}^S$, there exists $\lambda\geq 0$ big enough such that $V'(x,w) = U'(x+\lambda w)$.

Suppose that H is an affine space of $\mathbb{R}^S$, and $\sup_{x\in H} V(x,w) = +\infty$. We consider two cases:

1. H is parallel to w: $\forall\, x\in H$, $x+w\in H$.
2. H is not parallel to w.

Denote $\sigma_\alpha^{V,H} := \{x\in H \mid V(x,w)=\alpha\}$ and consider the sequence $\{x_n\}_n \subset \sigma_\alpha^{w,H}$ such that $\lim_n \frac{V'_H(x_n,w)}{\|V'_H(x_n,w)\|} = p$. For each n, there exists $\lambda_n\geq 0$ such that $V(x_n,w) = U(x_n+\lambda_n w)$, or $x_n+\lambda_n w\in\sigma_\alpha$, and using Lemma 6, $\lim_n \frac{U'_H(x_n+\lambda_n w)}{\|U'_H(x_n+\lambda_n w)\|} = p$.
For any affine subspace F, π_F denotes the projection on F.

Consider the first case. In this case, $x_n+\lambda_n w\in H$ for all n, and $\sup_{x\in H} U(x) = \sup_{x\in H} V(x,w) = +\infty$. By applying the closed gradients condition for the affine space H, there exists $\hat{x}\in\sigma_\alpha^{V,H}$ such that $\frac{U'_H(\hat{x})}{\|U'_H(\hat{x})\|} = p$. Observe that $U'(x_n+\lambda_n w)\cdot w = 0 \Rightarrow \pi_H[U'(x_n+\lambda_n w)]\cdot w = 0 \Rightarrow \pi_H[U'(\hat{x})]\cdot w = 0 \Rightarrow U'(\hat{x})\cdot w = 0$ and so $V(\widehat{x},w) = U(\widehat{x}) = \alpha$, or $U(\widehat{x}+\lambda w) = U(\widehat{x}) = \alpha$ for all $\lambda\geq 0$, and $V'(\widehat{x},w) = U'(\widehat{x})$. So we have

$$\frac{V'_H(\hat{x},w)}{\|V'_H(\widehat{x},w)\|} = \frac{\pi_H(V'(\widehat{x},w))}{\|\pi_H(V'(\widehat{x},w))\|} = \frac{\pi_H(U'(\widehat{x}))}{\|\pi_H(U'(\widehat{x}))\|} = \frac{U'_H(\widehat{x})}{\|U'_H(\widehat{x})\|} = p.$$

Consider the second case. Let $G := H + \{\lambda w\}_{\lambda\in\mathbb{R}}$. Then G is parallel to w. Observe that $\sup_{x\in G} U(x) = \sup_{x\in H} V(x,w) = +\infty$.

Suppose that $\frac{V'_H(x_n,w)}{\|V'_H(x_n,w)\|} = p$ with $x_n\in H$ and $V(x_n,w)=\alpha$ for all n. Without loss of generality, we can suppose that $\frac{U'_G(x_n+\lambda_n w)}{\|U'_G(x_n+\lambda n w)\|}\to\overline{p}$. Observe that $H\subset G$, so $\pi_H(\pi_G(q)) = \pi_H(q)$ for all vector $q\in\mathbb{R}^S$.

Firstly, we prove that $\overline{p}$ is not orthogonal to H. Using the same argument as the first case, applying closed gradient condition for the affine space G, we know that there exists $\hat{x} \in G$ such that

$$\frac{U'_G(\hat{x})}{\|U'_G(\hat{x})\|} = \overline{p} \text{ and } U'(\hat{x}) \cdot w = 0$$

or we can write

$$\frac{V'_G(\hat{x}, w)}{\|V'_G(\hat{x}, w)\|} = \overline{p}.$$

H is not parallel to w, so there exists $\lambda \in \mathbb{R}$ satisfies $\hat{x} + \lambda w \in H$. Denote $\overline{x} = \hat{x} + \lambda w$. Observe that $V(\overline{x}, w) = V(\hat{x}, w)$ and $V'(\overline{x}, w) = V'(\widehat{x}, w)$.

If $\overline{p}$ is orthogonal to H then $V'(\overline{x}, w)$ is orthogonal to H, then for all $y \in H$ we have:

$$V(y, w) - V(\overline{x}, w) \leq V'(\overline{x}, w) \cdot (y - x) = 0.$$

Thus $V(\overline{x}, w) = \sup_{y \in H} V(y, w) = +\infty$: a contradiction.

