

Arbitrage and equilibrium in economies with short-selling and ambiguity

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Abstract

We consider a model with a finite number of states of nature where short sells are allowed. We present a notion of no-arbitrage price weaker than the one of Werner [26], and prove that in the case of separable averse at risk utility functions, the existence of one common weak no-arbitrage price is equivalent to the existence of equilibrium.

Keywords: asset market equilibrium, individually rational attainable allocations, individually rational utility set, no-arbitrage prices, no-arbitrage condition.

JEL Classification: C62, D50, D81, D84, G1.

1 Introduction

Equilibrium conditions on financial markets differ with the ones on good market when short-selling is accepted. This assumption makes useless traditional techniques using fixed point theory. In the finite dimension case, there is a huge literature on well-known conditions called no arbitrage conditions. These conditions in general imply the compactness of the allocations set or the utilities set. We can classify them in three main categories. The first category is based on conditions on net trade, for example Hart [20], Page [22], Nielsen [21], Page and Wooders [23], Allouch [1], Page, Wooders and Monteiro [24]. We define Individual arbitrage opportunity as the set of directions along which the agent wants to trade with infinite quantities. In the case the agents disagree

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too much, some agents can make an arbitrage which is an opportunity with a mutually compatible set of net trades which are utility non-decreasing and, at most, costless to make (Hart [20]). Taking the fact that this opportunity can be repeated indefinitely, equilibrium may not exist. The Weak-no-market-arbitrage WNMA requires that that all mutually compatible net trades which are non-decreasing be useless. Page [22] proposes the no-unbounded-arbitrage NUBA, a situation in which there is no group of agents can make mutually compatible, unbounded and utility increasing trades. In 2000, Page Wooders and Monteiro [24], introduce Inconsequential arbitrage condition to ensure the existence for an equilibrium.

The second category is based on conditions on prices, for example Green [16], Grandmont [14], [15], Hammond [19] and Werner [26]. These authors define Non-arbitrage price as element in the strictly positive dual of the set of useful vectors. If the intersection of No-arbitrage price cones of all agents is non empty (existence of No-arbitrage-price-system NAPS), then there exists a general equilibrium.

The third category includes authors, like Brown and Werner [3], Dana, Le Van, Magnien [8], who assume the compactness of attainable utility set to ensure the existence of equilibrium.

In the case utility functions are quasi-concave, Allouch, Le Van and Page [1], Ha-Huy and Nguyen [18] by different approaches, prove the equivalence between the existence of No-arbitrage-price-system NAPS and NUBA or WNMA. If the agents in the economy have no trivial useless vectors, then NAPS and NUBA are equivalent and they imply existence of a general equilibrium. Obviously, we can wonder whether we can have equivalent conditions between the existence of general equilibrium and these no arbitrage conditions when the utility functions are not strictly concave. Unfortunately, the answer is no. In this paper, we give an example in which NAPS and NUBA, even WNMA conditions are violated, but a general equilibrium does exist (Subsection 3.3). In 2010, Dana and Le Van [9], by considering the relationships between the agents beliefs and risk when there is ambiguity, propose to use the set of derivatives of the utility function as no-arbitrage prices set. By using this trick, they give a description of weak no-arbitrage prices and useful vectors. Furthermore, they give an equivalence between non-emptiness of the intersection of interiors of no-arbitrage price cones and NUBA condition, or non-emptiness of the intersection of relative interiors of no-arbitrage prices cones and WNMA condition. Hence, if this intersection is non empty, existence of a general equilibrium is ensured. In this paper, we reconsider the equilibrium theory of assets with short-selling when there is risk and ambiguity. The agents have Von Neumann Morgenstern utility functions. They are not only risk averse but also ambiguity averse. We suppose the set of beliefs is a convex compact subset of the unit-simplex. Using the

notion of weak no-arbitrage prices proposed by Dana and Le Van [9], we prove the equivalence between existence of a general equilibrium and non-emptiness of the intersection of no-arbitrage prices cones. To the best of our knowledge, when the utility functions are not strictly concave, such a result does not exist in the literature. Hence, our result is stronger than in Dana and Le Van [9] when the sets of beliefs are convex compact. Our paper is organized as follows. In Section 2 we recall several well-known conditions (No unbounded arbitrage - NUBA, or Weak no market arbitrage - WNMA, or No arbitrage price system - NAPS) for the existence of an equilibrium in a general framework. In Section 3, we consider an economy in which any agent i has a Von Neumann Morgenstern utility function. The agents are ambiguous in the probabilities of the outcomes and they are averse at ambiguity. As in Gilboa and Schmeidler [17], each agent faces a set of subjective probabilities.

