Measurement of equality of opportunity: A normative approach*

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Abstract. We develop a normative approach to the measurement of inequality of opportunity. This means that we measure inequality of opportunity by the welfare gain in moving from the actual income distribution to the optimum income distribution with the given amount of total income. Our study combines several approaches present in the literature: we axiomatically characterize social welfare functions, we obtain several allocation rules as optima and we obtain several inequality measures. Our analysis reflects moreover the main distinctions in the literature: ex post versus ex ante compensation and liberal versus utilitarian reward.

Keywords. Equality of opportunity · Inequality

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1 Introduction

Equality of opportunity is an important concept. Political philosophers such as Rawls and Dworkin have criticized concepts of equality focusing on outcomes, whereas others such as Cohen and Arneson have proposed to make opportunity the focus of equality. Moral judgments by laymen as expressed in experiments (Cappelen, Hole, Sørensen and Tungodden, 2007) and surveys (Schokkaert and Devooght, 2003) have been shown to correspond well to ideas of equality of opportunity. But not only is it intrinsically suitable for describing moral objectives and judgments, it is also stressed by the World Bank and the OECD as a good instrument for growth that is superior to outcome equality.

A large and rapidly growing literature formalizes the idea of equality of opportunity to yield measures and objective functions. Formalizations center on the two principles on which the idea of equality of opportunity rests. The first is the compensation principle, which says individual circumstances are not legitimate sources of inequality. The second is the reward principle, which says that inequalities arising from the exercise of individual responsibility do not need correcting. The literature has studied combination of versions of these principles for various types of criteria: social welfare functions that provide complete rankings of income distributions, allocation rules that select the best income distribution dividing a fixed total income amount, and inequality of opportunity measures for empirical analysis that cleverly use the tools of unidimensional inequality measurement.

We use a normative approach to the measurement of inequality of opportunity. This means that inequality is identified with the welfare loss associated to being in the actual rather than the optimal distribution of the available total income. Because the welfare functions are characterized on the basis of axioms that express ethical values—most importantly the compensation and reward ideas—the derived inequality measures receive a normative basis.

Our approach encompasses the literature in two respects. First, we are broad in terms of the normative choices that we consider with regard to the compensation and reward principles. The former exists in an ex ante and ex post form. The ex post form stresses that individuals exercising the same responsibility should get the same incomes, thus neutralizing the effect of their different circumstances. This version of compensation has been stressed in the works of Fleurbaey and Roemer. The ex ante form says that income distributions of different circumstance groups (where within a group individuals have identical circumstances) should be as equal as possible. This version of compensation features in Van de gaer's work and is popular empirically, presumably because it does not require information on responsibility exercised. The reward principle also comes in two versions. The liberal reward principle, favored by Fleurbaey, says that income differences due to responsibility should respect the market returns to responsibility. The utilitarian reward principle, advocated by Roemer and Van de gaer, says that inequalities due to responsibility are a matter of neutral treatment. Our framework will deal with ex post and ex ante versions of compensation, liberal and utilitarian versions of reward, and several variants of these that have been studied in the literature. This allows considering all these different outlooks on equality of opportunity within a single framework, all neatly expressed as axioms imposed on welfare functions.

A second way in which the current analysis is broad in a very natural sense is with respect to the methodologies that have been used to approach equality of opportunity. It unifies and generalizes in several ways. It generalizes the classes of social welfare functions that have been proposed. In turn, these social welfare functions extend allocation rules that have been proposed. Finally it provides an axiomatic underpinning for many of the inequality measures that have been proposed in mainly the empirical literature.

The next section introduces notation and some basic axioms to be imposed on social welfare functions. Section 3 defines axioms that capture the ex post and ex ante versions of compensation and the liberal and utilitarian versions of reward. Section 4 characterizes classes of social welfare functions using different combinations of compensation and reward principles. Section 5 derives optimal distributions from the obtained social welfare functions. Section 6 explains the normative approach to measuring inequality and shows that inequality can be written as the welfare difference between the optimal and the actual distribution. The classes of inequality measures corresponding to the different classes of social welfare functions (embedding different combinations of compensation and reward) are derived and discussed in the light of the literature. Section 7 concludes.

2 Preliminaries

Each individual is characterized by his circumstance and responsibility characteristics. The set of all circumstance characteristics is $C = \{1, 2, \ldots, c\}$ and the set of all responsibility characteristics is $R = \{1, 2, \ldots, r\}$. For simplicity, we assume that each combination (i, k) in $C \times R$ occurs exactly once.[1] We refer to each (i, k) as an individual.

We use a $c \times r$ real-valued matrix X to represent an income distribution. The ikth entry of X, denoted by x_{ik}, is the income of individual (i, k). The

[1] The extension to the general case where some combinations do not occur or occur more than once is possible, but would require considerably heavier notation without adding real substance.

ith row of X is denoted by $x_{i\cdot}$ and the kth column is denoted by $x_{\cdot k}$. We write $1_{c\times r}$ for the $c \times r$ matrix with 1 at each entry and 1_r for the r-dimensional vector with 1 at each entry.

The hypothetical laissez-faire market incomes are ethically significant according to the liberal reward principle. We denote this income distribution by M in $\mathbb{R}^{c\times r}$ and refer to it simply as the market income distribution. We assume that M is fixed.

We use a social welfare function to compare income distributions. A social welfare function $W : \mathbb{R}^{c\times r} \to \mathbb{R}$ assigns to each income distribution X in $\mathbb{R}^{c\times r}$ a real number $W(X)$. The function W depends on M, but we suppress this dependency in the notation.

We impose axioms on the social welfare function to make concrete its normative properties. In the remainder of this section, we formulate three basic axioms. The next section discusses more substantive axioms representing the ideas of compensation and reward.

Monotonicity says that increasing the income of an individual is socially desirable provided that no other individual's income decreases.

Monotonicity. For all income distributions X and X' in $\mathbb{R}^{c\times r}$, if $x_{ik} \geq x'_{ik}$ for each individual (i, k) in $C \times R$ and $x_{jl} > x'_{jl}$ for some individual (j, l) in $C \times R$, then $W(X) > W(X')$.

Continuity ensures that social welfare comparisons are not overly sensitive to small changes in the income distributions.

Continuity. The function W is continuous.

Translation invariance demands that the social welfare ranking of two income distributions does not change if the same amount is added to each income in both income distributions.

Translation invariance. For all income distributions X and X' in $\mathbb{R}^{c\times r}$ and for each real number λ, we have $W(X) \geq W(X')$ if and only if $W(X+\lambda 1_{c\times r}) \geq W(X' + \lambda 1_{c\times r})$.

Translation invariance ensures that the inequality indices we derive later are absolute. That is, adding the same amount to each income does not change the level of inequality of opportunity in an income distribution.[2]

[2] An alternative to this axiom, scale invariance, says that the social welfare ranking of two income distributions should not change if we multiply each income with the same factor in both income distributions. In Section 7, we discuss how the results change if translation invariance is replaced by scale invariance.

3 Compensation and reward

3.1 Compensation axioms

The compensation principle says that income inequalities due to differences in circumstances ought to be redressed. There are two versions of compensation, ex post compensation and ex ante compensation. Imagine that circumstance characteristics are determined prior to responsibility characteristics. Ex ante compensation is defined in terms of the income possibilities of circumstance groups when responsibility characteristics are not yet determined, and ex post compensation is defined in terms of the actual incomes that arise after responsibility characteristics are also determined.

Ex post compensation comprises two components, a Pigou-Dalton transfer principle and a symmetry principle. Together, these components express the idea that individuals who exercise the same responsibility should be treated equally. Ex post Pigou-Dalton requires that an income transfer that widens the income gap between two individuals in the same responsibility group reduces social welfare.

Ex post Pigou-Dalton. For all income distributions X and X' in $\mathbb{R}^{c\times r}$, if there exist two individuals (i,k) and (j,k) in $C \times R$ such that $x_{ik} \geq x_{jk}$ and a positive real number δ such that $x'_{ik} = x_{ik} + \delta$ and $x'_{jk} = x_{jk} - \delta$ with X and X' coinciding everywhere else, then $W(X) > W(X')$.

Ex post symmetry demands that switching the incomes of two individuals in the same responsibility group does not change social welfare.

Ex post symmetry. For all income distributions X and X' in $\mathbb{R}^{c\times r}$, if there exist two individuals (i,k) and (j,k) in $C \times R$ such that $x_{ik} = x'_{jk}$ and $x_{jk} = x'_{ik}$ with X and X' coinciding everywhere else, then $W(X) = W(X')$.

We refer to the combination of ex post Pigou-Dalton and ex post symmetry as ex post compensation.

Ex post compensation. Both ex post Pigou-Dalton and ex post symmetry hold.

Next, we define the ex ante version of compensation. To understand ex ante compensation, interpret row i of an income distribution as the (income) opportunities of an individual with circumstance characteristic i. Ex ante compensation says that differences in circumstances do not justify differences in these opportunities. The axiom consists of, again, a Pigou-Dalton transfer and a symmetry component.

