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# AGGREGATION AND STABILITY IN NETWORKS

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Abstract. We show that a concept of aggregation can hold for games played on networks. We first provide a condition on a group of players in a network, called a module, which ensures that the group can behave like a single player. Furthermore, we show that a partition of players of a game into modules gives rise to an aggregate game, whose Nash equilibria, together with the Nash equilibria of the games played at the module level, correspond to Nash equilibria of the game. Then, we show that fitting aggregate games into each other in an appropriate way provides a hierarchical decomposition of the game, which can inform a recursive computation of Nash equilibria. Finally, we provide an application to the model of public goods in networks to illustrate the usefulness of our results.

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## 1. INTRODUCTION

The economic of networks, which focuses on modeling and understanding varied economic interactions, has recently become one of the most active and dynamic fields in economics, with the potential for important and lasting policy implications (see Goyal (2007) and Jackson (2008)). Nevertheless, it is notable that most economic interactions take place in large networks, whose sheer size and complex structures make economic analysis quite a challenging task. Throughout history, various concepts have been developed to reduce the inherent complexity found in large economic systems, thereby rendering them more amenable to economic analysis. A prominent example is aggregation, which aims to devise representative concepts that can be analyzed in a more tractable manner. For instance, a key question, which appeared in the seminal contributions of von Neumann and Morgenstern (1944), Chapter IX, Gorman (1953, 1961), and Shapley (1964, 1967), is: when does a group of individuals behave as if it was a single individual?

In this paper we investigate whether a similar concept of aggregation could hold for games played on networks, a subject of ongoing research (see Ballester, Calvó-Armengol, and Zenou (2006), for criminal activity, Bramoullé and Kranton (2007) for public good provision, and Bramoullé, Kranton, and D'Amours (2014) for various economic interactions). A key ingredient of our analysis is a group of players, called a module, such that players in the group have exactly the same neighbors outside the group. Obviously, both single players and the entire set of players are always modules, called trivial modules, which may well be the only modules for some networks.[1] Connected components are also always modules, and, in fact, a module can be thought of as a generalization of a connected component.

We first show that a partition of players of a network game into modules gives rise to a possibly smaller aggregate game, obtained by coalescing each module into a representative player. More specifically, since players in a module are indistinguishable by players outside the module in terms of their network position from outside players, outside players can then substitute them for a representative player with suitably defined payoffs. This aggregation procedure allows us to gain significant insights, since we establish that a Nash equilibrium of the aggregate game, together with Nash equilibria of the games played at

[1]The notion of a modular set has been rediscovered several times in many fields and appeared under various names, including committee in Shapley (1967).

modules level, corresponds to a Nash equilibrium of the (full) game. More specifically, the aggregate game determines the participation rates of the various modules, whose actions are the Nash equilibria of the games played at the module level. Thereby, we establish a systematic relationship between each player's network position in each module, and his Nash equilibrium actions in the full game

Furthermore, we show that a game on a network can be decomposed into a unique hierarchy of aggregate games. Key to this are the modules that overlap with no other modules, called strong modules, which, when ordered by inclusion, define a unique tree, called the modular decomposition tree, whose root is the set of players and whose leaves are the single players.[2] By fitting aggregate games into each other along the nodes of the modular decomposition tree we obtain a hierarchical decomposition of the game, which may be useful for the analysis of strategic interactions. First, it determines the nature of interactions between the strong modules, ranging from strategic complements to strategic substitutes. Second, it can be used to carry out a recursive computation of Nash equilibria, which could be of great algorithmic interest.

In the final part of the paper, we provide an application of our results to the model of public goods in networks, introduced in Bramoullé and Kranton (2007). The key question addressed in Bramoullé and Kranton (2007) is how the network architecture of spillovers influences public goods provision, in the absence of coordination or government provision. Our aggregation approach complements the analysis of Bramoullé and Kranton (2007), as it determines a necessary condition on the modular decomposition of the network in order to have a Nash equilibrium with strictly positive contributions by all players,[3] which despite the attractive normative feature of sharing the burden of public goods among all players, is not always guaranteed to exist. The necessary condition, which also becomes sufficient for a special class of networks, illustrates the role played by the intermediate structures of the network architecture in determining public goods provision.