Our proof proceeds in two stages. At the first stage we assume that the set of beliefs of each agent is the convex hull of a finite number of strictly positive probabilities. We introduce the cone of common weak no-arbitrage prices and state the equivalence between existence of equilibrium and existence of a weak no-arbitrage price common to all the agents. In the second we take a sequence of convex polyhedrons which converge to the initial sets of beliefs. With each set of polyhedrons we associate an appropriate equilibrium. We prove that the limit of this sequence is an equilibrium. In Subsection 3.3, we give an example of economy which does not satisfy either NUBA or WNMA or NAPS and however has an equilibrium since it satisfies our no-arbitrage condition.

2 Existence of equilibrium: the general case

We consider now the economy in which any agent i has a Von Neumann Morgenstern utility function. Define $\Delta = \{\pi \in \mathbb{R}_+^S \text{ such that } \sum_{s=1}^S \pi_s = 1\}$. The agents are ambiguous in the probabilities of the outcomes and they are averse at ambiguity. As in Gilboa and Schmeidler [17], each agent faces a set of subjective probabilities $\Delta^i \subset \Delta = \{\pi \in \mathbb{R}_+^S \text{ such that } \sum_{s=1}^S \pi_s = 1\}$ and their utility functions take the form

$$U^i(x) = \inf_{\pi \in \Delta^i} \sum_{s=1}^m \pi_s u^i(x_s),$$

where $u^i : \mathbb{R} \rightarrow \mathbb{R}$ is a concave, strictly increasing, differentiable function¹, and $\Delta^i \subset \Delta$ is a convex, compact subset in the interior of Δ .

¹For the sake of simplicity the presentation, we assume the differentiability. The results do not change for general case with sub-differentials.

Definition 1 An equilibrium is a list $((x^{i*})_{i=1,\dots,m}, p^*)$ such that $p^* \in \mathbb{R}_+^S \setminus \{0\}$ and

(a) For any i , $U^i(x) > U^i(x^{i*}) \Rightarrow p^* \cdot x > p^* \cdot x^{i*}$.

(b) $\sum_{i=1}^m x^{i*} = \sum_{i=1}^m e^i$.

Definition 2 An quasi equilibrium is a list $((x^{i*})_{i=1,\dots,m}, p^*)$ such that $p^* \in \mathbb{R}_+^S \setminus \{0\}$ and

(a) For any i , $U^i(x) > U^i(x^{i*}) \Rightarrow p^* \cdot x \geq p^* \cdot x^{i*}$.

(b) $\sum_{i=1}^m x^{i*} = \sum_{i=1}^m e^i$.

We recall the definitions of the attainable allocations set and the individually rational utility set.

Definition 3 1. The individually rational attainable allocations set \mathcal{A} is defined by

$$\mathcal{A} = \{(x^1, x^2, \dots, x^m) \in (\mathbb{R}^S)^m \mid \sum_{i=1}^m x^i = \sum_{i=1}^m e^i \text{ and } U^i(x^i) \geq U^i(e^i) \text{ for all } i\}.$$

2. The individually rational utility set \mathcal{U} is defined by

$$\mathcal{U} = \{(v^1, v^2, \dots, v^m) \in \mathbb{R}^m \mid \exists x \in \mathcal{A} \text{ s.t. } U^i(e^i) \leq v^i \leq U^i(x^i) \text{ for all } i\}.$$

Theorem 1 If \mathcal{U} is compact then there exists an equilibrium.

Proof: See Dana, Le Van and Magnien [10] for the existence of quasi-equilibrium. Since short sales are allowed, in our model quasi equilibrium is also equilibrium. For this result, see Florenzano [12]. ■

Theorem 2 We have

$NUBA \Rightarrow NAPS \Rightarrow \text{Inconsequential arbitrage} \Rightarrow \mathcal{U} \text{ is compact} \Rightarrow \text{There exists a general equilibrium.}$

Proof: See Allouch, Le Van and Page [1]. ■

Theorems 1 and 2 hold also in the general case.