Since $\overline{p}$ is not orthogonal to H, we have $\pi_H(\overline{p}) \neq 0$.

Secondly, we prove that $\frac{\pi_H(\overline{p})}{\|\pi_H(\overline{p})\|} = p$. Indeed, for all n we have

$$\frac{\pi_H(\pi_G(U'(x_n + \lambda_n w))}{\|\pi_H(\pi_G(U'(x_n + \lambda_n w))\|} = \frac{\pi_H(\frac{\pi_G(U'(x_n + \lambda_n))}{\|\pi_G(U'(x_n + \lambda_n))\|})}{\|\pi_H(\frac{\pi_G(U'(x_n + \lambda_n))}{\|\pi_G(U'(x_n + \lambda_n))\|})\|}.$$

And we let $n \to \infty$, the left hand side tends to p, and the right hand side tends to $\frac{\pi_H(\overline{p})}{\|\pi_H(\overline{p})\|}$. Hence, $\frac{\pi_H(\overline{p})}{\|\pi_H(\overline{p})\|} = p$.

Finally we have

$$\frac{V'_H(\overline{x}, w)}{\|V'_H(\overline{x}, w)\|} = \frac{\pi_H(V'(\widehat{x}, w))}{\|\pi_H(V'(\hat{x}, w))\|} = \frac{\pi_H(\pi_G(V'(\hat{x}, w)))}{\|\pi_H(\pi_G(V'(\widehat{x}, w)))\|})) = \frac{\pi_H(\frac{\pi_G(V'(\hat{x}, w))}{\|\pi_G(V'(\hat{x}, w))\|})}{\|\pi_H(\frac{\pi_G(V'(\widehat{x}, w))}{\|\pi_G(V'(\widehat{x}, w))\|})\|}$$

$\Rightarrow$

$$\frac{V'_H(\overline{x}, w)}{\|V'_H(\overline{x}, w)\|} = \frac{\pi_H(\overline{p})}{\|\pi_H(\overline{p})\|} = p.$$

$V(\cdot, w)$ satisfies **C** condition. ■

7.8 Proof of Proposition 6

Firstly, observe that from the property of condition **M**, if $\sup_\lambda U(x+\lambda w) < +\infty$ for some x, then there exists $\overline{\lambda}$ such that $U(x+\overline{\lambda}w) = \sup_\lambda U(x+\lambda w) = V(x, w)$.

Suppose that $\sup_H V(x, w) < \infty$. Let $G = H + \{\lambda w\}_{\lambda \in \mathbb{R}}$. It is an affine subspace. We can verify easily that $\sup_G U(x) = \sup_H V(x, w) < +\infty$, so there exists $\overline{x} \in G$ such that $U(\overline{x}) = \sup_G U(x)$. Observe that $V(\overline{x}, w) = \sup_\lambda U(\overline{x} +$

$\lambda w) \geq U(\overline{x}) = \sup_G U(x) = \sup_H V(x,w)$, so $V(\overline{x}, w) = \sup_H V(x,w)$. Since $\overline{x} \in H + \{\lambda w\}_\lambda$, there exists $\hat{x} \in H$, $\lambda \in \mathbb{R}$ such that $\overline{x} = \hat{x} + \lambda w$. We have $V(\hat{x}, w) = \sup_H V(x,w)$. ■

7.9 Proof of Proposition 7

Before beginning the proof, for each vector $w \in \mathbb{R}$, define $S_+(w) = \{s \text{ such that } w_s > 0\}$, $S_-(w) = \{s \text{ such that } w_s < 0\}$.

(a) Fix an affine subspace H. Denote by σ_α^H the set $\{x \in H \text{ such that } U(x) = \alpha\}$. Suppose that p is a limit point of $\{\frac{U'(x)}{\|U'(x)\|}\}_{\sigma_\alpha^H}$. Since $U'(x) \cdot w \geq 0$ we have $p \cdot w \geq 0$. For each n, define

$$A_n = \{x \in \sigma_\alpha^H \text{ such that } \|p - \frac{U'(x)}{\|U'(x)\|}\| \leq \frac{1}{n}\}.$$

Denote by x_n the element of A_n which has the smallest norm: $\|x_n\| \leq \|x\|$ for all $x \in A_n$. We prove that the set $\{x_n\}$ is bounded.