The paper is organized as follows. In Section 2, we present the basic model of network games. In Section 3, we introduce the concept of modular aggregation. In Section 4, we show that a game on a network can be decomposed into a unique hierarchy of aggregate games. In Section 5, we provide an application of our results to the model of public goods in networks. Section 6 concludes the paper.

[2]The concept of modular decomposition of networks was introduced in Gallai (1967).
[3]That is, with no free-riders.

## 2. The model

We consider a strategic form game $\mathbf{\Gamma}(\mathbf{g}, \delta)$ with $N = \{1, \dots, n\}$ players embedded on an undirected and unweighted network $\mathbf{g}$ of interactions and $\delta \in [0, 1]$ measures how much player $i$'s action is affected by his neighbors actions. Each player $i$ chooses an action $x_i \in \mathbb{R}_+$. Given a subset of players $I$ and a profile of actions $\mathbf{x} = (x_1, \dots, x_n)$, let $\mathbf{x}_I = (x_i)_{i \in I}$ denote the actions of the players in $I$ and $x_I = \sum_{i \in I} x_i$ denote their sum. As usual, let $\mathbf{x}_{-i} = \mathbf{x}_{N \setminus \{i\}}$ denote the actions of all other players than $i$. The payoffs of player $i$ for the profile of action $\mathbf{x} = (x_1, \dots, x_n)$ are

$$U_i(\mathbf{x}) = U_i(x_i, \mathbf{x}_{-i}).$$

Player $i$ seeks to maximize his payoffs and has a best-reply function

$$x_i = f_i(\mathbf{x}_{-i}) \stackrel{\text{def}}{=} \max\{1 - \delta \; x_{\mathcal{N}_i(\mathbf{g})}, 0\},$$

where $\mathcal{N}_i(\mathbf{g})$ denote $i$'s neighbors in $\mathbf{g}$ and 1 is the action player $i$ chooses in isolation.

As shown in Bramoullé, Kranton, and D'Amours (2014), this type of game, $\mathbf{\Gamma}(\mathbf{g}, \delta)$, can be used to represent various types of economic interactions including, the model of public goods in networks, introduced in Bramoullé and Kranton (2007), and the model of negative externalities with linear-quadratic payoffs, introduced in Ballester, Calvó-Armengol, and Zenou (2006).

At a Nash equilibrium $\mathbf{x}^* = (x_1^*, \dots, x_n^*)$ of the game $\mathbf{\Gamma}(\mathbf{g}, \delta)$, each player's action is a best-reply to his neighbors' actions, that is, $x_i^* = f_i(\mathbf{x}_{-i})$ for each player $i \in N$. The existence of a Nash equilibrium of $\mathbf{\Gamma}(\mathbf{g}, \delta)$ is guaranteed by Brouwer's fixed point theorem by restricting strategies of players to $[0, 1]^n$.

## 3. Modular aggregation

We now introduce a network structural similarity of a group of players, which ensures that it can behave like a single player. A group of players $M$ is called a module if they have exactly the same neighbors outside the module, that is, for any player $i \in N \setminus M$, either $i$ is adjacent to every member of $M$, or $i$ is adjacent to no member of $M$. It is easy to notice that each single player $\{1\}, \dots, \{n\}$, as well as, the entire set of players $N = \{1, \dots, n\}$ are always modules, called trivial modules. Connected components are also always modules.

A partition $\mathbf{p} = \{M_1, \ldots, M_K\}$ of the set of players $N$ is called a modular partition if $M_k$ is a module of $\mathbf{g}$, for each $k = 1, \ldots, K$.[4] Given two disjoint modules $M_k$, $M_h$ of $\mathbf{p}$ then either every player in $M_k$ is a neighbor of every player in $M_h$, or no player in $M_k$ is adjacent to a player in $M_h$. Thus, the relationship between two disjoint modules is either adjacent or nonadjacent. Hence the modular partition $\mathbf{p}$ gives rise to a new network, $\mathbf{g}/\mathbf{p}$, called the quotient network, whose vertices are the modules of the partition $\mathbf{p}$ and links are the adjacencies of these modules.