Easily, we can prove that U^i is concave. We define R^i the set of useful vectors of U^i .

2.1 Characterization of useful vectors

The following lemma characterizes the useful vectors set of agent i .

Denote ²

$$\begin{aligned} a^i &= \inf_{z \in \mathbb{R}} u^{i'}(z) = u^{i'}(+\infty) \\ b^i &= \sup_{z \in \mathbb{R}} u^{i'}(z) = u^{i'}(-\infty).^3 \end{aligned}$$

Lemma 1 *The vector $w \in \mathbb{R}^S$ is useful for agent i if and only if for any $x \in \mathbb{R}^S$, any $\pi \in \Delta^i$ we have*

$$\sum_{s=1}^S \pi_s u^{i'}(x_s) w_s \geq 0.$$

Proof: See Proposition 2 in Dana and Le Van [9]. ■

For each vector $w \in \mathbb{R}^S$, define $S_+(w) = \{s \text{ such that } w_s > 0\}$, and $S_-(w) = \{s \text{ such that } w_s < 0\}$. The following proposition is a direct consequence of Lemma 1.

Proposition 1 *The vector w is useful for agent i , if and only if, for any $\pi \in \Delta^i$ we have*

$$a^i \sum_{s \in S_+(w)} \pi_s w_s + b^i \sum_{s \in S_-(w)} \pi_s w_s \geq 0. \quad (1)$$

Proof: Suppose that w is a useful vector of agent i . We use Lemma 1. By letting x_s converge to $+\infty$ when $s \in S_+(w)$, and x_s converge to $-\infty$ when $s \in S_-(w)$, we obtain (1).

Now, we prove the converse. Suppose that the vector w satisfies (1). For any $x \in \mathbb{R}^S$ we have $a^i \leq u^{i'}(x_s) \leq b^i$. This implies

$$\sum_{s=1}^S \pi_s u^{i'}(x_s) w_s \geq a^i \sum_{s \in S_+(w)} \pi_s w_s + b^i \sum_{s \in S_-(w)} \pi_s w_s \geq 0.$$

From Lemma 1, the vector w is useful for agent i . ■

Corollary 1 *If $a^i = 0$ or $b^i = \infty$, then $R^i = \mathbb{R}_+^S$.*

²We rule out the case $a^i = b^i = 0$ which is not interesting. The utility function is constant in this case.

2.2 Weak No-Arbitrage prices and existence of equilibrium

Using the idea of Dana and Le Van [9], we define the set of weak no-arbitrage prices.

$$P^i = \{p \in \mathbb{R}^S \text{ such that } \exists \lambda \geq 0, x \in \mathbb{R}^S, \pi \in \Delta^i, \text{ satisfying } p_s = \lambda \pi_s u^{i'}(x_s) \forall s = 1, 2, \dots, S\}.$$

Lemma 2 *For all i , P^i is a convex cone.*

Proof: See Dana and Le Van [9]. ■

We have two cases.

Case 1: For all i except at most one agent, for any $z \in \mathbb{R}$, either $a^i < u^{i'}(z)$ or $b^i > u^{i'}(z)$.

Proposition 2 *Suppose for any $z \in \mathbb{R}$, either $a^i < u^{i'}(z)$ or $b^i > u^{i'}(z)$. Then P^i is open.*

Proof: See Dana and Le Van [9]. ■

Lemma 3 *Fix $x \in \mathbb{R}^S$. Then $\partial U^i(x)$ is the set*

$$Q = \{p : p = (\pi_1 u^{i'}(x_1), \pi_2 u^{i'}(x_2), \dots, \pi_S u^{i'}(x_S))\}$$

where $\pi \in \Delta^i$ satisfies $U^i(x) = \sum_{s=1}^S \pi_s u^i(x_s)$.