Ex ante Pigou-Dalton requires that increasing the gap between opportunities decreases social welfare. Assume that the minimum income in circumstance group i is greater than the maximum income in circumstance group j. We can then conclude that group i is unambiguously better off than group j. Now, imagine a transfer from each individual in j to each individual in i. Ex ante Pigou-Dalton requires that such a transfer reduces social welfare.

Ex ante Pigou-Dalton. For all income distributions X and X' in $\mathbb{R}^{c \times r}$, if there exist two circumstance groups i and j in C such that $\min_{k \in R} x_{ik} \geq \max_{k \in R} x_{jk}$ and a positive real number δ such that $x'_{i\cdot} = x_{i\cdot} + \delta 1_r$ and $x'_{j\cdot} = x_{j\cdot} - \delta 1_r$ with X and X' coinciding everywhere else, then $W(X) > W(X')$.

Ex ante symmetry requires that switching two rows of an income distribution—one row again unambiguously better than the other as defined above—does not change social welfare.

Ex ante symmetry. For all income distributions X and X' in $\mathbb{R}^{c \times r}$, if there exist two circumstance groups i and j in C such that $\min_{k \in R} x_{ik} \geq \max_{k \in R} x_{jk}$ and $x_{i\cdot} = x'_{j\cdot}$ and $x_{j\cdot} = x'_{i\cdot}$ with X and X' coinciding everywhere else, then $W(X) = W(X')$.

We refer to the combination of ex ante Pigou-Dalton and ex ante symmetry as ex ante compensation.

Ex ante compensation. Both ex ante Pigou-Dalton and ex ante symmetry hold.

Obviously, ex post Pigou-Dalton implies ex ante Pigou-Dalton and ex post symmetry implies ex ante symmetry. By consequence, ex post compensation implies ex ante compensation.[3]

3.2 Reward axioms

The reward principle complements the compensation principle. Whereas compensation aims to neutralize differences in circumstances, reward tells us whether and how to respect differences in responsibility. The literature considers two versions of reward, liberal reward and utilitarian reward.

Liberal reward states that differences in the market incomes of individuals in the same circumstance group should be respected. A useful restatement of this idea is that each individual in the same circumstance class should receive the same subsidy, where a subsidy is defined as the actual income minus the market income.

[3] Fleurbaey and Peragine (2013) consider alternative ex post and ex ante versions of compensation and find that they clash.

Liberal reward consists of two components, a Pigou-Dalton transfer principle and a symmetry principle. Consider two individuals in the same circumstance group i. The subsidies received by (i, k) and (i, l) in income distribution X are $x_{ik} - m_{ik}$ and $x_{il} - m_{il}$. Assume that the subsidy received by (i, k) is greater than the subsidy received by (i, l). Liberal Pigou-Dalton requires that transferring income from (i, l) to (i, k) reduces social welfare, as such an income transfer further widens the gap between the subsidies received by the two individuals.

Liberal Pigou-Dalton. For all income distributions X and X' in $\mathbb{R}^{c \times r}$, if there exist two individuals (i, k) and (i, l) in $C \times R$ such that $x_{ik} - m_{ik} \geq x_{il} - m_{il}$ and a positive real number δ such that $x'_{ik} = x_{ik} + \delta$ and $x'_{il} = x_{il} - \delta$ with X and X' coinciding everywhere else, then $W(X) > W(X')$.

We illustrate the axiom with an example. Imagine a society with one circumstance group and three responsibility groups. Consider the income distributions $X = (9, 9, 15)$ and $X' = (10, 8, 15)$. The market income distribution is $M = (7, 9, 14)$. The distributions of subsidies in X and X' are $X - M = (2, 0, 1)$ and $X' - M = (3, -1, 1)$. The gap between the subsidies received by the first two individuals is smaller in X than in X'. Thus, liberal Pigou-Dalton says that X is better than X'.[4]

Liberal symmetry demands that switching the subsidies of two individuals in the same circumstance group leaves social welfare unchanged. Note that such a switch does not alter the total income of the circumstance group.

Liberal symmetry. For all income distributions X and X' in $\mathbb{R}^{c \times r}$, if there exist two individuals (i, k) and (i, l) in $C \times R$ such that $x_{ik} - m_{ik} = x'_{il} - m_{il}$ and $x_{il} - m_{il} = x'_{ik} - m_{ik}$ with X and X' coinciding everywhere else, then $W(X) = W(X')$.

To illustrate the axiom, let $X'' = (7, 11, 15)$ and $M = (7, 9, 14)$. Liberal symmetry says that X'' and $X = (9, 9, 15)$ are equally good since $X'' - M = (0, 2, 1)$ is obtained from $X - M = (2, 0, 1)$ by switching the subsidies received by the first two individuals.

We refer to the combination of liberal Pigou-Dalton and liberal symmetry as liberal reward.

Liberal reward. Both liberal Pigou-Dalton and liberal symmetry hold.

Liberal reward says that, for an individual with circumstances i, the move from responsibility k to l should ideally be rewarded as it is rewarded by the

[4] Repeated use of liberal Pigou-Dalton yields the optimal income distribution $(8, 10, 15)$. In this distribution, each individual receives a subsidy of 1.

market, that is, by an income change of $m_{il} - m_{ik}$. One can imagine many other possible non-liberal reward principles that specify rewards for the exercise of responsibility that deviate from the market rewards.[5] Such alternative principles can be captured by letting M be, instead of the market income distribution, the income distribution featuring these alternative ideal income differences.

Next, we define utilitarian reward. Utilitarian reward takes the agnostic view that equality of opportunity should be silent on how to reward differences in responsibility. Accordingly, utilitarian reward requires the social welfare function to be neutral with respect to transfers within a circumstance group. There is no need to separately define a Pigou-Dalton transfer and a symmetry component for utilitarian reward since the axiom as stated includes both ideas.

Utilitarian reward. For all income distributions X and X' in $\mathbb{R}^{c\times r}$, if there exist two individuals (i,k) and (i,l) in $C \times R$ and a positive real number δ such that $x'_{ik} = x_{ik} + \delta$ and $x'_{il} = x_{il} - \delta$ with X and X' coinciding everywhere else, then $W(X) = W(X')$.

4 Social welfare functions

4.1 Compensation and liberal reward

We first focus on the combination of ex post compensation and liberal reward. As the following example shows, these two axioms clash. Assume that there are two circumstance groups and two responsibility groups. Consider

$$X = \begin{bmatrix} 3 & 5 \\ 3 & 5 \end{bmatrix}, \quad X' = \begin{bmatrix} 1 & 7 \\ 5 & 3 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 4 & 10 \\ 2 & 0 \end{bmatrix}.$$

Ex post Pigou-Dalton implies $W(X) > W(X')$, whereas liberal Pigou-Dalton implies $W(X') > W(X)$.

It is well known that ex post compensation and liberal reward can be combined only if market income can be written as an additively separable function of circumstance and responsibility characteristics.[6] We say that market incomes are additively separable if $m_{ik} - m_{il} = m_{jk} - m_{jl}$ for all circumstance groups i and j and all responsibility groups k and l.[7]

[5] See Roemer (2012, pp. 178-179) for a discussion.

[6] See, for example, Bossert (1995), Bossert and Fleurbaey (1996) and Fleurbaey (1994, 1995). For a survey, see Fleurbaey (2008).

[7] It is common in the literature to use a function f such that $f(i,k)$ is the market income of (i,k). The function f is additively separable if there exist functions g and h such that, for each (i,k) in $C \times R$, we have $f(i,k) = g(i) + h(k)$. This is equivalent to the condition that, for all i and j in C and k and l in R, we have $f(i,k) - f(i,l) = f(j,k) - f(j,l)$. This clearly corresponds to our definition of additive separability of market incomes.

Proposition 1. *If a social welfare function satisfies ex post Pigou-Dalton and liberal Pigou-Dalton, then market incomes must be additively separable.*

We will in two ways deal with the incompatibility between ex post Pigou-Dalton and liberal Pigou-Dalton. First, we combine the axioms under the restriction of additively separable market incomes (Theorem 1). Second, we consider weakenings of liberal reward and of ex post compensation (Theorems 2 and 3).

Theorem 1 restricts market incomes to be additively separable, and characterizes social welfare functions that satisfy ex post compensation and liberal reward in addition to the three basic axioms monotonicity, continuity and translation invariance. As we will see in Section 5, Theorem 1 extends the so-called natural rule. We denote the set of $c \times r$-dimensional real valued vectors by $\mathbb{R}^{cr}$.