Now we define an aggregate game played on the quotient network $\mathbf{g}/\mathbf{p}$, denoted by $\Gamma(\mathbf{g}/\mathbf{p}, \delta; \mathbf{z})$, where $\mathbf{z} = (z_1, \ldots, z_K) \in \mathbb{R}^K_+$ is a vector of weights determined exogenously. This set-up means that in the quotient network, player-positions are filled by representative players of the modules. For each module $M_k$, there is a representative player $k$, who chooses an action $r_k \in [0, 1]$. Representative player $k$'s payoffs depend on his own action $r_k$ and the actions of the other representative players $\mathbf{r}_{-k}$. We denote the payoffs of the representative player $k$ by $V_k$, which are assumed to yield the best-reply function:

$$r_k = F_k(\mathbf{r}_{-k}) \stackrel{\text{def}}{=} \max\{1 - \delta \sum_{h \in \mathcal{N}_h(\mathbf{g}/\mathbf{p})} z_h r_h, 0\}.$$

At a Nash equilibrium $\mathbf{r}^* = (r_1^*, \ldots, x_K^*)$, each representative player's action is a best-reply to the actions of his neighbors, that is, $r_k^* = F_k(\mathbf{r}_{-k})$, for each player $k \in K$.

The following theorem shows that a Nash equilibrium of the aggregate game, whose vector of weights are the aggregate actions of Nash equilibria of the games played within the modules, corresponds to a Nash equilibrium of the (full) game.

**Theorem 1.** *Given a modular partition* $\mathbf{p} = \{M_1, \ldots, M_K\}$ *of the set of players* $N$, *the following are equivalent:*

*(1)* $\mathbf{x}^*$ *is a Nash equilibrium of* $\mathbf{\Gamma}(\mathbf{g}, \delta)$

*(2)* $\mathbf{x}^* = (r_1^* \ \mathbf{y}^*_{M_1}, \ldots, r_K^* \ \mathbf{y}^*_{M_K})$ *such that*

*(a)* $\mathbf{y}^*_{M_k}$ *is a Nash equilibrium of* $\mathbf{\Gamma}(\mathbf{g}_{M_k}, \delta)$, *for each* $k \in K$, *and*

*(b)* $\mathbf{r}^*$ *is a Nash equilibrium of* $\mathbf{\Gamma}(\mathbf{g}/\mathbf{p}, \delta; y^*_{M_1}, \ldots, y^*_{M_K})$.

In interpretation, the aggregate game determines the rates of participation of the modules in a Nash equilibrium of the full game. Note also that in view of the equivalence in Theorem 1 a Nash equilibrium of the full game is always proportional to a Nash

[4]Note that this partition may not be unique.

equilibrium of each module. Hence, finding the Nash equilibria of the modules provides significant insights into the Nash equilibria of the full game. Understandably, the coarser is the modular partition, the larger are the modules, the more significant are the insights.[5]

## 4. Hierarchical decomposition

Now, we show that a game on a network can be decomposed into a unique hierarchy of aggregate games. A module $M$ is called a strong module, if for any module $M' \neq M$ it holds that either $M' \cap M = \emptyset$ or one module is included into the other. We say that a strong module $M$ is a descendant of another strong module $M'$ if $M \subset M'$ and there is no other strong module $M^*$ such that $M \subset M^* \subset M$.

The descendant relation yields a tree on strong modules, called the modular decomposition tree of the network, where the set of players $\{1, \ldots, N\}$ is the root, the single players $\{1\}, \ldots \{N\}$ are the leaves, and any other strong module is an internal node. The nodes of the modular decomposition tree are labeled in three ways: parallel when the descendants are all non-neighbors of each other, series when the descendants are all neighbors of each other, and prime otherwise. The modular decomposition tree is unique and constitutes an exact alternative representation of the network whenever the structure of each prime module is depicted. The following theorem relates the Nash equilibria of a strong module to the Nash equilibria of its direct descendants.

**Theorem 2.** *Given a strong module $M$ with direct descendants partition $\mathbf{p}_M = (D_1, \ldots, D_T)$. Then the following are equivalent:*

*(1)* $\mathbf{x}^*$ *is a Nash equilibrium of* $\mathbf{\Gamma}(\mathbf{g}_M, \delta)$

*(2)* $\mathbf{x}^* = (r_1^* \, \mathbf{y}_{D_1}^*, \ldots, r_T^* \, \mathbf{y}_{D_T}^*)$ *such that*

- *(i)* $\mathbf{y}_t^*$ *is a Nash equilibrium of* $\mathbf{\Gamma}(\mathbf{g}_{D_t}, \delta)$*, for each* $t = 1, \ldots, T$*, and*
- *(ii) If* $M$ *is parallel, then for each* $t \in T$*, it holds that* $r_t^* = 1$.