Proof: First observe $Q \subset \partial U^i(x)$. Conversely, from Clarke [7], $\partial U^i(x)$ is the convex hull of the derivatives $(\tilde{\pi}_1 u^{i'}(x_1), \tilde{\pi}_2 u^{i'}(x_2), \dots, \tilde{\pi}_S u^{i'}(x_S))$, the probabilities $(\tilde{\pi})$ satisfy $U^i(x) = \sum_{s=1}^S \tilde{\pi}_s u^i(x_s)$. Hence $\partial U^i(x) \subset Q$.⁴ ■

Proposition 3 *Suppose that for all i except at most one agent, we have either $a^i < u^{i'}(z), \forall z \in \mathbb{R}$, or $b^i > u^{i'}(z), \forall z \in \mathbb{R}$, then we have:*

$$\bigcap_{i=1}^m P^i \neq \emptyset \Leftrightarrow \text{NUBA condition} \Leftrightarrow \mathcal{U} \text{ is compact} \Leftrightarrow \text{there exists equilibrium.}$$

Proof: Consider first the case for all i , we have either $a^i < u^{i'}(z), \forall z \in \mathbb{R}$, or $b^i > u^{i'}(z), \forall z \in \mathbb{R}$. Since P^i is convex, the set of weak no-arbitrage prices coincides with the set of no-arbitrage prices S^i . See Dana and Le Van [9].

⁴A more simplified version of Clarke's one, which can be used for the case of utility functions in theorem 3, can be found in [13].

Consider the second case in which the above condition is satisfied for all $i \neq i_0$. We can chose $p \in \cap_{i \neq i_0} P^i$, and $p \in \text{int}P^{i_0}$. The argument is the same as in the first case. ■

Case 2: Now we consider the case where for some i , the utility function u^i becomes affine when the consumption is large enough, i.e. there exists \bar{z}^i such that

$$u^{i'}(z) = a^i \text{ for } z \geq \bar{z}^i \text{ and } u^{i'}(x) = b^i \text{ for } z \leq -\bar{z}^i.$$

Lemma 4 Fix $x \in \mathbb{R}^S$. Then $\partial U^i(x)$ is the set

$$Q = \{p : p = (\pi_1 u^{i'}(x_1), \pi_2 u^{i'}(x_2), \dots, \pi_S u^{i'}(x_S))\}$$

where $\pi \in \Delta^i$ satisfies $U^i(x) = \sum_{s=1}^S \pi_s u^i(x_s)$.

Proof: First observe $Q \subset \partial U^i(x)$. Conversely, from Clarke [7], $\partial U^i(x)$ is the convex hull of the derivatives $(\tilde{\pi}_1 u^{i'}(x_1), \tilde{\pi}_2 u^{i'}(x_2), \dots, \tilde{\pi}_S u^{i'}(x_S))$, the probabilities $(\tilde{\pi})$ satisfy $U^i(x) = \sum_{s=1}^S \tilde{\pi}_s u^i(x_s)$. Hence $\partial U^i(x) \subset Q$. ■

3 Existence of equilibrium when agents are risk averse and ambiguity averse

3.1 Step 1: The sets of beliefs are polyhedral

Theorem 3 Suppose that each probabilities set Δ^i is a convex hull of the set of M^i points, i.e. $\Delta^i = \text{conv}\{\pi_0^i, \pi_1^i, \dots, \pi_{M^i}^i\}$. Then:

$$\bigcap_{i=1}^m P^i \neq \emptyset \Leftrightarrow \mathcal{U} \text{ is compact} \Leftrightarrow \text{there exists a general equilibrium.}$$

Proof: See Ha-Huy and Le Van [18]. ■

The following Corollary is the direct consequence.

Corollary 2 If for any i , Δ^i is a singleton, i.e $\Delta^i = \{\pi^i\}$, then $P^i = \{p \in \mathbb{R}^S \text{ such that there exists } \lambda > 0, x \in \mathbb{R}^S : p_s = \lambda \pi_s^i u^{i'}(x_s), \text{ for any } 1 \leq s \leq S\}$. We also have

$$\bigcap_{i=1}^m P^i \neq \emptyset \Leftrightarrow \text{there exists general equilibrium.}$$

3.2 Step 2: the sets of beliefs are convex compact in the unit-simplex

In the following theorem, we prove the equivalent between the existence of a common weak no-arbitrage price the existence of general equilibrium. But we do not ensure the compactness of utility set \mathcal{U} .

