Optimal Income Taxation for the Alleviation of Working Poverty When Domestic Work is Rewarded [*]

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Abstract

The increase in income needed for working households to escape relative poverty may be achieved either by their supplying more hours of paid work on the labor market, or by policy makers adjusting income taxation, minimum wages, and social transfers targeted at such households. While the literature has paid considerable attention to the other two policy instruments, theoretical work on how income taxation could minimize working poverty is scarce. Our study aims to fill this gap. Unlike the traditional optimal income taxation literature, which considers that households allocate time only between consumption and leisure, we explicitly model the decision of households as including domestic work, which is a social contribution and should be rewarded. This new framework highlights (i) the importance of emphasizing the difference between domestic work and real leisure and, (ii) the policy implications of non-market time allocation.

Keywords: Working poverty, time allocation, optimal income taxation, household-public goods.

JEL classification: H21, I32, H24

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1 Introduction

In 2003, the European Union adopted the alleviation of in-work poverty as a policy objective. According to the national in-work poverty risk indicators published by the European Commission, progress in achieving this objective remains very limited[1]. These indicators identify the working poor as those individuals who have been employed or self-employed for at least 7 months during the reference year and whose household equivalent disposable income does not exceed 60 percent of the national median disposable household income. The attempt to understand how public policies (and specifically tax policy) may reduce in-work poverty is hindered by the lack of theoretical economic analysis on the topic. Only few studies have been devoted to tax policy aimed at reducing general income poverty (Kanbur et al. 1994; Wane 2001; Pritilla and Tuomala 2004; and Boone and Bovenberg 2004, among others).

This is striking because an estimated 9.3 percent of the members of the European Union workforce, or more than 15 million people, were affected by in-work poverty in 2012. In other words, one-third of adults (aged 18-64 years) who are at risk of poverty are employed (*Employment and Social Developments in Europe 2013*). Another alarming finding is that households with children always fare worse than their childless counterparts with comparable employment status. For example, in the U.K., the *Institute for Fiscal Studies annual poverty and inequality 2015 report* states that nearly two-thirds of British children in poverty live in working families (Belfield et al. 2015). Thus, working poverty cannot be dismissed as a negligible phenomenon that concerns a marginal population.

While public policies to reduce working poverty are necessarily a part of the policies to reduce general income poverty, working poverty cannot be assumed to mirror general income poverty. Generally, the households of employed persons are less likely to be poor; thus, securing employment is suggested as a key pathway for escaping poverty. However, whether obtaining employment is sufficient for lifting a household out of poverty depends on both the level of the wage earned net of income tax (and plus social transfers) and the household composition. The *OECD policy brief September 2009* states that *"working full-time does not always provide a solid pathway out of poverty"* and *"employment is not a panacea: on average across OECD countries, 7% of individuals*

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[1]See the data from Eurostat at http://ec.europa.eu/eurostat/data/database.

living in households with at least one worker are poor, and more than 10% of the working population in Japan, Mexico, Poland, Portugal, Turkey and the United States are poor." Even in Ireland, which has relatively low levels of in-work poverty, "*somewhat less than a half of those at risk of poverty in 2008 lived in households where at least one person was in employment*" (Daly, 2010, Page 4). A worker living alone may be protected against the risk of poverty even when he or she has a low net wage, whereas in a household with a large number of dependent persons, even a substantial net wage could be inadequate. This can be seen from the example given by Neumark (2015) that, in the U.S., " *A single working adult with two children earning the federal minimum wage of $7.25 and working full time earns about $14,500 a year, well below the U.S. poverty threshold of $19,073 for a family of this size* ". Even in the richest OECD country, Luxembourg, the same situation occurs. In 2014, the Luxembourg minimum wage of full-time work was 1874.19 euros per month, and the poverty threshold for a single adult household was 1716 euros and that of a two-adult household was 2427 euros[2]. Thus, households without dependents did not face the problem of poverty. However, the Luxembourg poverty threshold for families with two adults and two children was 3837 euros per month. Even if both adults were working full time, the family was under the risk of being in poverty, as were families with one adult and two children. These examples illustrate that the limited overlap between working poverty and general poverty stems from the use of two levels of analysis in working poverty: work is the attribute of an individual, whereas poverty is a household characteristic. Thus, theoretical results regarding reducing general income poverty cannot be readily applied to lifting households out of working poverty.

At face value, it may seem that increasing the labor supply (such as obtaining full-time jobs instead of part-time jobs) of in-work poor households could reduce or eliminate working poverty. This view has been disputed by the existing empirical work[3]. There are two flaws in this argument that are at the core of our study. First, households with dependents (especially children and/or aging parents) need to allocate at least some time to domestic work and, thus, cannot entirely devote their time to market

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[2]See EU-SILC survey, STATEC, Luxembourg.

[3]For example, as early as 2003, Gerfin and Leu (2003) conclude that "*... adding a minimum hours requirement to the current social assistance system is the most cost-efficient reform*" in the study of working poverty of Switzerland. See also the alternatives proposed by Neumark (2015) for the U.S.A. *The Economist* June 25th 2016 issue states the working poverty situation in the UK and present some possible reasoning.

work. Second, even when individuals work full time, low-earning households might still be poor, as is demonstrated by the examples above.

Drawing on these considerations and inspired by Becker's (1965) classical study "*A theory of allocation of time*", its further development Kleven (2004) and the *American Time Use Survey* and *EU Time Use Survey*,[4] we derive the optimal taxation of labor income assuming that working families must satisfy constraints when allocating their time to market work, domestic work, and leisure. Piachaud (1987) already address the time allocation problem in the definition of poverty and states that "*the treatment of time and home production: the time and ability of individuals to prepare food or to wash and feed without assistance, for example, vary greatly depending on circumstances and in turn affect income needs*". However, after 30 years, there is still no clear answer to this question. This paper is trying to provide some insight information by taking into account the time allocation and constraint, poverty and market work together. We study a poverty-averse society and assume that social welfare in this society is the aggregate of the welfare of its individual members. The individual's contribution to social welfare consists not only of market work, but also domestic work.[5]

Our study makes two main contributions to the literature: (i) households' decisions include domestic work, which represents a social contribution and should be rewarded; (ii) the household is considered as one social taxation unit, which differs from the classical taxation of couples.

The above-mentioned surveys clearly document that domestic work, family care, and informal help provided to other households, differ from leisure, which typically includes activities such as sports, outdoor activities, games, reading, and sleeping. Taking into consideration the concerns of a household, we explicitly model domestic work as a part of a households decision by assuming that domestic work occupies part of the households time, which is not assumed in most of the literature on optimal income taxation [6]. As Boadway and Tremblay (2013, page 103) mention, "*the standard model*

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[4]See, http : //www.bls.gov/tus/ and https : //www.h5.scb.se/tus/tus/

[5]Obviously, the time that parents spend monitoring their young children's completion of homework, engagement in sports, and so on, instead of drinking, smoking, or getting into trouble on the street has its own social value, although engagement in these activities reduces the amount of time parents spend on market work.