Theorem 1. *Let market incomes in M be additively separable. A social welfare function W satisfies monotonicity, continuity, translation invariance, ex post compensation and liberal reward if and only if there exists a strictly increasing, continuous, translatable*[8] *and strictly Schur-concave*[9] *function $f : \mathbb{R}^{cr} \to \mathbb{R}$ such that, for each X in $\mathbb{R}^{c\times r}$,*

$$W(X) = f(x_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, x_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, x_{cr} - m_{cr} + \bar{m}_{c\cdot}). \quad (1)$$

We explain why the social welfare function in Theorem 1 satisfies ex post compensation and liberal reward. (The intuition for Theorems 2 and 3 is similar.) For any two individuals in the same responsibility group, the same number is subtracted from their incomes. Indeed, additive separability of market incomes implies $m_{ik} - \bar{m}_{i\cdot} = m_{jk} - \bar{m}_{j\cdot}$ for all circumstance groups i and j and each responsibility group k. Strict Schur-concavity of f then ensures that ex post compensation is satisfied. For any two individuals in the same circumstance group, what goes into f is their subsidies plus a uniform constant. Again, strict Schur-concavity of f guarantees that liberal reward is satisfied.

Next, we drop the restriction that market incomes are additively separable, and consider weaker versions of the liberal reward and ex post compensation axioms.

[8] A function $f : \mathbb{R}^n \to \mathbb{R}$ is translatable if $f(x) \geq f(y)$ if and only if $f(x+\delta 1_n) \geq f(y+\delta 1_n)$ for all x and y in $\mathbb{R}^n$ and each real number δ.

[9] A bistochastic matrix is a nonnegative square matrix of which each row sums to 1 and each column sums to 1. A permutation matrix is a bistochastic matrix of which each component is either 0 or 1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is Schur concave if $f(Bx) \geq f(x)$ for each x in $\mathbb{R}^n$ and each $n \times n$ bistochastic matrix B. If, in addition, $f(Bx) > f(x)$ whenever B is not a permutation matrix, then f is strictly Schur-concave. Note that (strict) Schur-concavity of f implies symmetry of f.

We start by weakening liberal reward. The idea is to use for each circumstance group the market incomes of a predetermined circumstance group $\hat{c}$ instead of the group's actual market incomes. Liberal reward is then guaranteed only with respect to the reference circumstance group $\hat{c}$. The weaker version of liberal Pigou-Dalton is as follows.

Liberal Pigou-Dalton for $\hat{c}$. Let $\hat{c}$ be a circumstance characteristic in C. For all income distributions X and X' in $\mathbb{R}^{c \times r}$, if there exist two individuals (i, k) and (i, l) in $C \times R$ such that $x_{ik} - m_{\hat{c}k} \geq x_{il} - m_{\hat{c}l}$ and a positive real number δ such that $x'_{ik} = x_{ik} + \delta$ and $x'_{il} = x_{il} - \delta$ with X and X' coinciding everywhere else, then $W(X) > W(X')$.

The corresponding weaker version of the liberal symmetry axiom is as follows.

Liberal symmetry for $\hat{c}$. Let $\hat{c}$ be a circumstance characteristic in C. For all income distributions X and X' in $\mathbb{R}^{c \times r}$, if there exist two individuals (i, k) and (i, l) in $C \times R$ such that $x_{ik} - m_{\hat{c}k} = x'_{il} - m_{\hat{c}l}$ and $x_{il} - m_{\hat{c}l} = x'_{ik} - m_{\hat{c}k}$ with X and X' coinciding everywhere else, then $W(X) = W(X')$.

Liberal reward for $\hat{c}$ combines liberal Pigou-Dalton for $\hat{c}$ and liberal symmetry for $\hat{c}$. Note that if market incomes are additively separable, then liberal reward for $\hat{c}$ is equivalent to liberal reward.

Liberal reward for $\hat{c}$. Both liberal Pigou-Dalton for $\hat{c}$ and liberal symmetry for $\hat{c}$ hold.

Theorem 2 characterizes the social welfare functions that satisfy ex post compensation and liberal reward for $\hat{c}$ in addition to the three basic axioms. The result follows easily from Theorem 1[10] and we state it without proof. In Section 5 we will see that the theorem extends the egalitarian equivalence allocation rule.

Theorem 2. *A social welfare function W satisfies monotonicity, continuity, translation invariance, ex post compensation and liberal reward for $\hat{c}$ if and only if there exists a strictly increasing, continuous, translatable and strictly Schur-concave function $f : \mathbb{R}^{cr} \to \mathbb{R}$ such that, for each X in $\mathbb{R}^{c \times r}$,*

$$W(X) = f(x_{11} - m_{\hat{c}1}, \ldots, x_{ik} - m_{\hat{c}k}, \ldots, x_{cr} - m_{\hat{c}r}). \tag{2}$$

Next, we weaken ex post compensation. The weakening guarantees ex post Pigou-Dalton only with respect to a chosen responsibility group $\hat{r}$. The weaker version of ex post Pigou-Dalton is as follows.

[10] Replace M by the income distribution of which each row equals $m_{\hat{c}\cdot}$ and apply Theorem 1.

Ex post Pigou-Dalton for $\hat{r}$. Let $\hat{r}$ be a responsibility characteristic in R. For all income distributions X and X' in $\mathbb{R}^{c\times r}$, if there exist two individuals (i,k) and (j,k) in $C \times R$ such that $x_{ik} - m_{ik} + m_{i\hat{r}} \geq x_{jk} - m_{jk} + m_{j\hat{r}}$ and a positive real number δ such that $x'_{ik} = x_{ik} + \delta$ and $x'_{jk} = x_{jk} - \delta$ with X and X' coinciding everywhere else, then $W(X) > W(X')$.

The corresponding weaker version of ex post symmetry is as follows.

Ex post symmetry for $\hat{r}$. Let $\hat{r}$ be a responsibility characteristic in R. For all income distributions X and X' in $\mathbb{R}^{c\times r}$, if there exist two individuals (i,k) and (j,k) in $C \times R$ such that $x_{ik} - m_{ik} + m_{i\hat{r}} = x'_{jk} - m_{jk} + m_{j\hat{r}}$ and $x_{jk} - m_{jk} + m_{j\hat{r}} = x'_{ik} - m_{ik} + m_{i\hat{r}}$ with X and X' coinciding everywhere else, then $W(X) = W(X')$.

Ex post compensation for $\hat{r}$ combines ex post Pigou-Dalton for $\hat{r}$ and ex post symmetry for $\hat{r}$. Note that if market incomes are additively separable, then ex post compensation for $\hat{r}$ is equivalent to ex post compensation.

Ex post compensation for $\hat{r}$. Both ex post Pigou-Dalton for $\hat{r}$ and ex post symmetry for $\hat{r}$ hold.

Theorem 3 characterizes the social welfare functions that satisfy ex post compensation for $\hat{r}$ and liberal reward in addition to the three basic axioms. The proof is similar to that of Theorem 1 and is therefore omitted. Section 5 will show that the theorem extends the conditional equality allocation rule.

Theorem 3. *A social welfare function W satisfies monotonicity, continuity, translation invariance, ex post compensation for $\hat{r}$ and liberal reward if and only if there exists a strictly increasing, continuous, translatable and strictly Schur-concave function $f : \mathbb{R}^{cr} \to \mathbb{R}$ such that, for each X in $\mathbb{R}^{c\times r}$,*

$$W(X) = f(x_{11} - m_{11} + m_{1\hat{r}}, \ldots, x_{ik} - m_{ik} + m_{i\hat{r}}, \ldots, x_{cr} - m_{cr} + m_{c\hat{r}}). \quad (3)$$

We now move on to the combination of ex ante compensation and liberal reward. As the following example shows, ex ante symmetry and liberal Pigou-Dalton clash. Consider the income distributions

$$X = \begin{bmatrix} 10 & 14 \\ 1 & 3 \end{bmatrix}, \; X' = \begin{bmatrix} 1 & 3 \\ 10 & 14 \end{bmatrix}, \; X'' = \begin{bmatrix} 0 & 4 \\ 11 & 13 \end{bmatrix} \text{ and } \; X''' = \begin{bmatrix} 11 & 13 \\ 0 & 4 \end{bmatrix}$$

and the market income distribution

$$M = \begin{bmatrix} 10 & 14 \\ 1 & 3 \end{bmatrix}.$$

We have $W(X) = W(X')$ by ex ante symmetry, $W(X') < W(X'')$ by liberal Pigou-Dalton and $W(X'') = W(X''')$ by ex ante symmetry. Hence, $W(X) < W(X''')$. However, we have $W(X''') < W(X)$ by liberal Pigou-Dalton.

We again obtain that a necessary condition to avoid a clash is that market incomes are additively separable.

Proposition 2. *If a social welfare function W satisfies ex ante symmetry and liberal Pigou-Dalton, then market incomes must be additively separable.*

For the case of ex ante compensation, we will not explore domain restrictions and axiom weakenings. We suffice instead by remarking that all the social welfare functions in Theorems 1 and 2 satisfy ex ante compensation, as the latter is implied by ex post compensation.