Theorem 4 *We have*

$$\bigcap_{i=1}^m P^i \neq \emptyset \Leftrightarrow \text{there exists a general equilibrium.}$$

Proof: Suppose that $\bigcap_{i=1}^m P^i \neq \emptyset$. Take any $\bar{p} \in \bigcap_{i=1}^m P^i$. There exist $\pi_0^i \in \Delta^i$, $\bar{x}^i \in \mathbb{R}^S$, $\lambda_i > 0$ for any i satisfying

$$\bar{p}_s = \lambda_i \pi_0^i u^i(\bar{x}_s^i) \text{ for any } s = 1, 2, \dots, S.$$

For each i , we can construct a sequence of subset $\{\Delta_n^i\}_{n=1}^\infty \subset \Delta^i$ satisfying

- For any n , Δ_n^i is a convex hull of finite number of elements of Δ^i .
- For any n , $\pi_0^i \in \Delta_n^i$.
- For any $\pi \in \text{ri}(\Delta^i)$, the relative interior of Δ^i , there exists N big enough such that $\pi \in \Delta_n^i$ for any $n \geq N$.

For each n , define real function U_n^i on \mathbb{R}^S :

$$U_n^i(x) = \inf_{\pi \in \Delta_n^i} \sum_{s=1}^S \pi_s^i u^i(x_s).$$

Claim 1 *Let $\{x_n^i\}_n$ be a sequence which converges to x^i . We have $\lim_{n \rightarrow \infty} U_n^i(x_n^i) = U^i(x^i)$.*

Proof:

- We have

$$\begin{aligned} U_n^i(x_n^i) &= \sum_s \pi_s^i(n) u^i(x_{n,s}^i), \text{ for some } \pi^i(n) \in \Delta_n^i \\ &\geq U^i(x_n^i) \\ \implies \liminf_n U_n^i(x_n^i) &\geq \lim_n U^i(x_n^i) = U^i(x^i) \end{aligned}$$

- We also have $U^i(x^i) = \sum_s \pi_s^i u^i(x_s^i)$ for some $\pi^i \in \Delta^i$.
There exists a sequence $\{\pi^i(n)\}_n \subset \Delta_n^i, \forall n$ which converges to π^i .
Then

$$\begin{aligned}
U^i(x^i) &= \lim_n \sum_s \pi_s^i(n) u^i(x_{n,s}^i) \\
\text{but } \sum_s \pi_s^i(n) u^i(x_{n,s}^i) &\geq U_n^i(x_n^i) \\
\implies U^i(x^i) &\geq \limsup_n U_n^i(x_n^i)
\end{aligned}$$

■

Observe that for any $x \in \mathbb{R}^S$, we have $\lim_{n \rightarrow \infty} U_n^i(x) = U^i(x)$.

We consider now the economy \mathcal{E}_n in which the agent i has utility function U_n^i , endowment e^i .

Denote by P_n^i and R_n^i the set of weak no-arbitrage prices of agent i in economy \mathcal{E}_n :

$$P_n^i = \{p \in \mathbb{R}^S \text{ such that } \exists \lambda \geq 0, x \in \mathbb{R}^s, \pi \in \Delta_n^i, \text{ satisfying } p_s = \lambda \pi_s u^i(x_s) \forall s = 1, 2, \dots, S\}.$$

Denote by R_n^i the set of useful vectors of agent i in economy \mathcal{E}_n . We have R_n^i is a positive polar cone of P_n^i .

Observe that $P_n^i \subset P^i$, and hence $R^i \subset R_n^i$.

Since $\pi_0^i \in \Delta_n^i$ for any n , we have $\bigcap_{i=1}^m P_n^i \neq \emptyset$. Hence the economy \mathcal{E}_n has general equilibrium. Denote by G_n the set $\{x^* = (x^{*1}, x^{*2}, \dots, x^{*m}) \in (\mathbb{R}^S)^m\}$ such that there exists $p^* \in (\mathbb{R}_{++}^S)^m$ satisfying (p^*, x^*) is a general equilibrium of economy \mathcal{E}_n .

Firstly, observe that G_n is closed. Indeed, suppose that x^* is a limit of a sequence $\{x^*(k)\} \subset G_n$. Define $p^*(k)$ the sequence associated equilibrium prices, which are (without loss of generality) normalized: $\sum_{s=1}^S p_s^*(k) = 1$. We can assume that $p^*(k)$ converges to p^* . If $U_n^i(x) > U_n^i(x^{*i})$, then for k big enough we have $U_n^i(x) > U_n^i(x^{*i}(k))$, hence $p^*(k) \cdot x > p^*(k) \cdot x^{*i}(k)$. Let k converges to infinity we get: if $U_n^i(x) > U_n^i(x^{*i})$, we have $p^* \cdot x \geq p^* \cdot x^{*i}$. This implies (p^*, x^*) is a quasi-equilibrium of the economy \mathcal{E}_n . Since short-sales are allowed, quasi-equilibrium is equilibrium, see [12]. We have $x^* \in G_n$.