[6]See Boadway (2012) for the recent and complete survey.

assumes time is devoted either to work or to pure leisure. In fact, non-work time takes a variety of forms". One exception to the work-leisure pattern in the optimal income taxation literature is Beaudry et al. (2009), who assume that workers can divide their time between market and non-market activities with different productivities. However, they do not consider leisure, which is traditionally examined in the literature.

Following the literature, such as Atkinson and Stiglitz (2015) and Kleven et al (2000), we label domestic work as *household production* and label the corresponding outcome as *household-public goods*, which includes taking care of aged parents, nursing young children, cleaning, and cooking, and can be completed only through time contributed by family members. Notably, Beaudry et al. (2009) illustrate some situations in which social programs use information on time worked; therefore, it is relevant to consider time as an observable variable. This is especially true for the case of working poverty.

The second contribution is related to the taxation of households. In particular, we do not distinguish between the first and second earners in one family as is the case in the standard taxation of couples literature (Boskin and Sheshinski, 1983; Apps and Rees, 2009, 2011; Kleven et al. 2009). More generally, in their survey, Boadway and Tremblay (2013, Page 113) discuss four particular problems when considering the taxation of couples. Their second problem states "*some family goods and services are provided through household production. In principle, this should be treated like market income for tax purpose, but this is obviously difficult because of the absence of market prices*". We take this kind of household production and its social value into consideration.

From a technical point of view, we employ a modified version of The Mirrlees (1971) optimal taxation model, which considers poverty as a public bad that has a negative externality to the aggregate utility (Wane, 2001). In this study, we propose an extension of this model by explicitly taking into account individuals' time allocation among market work, household production, and leisure. Compared with the standard Mirrlees (1971) model, our model includes two additional features: poverty alleviation and domestic work. In our model, individuals differ not only in their innate ability to earn an income from the labor market, but also in their social statuses (being poor or rich) and in their valuation of non-market time. The additional dimensions of individual heterogeneity affect the social weights in the government's objective function. The analytic and numerical results of this study show that the optimal tax rates of different types of individuals differ dramatically. For instance, the lowest-skilled working

population benefits from a significant negative marginal tax rate. In this respect, our study is closely related to a growing body of literature that focuses on the implications of multiple heterogeneity and negative marginal taxation (Diamond, 1980; Laroque, 2005; Choné and Laroque 2010; Lockwood and Weinzierl, 2015, among others).

The remainder of this paper is organized as follows. We present the theoretical model in Sections 2 and 3 by modifying the Wane (2001) model in the following ways: an individuals utility includes real leisure and two goods, consumption goods and household-public goods, where consumption goods come from labor income and household-public goods come from household production. To obtain explicit results and useful insights for policy recommendations, we devote Section 4 to numerical exercises and the presentation of simulation results. Concluding remarks are presented in Section 5.

2 Household production and incentive concerns

We consider a nonlinear optimal taxation model *à la* Mirrlees (1971), which reflects an imperfect information game between taxpayers and the social planner. In this model, the tax units are households, which differ in their innate market ability, n. We assume that this economy consists of a continuum of the working population over the support $n \in [\underline{n}, \overline{n}]$ with distribution function $F(n)$ and density function $f(n)$. A household with skills n supplies $y(n)$ units of time in the labor market and, hence, gains $z(n) = ny(n)$ units of final output, which we use as the numeraire. The planner can observe household incomes, but cannot directly observe their ability. Given this unobserved heterogeneity, the policy maker maximizes a social welfare function by proposing an income tax scheme $T(z)$ subject to the budget constraint, and guarantees sufficient incentive for households to maintain their productivity.

In the classical income taxation theory, household utility depends on: (i) consumption, which is equal to the after-tax income, $c(n) = z(n) - T(z(n))$, and (ii) leisure, which is $1 - y$. Our main departure from the classical literature is that we introduce a new dimension of productive activity, namely household production or domestic work. Thus, in this economy, there are two goods: the consumption normal good, which can be purchased in the market, and the household good, which can be obtained only via domestic labor. The household production technology is assumed to be homogeneous

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and independent of market productivity n. The only input of this production is time, denoted by x (unlike Becker's (1965) model, our model does not consider domestic capital). Domestically produced goods are directly consumed within the household, cannot be resold in the market, and, therefore, are not subject to any market taxes. Assuming unitary marginal productivity, the household produced good equals x.

In the standard Mirrlees framework, there is no explicit distinction between pure leisure time and the time spent on household production. We make this distinction for two reasons. First, there are more responsibilities and duties associated with domestic work as compared to pure leisure. For instance, domestic work includes educating young children and taking care of aged parents, from which some households cannot be exempt. In contrast, no obvious constraints prevent individuals from reducing their leisure time. Second, domestic goods and pure leisure may have different social values, and more importantly, have different substitutability with consumption goods. Domestic goods usually can be substituted by consumption, which is paid by the market labor income, such as placing aged parents in retirement houses, sending young children to kindergarten or to nannies, employing individuals to clean the house, and dining out. The direct utility obtained from real leisure is different in this respect; for example, enjoying a beer or a nap cannot be replaced by someone else's beer or nap. Hence, by treating domestic work and real leisure differently in our taxation model, we can design the tax policy such that labor income tax can be used to help the working households that are really in need of social help.

2.1 The household's incentive concerns

The utility of an $n-$household is given by $u(c, x, y)$, which is at least twice continuously differentiable, increasing in consumption and household goods, $u_x \geqslant 0, \quad u_c > 0$, and decreasing in working time, $u_y < 0$. We also assume that the utility function is concave in the $x-$space and strictly concave in the (c, y)-space. Furthermore, we assume that both income and consumption increase with skill: $z'(n) > 0$ and $c'(n) > 0$. [7] The household's problem is choosing market labor supply y, consumption of market goods

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[7] As mentioned by Diamond (1980, page 104), *"in general there is no reason for consumption to necessarily increase with income"*. However, this assumption helps us focus on the main issues of poverty alleviation and domestic production.

c, and contribution to domestic labor x to obtain the maximum utility level:

$$\max_{c,\,x,\,y} u(c, x, y) = \max_{x,\,z} U\left(z - T(z), x, z, n\right),$$

subject to the time constraint:

$$x + y = x + \frac{z}{n} \leq 1. \tag{1}$$

Note the difference between x and $1 - x - y$, the time spent on household production and leisure, respectively, are not the same under the current setting, which is the key difference between this study and the previous literature at this stage.