4.2 Compensation and utilitarian reward

We begin with the combination of ex post compensation and utilitarian reward. The following example shows that the two axioms clash.[11] Consider the income distributions

$$X = \begin{bmatrix} 8 & 7 \\ 6 & 9 \end{bmatrix} \quad \text{and} \quad X' = \begin{bmatrix} 7 & 8 \\ 7 & 8 \end{bmatrix}.$$

Ex post Pigou-Dalton implies $W(X') > W(X)$, whereas utilitarian reward implies $W(X) = W(X')$.

We consider a weakening of utilitarian reward and combine it with ex post compensation.[12] Uniform utilitarian reward says that transferring the same amount δ from each individual in a responsibility group to each individual in another responsibility group should not alter social welfare.

Uniform utilitarian reward. For all income distributions X and X' in $\mathbb{R}^{c \times r}$, if there exist two responsibility groups k and l in R and a positive real number δ such that $x'_{\cdot k} = x_{\cdot k} + \delta 1_c$ and $x'_{\cdot l} = x_{\cdot l} - \delta 1_c$ with X and X' coinciding everywhere else, then $W(X) = W(X')$.

We impose an additional axiom that puts structure on social welfare comparisons. The axiom requires that the social welfare function first aggregates across responsibility groups and second aggregates the obtained values across circumstance groups. Because this order of aggregation requires knowledge of individuals' responsibility characteristics, we refer to the axiom as ex post aggregation.

[11] Fleurbaey and Peragine (2013) find a similar incompatibility.

[12] Since utilitarian reward does not take market incomes into account, a restriction on the domain of market income distributions is not an option in this case.

Ex post aggregation. There exist a function $\phi : \mathbb{R}^r \to \mathbb{R}$ and functions $\gamma_1, \ldots, \gamma_r : \mathbb{R}^c \to \mathbb{R}$ such that, for each income distribution X in $\mathbb{R}^{c\times r}$, we have $W(X) = \phi(\gamma_1(x_{\cdot 1}), \ldots, \gamma_r(x_{\cdot r}))$.

Theorem 4 characterizes social welfare functions that satisfy ex post compensation and uniform utilitarian reward in addition to ex post aggregation and the three basic axioms monotonicity, continuity and translation invariance.

Theorem 4. *A social welfare function W satisfies monotonicity, continuity, translation invariance, ex-post aggregation, ex post compensation and uniform utilitarian reward if and only if there exist a strictly increasing and continuous function $F : \mathbb{R} \to \mathbb{R}$ and a strictly increasing, continuous, unit-translatable*[13] *and strictly Schur-concave function $f : \mathbb{R}^c \to \mathbb{R}$ such that, for each X in $\mathbb{R}^{c\times r}$,*

$$W(X) = F\Big(\frac{1}{r}\sum_{k=1}^{r} f(x_{\cdot k})\Big). \tag{4}$$

The social welfare functions in the theorem first aggregate the incomes in each responsibility group using the function f. Strict Schur-concavity of f ensures that ex post compensation is satisfied. The obtained values are then aggregated by averaging, which ensures satisfaction of uniform utilitarian reward.

We now turn to the combination of ex ante compensation and utilitarian reward. Theorem 5 characterizes social welfare functions that satisfy ex ante compensation and utilitarian reward in addition to the three basic axioms.

Theorem 5. *A social welfare function W satisfies monotonicity, continuity, translation invariance, ex ante compensation and utilitarian reward if and only if there exists a strictly increasing, continuous, translatable and strictly Schur-concave function $f : \mathbb{R}^c \to \mathbb{R}$ such that, for each X in $\mathbb{R}^{c\times r}$,*

$$W(X) = f\Big(\frac{1}{r}\sum_{k=1}^{r} x_{1k}, \frac{1}{r}\sum_{k=1}^{r} x_{2k}, \ldots, \frac{1}{r}\sum_{k=1}^{r} x_{ck}\Big). \tag{5}$$

The social welfare functions in the theorem first aggregate the incomes of each circumstance group by averaging, thus ensuring satisfaction of utilitarian reward. The obtained values are then aggregated using the strictly Schur-concave function f, which ensures satisfaction of ex ante compensation. Ooghe, Schokkaert and Van de gaer (2007) have characterized classes of social welfare functions similar to those in Theorems 4 and 5.

[13] A function $f : \mathbb{R}^n \to \mathbb{R}$ is unit-translatable if $f(x + \delta 1_n) = f(x) + \delta$ for each real number δ.

5 Allocation rules

An allocation rule determines how a given amount of income should be divided among the individuals based on their circumstance and responsibility characteristics. The optimal distribution(s) chosen by each of the above social welfare functions naturally defines an allocation rule. In section 6, we use these optima to derive our inequality measures.

Proposition 3 shows the optimal distributions for the classes of social welfare functions characterized in the previous section. In each case, the whole class settles on the same distributions. The proof of the proposition is straightforward and is therefore omitted.

Proposition 3. *Let W be a social welfare function that satisfies monotonicity, continuity and translation invariance.*

(i) Let market incomes in M be additively separable and let W satisfy, in addition, ex post compensation and liberal reward (Theorem 1). For each income distribution X in $\mathbb{R}^{c\times r}$, the unique optimal distribution of the total income in X, denoted by X^*, is such that

$$x^*_{ik} = m_{ik} - \bar{m}_{i\cdot} + \bar{X} \qquad \textit{for each } (i,k) \textit{ in } C \times R.$$

(ii) Let W satisfy, in addition, ex post compensation and liberal reward for $\hat{c}$ (Theorem 2). For each income distribution X in $\mathbb{R}^{c\times r}$, the unique optimal distribution of the total income in X, denoted by X^*, is such that

$$x^*_{ik} = m_{\hat{c}k} - \bar{m}_{\hat{c}\cdot} + \bar{X} \qquad \textit{for each } (i,k) \textit{ in } C \times R.$$

(iii) Let W satisfy, in addition, ex post compensation for $\hat{r}$ and liberal reward (Theorem 3). For each income distribution X in $\mathbb{R}^{c\times r}$, the unique optimal distribution of the total income in X, denoted by X^*, is such that

$$x^*_{ik} = m_{ik} - m_{i\hat{r}} - (\bar{M} - \bar{m}_{\cdot\hat{r}}) + \bar{X} \qquad \textit{for each } (i,k) \textit{ in } C \times R.$$

(iv) Let W satisfy, in addition, ex post compensation and uniform utilitarian reward (Theorem 4). For each income distribution X in $\mathbb{R}^{c\times r}$, a distribution X^* is an optimal distribution of the total income in X if and only if

$$x^*_{ik} = x^*_{jk} \qquad \textit{for all } i \textit{ and } j \textit{ in } C \textit{ and each } k \textit{ in } R.$$

(v) Let W satisfy, in addition, ex ante compensation and utilitarian reward (Theorem 5). For each income distribution X in $\mathbb{R}^{c\times r}$, a distribution X^* is an optimal distribution of the total income in X if and only if

$$\bar{x}^*_{i\cdot} = \bar{x}^*_{j\cdot} \qquad \textit{for all } i \textit{ and } j \textit{ in } C.$$

Proposition 3 links our social welfare classes to established rules. If $\bar{X} = \bar{M}$, then the allocation rules derived from Theorems 1, 2 and 3 coincide with the "natural" rule, egalitarian-equivalent and conditional equality rules studied by Bossert (1995) and Bossert and Fleurbaey (1996). The natural rule applies only if market incomes in M are additively separable. Additive separability implies that there exists a real number a_i for each circumstance group i in C and a real number b_k for each responsibility group k in R such that $m_{ik} = a_i + b_k$. The natural rule assigns to each individual (i,k) the income $b_k + \bar{a}$, i.e., the part of his market income determined by responsibility plus the average of the part determined by circumstances. The rule in Proposition 3(i) indeed coincides with the natural rule since $m_{ik} = a_i + b_k$, $\bar{m}_i = a_i + \bar{b}$ and $\bar{X} = \bar{M} = \bar{a} + \bar{b}$. The egalitarian-equivalent rule assigns to each individual (i,k) the income $m_{\hat{c}k} - \bar{m}_{\hat{c}\cdot} + \bar{M}$, i.e., the market income she would have received if her circumstance were $\hat{c}$ plus a uniform amount. The conditional equality rule assigns to each individual (i,k) the income $m_{ik} - m_{i\hat{r}} + \bar{m}_{\cdot\hat{r}}$, i.e., the average market income of the responsibility group $\hat{r}$ plus the amount by which the individual's market income deviates from the market income he would have had were k equal to $\hat{r}$. These two rules are obtained from Proposition 3(ii) and (iii) by setting $\bar{X} = \bar{M}$. If $\bar{X} \neq \bar{M}$, then every individual receives what they would have received under the three established rules plus the difference $(\bar{X} - \bar{M})$.