[5]In particular, the coarsest modular partition consisting of just one module $N$ corresponds to the full game.

*(iii) If $M$ is series, then for almost every $\delta$,*[6] *either for each $t \in A$ it holds that $y^*_{D_t} > \frac{1}{\delta}$ (or, for each $t \in A = T$ it holds that $y^*_{D_t} < \frac{1}{\delta}$), and*

$$r^*_t = \frac{\frac{1}{1-\delta y^*_{D_t}}}{1 + \sum_{s \in A} \frac{\delta y^*_{D_s}}{1-\delta y^*_{D_s}}}.$$

*(iv) If $M$ is prime, then $\mathbf{r}^*_M$ is a Nash equilibrium of $\Gamma(\mathbf{g}/\mathbf{p}, \delta; y^*_{D_1}, \ldots, y^*_{D_T})$.*

Theorem 2 shows the relationship between the Nash equilibria of a module and the Nash equilibria of its descendants. In interpretation, the descendants of parallel modules can be thought of as strategic complements, series modules can be thought of as strategic substitutes, and prime modules can be thought of as a combination of strategic substitutes and complements.

Given that a game on a network can be decomposed, along the nodes of the modular decomposition tree, into a unique hierarchy of aggregate games, Theorem 2 can be used to carry out a recursive computation of Nash equilibria of $\mathbf{\Gamma}(\mathbf{g}, \delta)$. In particular, for the special class of networks known as cographs, which consist of networks with only parallel and series modules in their modular decomposition tree, Theorem 2 provides an algorithm to find all Nash equilibria of $\mathbf{\Gamma}(\mathbf{g}, \delta)$, for almost every $\delta$.

[6]We say that a property holds for almost every $\delta$ if it holds for every $\delta$ except may be finite number of values.