Similar to Kanbur et al. (1994) and Wane (2001), we define the marginal rate of substitution between gross income and consumption goods as

$$s(c, x, z, n) = -\frac{U_z(c, x, z, n)}{U_c(c, x, z, n)} > 0.$$

Furthermore, we impose Seade's (1982) Agent Monotonicity Condition on the derivative of s with respect to n.[8]

$$s_n = \frac{\partial s(c, x, z, n)}{\partial n} < 0, \forall(c, x, y).$$

The marginal rate of substitution between household-public goods and consumption goods is defined as:

$$\varepsilon(c, x, z, n) = \frac{U_x(c, x, z, n)}{U_c(c, x, z, n)} > 0.$$

A similar condition to $s_n(n) < 0$ should be imposed on $\varepsilon_n(c, x, z, n)$. Following the arguments of Beaudry et al. (2009, Page 220), *"one may be very productive in the formal/market sector but have low productivity in the informal sector, or vice versa,"* we make the following assumption:

$$\varepsilon_n = \frac{\partial \varepsilon(c, x, z, n)}{\partial n} = 0, \forall(c, x, y).$$

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[8]As clearly stated by Kanbur et al. (1994), *"this is Assumption B of Mirrlees (1971)...It implies that indifference curves in the consumption-gross-income space become flatter as the individual's wage rate increases, which in turn ensures that both consumption and gross earnings increase with the wage rate."*

That is, the marginal rate of substitution between household goods and consumption goods is productivity independent. In other words, households with higher market productivity, n, do not produce more (or less) domestically than less productive households.[9]

Given the utility maximization problem with the inequality time constraint, the standard Kuhn-Tucker condition yields (see, Theorem 7.16 of Sundaram, 2009):

$$\begin{cases} U_x = \lambda(n); \\ U_c \left(1 - T'(z)\right) + U_z = \frac{\lambda(n)}{n}; \\ \lambda(n) \geq 0, \ 1 - x - \frac{z}{n} \geq 0, \ \lambda \cdot \left(1 - x - \frac{z}{n}\right) = 0, \end{cases} \tag{2}$$

where λ is the Kuhn-Tucker multiplier that measures the optimal individual utility gain due to one extra unit of time. We distinguish between the effective constraint case, $x + \frac{z}{n} = 1$ with $\lambda > 0$, and the ineffective constraint case, $x + \frac{z}{n} < 1$ with $\lambda = 0$.[10]

In the case of fully effective constraint, time is fully spent on working and domestic production, and there is no real leisure time. Thus, if there is extra time for household production, it will increase the marginal utility of the household. This implies that the marginal utility of domestic activities is strictly positive $U_x = \lambda(n) > 0$, which in turn gives $U_c \left(1 - T'(z)\right) + U_z = \frac{\lambda(n)}{n} = \frac{U_x}{n}$. It is also clear that the marginal tax rate of household n is as follows:

$$t(z(n)) = T'(z(n)) = 1 + \frac{U_z}{U_c} - \frac{U_x}{U_c n} = 1 - s(n) - \frac{\varepsilon(n)}{n}. \tag{3}$$

Furthermore, Seade's (1982) Monotonicity condition and the positive substitution between c and x implies the following:

$$\frac{\partial t(z(n))}{\partial n} = -s_n + \frac{\varepsilon(n)}{n^2} > 0,$$

which confirms the classical intuition that the marginal tax rate increases as skill increases. Nonetheless, in this model, the marginal tax rate also depends on the marginal

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[9]Alternatively, following the law of diminishing marginal rate of substitution, we may also impose that $\varepsilon_n \leq 0$, $\forall (c, x, y)$, which states that more productive individuals would like to give up more consumption goods to have more time taking care of their family than less productive individuals would, but it may be too costly for them to do so.

[10]An inequality constraint is *effective* at some point if the constraint holds with equality at this special point.

rate of substitution between household goods and consumption. Finally, combining the Kuhn-Tucker conditions, we can rewrite the incentive concern in a more compact manner (proof is given in Appendix B.1. step 1):

$$\frac{du(n)}{dn} = U_x \frac{z}{n^2} + U_n. \tag{4}$$

If households engage in pure leisure activity, we have an ineffective constraint, that is, $x + \frac{z}{n} < 1$. In this case, both x and z are independent strategic variables. Thus, the *constraint qualification* in the inequality constraint of the Kuhn-Tucker condition fails. Then, we obtain $U_x = \lambda(n) = 0$ and $U_c (1 - T'(z)) + U_z = 0$, which implies that no shadow value is added to the utility from extra time allocated to domestic work. In this case, the marginal tax rate has the standard form as in Wane (2001):

$$t(z(n)) = 1 + \frac{U_z}{U_c} = 1 - s(n), \tag{5}$$

with $\frac{\partial t(z(n))}{\partial n} = -s_n > 0$. Finally, the household's concern is as follows:

$$\frac{du(n)}{dn} = U_n. \tag{6}$$

2.2   The household's time allocation

In reality, some households can afford to engage in leisure activities, while others cannot. In other words, a fraction of the population's utility concern checks (4), while the other fraction checks (6).

Who should belong to which group? To answer this question, we rely on the Time Use Survey data, which clearly show that individuals differ in terms of how they allocate their time among market work, household product, and pure leisure. Before proceeding to link the mathematical analysis in the last subsection and the time allocation, we must first state that following the American Time Use Surveys in recent decades, generally, low-skilled labor enjoy more leisure than the high-skilled ones, especially passive leisure, such as watching TV (Aguiar and Hurst, 2007 and 2009; The Economist, April 19th 2014). The EU Time Use Surveys present some similar patterns.

Nevertheless, from the more recent American Time Use Survey in 2012 and 2014, the following trends emerge. First, employed adults living in households with no children under 18 years of age engaged in leisure activities for 4.7 hours per day, which

approximately an hour more than employed adults living with a child under 6 years of age (See Table 8 of the 2012 survey). Second, the survey shows that less-qualified individuals have higher opportunity costs for participating in the labor market, because they lose the possibility of participating in household production. For example, on the days they worked, 38% of employed people aged 25 years and above with a bachelor's degree or higher could work at home, compared with only 5% of those with less than a high school degree (see Table 6 in the 2012 survey report). This result shows that the high-skilled population has fewer time constraints. Third, the survey also shows that the choice of allocating time to leisure or to household production depends on an individuals market productivity. For example, the data suggest that high-income households or better-educated households worked less on Saturdays, Sundays, and holidays and, hence had more opportunities to enjoy pure leisure time (see Appendix A, some data from the 2012 survey). The rationale behind this observation is that since some household goods can be substituted with market goods, individuals with higher abilities (that are therefore richer) can reduce their household production and have more time for leisure (sometimes creative intellectual work itself is leisure).

In advanced economies, there are laws that clearly define the maximum length of the workweek. Thus, in the following, *our notation of time constraint is precisely defined as the inconvenient time for the household product*. For example, during school holidays, or on the weekend, or after school, working parents have to work, instead of staying home to take care of their children.

Combining this definition with time use survey data and the analysis in the previous section, we obtain the first result:

Proposition 1 *Assume there is a threshold value of consumption $\widetilde{c}$. Then,*

- *if $c(n) < \widetilde{c}$, the marginal utility and optimal time allocation of household-n satisfies the first order conditions (4);*

- *if $c(n) \geqslant \widetilde{c}$, the marginal utility and optimal time allocation of household-n satisfies the first order conditions (6).*

Proposition 1 implies that there is an ability threshold $\widetilde{n}$ that corresponds to the

skill level of a household that has a consumption level of $\widetilde{c}$, that is, $c(\widetilde{n}) = \widetilde{c}$.[11] In other words, the $\widetilde{n}-$individual has the lowest skill level and consumption among the group of individuals who can afford time for real leisure. Technically, Proposition 1 introduces an additional dimension of individual heterogeneity into the utility function. Depending on their ability, some members of the working population have a utility function that satisfies $U_x \neq 0$, while others have $U_x = 0$. In the previous literature on multiple heterogeneity (Chon and Laroque 2010 and Lockwood and Weinzierl, 2015), the additional individual heterogeneity is typically modeled as a continuous and exogenous variable, and is therefore independent of ability. In contrast, we consider the additional heterogeneity to be *endogenous* (in the sense that it depends on ability) and *discrete* (in the sense that it divides the population into two types).