The allocation rules derived from Theorems 4 and 5 coincide with the rules by Roemer (1993) and Van de gaer (1993). Indeed, the allocations in Proposition 3(iv) maximize Roemer's mean of mins, $\frac{1}{r}\sum_{k\in R}\min_{i\in C} x_{ik}$, and those in (v) maximize Van de gaer's min of means, $\min_{i\in C}\frac{1}{r}\sum_{k\in R} x_{ik}$.[14]

6 Inequality measures

The normative approach to inequality measurement identifies inequality with the welfare loss incurred by having the actual rather than the optimal distribution of the available income. First, we review the procedure proposed by Kolm (1969), Atkinson (1970) and Sen (1973) to derive inequality measures in the unidimensional setting. Next, we extend this procedure to our setting. As we will see later, the unidimensional Kolm-Atkinson-Sen (KAS) inequality measure constitutes the basic building block of our measures of inequality of opportunity.

Consider the unidimensional setting in which all individuals are identical. Let x in $\mathbb{R}^n$ be an income distribution for n individuals, and let $w : \mathbb{R}^n \to \mathbb{R}$ be

[14]The mean of mins and the min of means naturally define social welfare functions. These social welfare functions are not members of the classes in Theorems 4 and 5, but can be approached arbitrarily closely by choosing f sufficiently concave.

a strictly increasing, continuous, translatable and strictly Schur-concave social welfare function. The equally distributed equivalent income $\xi(x)$ associated with w is the income that, if received by each individual, would yield the same welfare level as x. Formally, $\xi(x)$ is the real number such that $w(\xi(x)1_n) = w(x)$. The KAS inequality measure $J : \mathbb{R} \to \mathbb{R}$ associated with w is such that, for each x in $\mathbb{R}^n$, we have

$$J(x) = \bar{x} - \xi(x). \tag{6}$$

The KAS measure has an intuitive interpretation. For each x in $\mathbb{R}^n$, $J(x)$ is the per capita income that could be destroyed if incomes are equalized while maintaining the same level of welfare. In other words, it is a measure of waste due to inequality.

Next, we extend the KAS procedure to our setting. The difference with the unidimensional setting is that the equal distribution is not necessarily optimal. Let X in $\mathbb{R}^{c\times r}$ be an income distribution, and let $W : \mathbb{R}^{c\times r} \to \mathbb{R}$ be a strictly increasing and continuous social welfare function. The optimally distributed equivalent average income $\Xi(X)$ is the average income that, if distributed optimally among the individuals, would yield the same welfare level as X. Formally, $\Xi(X) = \bar{Y}$ with Y an optimal distribution such that $W(Y) = W(X)$. The inequality of opportunity measure I associated with W is such that, for each X in $\mathbb{R}^{c\times r}$,

$$I(X) = \bar{X} - \Xi(X).$$

For each X in $\mathbb{R}^{c\times r}$, $I(X)$ is the per capita income that could be destroyed if income is optimally distributed while maintaining the same level of welfare.

Proposition 4 presents the inequality of opportunity measures corresponding to the social welfare functions described in the five theorems in Section 4.

Proposition 4. *Let W be a social welfare function that satisfies monotonicity, continuity and translation invariance.*

(i) Let market incomes in M be additively separable, and let W satisfy, in addition, ex post compensation and liberal reward (Theorem 1). For each income distribution X in $\mathbb{R}^{c\times r}$, we have

$$I(X) = J(x_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, x_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, x_{cr} - m_{cr} + \bar{m}_{c\cdot}),$$

where $J : \mathbb{R}^{cr} \to \mathbb{R}$ is the KAS inequality measure associated with f in equation (1).

(ii) Let W satisfy, in addition, ex post compensation and liberal reward for $\hat{c}$ (Theorem 2). For each income distribution X in $\mathbb{R}^{c\times r}$, we have

$$I(X) = J(x_{11} - m_{\hat{c}1}, \ldots, x_{ik} - m_{\hat{c}k}, \ldots, x_{cr} - m_{\hat{c}r}),$$

where $J : \mathbb{R}^{cr} \to \mathbb{R}$ *is the KAS inequality measure associated with* f *in equation* (2).

(iii) Let W *satisfy, in addition, ex post compensation for* $\hat{r}$ *and liberal reward (Theorem 3). For each income distribution* X *in* $\mathbb{R}^{c\times r}$*, we have*

$$I(X) = J(x_{11} - m_{11} + m_{1\hat{r}}, \ldots, x_{ik} - m_{ik} + m_{i\hat{r}}, \ldots, x_{cr} - m_{cr} + m_{c\hat{r}}),$$

where $J : \mathbb{R}^{cr} \to \mathbb{R}$ *is the KAS inequality measure associated with* f *in equation* (3).

(iv) Let W *satisfy, in addition, ex post compensation and uniform utilitarian reward (Theorem 4). For each income distribution* X *in* $\mathbb{R}^{c\times r}$*, we have*

$$I(X) = \frac{1}{r}\sum_{k\in R} J(x_{\cdot k}),$$

where $J : \mathbb{R}^{c} \to \mathbb{R}$ *is the KAS inequality measure associated with* f *in equation* (4).

(v) Let W *satisfy, in addition, ex ante compensation and utilitarian reward (Theorem 5). For each income distribution* X *in* $\mathbb{R}^{c\times r}$*, we have*

$$I(X) = J\Big(\frac{1}{r}\sum_{k=1}^{r} x_{1k}, \frac{1}{r}\sum_{k=1}^{r} x_{2k}, \ldots, \frac{1}{r}\sum_{k=1}^{r} x_{ck}\Big),$$

where $J : \mathbb{R}^{c} \to \mathbb{R}$ *is the KAS inequality measure associated with* f *in equation* (5).

Proposition 4 reveals that inequality of opportunity measurement reduces to the application of a unidimensional inequality measure to an appropriately adjusted income distribution. Our approach singles out the absolute KAS inequality measure as the unidimensional inequality measure to be employed. This use of unidimensional inequality measures as a basic building block is ubiquitous in the equality of opportunity literature, not surprisingly without the restriction to absolute KAS inequality measures.[15] We now discuss the five parts of Proposition 4 in connection with the previous literature.

The measures in Proposition 4(i)-(iii) apply a unidimensional inequality measure to a distribution of corrected incomes where the correction term is determined by the market income distribution. The measures in Proposition 4(ii) and (iii) correspond, respectively, to the fairness gap and direct unfairness measures proposed by Fleurbaey and Schokkaert (2009).

[15] In section 7, we discuss variations of our approach that would warrant the use of a wider class of unidimensional inequality measures.

Alternatively, the measures in Proposition 4(i)-(iii) can be written as a measure of distance between the vector of actual incomes and the vector of optimal incomes. Indeed, they are equivalent to the application of J to the vector $(x_{11} - x^*_{11}, \ldots, x_{cr} - x^*_{cr})$, where x^*_{ik} is the optimal income as given in Proposition 3(i)-(iii).[16] The measures proposed by Devooght (2008) and Almås, Cappelen, Lind, Sørensen and Tungodden (2011) use this idea of distance between the actual and the optimal. Devooght (2008) uses the optima corresponding to the natural, the egalitarian-equivalent and conditional equality rules as in Proposition 4(i)-(iii), but uses Cowell's (1985) measure of distributional change as a measure of distance. Almås et al. (2011) uses the optimum corresponding to the so-called generalized proportionality rule (Cappelen and Tungodden, 2016) and adopt the relative Gini index as a measure of distance. The advantage of our approach is that both the optimum and the distance measure follow from the axioms imposed on the social welfare function.

The measure in Proposition 4(iv) measures inequality of opportunity by the sum of the inequality levels of the responsibility groups. Aaberge, Mogstad and Peragine (2011) propose a measure in this form with J a rank-dependent inequality measure. The measure in Proposition 4(iv) can be interpreted as measuring the inequality within responsibility groups while disregarding the (unproblematic) inequality between responsibility groups. This interpretation has been exploited by Checchi and Peragine (2010), who propose, among others, the within responsibility group component of the mean logarithmic deviation as a measure of inequality of opportunity. Because responsibility characteristics are difficult to observe, this approach is not common in the empirical literature.

The measure in Proposition 4 part (iii) is a measure of inequality between average incomes of circumstance groups. Here, one can consider the average income of a circumstance group to represent the value of the group's opportunity set. This approach has been extensively used in empirical analysis. The specific inequality index applied changes across studies. We do not attempt to review all inequality indices employed within this approach but, our measure is in line with the measures employed by, among others, Peragine (2002), Cogneau and Mesplé-Somps (2008), Bourguignon, Ferreira and Menendez (2007), Pistolesi (2009), Checchi and Peragine (2010), Ferreira and Gignoux (2011), Aaberge et al. (2011), Hassine (2011), Singh (2012).

[16] Consider, for example, the measure in Proposition 4(iii). Because J is absolute, we have $J(x_{11} - m_{11} + m_{1\hat{r}}, \ldots, x_{cr} - m_{cr} + m_{c\hat{r}}) = J(x_{11} - x^*_{11}, \ldots, x_{cr} - x^*_{cr})$, where $x^*_{ik} = m_{ik} - m_{i\hat{r}} - (\bar{M} - \bar{m}_{\cdot\hat{r}}) + \bar{X}$.