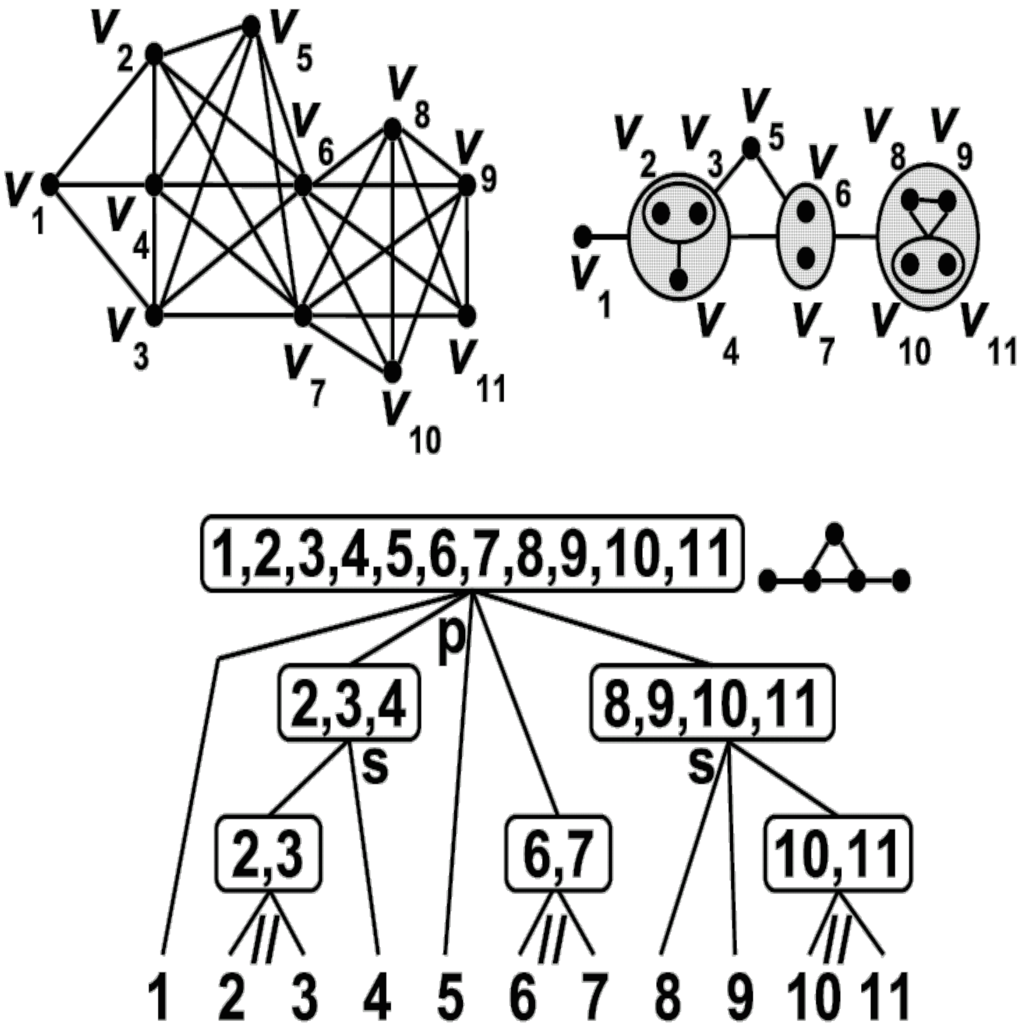


Figure 1: Modular decomposition of a network.

## 5. An application: public goods in networks

In this section, we provide an application of our results to the model of public goods in networks, introduced in Bramoullé and Kranton (2007), which can be investigated as a $\mathbf{\Gamma}(\mathbf{g}, 1)$ game.

Recall that for a profile of contributions to be a Nash equilibrium, it has to be the case that every player contributes nothing to the public good if the sum of his neighbor's contributions exceeds 1 or contributes exactly the difference between 1 and the sum of his neighbor's contributions. Therefore, at a Nash equilibrium we may distinguish three types of players: free-riders, who contribute nothing, experts, who make full contributions, and the others. Bramoullé and Kranton (2007) insightfully show that specialized equilibria, that is, equilibria with only experts and free-riders, correspond to maximal independent sets of the network and therefore are always guaranteed to exist.

Specialized equilibria are of interest as they illustrate in an acute form how the network can lead to specialization. However, beyond specialized equilibrium, very little is known about other equilibria such as distributed equilibria, where all players make positive contributions, and hybrid equilibria, which are are neither specialized nor distributed. Distributed equilibria can be also of interest given their normative importance, because all players share the burden of contributing to the public good, but they are not always guaranteed to exist. For instance, distributed equilibria are not possible in star networks. Moreover, even when distributed equilibria exist very little is known about their properties beyond the symmetric contribution equilibrium in regular networks.

In the following we will provide a condition on the modular decomposition of the network that is necessary for the existence of distributed equilibrium. We say that a series model is uncentered if all (or, none) of its descendants are single players.

**Proposition 1.** *If a distributed equilibrium exists then all series modules are uncentered.*

The next result shows that the necessary condition becomes also sufficient for a special class of networks.

**Proposition 2.** *If the network is a cograph, then a distributed equilibrium exists if and only if all series modules are uncentered.*

## 6. Conclusion

In this paper, we have investigated a concept of aggregation in games played on networks based on a hierarchical structural decomposition of the network. The concept of aggregation allows us to gain significant insights into strategic interactions by investigating smaller networks.

Understanding, and making sense, of large networks is an increasingly important problem from an economic perspective, due to the ever-widening gap between technological advances in constructing such networks, and our ability to predict and estimate their properties. In this regard modular decomposition, by breaking up large networks into smaller pieces, seems to provide an interesting method for summarizing complex strategic interactions by simple ones. While our findings could potentially have applications to many network models in economics, it remains to be seen whether other approaches from the vast and important literature on structural decompositions of networks across myriad disciplines, ranging from biology to computer science,[7] could be useful to analyze complex strategic interactions.