Now, we can study how the two types of working populations that have different utility concerns (4) and (6) will react to the government's income tax policy. The advantage of introducing this new dimension of non-market time allocation is that it leaves space for policy interpretation. If we consider taking care of aged parents and nursing young children to be a social contribution (that can therefore be paid activities), then a tax transfer or tax reduction can be helpful for the low-income working population. In the following section, we formally describe the government's optimization problem that maximizes social welfare and minimizes aggregate poverty.

3   Poverty alleviation and optimal income taxation

Following the arguments of Wane (2001), income poverty is considered as a public bad and government's objective is to alleviate this poverty via taxation. Here, poverty is measured in terms of disposable income as $P(c, c^*)$ with $\underline{c}$ the minimum income and $c^* \in (0, \infty)$ the income poverty threshold. When disposal income is below the poverty line, $\forall c \in [\underline{c}, c^*[$, this poverty function satisfies: $P(c, c^*) \geq 0$ and $P_c(c, c^*) < 0$. When disposable income is above the poverty line, $\forall c \geq c^*$, $P(c, c^*) = 0$, $P(c^*, c^*) = P_c(c^*, c^*) = 0$. The aggregate poverty measure of the economy is:

$$\mathcal{P}(c^*) = \int_{\underline{n}}^{\overline{n}} P(c(n), c^*) f(n) dn.$$

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[11]The threshold $\widetilde{n}$ corresponds to the cut-off individual in Beaudry et al. (2009).

The government's objective is choosing the optimal income tax rate to

$$\max_{t(z(n))} \int_{\underline{n}}^{\overline{n}} [u(n) - \beta P(c(n), c^*)] f(n) dn, \tag{7}$$

subject to the government tax revenue budget constraint:

$$\int_{\underline{n}}^{\overline{n}} [z(n) - c(n)] f(n) dn = \overline{R} \tag{8}$$

and the household's utility concern, that is, (4) or (6). Constant $\overline{R}$ is the desired level of tax revenue, which can serve as a public good or source of redistribution. The constant parameter β measures the social aversion to poverty. Obviously, if $\beta = 0$, the government is not concerned about poverty. Thus, in what follows, we assume $\beta > 0$ and that a large poor population would lead to a larger social welfare loss. Denote the multiplier of the budget constraint γ and the co-state variable of the individuals utility concern as $\mu(n)$. In other words, γ measures the value added to the social welfare due to an extra unit of tax revenue and $\mu(n)$ represents the increase in social welfare if the marginal type of an individual's utility increases by one unit. Therefore, these two variables are the key variables in the "equity-efficiency" study of poverty alleviation.

3.1 Optimal marginal taxation

In this subsection, we first present the government's optimization problem under the incentive concern (4) or under (6). This problem is a standard dynamic control system, which can be solved using the Pontryagin maximum principle (see Appendix B.1).

Given the maximization problem (7) with constraints (4) and (8), the first order condition with respect to z yields:

$$\left[\gamma \left(1 - s(n) - \frac{\varepsilon(n)}{n} \right) - \beta P_c \left(s(n) + \frac{\varepsilon(n)}{n} \right) \right] f(n) = -\mu(n) \left[U_{nz} + \frac{U_x}{n^2} \right]. \tag{9}$$

The left hand side is the net social loss—increase in budget net of poverty—due to allocating more income to an individual n within the population $n < \widetilde{n}$. The right hand side is net social gain induced by an increase in the income of individual n, which is measured by the gain in the individual's marginal utility from consumption net of

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sacrificing in domestic-work, and multiplied by its social value $\mu(n)$. This co-state variable checks:

$$\mu'(n) + \mu(n) \left[\frac{U_{nz} + U_x/n^2}{U_z - U_x/n} \right] + \left[1 + \frac{\gamma}{U_z - U_x/n} \right] f(n) = 0, \tag{10}$$

with the transversality condition:

$$\mu(\underline{n}) = \mu(\overline{n}) = 0.$$

The social value of n-household's utility in terms of income is:

$$\mu(n) = \int_n^{\tilde{n}} \left(1 + \frac{\gamma}{U_z - U_x/p} \right) \exp \left(\int_n^p \frac{U_{nz} + U_x/m^2}{U_z - U_x/m} dm \right) f(p) dp \tag{11}$$
$$+ \int_{\tilde{n}}^{\overline{n}} \left(1 + \frac{\gamma}{U_z} \right) \exp \left(\int_{\tilde{n}}^p \frac{U_{nz}}{U_z} dm \right) f(p) dp.$$

From the transversality condition $\mu(\underline{n}) = 0$, (11) yields the shadow value of the government budget:

$$\gamma = - \frac{ \int_{\underline{n}}^{\tilde{n}} \exp \left(\int_{\underline{n}}^p \frac{U_{nz} + U_x/m^2}{U_z - U_x/m} dm \right) f(p) dp + \int_{\tilde{n}}^{\overline{n}} \exp \left(\int_{\tilde{n}}^p \frac{U_{nz}}{U_z} dm \right) f(p) dp }{ \int_{\underline{n}}^{\tilde{n}} \frac{1}{U_z - U_x/p} \exp \left(\int_{\underline{n}}^p \frac{U_{nz} + U_x/m^2}{U_z - U_x/m} dm \right) f(p) dp + \int_{\tilde{n}}^{\overline{n}} \frac{1}{U_z} \exp \left(\int_{\tilde{n}}^p \frac{U_{nz}}{U_z} dm \right) f(p) dp }, \tag{12}$$

which is, naturally, always positive given $U_z < 0$. Combining (11) and (12) together, we can prove that $\mu(n) < 0$ for $\underline{n} < n < \overline{n}$ with the transversality condition $\mu(\underline{n}) = \mu(\overline{n}) = 0$. In other worlds, households' concerns always have a negative social value in the government's aggregate concerns. The details of the calculations done are reported in Appendix B.1. step 3.

Finally, it is clear that the marginal tax rate of n-household is

$$t(z(n)) = 1 - s(n) - \frac{\varepsilon(n)}{n}$$
$$= \frac{\beta P_c(\cdot, c^*)}{\gamma} \left(s(n) + \frac{\varepsilon(n)}{n} \right) - \frac{\mu(n)}{\gamma f(n)} \left(U_{nz} + \frac{U_x}{n^2} \right). \tag{13}$$

This expression shows that government's optimal taxation policy is a trade-off between reducing aggregate poverty and maximizing utilitarian welfare. (13) states that if poverty is part of the government's concern, given $P_c(c, c^*) < 0$ for $c < c^*$, it would

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reduce the tax rate of low-income individuals (a negative marginal taxation). For the higher income households ($c > c^*$), the marginal poverty is $P_c(c, c^*) = 0$. Thus, the higher income households face a strictly positive tax rate of $-\frac{\mu(n)}{\gamma f(n)}\left(U_{nz} + \frac{U_x}{n^2}\right)$.