7 Conclusion

We conclude the paper with two remarks.

First, the inequality measures in this paper are absolute due to translation invariance. Alternatively, one could impose scale invariance on the social welfare function. The scale invariance axiom would read "For all income distributions X and X' in $\mathbb{R}^{c\times r}$ and for each real number λ, we have $W(X) \geq W(X')$ if and only if $W(\lambda X) \geq W(\lambda X')$." However, scale invariance clashes with liberal reward. To see that, imagine a society with one circumstance group and two responsibility groups. Let $M = (0, 4)$, $X = (2, 6)$ and $X' = (3, 5)$. Liberal Pigou-Dalton implies $W(X) > W(X')$. Next, let $Y = (8, 24)$ and $Y' = (12, 20)$. Liberal Pigou-Dalton implies $W(4X') > W(4X)$ whereas scale invariance implies $W(4X') > W(4X)$. Let us examine a proportional version of liberal reward that is compatible with scale invariance. The only change to the assumptions in Section 2 is that incomes and market incomes can only take positive values. That is, $W : \mathbb{R}^{c\times r}_{++} \rightarrow \mathbb{R}$ and M is in $\mathbb{R}^{c\times r}_{++}$, where $\mathbb{R}^{c\times r}_{++}$ is the set of all positive $c \times r$ matrices. Proportional liberal Pigou-Dalton says that, for all income distributions X and X' in $\mathbb{R}^{c\times r}_{++}$, if there exist two individuals (i, k) and (i, l) in $C \times R$ such that $\frac{x_{ik}}{m_{ik}} \geq \frac{x_{il}}{m_{il}}$ and a positive real number δ such that $x'_{ik} = x_{ik} + \delta$ and $x'_{il} = x_{il} - \delta$ with X and X' coinciding everywhere else, then $W(X) > W(X')$. Proportional liberal symmetry says that for all income distributions X and X' in $\mathbb{R}^{c\times r}_{++}$, if there exist two individuals (i, k) and (i, l) in $C \times R$ such that $\frac{x_{ik}}{m_{ik}} = \frac{x'_{il}}{m_{il}}$ and $\frac{x_{il}}{m_{il}} = \frac{x'_{ik}}{m_{ik}}$ with X and X' coinciding everywhere else, then $W(X) = W(X')$. Finally, proportional liberal holds if both proportional liberal Pigou-Dalton and proportional liberal symmetry hold. In this case, we find that ex post compensation and proportional liberal reward clash unless M is such that $\frac{m_{ik}}{m_{il}} = \frac{m_{jk}}{m_{jl}}$ for all circumstance groups i and j and all responsibility groups k and l. Let $\mu = (\mu_1, \mu_2, \dots, \mu_r)$ be such that $\frac{\mu_k}{\mu_l} = \frac{m_{ik}}{m_{il}}$ for all k and l in R. Replacing additive separability of M with this alternative assumption, translation invariance with scale invariance and liberal reward with proportional liberal reward in Theorem 1 yields the following class of social welfare functions: for each X in $\mathbb{R}^{c\times r}_{++}$,

$$W(X) = \phi\big(f((\frac{x_{ik}}{\mu_k})_{i\in C, k\in R})\big),$$

where $\mu = (\mu_1, \mu_2, \dots, \mu_r)$, $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly increasing, continuous function and $f : \mathbb{R}^{cr} \rightarrow \mathbb{R}$ is a strictly increasing, continuous, homogeneous of degree one,[17] strictly Schur-concave function. Proportional version of liberal reward for $\hat{c}$ can be defined similarly and relative versions of Theorems 2 and 3 can be obtained in a similar way.

[17] A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is homogeneous of degree one if $f(\delta x) = \delta f(x)$ for each real number δ.

Second, the market income distribution M is fixed throughout this paper. Alternatively, one might want to compare income distributions with different market income distributions, such as (X, M) and (X', M'). Such a change in the domain of the social welfare function would require a change in the monotonicity axiom. The new axiom would read "For all social states (X, M) and (X', M') in $\mathbb{R}^{c\times r} \times \mathbb{R}^{c\times r}$, if $x_{ik} \geq x'_{ik}$ for each individual (i, k) in $C \times R$ and $x_{jl} > x'_{jl}$ for some individual (j, l) in $C \times R$, then $W(X, M) > W(X', M')$." Liberal reward would not need to be changed. Surprisingly, in an accompanying paper (Bosmans and Ozturk, 2015), we show that we cannot combine monotonicity with liberal reward. A natural extension of our paper is therefore to characterize the domain—if any—on which monotonicity is compatible with liberal reward.

Appendix

Proof of Proposition 1. Let W be a social welfare function that satisfies ex post Pigou-Dalton and liberal Pigou-Dalton.

Assume to the contrary that market incomes are not additively separable. That is, there exist i and j in C and k and l in R such that $m_{ik}-m_{il} \neq m_{jk}-m_{jl}$. Let X be an income distribution such that $x_{ik} = x_{jk} = (m_{ik} + m_{jk})/2$ and $x_{il} = x_{jl} = (m_{il}+m_{jl})/2$. Let X' be an income distribution such that $x'_{ik}+x'_{il} = x_{ik} + x_{il}$, $x'_{ik} - x'_{il} = m_{ik} - m_{il}$, $x'_{jk} + x'_{jl} = x_{jk} + x_{jl}$ and $x'_{jk} - x'_{jl} = m_{jk} - m_{jl}$ with X' and X coinciding everywhere else. Ex post Pigou-Dalton implies $W(X) > W(X')$, whereas liberal Pigou-Dalton implies $W(X') > W(X)$. We have a contradiction. □

The following two lemmas are used throughout the proofs. A progressive transfer is a transfer of income from a richer to a poorer individual such that the one that starts out with less money does not end up with more than the other. We say that a function is Pigou-Dalton consistent if its value increases as a result of a progressive transfer. See Olkin and Marshall (1979, pp. 10-12) for the first lemma and Dasgupta, Sen and Starrett (1973, p. 183) for the second lemma.

Lemma 1. *For all vectors a and b in $\mathbb{R}^n$, a is obtained from b by a finite sequence of progressive transfers and permutations if and only if $a = bB$ for some $n \times n$ bistochastic matrix B.*

Lemma 2. *Each symmetric and Pigou-Dalton consistent function $f : \mathbb{R}^n \to \mathbb{R}$ is Schur-concave.*

Proof of Theorem 1. It is easy to verify that the specified social welfare function satisfies the axioms in the case of additively separable market incomes. We focus on the reverse implication.

Let W be a social welfare function that satisfies the axioms.

By monotonicity and continuity, there exists a strictly increasing and continuous function $\hat{f} : \mathbb{R}^{cr} \to \mathbb{R}$ such that, for each X in $R^{c\times r}$, we have $W(X) = \hat{f}((x_{ik})_{(i,k)\in C\times R})$. Translation invariance implies that, for all x and x' in $\mathbb{R}^{cr}$ and each real number λ, we have $\hat{f}(x) \geq \hat{f}(x')$ if and only if $\hat{f}(x + \lambda 1_{cr}) \geq \hat{f}(x'+\lambda 1_{cr})$, i.e., $\hat{f}$ is a translatable function. Let f be the function $f : \mathbb{R}^{cr} \to \mathbb{R}$ such that, for each vector $(x_{ik})_{(i,k)\in C\times R}$, we have $f((x_{ik}-m_{ik}+\bar{m}_{i\cdot})_{(i,k)\in C\times R}) = \hat{f}((x_{ik})_{(i,k)\in C\times R})$. It follows that, for each X in $\mathbb{R}^{c\times r}$, we have $W(X) = f(x_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, x_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, x_{cr} - m_{cr} + \bar{m}_{c\cdot})$. The function f is strictly increasing, continuous and translatable since $\hat{f}$ is strictly increasing, continuous and translatable.