## 7. Appendix

**Proof of Theorem ??.** First, observe that a profile of actions $\mathbf{x}^* = (x_1^*, \dots, x_n^*)$ is a Nash equilibrium of $\mathbf{\Gamma}(\mathbf{g}, \delta)$ if and only if for each player $i \in N$

$$x_i^* = \begin{cases} 1 - \delta x^*_{\mathcal{N}_i(\mathbf{g})} & \text{if } \delta x^*_{\mathcal{N}_i(\mathbf{g})} \leq 1, \\ 0 & \text{if } \delta x^*_{\mathcal{N}_i(\mathbf{g})} > 1. \end{cases} \tag{7.1}$$

Since $M_k$ is a module, for each $i \in M_k$ and for each $h \neq k$, it holds that the set of neighbors of $i$ in $M_h$, that is, $\mathcal{N}_i(\mathbf{g}_{M_h})$, is independent of the choice of $i \in M_k$. Let's posit

$$r_k^* \overset{\text{def}}{=} \max\{1 - \delta \sum_{h \in \mathcal{N}_h(\mathbf{g}/\mathbf{p})} x^*_{\mathcal{N}_i(\mathbf{g}_{M_h})}, 0\}$$

Then, since for each $i \in M_k$

$$\mathcal{N}_i(\mathbf{g}) = \bigcup_{h \in k \cup \mathcal{N}_h(\mathbf{g}/\mathbf{p})} \mathcal{N}_i(\mathbf{g}_{M_h})$$

[7]See, for example, Gagneur et al (2004) and Newman (2006).

it holds that

$$\delta\ x^*_{\mathcal{N}_i(\mathbf{g})} = \delta \sum_{h\in k\cup\mathcal{N}_h(\mathbf{g}/\mathbf{p})} x^*_{\mathcal{N}_i(\mathbf{g}_{M_k})} = \delta\ x^*_{\mathcal{N}_i(\mathbf{g}_{M_k})} + r^*_k. \tag{7.2}$$

Let also

$$\mathbf{y}^*_{M_k} \stackrel{\text{def}}{=} \begin{cases} \frac{\mathbf{x}^*_{M_k}}{r_{M_k}} & \text{if } r^*_k > 0, \\ \text{a Nash equilibrium of } \mathbf{\Gamma}(\mathbf{g}_{M_k}, \delta) & \text{if } r^*_k = 0, \end{cases}$$

Hence, in view of (**??**) and (**??**) , $\mathbf{x}^*$ is a Nash equilibrium of $\mathbf{\Gamma}(\mathbf{g}, \delta)$ if and only if for each module $k = 1, \dots, K$,

$$r^*_k = \max\{1 - \delta \sum_{h\in\mathcal{N}_h(\mathbf{g}/\mathbf{p})} y^*_{\mathcal{N}_i(\mathbf{g}_{M_h})} r^*_h, 0\}$$

and for each player $i \in M_k$ it holds that

$$x^*_i = \begin{cases} r^*_k - \delta\ x^*_{\mathcal{N}_i(\mathbf{g}_{M_k})} & \text{if } \delta\ x^*_{\mathcal{N}_i(\mathbf{g}_{M_k})} \le r^*_k, \\ 0 & \text{if } \delta\ x^*_{\mathcal{N}_i(\mathbf{g}_{M_k})} > r^*_k, \end{cases}$$

or, equivalently,

$$y^*_i = \begin{cases} 1 - \delta\ y^*_{\mathcal{N}_i(\mathbf{g}_{M_k})} & \text{if } \delta\ y^*_{\mathcal{N}_i(\mathbf{g}_{M_k})} \le 1, \\ 0 & \text{if } \delta y^*_{\mathcal{N}_i(\mathbf{g}_{M_k})} > 1. \end{cases}$$

Therefore, $\mathbf{x}^*$ is a Nash equilibrium of $\mathbf{\Gamma}(\mathbf{g}, \delta)$ if and only if $\mathbf{x}^* = (r^*_k\ \mathbf{y}^*_{M_k})_{k\in K}$ such that $\mathbf{r}^*$ is a Nash equilibrium of $\mathbf{\Gamma}(\mathbf{g}/\mathbf{p}, \delta; y^*_{M_1}, \dots, y^*_{M_K})$ and $\mathbf{y}^*_{M_k}$ is a Nash equilibrium of $\mathbf{\Gamma}(\mathbf{g}_{M_k}, \delta)$, for each $k = 1, \dots, K$.□