When the constraints of government's maximization problem follow (6) and (8), our model coincides with the standard situation in the literature. The first order conditions are identical to those in the Wane (2001) model, except that we have one more first order condition (with respect to x) for determining the optimal household production. In other words, x is an additional control variable in our optimization problem. In this case, the co-state equation in terms of income becomes:

$$\mu(n) = \int_n^{\overline{n}} \left(1 + \frac{\gamma}{U_z}\right) \exp\left(\int_n^p \frac{U_{nz}}{U_z} dm\right) f(p) dp. \tag{14}$$

The standard first order condition with respect to z yields:

$$[\gamma(1 - s(n)) - \beta P_c s(n)] f(n) = -\mu(n) U_{nz}. \tag{15}$$

Given that $\varepsilon_n = 0$, the first order condition with respect to c implies $U_x = 0$. The marginal tax rate of n-household is:

$$t(z(n)) = 1 - s(n) \quad = \frac{\beta P_c(\cdot, c^*) s(n)}{\gamma} + \frac{-\mu(n) U_{nz}}{\gamma f(n)}. \tag{16}$$

The above analysis is concluded in Proposition 2.

Proposition 2 *For any individual concern and government's budget constraint given by (4) (or (6)) and (8) that corresponds to an optimal choice, there exist a piecewise absolutely continuous co-state variable $\mu(n)$ and a multiplier γ, such that, the optimal choice of the marginal tax rate $t(z)$ is given by (13) or (16), and the co-state is given by (11) (or (14)), and the multiplier is (12).*

To close this section, we provide a discussion of the tax implications of the domestic work and the income poverty alleviation. Then, in the next section, we use numerical simulations to present more explicit results.

3.2 Impacts of household production and poverty consideration on optimal taxation

In this model, there is a jump in the optimal marginal tax rate around the threshold ability value, $\widetilde{n}$, due to the binding constraint. Denote n^* as $c(n^*) = c^*$. Thus, depending on the relative location of n^* and $\widetilde{n}$, three cases appear.

First, the time constraint binding individuals are the ones above the poverty threshold, that is, $\widetilde{c} > c^*$ or $n^* < \widetilde{n}$. In this case, the optimal taxation would be given by:

$$
t(z(n)) = \begin{cases}
(i) & \dfrac{\beta P_c}{\gamma}\left(s(n) + \dfrac{\varepsilon(n)}{n}\right) - \dfrac{\mu(n)}{\gamma f(n)}\left(U_{nz} + \dfrac{U_x}{n^2}\right), & \underline{n} \leq n < n^*; \\[3ex]
(ii) & -\dfrac{\mu(n)}{\gamma f(n)}\left(U_{nz} + \dfrac{U_x}{n^2}\right), & n^* \leq n < \widetilde{n}; \\[3ex]
(iii) & -\dfrac{\mu(n)}{\gamma f(n)}U_{nz}, & \widetilde{n} \leq n \leq \overline{n}.
\end{cases}
$$

Second, if the individuals from whom the time constraint is binding are exactly the ones on the poverty threshold, that is, $\widetilde{c} = c^*$ or $n^* = \widetilde{n}$, then the optimal marginal tax rate remains the same as above, except that (ii) vanishes.

The last case is that the individuals for which the time constraint is binding are the ones below poverty threshold, $n^* > \widetilde{n}$. Then, the optimal taxation would be

$$
t(z(n)) = \begin{cases}
(i) & \dfrac{\beta P_c}{\gamma}\left(s(n) + \dfrac{\varepsilon(n)}{n}\right) - \dfrac{\mu(n)}{\gamma f(n)}\left(U_{nz} + \dfrac{U_x}{n^2}\right), & \underline{n} \leq n < \widetilde{n}; \\[3ex]
(ii) & \dfrac{\beta P_c}{\gamma}s(n) - \dfrac{\mu(n)}{\gamma f(n)}U_{nz}, & \widetilde{n} \leq n < n^*; \\[3ex]
(iii) & -\dfrac{\mu(n)}{\gamma f(n)}U_{nz}, & n^* \leq n \leq \overline{n}.
\end{cases}
$$

The first case states that even some households that are not under the income poverty threshold cannot afford to engage in leisure. Nevertheless, the Europe- and American-Time-Use-Survey suggest that even the poor households take some time off to enjoy some pure leisure, although this time may be less than that of individuals that are more productive. The second case can only occur by accident. In other words, the last case is closer to reality in most Western European countries and the United States. Thus, we will only focus on the last case where $\underline{n} < \widetilde{n} < n^* < \overline{n}$.

The expression in the last case shows that the three groups of individuals face different tax regimes. This result is due to multiple heterogeneity. In our model, both household-production and poverty considerations introduce additional dimensions to individual heterogeneity. First, our model distinguishes the poor and the rich populations by their time allocation. Second, it distinguishes the poor and the rich populations by their consumption. On the one hand, the low-skilled individuals with $n < \widetilde{n}$ have a strictly positive marginal utility, $U_x > 0$, thus, the outcome of their household-production contributes to the social welfare. On the other hand, the low-skilled individuals with $n < n^*$, also cause higher income poverty, which results as a negative externality on the social welfare. In response, the government assigns different social weights to different types of individuals.

Formally, the individuals below and above $\widetilde{n}$ differ in their marginal value of labor or in their valuation of non-market time. For households with $n < \widetilde{n}$, the marginal utility (or disutility) of work is $-U_z + \frac{U_x}{n}$ (or $U_z - \frac{U_x}{n}$). For households with $n \geq \widetilde{n}$, the marginal utility (or disutility) is $-U_z$ (or U_z). As a consequence of this difference, all other things being equal, the tax rate in (i) differs from (ii) in two factors: the substitution rate between labor and consumption, that is, $s(n) + \frac{\varepsilon(n)}{n} = \frac{-U_z + U_x/n}{U_c}$; and the derivative of disutility of labor with respect to n, that is, $U_{nz} + \frac{U_x}{n^2} = \frac{\partial(U_z - U_x/n)}{\partial n}$. When individuals below and above n^* are compared, the main difference is in the value of marginal poverty: for the high-skilled population, $P_c(c, c^*) = 0$. Thus, the tax rate for the high-skilled population in (iii) is the standard welfarist result, which is non-negative.

The main policy concerns from the above analysis is, considering working-poverty alleviation, that it is important to emphasize on the difference between domestic work and real leisure, especially in the differences in their social value. These results show that optimal tax policy should help to alleviate some burden of the poorest working households by proving more generous tax credits, such as offering more or free child-care and/or old age support, thereby freeing individuals from domestic work. This would allow them to have more time to work outside of the house (or obtain some extra training to improve their ability) and earn some extra income.