Next, we show that f is symmetric. Let X and X' be income distributions such that the vector $(x'_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, x'_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, x'_{cr} - m_{cr} + \bar{m}_{c\cdot})$ is obtained from the vector $(x_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, x_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, x_{cr} - m_{cr} + \bar{m}_{c\cdot})$ by a switch of two components. First, assume the switch is between the components corresponding to individuals (i, k) and (j, k). Note that $m_{ik} - \bar{m}_{i\cdot} = m_{jk} - \bar{m}_{j\cdot}$ by additive separability of M. Because the same value $(m_{ik} - \bar{m}_{i\cdot} = m_{jk} - \bar{m}_{j\cdot})$ is subtracted from the incomes x_{ik} and x_{jk}, the switch is equivalent to a switch of these incomes. Hence, $W(X) = W(X')$ by ex post symmetry. Second, assume the switch is between the components corresponding to individuals (i, k) and (i, l). This switch is equivalent to a switch of the subsidies $x_{ik} - m_{ik}$ and $x_{il} - m_{il}$. Hence, we have $W(X) = W(X')$ by liberal symmetry. Third, assume the switch is between the components corresponding to individuals (i, k) and (j, l). Let Y be the income distribution such that $(y_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, y_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, y_{cr} - m_{cr} + \bar{m}_{c\cdot})$ is obtained from the vector $(x_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, x_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, x_{cr} - m_{cr} + \bar{m}_{c\cdot})$ by a switch of the components corresponding to individuals (i, l) and (j, l). Using the same reasoning as above, by ex post symmetry, we have $W(X) = W(Y)$. Let Y' be the income distribution such that $(y'_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, y'_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, y'_{cr} - m_{cr} + \bar{m}_{c\cdot})$ is obtained from the vector $(y_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, y_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, y_{cr} - m_{cr} + \bar{m}_{c\cdot})$ by a switch of the components corresponding to individuals (i, k) and (i, l). Using the same reasoning as above, by liberal symmetry, we have $W(Y) = W(Y')$. The vector $(x'_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, x'_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, x'_{cr} - m_{cr} + \bar{m}_{c\cdot})$ is obtained from the vector $(y'_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, y'_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, y'_{cr} - m_{cr} + \bar{m}_{c\cdot})$ by a switch of the components corresponding to individuals (i, l) and (j, l). Using the same reasoning as above, by ex post symmetry, we have $W(X') = W(Y')$. Thus, we obtain $W(X) = W(X')$.

Finally, we show that f is strictly Schur-concave. Since f is symmetric, it suffices to show that f is Pigou-Dalton consistent. Let X and X' be income distributions such that the vector $(x'_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, x'_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, x'_{cr} - m_{cr} + \bar{m}_{c\cdot})$ is obtained from the vector $(x_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, x_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, x_{cr} - m_{cr} + \bar{m}_{c\cdot})$ by a single progressive transfer. First, assume the transfer is from the component corresponding to individual (i, k) to the component corresponding to individual (j, k). Because the same value is subtracted from the incomes x_{ik} and x_{jk}, this transfer is equivalent to a progressive transfer of income between (i, k) and (j, k). Hence, $W(X) > W(X')$ by ex post Pigou-Dalton. Second, assume the transfer is from the component corresponding to individual (i, k) to the component corresponding to individual (i, l). This transfer is equivalent to a progressive transfer from the subsidy $x_{ik} - m_{ik}$ to the subsidy $x_{il} - m_{il}$. Hence, we have $W(X) > W(X')$ by liberal Pigou-Dalton. Third, assume the transfer is from the component corresponding to individual (i, k) to the component corresponding to (j, l). Let Y be the income distribution such that $(y_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, y_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, y_{cr} - m_{cr} + \bar{m}_{c\cdot})$ is obtained from the vector $(x_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, x_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, x_{cr} - m_{cr} + \bar{m}_{c\cdot})$ by a switch of the components corresponding to individuals (i, l) and (j, l). Using the same reasoning as above, by ex post symmetry, we have $W(X) = W(Y)$. Let Y' be the income distribution such that $(y'_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, y'_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, y'_{cr} - m_{cr} + \bar{m}_{c\cdot})$ is obtained from the vector $(y_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, y_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, y_{cr} - m_{cr} + \bar{m}_{c\cdot})$ by a transfer from the component corresponding to (i, k) to the component corresponding to (i, l). Using the same reasoning as above, by liberal Pigou-Dalton, we have $W(Y') > W(Y)$. The vector $(x'_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, x'_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, x'_{cr} - m_{cr} + \bar{m}_{c\cdot})$ is obtained from the vector $(y'_{11} - m_{11} + \bar{m}_{1\cdot}, \ldots, y'_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, y'_{cr} - m_{cr} + \bar{m}_{c\cdot})$ by a switch of the components corresponding to individuals (i, l) and (j, l). Using the same reasoning as above, by ex post symmetry, we have $W(X') = W(Y')$. Thus, we obtain $W(X') > W(X)$. □

Proof of Proposition 2. Let W be a social welfare function that satisfies liberal Pigou-Dalton and ex ante symmetry.

Assume to the contrary that market incomes are not additively separable. That is, there exist i and j in C and k and l in R such that $m_{ik} - m_{il} \neq m_{jk} - m_{jl}$. Let X be an income distribution such that there exist positive real numbers α and β such that $x_{i\cdot} = m_{i\cdot} + \alpha 1_r$ and $x_{j\cdot} = m_{j\cdot} + \beta 1_r$ with $\min_{k \in R} x_{ik} > \max_{k \in R} x_{jk}$. Let X' be the income distribution obtained from X by switching the ith and the jth rows of X. Let X'' be the income distribution obtained from X' by an income transfer of an amount ϵ between the individuals in columns k and l of row i that corresponds to a progressive transfer in their subsidies. Moreover, let $\epsilon < \min_{k \in R} x_{ik} - \max_{k \in R} x_{jk}$, which implies that $\min_{k \in R} x''_{jk} > \max_{k \in R} x''_{ik}$. Let X''' be the income distribution obtained from X'' by switching the ith and the jth rows of X''.

We have $W(X) = W(X')$ by ex ante symmetry, $W(X') < W(X'')$ by liberal Pigou-Dalton and $W(X'') = W(X''')$ by ex ante symmetry. Hence, $W(X) < W(X''')$. However, we have $W(X) > W(X''')$ by liberal Pigou-Dalton. We have a contradiction. □

Proof of Theorem 4. It is easy to verify that the specified social welfare function satisfies the axioms. We focus on the reverse implication.

Let W be a social welfare function that satisfies the axioms. By monotonicity, continuity and ex post aggregation, there exist a strictly increasing and continuous function $h : \mathbb{R}^r \to \mathbb{R}$ and strictly increasing and continuous functions $f_1, f_2, \ldots, f_r : \mathbb{R}^c \to \mathbb{R}$ such that, for each X in $\mathbb{R}^{c \times r}$, we have

$W(X) = h(g_1(x_{\cdot 1}), g_2(x_{\cdot 2}), \ldots, g_r(x_{\cdot r}))$. Using the symmetry imposed by uniform utilitarian reward, we can define strictly increasing and continuous functions $\phi : \mathbb{R}^r \to \mathbb{R}$ and $\hat{f} : \mathbb{R}^c \to \mathbb{R}$ such that, for each X in $\mathbb{R}^{c\times r}$, we have $W(X) = \hat{\phi}(\hat{f}(x_{\cdot 1}), \hat{f}(x_{\cdot 2}), \ldots, \hat{f}(x_{\cdot r}))$.

Translation invariance implies that, for all x and x' in $\mathbb{R}^c$ and each real number λ, we have $\hat{f}(x) \geq \hat{f}(x')$ if and only if $\hat{f}(x + \lambda 1_c) \geq \hat{f}(x' + \lambda 1_c)$, i.e., $\hat{f}$ is a translatable function. Hence, there exist a strictly increasing and continuous function $\psi : \mathbb{R} \to \mathbb{R}$ and a unit-translatable function $f : \mathbb{R}^c \to \mathbb{R}$ such that $\hat{f} = \psi \circ f$. Define the strictly increasing and continuous function $\phi : \mathbb{R}^c \to \mathbb{R}$ such that, for each $(t_1, t_2, \ldots, t_r)$ in $\mathbb{R}^r$, we have $\phi(t_1, t_2, \ldots, t_r) = \hat{\phi}(\psi(t_1), \psi(t_2), \ldots, \psi(t_r))$. It follows that, for each X in $\mathbb{R}^{c\times r}$, we have $W(X) = \phi(f(x_{\cdot 1}), f(x_{\cdot 2}), \ldots, f(x_{\cdot r}))$. The function f is strictly increasing and continuous because h and $\hat{f}$ are strictly increasing and continuous. Moreover, f is symmetric by ex post symmetry and Pigou-Dalton consistent by ex post Pigou-dalton, and hence strictly Schur-concave.