Suppose the contrary, $\lim_n \|x_n\| = \infty$. Without loss of generality, we can suppose that $\lim_n \frac{x_n}{\|x_n\|} = w$. Observe that w is parallel to H. Since $U(x_n) = \alpha$ for all n, the vector w is a useful vector. We have also $p \cdot w = 0$. Indeed, denote by $\mathbf{0}$ the vector $(0, 0, \ldots, 0)$. We have $U(x_n) - U(\mathbf{0}) \geq U'(x_n) \cdot x_n$. Hence

$$\frac{\alpha - U(\mathbf{0})}{\|x_n\|} \geq U'(x_n) \cdot \frac{x_n}{\|x_n\|}.$$

Let n converges to infinity, the LHS converges to 0. Observe that since $0 < a \leq b < +\infty$, we have $\{U'(x)\}$ is bounded. Hence without loss of generality, assume that $U'(x_n)$ converges to a vector q, with $\frac{q}{\|q\|} = p$. The inequality above implies that $p \cdot w \leq 0$, that implies $p \cdot w = q \cdot w = 0$.

Dana and Le Van in [7] prove that for all x we have

$$U'(x) \cdot w \geq a \sum_{s \in S_+(w)} \pi_s w_s + b \sum_{s \in S_-(w)} \pi_s w_s \geq 0.$$

Since $\lim_n U'(x_n) \cdot w = 0$, we have $a \sum_{s \in S_+(w)} \pi_s w_s + b \sum_{s \in S_-(w)} \pi_s w_s = 0$.

Fix $0 < \epsilon < 1$. Take N big enough such that for all $n \geq N$ we have $x_{n,s} - \epsilon w_s > z$ for $s \in S_+(w)$ and $x_{n,s} - \epsilon w_s < -z$ for $s \in S_-(w)$. This implies

$\|x_n - \epsilon w\| < \|x_n\|$. We will prove that $U(x_n - \epsilon w) = \alpha$, $\forall\ n \geq N$. Indeed,

$$\begin{aligned} U(x_n - \epsilon w) - U(x_n) &\geq -\epsilon U'(x_n - \epsilon w) \cdot w \\ &= -\epsilon \sum_{s=1}^{S} \pi_s u'(x_{n,s} - \epsilon w_s) w_s \\ &= -\epsilon \sum_{s \in S_+(w)} \pi_s u'(x_{n,s} - \epsilon w_s) w_s - \epsilon \sum_{s \in S_-(w)} \pi_s u'(x_{n,s} - \epsilon w_s) w_s \\ &= -\epsilon a \sum_{s \in S_+(w)} \pi_s w_s - \epsilon b \sum_{s \in S_-(w)} \pi_s w_s \\ &= -\epsilon [a \sum_{s \in S_+(w)} \pi_s w_s + b \sum_{s \in S_-(w)} \pi_s w_s] \\ &= 0. \end{aligned}$$

Hence $U(x_n - \epsilon w) \geq U(x_n) \geq U(x_n - \epsilon w)$. This implies $U(x_n - \epsilon w) = \alpha$.

Observe that for $s \in S_+ \cup S_-$, we have $u'(x_{n,s} - \epsilon w_s) = u'(x_{n,s})$, which is equal to a if $s \in S_+(w)$, and to b if $s \in S_-$, so $U'(x_n - \epsilon w) = U'(x_n)$. Since w is parallel to H, $x_n - \epsilon w \in H$. We have proved above $\|x_n - \epsilon w\| < \|x_n\|$. This is a contradiction to the definition of x_n.

So, the set $\{x_n\}$ is bounded. Take a limit point x of this set, we have $x \in H$, $U(x) = \alpha$, and $\frac{U'(x)}{\|U'(x)\|} = p$. We have proved that **C** condition holds.

(b) Fix an affine space H such that $\sup_H U(x) = a < +\infty$. For each n, define

$$A_n = \{x \in H \text{ such that } U(x) \geq a - \frac{1}{n}\}.$$

Take $x_n \in A_n$ such that $\|x_n\| \leq \|x\|$ for all $x \in A_n$. We prove that the set $\{x_n\}$ is bounded. Suppose the contrary. We can suppose that $\lim_n \frac{x_n}{\|x_n\|} = w$. Since $\{U(x_n)\}$ is bounded below, w is a useful vector. We will prove that $a \sum_{s \in S_+(w)} \pi_s w_s + b \sum_{s \in S_-(w)} \pi_s w_s = 0$. Indeed, suppose that $a \sum_{s \in S_+(w)} \pi_s w_s + b \sum_{s \in S_-(w)} \pi_s w_s > 0$. Fix $x \in H$. For $\lambda > 0$ big enough we have $x_s + \lambda w_s > z$, $\forall s \in S_+(w)$, and $x_s + \lambda w_s < -z$, $\forall s \in S_-$. Hence

$$\begin{aligned} U(x + \lambda w) - U(x) &\geq \lambda U'(x + \lambda w) \cdot w \\ &= \lambda \left[a \sum_{s \in S_+(w)} \pi_s w_s + b \sum_{s \in S_-(w)} \pi_s w_s \right]. \end{aligned}$$

Let λ converges to infinity, the RHS tends to $+\infty$. Observe that since w is parallel to H, $x + \lambda w \in H$ for all λ. This implies $\sup_H U(x) = +\infty$, a contradiction.