**Proof of Theorem ??.** From Theorem **??** it holds that $\mathbf{x}^*_M$ is a Nash equilibrium of $\mathbf{\Gamma}(\mathbf{g}_M, \delta)$ if and only if $\mathbf{x}^*_M = (r^*_1\ \mathbf{y}^*_{D_1}, \dots, r^*_T\ \mathbf{y}^*_{D_T})$ such that $\mathbf{r}^*_M$ is a Nash equilibrium of $\mathbf{\Gamma}(\mathbf{g}_M/\mathbf{p}_M, \delta; y^*_{D_1}, \dots, y^*_{D_T})$ and $\mathbf{y}^*_{D_t}$ is a Nash equilibrium of $\mathbf{\Gamma}(\mathbf{g}_{D_t}, \delta)$, for each $t = 1, \dots, T$.

If $M$ is prime, then the equivalence follows from the result above.

If $M$ is parallel, then $\mathbf{r}^*_M$ is a Nash equilibrium of $\mathbf{\Gamma}(\mathbf{g}_M/\mathbf{p}_M, \delta; y^*_{D_1}, \dots, y^*_{D_T})$ is equivalent to

$$r^*_{D_t} = 1, \text{ for each } t = 1, \dots, T,$$

since $\mathcal{N}_t(\mathbf{g}_M/\mathbf{p}_M) = \emptyset$ for each $t = 1, \dots, T$.

If $M$ is series, then $\mathbf{r}_M^*$ is a Nash equilibrium of $\boldsymbol{\Gamma}(\mathbf{g}_M/\mathbf{p}_M, \delta; y_{D_1}^*, \ldots, y_{D_T}^*)$ is equivalent to

$$r_t^* = 1 - \delta \sum_{s \in A \setminus \{t\}} y_{D_s}^* r_s^* \text{ for each } t \in A \tag{7.3}$$

and

$$\delta \sum_{s \in A} y_{D_s}^* r_s^* \geq 1, \quad \text{if } A \neq T. \tag{7.4}$$

Let

$$\mathbf{v} \stackrel{\text{def}}{=} (\frac{\delta y_{D_s}^*}{1 - \delta y_{D_s}^*})_{s \in A} \text{ and } \mathbf{U} \stackrel{\text{def}}{=} \operatorname{diag}(1 - \delta y_{D_s}^*)_{s \in A}.$$

Then, (**??**) is equivalent that

$$(\mathbf{I} + \mathbf{1}\mathbf{v}^T)\mathbf{U}\mathbf{r}_A^* = \mathbf{1}.$$

From the Sherman–Morrison formula, provided that $1 + \mathbf{v}^T\mathbf{1} \neq 0$, it holds that

$$\mathbf{r}_A^* = \mathbf{U}^{-1}(\mathbf{I} + \mathbf{1}\mathbf{v}^T)^{-1}\mathbf{1} = \mathbf{U}^{-1}(\mathbf{I} - \frac{\mathbf{1}\mathbf{v}^T}{1 + \mathbf{v}^T\mathbf{1}})\mathbf{1} = \mathbf{U}^{-1}(\mathbf{1} - \frac{\mathbf{v}^T\mathbf{1}}{1 + \mathbf{v}^T\mathbf{1}}\mathbf{1}) = \frac{1}{1 + \mathbf{v}^T\mathbf{1}} \mathbf{U}^{-1}\mathbf{1}.$$

Hence, or each $t \in A$, it holds that

$$r_t^* = \frac{\frac{1}{1 - \delta y_{D_1}^*}}{1 + \sum_{s \in A} \frac{\delta y_{D_s}^*}{1 - \delta y_{D_s}^*}},$$

Note that since $r_t^* > 0$ for each $t \in A$, it follows from above that either $y_{D_t}^* > \frac{1}{\delta}$, for each $t \in A$ or $y_{D_t}^* < \frac{1}{\delta}$ for each $t \in A$. Moreover, in view of (**??**), if $A \neq T$ then

$$\frac{\sum_{s \in A} \frac{\delta y_{D_s}^*}{1 - \delta y_{D_1}}}{1 + \sum_{s \in A} \frac{\delta y_{D_s}^*}{1 - \delta y_{D_s}^*}} = 1 - \frac{1}{1 + \sum_{s \in A} \frac{\delta y_{D_s}^*}{1 - \delta y_{D_s}^*}} \geq 1,$$

which implies that

$$\sum_{s \in A} \frac{\delta y_{D_s}^*}{1 - \delta y_{D_s}^*} < -1.$$

Hence if $A \neq T$, then it holds that $y_{D_t}^* > \frac{1}{\delta}$, for each $t \in A$.