4 Explicit solution and numerical analysis

Due to its complexity, it is impossible to obtain an explicit solution for the general model. In this section, we use specific functions as an example to show some explicit results. Following Mirrlees (1971) and others such as Kanbur et al (1994), Wane (2001), and Boone and Bovenberg (2004), we consider quasi-linear utility, which is a logarithmic function of c and is linear in z and x. The implication of this setting is, as mentioned by Boone and Bovenberg (2004), that *"the strict concavity of the logarithm function implies that poverty alleviation is an objective of a utilitarian government. Therefore, such a government thus aims at achieving an equal distribution of consumption (that is, the alleviation of poverty)"*. The utility function is given by:

$$U(c, x, z, n) = \begin{cases} \ln c(n) - \frac{z}{n} + \sigma x, & \text{for } c(n) < \widetilde{c}, \\ \ln c(n) - \frac{z}{n} + \sigma \widetilde{x}, & \text{for } c(n) \geqslant \widetilde{c}. \end{cases} \tag{17}$$

The economic interpretation of this functional form is as follows: for the relative high-income population ($c(n) \geqslant \tilde{c}$), the household production ($\widetilde{x}$, a fixed value) is neutral and there is no incentive to allocate more time to household production, $U_x = 0$. For this part of population, the model is solved in the same way as in Wane (2001). The relative high-income households simply set their $x = \widetilde{x}$, which is time needed for domestic work; therefore, the marginal utility $U_x = 0$ (Proposition1). For the relative low-income households ($c(n) < \widetilde{c}$), the domestic production is non-neutral with a positive marginal utility $U_x = \sigma > 0$. In this case, the relative low-skill households will set their household production to the maximum level given the time constraint, namely, $1 - \frac{z}{n}$.

4.1 Solving the model

Following Kanbur et al. (1994) and Wane (2001), we take the poverty function as:

$$P(c(n), c^*) = \left(\frac{c(n) - c^*}{c^*} \right)^{\alpha}, \ \alpha > 1.$$

Given the explicit utility function, the first order conditions with respect to z are

$$\begin{cases} c(n)(\gamma + \beta P_c) = \dfrac{n\gamma}{1+\sigma} + \dfrac{\mu(n)}{f(n)n}, & \underline{n} \leq n < \widetilde{n}; \\[3mm] c(n)(\gamma + \beta P_c) = n\gamma + \dfrac{\mu(n)}{f(n)n}, & \widetilde{n} \leq n < n^*; \\[3mm] c(n) = n + \dfrac{\mu(n)}{f(n)n\gamma}, & n^* \leq n \leq \overline{n}, \end{cases} \qquad (18)$$

where $\widetilde{n}$ is determined by solving $c(\widetilde{n}) = \widetilde{c}$:

$$\widetilde{c}\,[\gamma + \beta P_c(\widetilde{c}, c^*)] = \frac{\widetilde{n}\gamma}{1+\sigma} + \frac{\mu(\widetilde{n})}{f(\widetilde{n})\widetilde{n}} \ .$$

Then, we can calculate the marginal tax rate as:

$$t(z(n)) = \begin{cases} (1+\sigma)\left[\dfrac{\beta P_c c(n)}{\gamma n} - \dfrac{\mu(n)}{\gamma f(n)n^2}\right], & \underline{n} \leq n < \widetilde{n}; \\[3mm] \dfrac{\beta P_c c(n)}{\gamma n} - \dfrac{\mu(n)}{\gamma f(n)n^2}, & \widetilde{n} \leq n < n^*; \\[3mm] -\dfrac{\mu(n)}{\gamma f(n)n^2}, & n^* \leq n \leq \overline{n}. \end{cases} \qquad (19)$$

where the expression of $\mu(n)$ and γ are given in Appendix B.2. The marginal tax rate in our model differs from that in Wane (2001) because of σ and $\widetilde{n}$. When $\sigma = 0$, that is, the when the marginal household public good is equivalent to leisure, our model is identical to that of Wane (2001). However, with $\sigma \neq 0$, the outcomes are quite different. These findings are summarized as follows.

Proposition 3 *Given the assumption that $\widetilde{n} < n^*$, and the quasi-linear utility function defined in (17), the optimal marginal tax rate is given by (19). Furthermore, there is jump in the marginal tax rate.*

Another variable of interest is the labor supply $z(n)$, which can be obtained by combing the individual's utility concern (4) (or (6)) and the government's budget constraint (8).

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Proposition 4 *Given the quasi-linear utility function defined in (17), the optimal aggregate labor supply is given by*

$$z(n) = \int_{\underline{n}}^{n} z'(m)dm - K$$

with

$$z'(n) = \begin{cases} \dfrac{nU_c c'(n)}{1 + U_x}, & n < \widetilde{n}; \\[2ex] nU_c c'(n), & \widetilde{n} \le n. \end{cases} \tag{20}$$

The calculation of labor supply and the expression for constant term K are reported in Appendix B.2. (20) shows that household labor supply depends on the individual's utility concern. Since it changes before and after the ability threshold point, so does the aggregate labor supply. Due to $U_x > 0$, the low-skilled working households have a smaller marginal labor supply.

4.2 Numerical findings

In order to compare with the existing literature, we follow the numerical example in Wanes (2001) with density function $f(n) = \frac{5}{6} - \frac{n}{5}, n \in [1, 4]$. We assume that the government is targeting a balanced budget $R = 0$. The calibration of other parameters are as follows: $\beta = 0.5$, $\sigma = 0.2$, $x^* = 2$, and $\widetilde{x} = 1.7$. This calibration implies that $n^* = 2.32$ and $\widetilde{n} = 1.75$.

In each of Figure 1, 2 and 3, we have two groups of results. The Figures on the left show the case where there is no poverty consideration at all. Thus, we study, from a welfarist planner's point of view, the impacts of household production on the optimal income taxation. In the figures on the right, we compared our model with the poverty alleviation literature. In both the figures on the left and on the right, the black lines are the numerical results in Wane (2001) under the welfarist planner without poverty consideration (solid line) and the poverty-as-public-bad (in short PPB) planner (dash line). The red line corresponds to the outcomes when $n < \widetilde{n}$. The blue line corresponds to the outcomes when $n \geqslant \widetilde{n}$. The following tables summarize the numerical results for a selection of individual types.