We have $a \sum_{s \in S_+(w)} \pi_s w_s + b \sum_{s \in S_-(w)} \pi_s w_s = 0$. By using the same arguments in the part (a), we have $U(x_n - \epsilon w) \geq U(x_n) \geq a - \frac{1}{n}$, and $\|x_n - \epsilon w\| <$

$\|x_n\|$, a contradiction. Hence the set $\{x_n\}$ is bounded. Since $\lim_n U(x_n) = a$, we have that $\max_H U(x)$ exists. We have proved that **M** condition holds. ■

References

[1] Allouch, N. (2002): An equilibrium result with short selling. *Journal of Mathematical Economics* **37**, 81-94 .

[2] Allouch, N., C. Le Van and F.H. Page (2002): The geometry of arbitrage and the existence of competitive equilibrium. *Journal of Mathematical Economics* **38**, 373-391.

[3] Arrow, K. J., L. Hurwicz, and H. Uzawa (1958): Programming in Linear spaces. *Stanford University Press, Stanford, California.*

[4] Brown, D. and J. Werner (1995): Existence theorems in the classical asset pricing model. *Econometrica* **59**, 1169-1174.

[5] Brown, D. and J. Werner (1995): Arbitrage and existence of equilibrium in infinite asset markets. *Review of Economic Studies*, **62**, 101-114.

[6] Chichilnisky, G. (1997): Topological invariant for competitive markets, *Journal of Mathematical Economics*, **28**, 445-469.

[7] Dana, R. A and C. Le Van (2010): Overlapping risk adjusted sets of priors and the existence of efficient allocations and equilibria with short-selling. *Journal of Economic Theory*, **145**, 2186-2202.

[8] Dana, R.A, C. Le Van and F. Magnien (1997): General Equilibirum in asset markets with or without short-selling. *Journal of Mathematical Analysis and Applications* **206**, 567-588.

[9] Dana, R.A, C.Le Van and F.Magnien (1999): On the different notions of arbitrage and existence of equilibrium. *Journal of Economic Theory* **87**, 169-193.

[10] Geistdorfer-Florenzano, M. (1982): The Gale-Nikaido-Debreu Lemma and the Existence of Transitive Equilibrium With or Without the Free Disposal Assumption. *Journal of Mathematical Economics* **9**, 113-134.

[11] Ha-Huy, T., C. Le Van (2017): Arbitrage and equilibrium in economies with short-selling and ambiguity. *Working paper.*

[12] P. Hammond (1983): Overlapping expectations and Hart's conditions for equilibrium in a securities models. *Journal of Economic Theory* **31**, 170-175.

[13] Hart, O. (1974): On the Existence of an Equilibrium in a Securities Model. *Journal of Economic Theory* **9**, 293-311.

[14] J.M. Grandmont (1972): Continuity properties of a von Neumann-Morgenstern utility. *Journal of Economic Theory*, **5**, 45-57.

[15] J.M. Grandmont (1977): Temporary general equilibrium theory. *Econometrica* **45**, 535-572.

[16] J.R. Green (1973): Temporary general equilibrium in a sequential trading model with spot and futures transactions. *Econometrica* **41**, 1103-1124.

[17] Nielsen, L. T (1989): Asset market equilibrium with short-selling. *Review of Economic Studies*, **41**, 392-469.

[18] Page, F.H Jr (1987): On equilibrium in Hart's securities exchange model. *Journal of Economic Theory* **41**, 392-404.

[19] Page, F.H. Jr, and M.H. Wooders, (1996): A necessary and sufficient condition for compactness of individually rational and feasible outcomes and existence of an equilibrium. *Economics Letters* **52**, 153-162.

[20] Page, F.H. Jr, M.H. Wooders and P.K. Monteiro (2000): Inconsequential arbitrage. *Journal of Mathematical Economics* **34**, 439-469.

[21] Werner, J. (1987): Arbitrage and the Existence of Competitive Equilibrium. *Econometrica* **55**, 1403-1418.