Next we show that W is a strictly increasing function of $\frac{1}{r}\sum_{k\in R} f(x_{\cdot k})$. Let X and X' in $\mathbb{R}^{c\times r}$ be such that $\frac{1}{r}\sum_{k\in R} f(x_{\cdot k}) \geq \frac{1}{r}\sum_{k\in R} f(x'_{\cdot k})$. Let Y and Y' in $\mathbb{R}^{c\times r}$ be income distributions such that $y_{ik} = f(x_{\cdot k})$ and $y'_{ik} = f(x'_{\cdot k})$ for each (i,k) in $C \times R$. We have $W(Y) = \phi(f(f(x_{\cdot 1})), f(f(x_{\cdot 2})), \ldots, f(f(x_{\cdot r})))$. Since f is unit-translatable, there exists a real number α such that $W(Y) = \phi(f(x_{\cdot 1})+\alpha, f(x_{\cdot 2})+\alpha, \ldots, f(x_{\cdot r})+\alpha)$. Similarly, there exists a real number α' such that $W(Y') = \phi(f(x'_{\cdot 1})+\alpha', f(x'_{\cdot 2})+\alpha', \ldots, f(x'_{\cdot r})+\alpha')$. Thus, by translation invariance, we have $W(X) \geq W(X')$ if and only if $W(Y) \geq W(Y')$. Next, let Z and Z' be income distributions such that $z_{ik} = \frac{1}{r}\sum_{k\in R} f(x_{\cdot k})$ and $z'_{ik} = \frac{1}{r}\sum_{k\in R} f(x'_{\cdot k})$ for each (i,k) in $C \times R$. By monotonicity, $W(Z) \geq W(Z')$ if and only if $\frac{1}{r}\sum_{k\in R} f(x_{\cdot k}) \geq \frac{1}{r}\sum_{k\in R} f(x'_{\cdot k})$ with equality holding whenever $\frac{1}{r}\sum_{k\in R} f(x_{\cdot k}) = \frac{1}{r}\sum_{k\in R} f(x'_{\cdot k})$. By uniform utilitarian reward, $W(Y) = W(Z)$ and $W(Y') = W(Z')$. Hence, $W(Y) \geq W(Y')$ if and only if $\frac{1}{r}\sum_{k\in R} f(x_{\cdot k}) \geq \frac{1}{r}\sum_{k\in R} f(x'_{\cdot k})$. We have already established that $W(X) \geq W(X')$ if and only if $W(Y) \geq W(Y')$. That is, $W(X) \geq W(X')$ if and only if $\frac{1}{r}\sum_{k\in R} f(x_{\cdot k}) \geq \frac{1}{r}\sum_{k\in R} f(x'_{\cdot k})$ with equality holding whenever $\frac{1}{r}\sum_{k\in R} f(x_{\cdot k}) = \frac{1}{r}\sum_{k\in R} f(x'_{\cdot k})$. It follows that there exists a strictly increasing and continuous function $F : \mathbb{R} \to \mathbb{R}$ such that $W(X) = F(\frac{1}{r}\sum_{k\in R} f(x_{\cdot k}))$. □

Proof of Theorem 5. It is easy to verify that the specified social welfare function satisfies the axioms. We focus on the reverse implication.

Let W be a social welfare function that satisfies the axioms. First, we show that, for each X and X' in $\mathbb{R}^{c\times r}$, if $\sum_{k\in R} x_{ik} = \sum_{k\in R} x'_{ik}$ for each i in C, then we have $W(X) = W(X')$. Let Y be the income distribution obtained from X such that for each individual (i,k), we have $y_{ik} = \bar{x}_{i\cdot}$ and let Y' be the income distribution obtained from X' such that for each individual

(i,k), we have $y'_{ik} = \bar{x}'_{i\cdot}$. By utilitarian reward, we have $W(X) = W(Y)$ and $W(X') = W(Y')$. By construction, $Y = Y'$ and hence $W(X) = W(X')$.

Furthermore, if $\sum_{k\in R} x_{ik} \geq \sum_{k\in R} x'_{ik}$ for each i in C with at least one inequality holding strictly, then we have $W(X) > W(X')$. This follows using monotonicity and the reasoning above. It follows that there exists a strictly increasing function $f: R^c \to \mathbb{R}$ such that, for each X in $\mathbb{R}^{c\times R}$, we have $W(X) = f(\bar{x}_{1\cdot}, \bar{x}_{2\cdot}, \ldots, \bar{x}_{c\cdot})$. The function f is continuous by continuity, symmetric by ex ante symmetry and translatable by translation invariance.

Next, we show that f is strictly Schur-concave. Let X and X' in $\mathbb{R}^{c\times r}$ be such that the vector $(\bar{x}_{1\cdot}, \bar{x}_{2\cdot}, \ldots, \bar{x}_{c\cdot})$ is obtained from the vector $(\bar{x}'_{1\cdot}, \bar{x}'_{2\cdot}, \ldots, \bar{x}'_{c\cdot})$ by a progressive transfer. Let Y be an income distribution such that $y_{i\cdot} = \bar{x}_{i\cdot}1_r$ for each i in C, and let Y' be an income distribution such that $y'_{i\cdot} = \bar{x}'_{i\cdot}1_r$ for each i in C. Utilitarian reward implies that $W(X) = W(Y)$ and $W(X') = W(Y')$. Ex ante Pigou-Dalton implies that $W(Y) > W(Y')$. Hence, we have $W(X) > W(X')$. That is, f is a Pigou-Dalton consistent function. Since it is also symmetric, by lemma 2, f is Schur-concave. □

Proof of Proposition 4. Let W be a social welfare function that satisfies monotonicity, continuity and translation invariance.

(i) Let market incomes in M be additively separable and let W satisfy, in addition, ex post compensation and liberal reward (Theorem 1). First, to find $\Xi(X)$, we look for the optimal income distribution Y such that $W(Y) = W(X)$. Since Y is optimal, we have $y_{ik} = m_{ik} - \bar{m}_{i\cdot} + \bar{Y}$ for each (i,k) in $C \times R$ by Proposition 3(i). By Theorem 1, $W(Y) = f(\bar{Y}, \ldots, \bar{Y})$ and $W(X) = f(x_{11} - m_{11} + \bar{m}_{1\cdot}, \ \ldots, \ x_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, \ x_{cr} - m_{cr} + \bar{m}_{c\cdot})$. Since $W(Y) = W(X)$, $f(\bar{Y}, \ldots, \bar{Y}) = f(x_{11} - m_{11} + \bar{m}_{1\cdot}, \ \ldots, \ x_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, \ x_{cr} - m_{cr} + \bar{m}_{c\cdot})$. Hence, $\bar{Y} = \xi(x_{11} - m_{11} + \bar{m}_{1\cdot}, \ \ldots, \ x_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, \ x_{cr} - m_{cr} + \bar{m}_{c\cdot})$, where ξ is the equally distributed equivalent income associated with f. Since $\bar{Y} = \Xi(X)$, we obtain $I(X) = \bar{X} - \xi(x_{11} - m_{11} + \bar{m}_{1\cdot}, \ \ldots, \ x_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, \ x_{cr} - m_{cr} + \bar{m}_{c\cdot})$, i.e., $I(X) = J(x_{11} - m_{11} + \bar{m}_{1\cdot}, \ \ldots, \ x_{ik} - m_{ik} + \bar{m}_{i\cdot}, \ldots, \ x_{cr} - m_{cr} + \bar{m}_{c\cdot})$, where J is the KAS inequality measure associated with f in equation (1). The proofs of (ii) and (iii) are similar and are therefore omitted.

(iv) Let W satisfy, in addition, ex post compensation and uniform utilitarian reward (Theorem 4). Let X be an income distribution in $\mathbb{R}^{c\times r}$. Using part

(iv) of Proposition 3, we find

$$\begin{aligned} I(X) &= \dot{W}(X^*) - \dot{W}(X) \\ &= 1/r \sum_{k \in R} g(\bar{x}_{\cdot k} 1_c) - 1/r \sum_{k \in R} g(x_{\cdot k}) \\ &= 1/r \sum_{k \in R} (\bar{x}_{\cdot k} - g(x_{\cdot k})) \\ &= 1/r \sum_{k \in R} g(\bar{x}_{\cdot k} 1_c - x_{\cdot k}). \end{aligned}$$

since g is unit-translatable.

(v) Let W satisfy, in addition, ex ante compensation and utilitarian reward (Theorem 5). Let X be an income distribution in $\mathbb{R}^{c \times r}$. Again, we look for an optimal distribution Y such that $W(Y) = W(X)$. Since Y is optimal, we have $\bar{y}_{i\cdot} = \bar{y}_{j\cdot}$ for all circumstance groups i and j in C by Proposition 3(v). By Theorem 5, $W(Y) = f(\bar{Y}, \ldots, \bar{Y})$ and $W(X) = f(\bar{x}_{1\cdot}, \bar{x}_{2\cdot}, \ldots, \bar{x}_{c\cdot})$. Since $W(Y) = W(X)$, we have $f(\bar{Y}, \ldots, \bar{Y}) = f(\bar{x}_{1\cdot}, \bar{x}_{2\cdot}, \ldots, \bar{x}_{c\cdot})$. Hence, $\bar{Y} = \xi(\bar{x}_{1\cdot}, \bar{x}_{2\cdot}, \ldots, \bar{x}_{c\cdot})$, where ξ is the equally distributed equivalent income associated with f. We obtain that $I(X) = \bar{X} - \xi(\bar{x}_{1\cdot}, \bar{x}_{2\cdot}, \ldots, \bar{x}_{c\cdot})$, i.e., $I(X) = J(\bar{x}_{1\cdot}, \bar{x}_{2\cdot}, \ldots, \bar{x}_{c\cdot})$, where J is the KAS inequality measure associated with f in equation (5). □

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