Let $d_n = \inf_{x^* \in G_n} \sum_{i=1}^m \|x^{*i}\|$. Let $\epsilon > 0$. The set x^* in G_n such that $\sum_{i=1}^m \|x^{*i}\| \leq d_n + \epsilon$ is non empty. This set is compact since G_n is closed. Minimizing over this set we get $x_n^* = \operatorname{argmin}_{x^* \in G_n} \sum_{i=1}^m \|x^{*i}\|$ or

$$\sum_{i=1}^m \|x_n^{*i}\| = \min_{x^* \in G_n} \sum_{i=1}^m \|x^{*i}\|.$$

Denote by p_n^* the associated equilibrium price, with $\sum_{s=1}^S p_{n,s}^* = 1$.

We will prove that $\{x_n^*\}$ is bounded. Suppose the contrary, $\lim_{n \rightarrow \infty} \sum_{i=1}^m \|x_n^{*i}\| = +\infty$. Without loss of generality, we can assume that for any i , there exists

$$\lim_{n \rightarrow \infty} \frac{x_n^{*i}}{\sum_{j=1}^m \|x_n^{*j}\|} = w^i.$$

Since for any n , $\sum_{i=1}^m x_n^{*i} = \sum_{i=1}^m e^i$, we have $\sum_{i=1}^m w^i = 0$. Observe that $\sum_{i=1}^m \|w^i\| = 1$.

Firstly, we will prove that w^i is a useful vector of agent i in the initial economy \mathcal{E} : $w^i \in R^i$. Indeed, fix $\pi \in \text{ri}(\Delta^i)$. For any n big enough such that $\pi \in \Delta_n^i$, we have $\{\pi\} \subset \Delta_n^i \subset \Delta^i$, hence

$$\sum_{s=1}^S \pi_s u^i(x_{n,s}^{*i}) \geq U_n^i(x_n^{*i}) \geq U_n^i(e^i) \geq U^i(e^i) \text{ for large } n.$$

Since w^i is a limit of the sequence $\left\{ \frac{x_n^{*i}}{\sum_{j=1}^m \|x_n^{*j}\|} \right\}$, this inequality implies w^i is a useful vector of the function $U_\pi(x) = \sum_{s=1}^S \pi_s u^i(x_s)$, hence we have for any $x \in \mathbb{R}^S$:

$$\sum_{s=1}^S \pi_s u^{i'}(x_s) w_s^i \geq 0.$$

Since $\sum_{s=1}^S \pi_s u^{i'}(x_s) w_s^i \geq 0$ for any $\pi \in \text{ri}(\Delta^i)$, we have $\sum_{s=1}^S \pi_s u^{i'}(x_s) w_s^i \geq 0$ for any $\pi \in \Delta^i$. We have proved that $w^i \in R^i$, and hence $w^i \in R^i \subset R_n^i$ for any n .

For each n , denote by $\hat{\Delta}_n^i$ the set of extreme points of Δ_n^i . By the definition of Δ_n^i , the set $\hat{\Delta}_n^i$ has a finite number of elements. Define $\tilde{\Delta}_n^i$ be the subset of $\hat{\Delta}_n^i$ satisfying for $\pi \in \tilde{\Delta}_n^i$, we have

$$U_n^i(x_n^{*i}) = \sum_{s=1}^S \pi_s u^i(x_{n,s}^{*i}).$$

Denote by \mathcal{I} the set of agents such that: there exists $z^i > 0$ satisfying $u^{i'}(z) = a^i$ for $x \geq z^i$ and $u^{i'}(z) = b^i$ for $x \leq -z^i$.

Using the same arguments in [18], we have for $i \notin \mathcal{I}$, $w^i = 0$. Fix n sufficiently big such that for any $i \in \mathcal{I}$, we have $x_{n,s}^{*i} > z^i$ if $s \in S_+(w_s^i)$ and $x_{n,s}^{*i} < -z^i$ if $s \in S_-(w_s^i)$.