Figure 1 invites at least three comments. First, there is a jump in consumption

Table 1: Simulation Results with Welfarist Planner ($\beta = 0$)

n	Our model			Welfarist model		
	$c(n)$	$t(n)$	$z(n)$	$c(n)$	$t(n)$	$z(n)$
1	0.83	0	0.90	1	0	0.87
1.50	1.02	0.18	1.12	1.24	0.17	1.15
1.75	1.18 to 1.47	0.1 to 0.16	1.31	1.44	0.18	1.39
2	1.69	0.16	1.57	1.66	0.17	1.66
2.50	2.19	0.13	2.15	2.17	0.13	2.27
3	2.75	0.09	2.78	2.74	0.09	2.91
3.50	3.35	0.04	3.43	3.34	0.05	3.57
4	4	0	4.11	4	0	4.26

Table 2: Simulation results with PPB planner ($\beta = 0.5$)

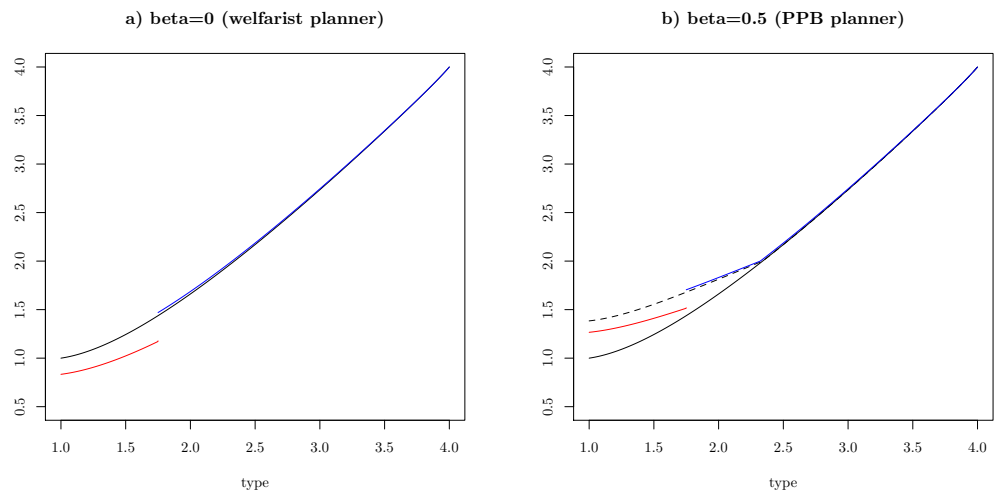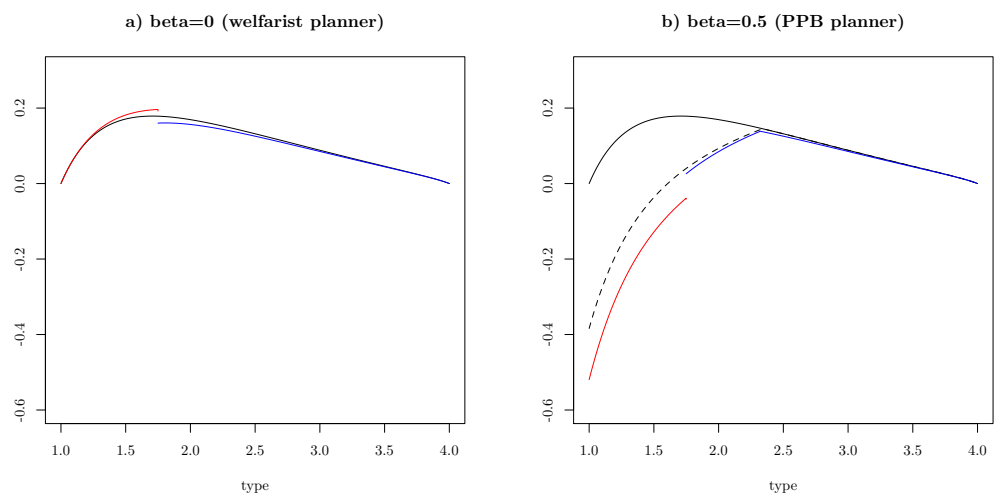| n	Our model			Wane (2001)'s model		
	$c(n)$	$t(n)$	$z(n)$	$c(n)$	$t(n)$	$z(n)$
1	1.27	-0.52	1.38	1.38	-0.38	1.37
1.50	1.42	-0.13	1.50	1.56	-0.04	1.52
1.75	1.52 to 1.70	-0.04 to 0.03	1.60	1.68	0.03	1.65
2	1.84	0.09	1.74	1.81	0.09	1.79
2.50	2.19	0.13	2.14	2.17	0.13	2.20
3	2.75	0.09	2.77	2.74	0.09	2.84
3.50	3.35	0.04	3.41	3.34	0.05	3.49
4	4	0	4.09	4	0	4.18

Figure 1: Consumption
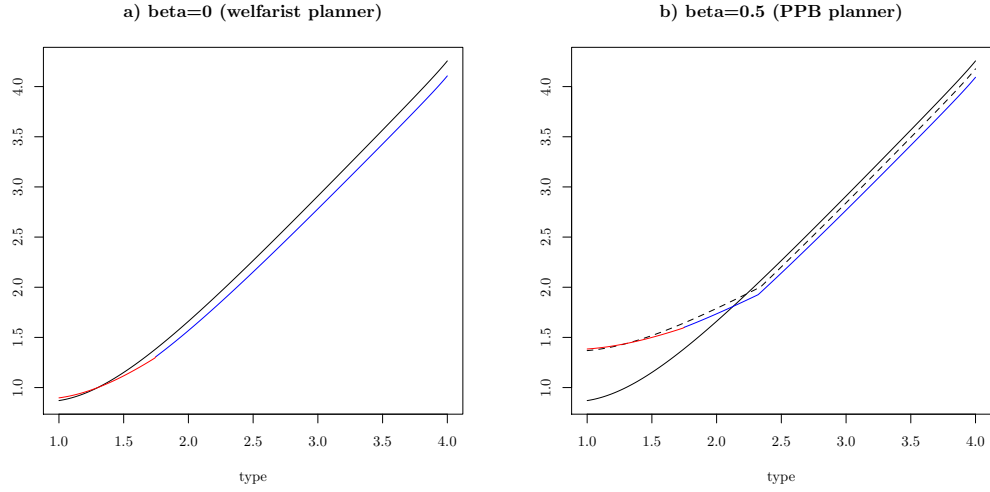

Figure 2: Marginal tax rate

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Figure 3: Labor supply

before and after $\widetilde{n}$ under both the welfarist planner and the PPB planner assumptions. This jump prevents the high-ability household from mimicking the low-ability ones. The households with $n < \widetilde{n}$ consume less than the higher-skilled households. This is because market purchased goods and domestic goods are substitutes. Thus, the low-skilled households can substitute their needs by relying on domestic production. Second, after $\widetilde{n}$, our model does not significantly differ from that in Wane (2001). The small differences are due to the values of γ and $\mu(n)$. Third, under the PPB planner, the consumption of individuals with $n < n^*$ is higher than the consumption under the welfarist planner. This is because the PPB planner is more concerned about stimulating the consumption of low-skilled population.

Figure 2 depicts our main variable of interest- the marginal tax rate. Similar to the consumption pattern, there is a jump in the marginal tax rate around the point of $\widetilde{n}$. However, depending on the government's objective, the nature of this jump is quite different. Under the welfarist planner, there is an upward jump for the low-skilled households, and a downward jump for high- skilled households. The opposite situation occurs under the PPB planner. This observation can be explained by examining (19). Under the welfarist planner, the government's sensibility to income poverty is null, $\beta = 0$. In this case, with other things being equal, the marginal tax rate faced by the $n < \widetilde{n}$ households is higher than that faced by the $n \geq \widetilde{n}$ households. Intuitively,

this is because the $n < \widetilde{n}$ households have a higher opportunity cost to work. Thus, in order to maximize social welfare, there is no need to encourage the $n < \widetilde{n}$ households to work longer. Therefore, it is optimal to set a higher tax rate for the $n < \widetilde{n}$ households. The situation is dramatically different for the PPB planner with a poverty consideration, $\beta = 0.5$. In this case, the government also wishes to help the low-skilled households by providing them a subsidy (or negative tax rate). The size of this subsidy for the $n < \widetilde{n}$ households is magnified by $(1 + \sigma)$, which draws the marginal tax rate downward in Figure 2 b).