Conversely, it is easy to check that if either for each $t \in A$ it holds that $y_{D_t}^* > \frac{1}{\delta}$ (or, for each $t \in A = T$ it holds that $y_{D_t}^* < \frac{1}{\delta}$), and

$$r_t^* = \frac{\frac{1}{1 - \delta y_{D_t}^*}}{1 + \sum_{s \in A} \frac{\delta y_{D_s}^*}{1 - \delta y_{D_s}^*}}$$

then $\mathbf{r}_M^*$ is a Nash equilibrium of $\boldsymbol{\Gamma}(\mathbf{g}/\mathbf{p}, \delta; y_{D_1}^*, \ldots, y_{D_T}^*)$.□

**Proof of Proposition ??.** Let $\mathbf{x}^*$ be a Nash equilibrium of $\mathbf{\Gamma}(\mathbf{g}, 1)$ such that $x_i^* > 0$, for each $i \in N$. Let $M$ be a series module. From Theorem 1 there exists a real number $r_M > 0$ such that $\mathbf{x}^* = r_M \mathbf{y}_M^*$, where $\mathbf{y}_M{}^*$ is a Nash equilibrium of $\mathbf{\Gamma}(\mathbf{g}_M, 1)$. Suppose that $M$ is not uncentered. Let $\mathbf{p}_M = (D_1, \ldots, D_T)$ denote the direct descendants partition of $M$. Then, there exists $1 \leq t_1 \neq t_2 \leq T$ such that $D_{t_1} = \{i_1\}$ is a single player and $D_{t_2}$ is not a single player. Note that there is (at least) one player $i_2 \in D_{t_2}$, that is not connected to all players in $D_{t_2}$. Otherwise, all players in $D_{t_2}$ are connected, which implies $D_{t_2}$ is not a direct descendant of $M$.

At the Nash equilibrium $\mathbf{y}_M^*$, each player's action is a best-reply to his neighbors' actions. In particular, it holds for player $i_1$

$$y_{i_1}^* + \sum_{i \in D_{t_2}} y_i^* + \sum_{t \neq t_1, t_2} y_{D_t}^* = 1,$$

and for player $i_2$

$$y_{i_2}^* + \sum_{i \in \mathcal{N}_{i_2}(\mathbf{g_M}) \cap D_{t_2}} y_i^* + y_{i_1}^* + \sum_{t \neq t_1, t_2} y_{D_t} = 1,$$

which together imply

$$\sum_{i \in \{i_2 \cup \mathcal{N}_{i_2}(\mathbf{g_M})\}^c \cap D_{t_2}} y_i^* = 0.$$

This is a contradiction since $\{i_2 \cup \mathcal{N}_{i_2}(\mathbf{g_M})\}^c \cap D_{t_2} \neq \emptyset$ and $y_i^* > 0$, for each $i \in M$.□

**Proof of Proposition 2.** Suppose the network $\mathbf{g}$ is a cograph. Therefore, the network $\mathbf{g}$ has only parallel and series modules in its modular decomposition tree. If no series module is uncentered. Then, given a series module $M$, with direct descendants partition $\mathbf{p}_M = (D_1, \ldots, D_T)$ either all or none of direct descendants are single players. If all direct descendants are single players then the symmetric contribution $\frac{\mathbf{1}}{n+1}$ is a Nash equilibrium of $\mathbf{\Gamma}(\mathbf{g}_M, 1)$. If none of $M$ direct descendants is a single player, then for each $t = 1, \ldots, T$, and for any Nash equilibrium $\mathbf{y}_t^*$ of $\mathbf{\Gamma}(\mathbf{g}_{D_t}, \delta)$ it holds that $y_{D_t}^* \geq 2$ since $D_t$ is a parallel module with at least two direct descendants. Therefore one can use $(ii)$ and $(iii)$ in Theorem 2 recursively along the nodes of the modular decomposition tree in order to construct a distributed Nash equilibrium.□

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