Recall that $(x_n^{*i})_{i=1}^m$ is an equilibrium of \mathcal{E}_n , and $w^i \in R_n^i$ with $\sum_{i=1}^m w^i = 0$, we have for any i , for any $\lambda \geq 0$,

$$U_n^i(x_n^{*i} + \lambda w^i) = U_n^i(x_n^{*i}),$$

since in the contrary case, there exists $\lambda > 0$ with $U_n^i(x_n^{*i} + \lambda w^i) > U_n^i(x_n^{*i})$, the allocation $(x_n^{*i})_{i=1}^m$ can not be Pareto optimal.

The equality implies that for any $p \in \partial U^i(x_n^{*i})$, we have $p \cdot w^i = 0$. This implies for any $\pi \in \tilde{\Delta}_n^i$ we have

$$\sum_{s=1}^S \pi_s u^i(x_{n,s}^{*i}) w_s^i = 0.$$

The equality is equivalent to

$$a^i \sum_{s \in S_+(w^i)} \pi_s w_s^i + b^i \sum_{s \in S_-(w^i)} \pi_s w_s^i = 0$$

for any $\pi \in \tilde{\Delta}_n^i$.

Since $w^i \in R_n^i$, we have $p_n^* \cdot w^i = 0$ for any i .

Observe that for any $\hat{\pi} \in \hat{\Delta}_n^i \setminus \tilde{\Delta}_n^i$, and any $\pi \in \tilde{\Delta}_n^i$, we have

$$\sum_{s=1}^S \hat{\pi}_s u^i(x_{n,s}^{*i}) > \sum_{s=1}^S \pi_s u^i(x_{n,s}^{*i}) = U_n^i(x_n^{*i}).$$

We can fix $\epsilon_n > 0$ satisfying the following properties:

- For any $\hat{\pi} \in \hat{\Delta}_n^i \setminus \tilde{\Delta}_n^i$, and any $\pi \in \tilde{\Delta}_n^i$, we have

$$\sum_{s=1}^S \hat{\pi}_s u^i(x_{n,s}^{*i} - \epsilon_n w_s^i) > \sum_{s=1}^S \pi_s u^i(x_{n,s}^{*i} - \epsilon_n w_s^i). \quad (2)$$

- If $s \in S_+(w^i)$, $x_{n,s}^{*i} - \epsilon_n w_s^i > z^i$, and if $s \in S_-(w^i)$, $x_{n,s}^{*i} - \epsilon_n w_s^i < -z^i$.

Since $\hat{\Delta}_n^i$ is finite set, the function $\hat{\Delta}_n^i \in \pi \mapsto \sum_{s=1}^S \pi_s u^i(x_{n,s}^{*i} - \epsilon_n w_s^i)$ attaches minimum. By (2),

$$\operatorname{argmin}_{\pi \in \hat{\Delta}_n^i} \sum_{s=1}^S \pi_s u^i(x_{n,s}^{*i} - \epsilon_n w_s^i) \in \tilde{\Delta}_n^i.$$

Using the same calculus in [18], we have $U_n^i(x_n^{*i} - \epsilon_n w^i) = U_n^i(x_n^{*i})$.

Since $p_n^* \cdot w^i = 0$, if $U_n^i(x) > U_n^i(x_n^{*i} - \epsilon_n w^i) = U_n^i(x_n^{*i})$, we have $p_n^* \cdot x > p_n^* \cdot x_n^{*i} = p_n^* \cdot (x_n^{*i} - \epsilon_n w^i)$. This implies $(p_n^*, (x_n^{*i} - \epsilon_n w^i)_{i=1}^m)$ is also an equilibrium of \mathcal{E}_n .

Claim 2 *We have*

$$\sum_{i=1}^m \|x_n^{*i} - \epsilon_n w^i\| < \sum_{i=1}^m \|x_n^{*i}\|,$$

a contradiction with the definition of x_n^ .*

Proof: For $s \in S_+(w^i)$, we have

$$0 < z^i < x_{n,s}^{*i} - \epsilon_n w_s^i \leq x_{n,s}^{*i} \quad (3)$$

For $s \in S_-(w^i)$, we have

$$0 > -z^i > x_{n,s}^{*i} - \epsilon_n w_s^i \geq x_{n,s}^{*i} \quad (4)$$

Since $i \in \mathcal{I}$, we have $w^i \neq 0$. Hence, in (3) or (4) at least one strict inequality must hold for the last RHS inequalities. Therefore

$$\sum_{i=1}^m \|x_n^{*i} - \epsilon_n w^i\| < \sum_{i=1}^m \|x_n^{*i}\|$$

■

The contradiction means that the sequence $\{x_n^*\}$ must be bounded. Without loss of generality, suppose that it converges to x^* , and the equilibrium price sequence $\{p_n^*\}$ converges to p^* .

Apply Claim 1 to have $\lim_{n \rightarrow \infty} U_n^i(x_n^{*i}) = U^i(x^{*i})$.

Suppose that $U^i(x) > U^i(x^{*i})$. For n big enough we have $U_n^i(x) > U_n^i(x^{*i})$, which implies $p_n^* \cdot x > p_n^* \cdot x^{*i}$. Let n converges to infinity we have $p^* \cdot x \geq p^* \cdot x^{*i}$. Hence (p^*, x^*) is a quasi-equilibrium. Using [12], this quasi-equilibrium is an equilibrium. ■

3.3 Example

We present here an example in which the weak no-arbitrage prices cones are closed, their intersection is non empty, the model does not satisfy *NAPS*, *NUBA* or *WNMA* conditions, but there exists an equilibrium.

We consider an economy with two agents, the number of states is $S = 2$. The belief of agent 1 is represented by the probability $\pi_1^1 = \pi_2^1 = \frac{1}{2}$. The belief of agent 2 is $\pi_1^2 = \frac{1}{3}, \pi_2^2 = \frac{2}{3}$. Their endowments are 0. Their utility functions are defined as follows:

$$u^1(x) = \begin{cases} \ln(x) & \text{if } x \in [1/3, 1/2] \\ 2x - 1 - \ln 2 & \text{if } x \geq 1/2 \\ 3x - 1 - \ln 3 & \text{if } x \leq 1/3 \end{cases}$$

$$u^2(x) = \begin{cases} \ln(x) & \text{if } x \in [1/3, 4/9] \\ \frac{9}{4}x - 1 + \ln(\frac{4}{9}) & \text{if } x \geq \frac{4}{9} \\ 3x - 1 - \ln 3 & \text{if } x \leq \frac{1}{3} \end{cases}$$

We have $u^{1'}(+\infty) = 2, u^{1'}(-\infty) = 3$ and $u^{2'}(+\infty) = \frac{9}{4}, u^{2'}(-\infty) = 3$. Therefore, the cone of no-arbitrage prices of agent 1 is $P^1 = \{\lambda(\zeta_1, \zeta_2)\}_{\lambda > 0}$

with $1 \leq \zeta_1 \leq \frac{3}{2}$, $1 \leq \zeta_2 \leq \frac{3}{2}$. The one of agent 2 is $P^2 = \{\lambda(\zeta_1, \zeta_2)\}_{\lambda>0}$ with $\frac{3}{4} \leq \zeta_1 \leq 1$, $\frac{3}{2} \leq \zeta_2 \leq 2$.

The set of common weak no-arbitrage prices is the intersection of the two cones $P^1 \cap P^2 = \{\lambda(1, \frac{3}{2})\}_{\lambda>0}$. If S^1, S^2 are the interiors of P^1 and P^2 , then $S^1 \cap S^2 = \emptyset$.

Our economy does not satisfies either *NUBA* or *WNMA* conditions. Indeed, if we consider the useful vector $w^1 = (1, -\frac{2}{3})$ of agent 1, the useful vector $w^2 = (-1, \frac{2}{3})$ of agent 2. We obtain that $w^1 + w^2 = 0$. That means *NUBA* is not satisfied. But $-w^1, -w^2$ are not useful vectors of agent 1 and agent 2. These vectors are not in the linearity space. That means *WNMA* does not hold. However, from our main Theorem ??, an equilibrium exists in this model. The equilibrium allocations are

$$x_1^{*1} = 1, x_2^{*1} = -2/3, x_1^{*2} = -1, x_2^{*2} = 2/3$$

The equilibrium prices are $p_1^* = 1, p_2^* = 3/2$.

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