Figure 3 presents the outcome of labor supply. Given our calibration of the model, the labor supply is relatively smooth around the point $\widetilde{n}$. However, The marginal labor supply $z'(n)$ are slightly different before and after $\widetilde{n}$, because of U_c as showed in Proposition 4. We also observe that the high-skilled households slightly decreases their market work as compared to the case in Wane (2001).

5   Conclusion

In this study, we investigate the ways in which income taxation policy could alleviate working poverty when domestic work is considered. Formally, we extend the optimal taxation model in Wane (2001) by explicitly considering the time allocation by households between market labor, domestic work, and pure leisure. In our model, households differ not only in their innate ability to earn an income from labor market participation, but also in their social statuses (being poor or rich) and their valuation of non-market time. Thus, this study contributes to the debate on the optimal design of income taxation by combining two strands of literature: (i) alleviation of working poverty (Kanbur et al. 1994; Wane 2001; Pritilla and Tuomala 2004; Boone and Bovenberg 2004) and (ii) multiple heterogeneity (Diamond, 1980; Laroque, 2005; Choné and Laroque 2010; Lockwood and Weinzierl, 2015). This new approach also provides some useful insights for policy makers.

This studys analytic and numerical results show that the taxation treatments of different types of populations should differ according to their social statuses. Therefore, the design of an optimal taxation policy should consider not only household income level, but also other characteristics. From the planner's perspective, this indicates that

the social weights should depend not only on labor market outcomes, but also on the valuation of domestic work. The optimal taxation policy is a result of the trade-off between different driving forces. For instance, our model shows that the outcomes of household production could directly benefit the poorest working population by allowing them to substitute their consumption. Thus, the tax policy does not need to encourage the poorest working population to work even harder on the market. In the case that the planner is willing to alleviate working poverty, which is measured in terms of consumption, the optimal tax policy should help the poorest working households by providing them with more generous tax credits or social support to free individuals from domestic work to have more time to work on the market (or obtain more training to improve their market productivity), earn extra income and, in turn, consume more goods. Our findings are based on theoretical analysis and numerical simulations. Arguably, empirical tests and calibration are needed before any proper policy recommendations can be made.

References

[1] Aguiar M. and E. Hurst(2007). *Measuring trends in leisure: The allocation of time over five decades*. Quarterly Journal of Economics, 122(3), 969-1006.

[2] Aguiar M. and E. Hurst(2009). *A summary of trends in American time allocation: 1965-2005*. Social Indicators Research, 93, 57-64.

[3] Apps P., R. Rees (2009). *Public economics and the household*, Cambridge University Press, Cambridge.

[4] Apps P., R. Rees (2011).*Optimal taxation and tax reform for two-earner household*, CE-Sifo Economic Studies 57(2), 283-304.

[5] Atkinson A.B. and J. Stiglitz (2015). *Lectures on Public Economics*. Princeton University Press; Originally published in 1980 by MacGraw-Hill.

[6] Beaudry P., C. Blackorby and D. Szalay (2009). *Taxes and employment subsidies in optimal redistribution programs*. The American Economic Review, 99 (1), 216-242.

[7] Becker G. (1965). *A theory of the allocation of time*. The Economic Journal, 75 (299), 493-517.

[8] Belfield C., J. Cribb, A. Hood and R. Joyce (2015). *Living standards, poverty and inequality in the UK: 2015*. Institute for Fiscal Studies, July 2015.

[9] Boadway R. (2012). *Recent advances in optimal income taxation*. Hacienda Publica Espanola/ Review of Public Economics, 200(1). 15-39.

[10] Boadway R., J-F. Tremblay (2013). *Optimal income taxation and the labor market: an overview.* CESifo Economic Studies, 59(1), 93-148.

[11] Boone J., L. Bovenberg (2004). *The optimal taxation for unskilled labor with job search and social assistance*. Journal of Public Economics, 88, 2227-2258.

[12] Boskin M., E. Sheshinski (1983). *The Optimal tax treatmentof the family*. Journal of Public Economics, 20, 281-297.

[13] Choné P., G. Laroque (2010). *Negative Marginal Tax Rates and Heterogeneity*, American Economic Review, 100, 2532-2547.

[14] Daly M. (2010). *Ireland In-work poverty and labour market segmentation. A study of national policies*. European Commission DG Employment, Social Aggairs and Equal Opportunities, May 2010.

[15] Diamond P. (1980). *Income taxation with fixed hours of work*. Journal of Public Economics, 13, 101-110.

[16] Gerfin M. and R. Leu (2003). *The impact of in-work benefits on poverty and household labor supply–a simulation study for Switzerland*. IZA DE. 762.

[17] Kanbur R., M. Keen and M. Tuomala (1994 a). *Optimal non-linear income taxation for the alleviation of income-poverty*, European Economic Review, 38, 1613-1632.

[18] Kanbur R., M. Keen and M. Tuomala (1994 b). *Labor supply and targeting in poverty aaleviation programs*, The World Bank Economic Review, 8(2), 191-211.

[19] Kleven H., C. Kreiner and E. Saez (2009). *The optimal income taxation of couple*. Econometrica, 77, 537-560.

[20] Kleven H., W. Richter and P. Sorensen (2000). *Optimal taxation with household production*. Oxford Economic Paper, 52, 584-594.

[21] Kleven H. (2004). *Optimal taxation and the allocation of time*. Journal of Public Economics, 88, 545-557.

[22] Laroque G. (2005). *Income Maintenance and Labor Force Participation*, Econometrica, 73, 341-376.

[23] Lockwood B., M. Weinzierl (2015). *De Gustibus non est Taxandum: Heterogeneity in preferences and optimal redistribution*, Journal of Public Economics, 124, 74-80.

[24] Mirrlees J. (1971). *An exploration in the theory of optimum income taxation*, Review of Economic Studies, 38, 175-208.

[25] Neumark D. (2015). *Reducing Poverty via Minimum Wages, Alternatives*, FRBSF ECONOMIC LETTER, 2015-38.

[26] Piachaud D. (1987). *Problems in the Definition and Measurement of Poverty*. Journal of Social Policy, 16, 147-164.

[27] Prittila J. and M. Tuomala (2004). *Poverty alleviation and tax policy*, European Economic Review, 48, 1075–1090.

[28] Seade J. (1982). *On the sign of the optimum marginal income tax*, Review of Economic Studies, 49, 637-643.

[29] Sundaram R. (2009). *A first course in optimization theory*, Cambridge University Press, 15th Printing.

[30] Wane (2001). *The optimal income tax when poverty is a public "bad"*, Journal of Public Economics, 82, 271-299.

Appendix A   Excerpt of American Time Use Survey 2012

Table 3: Employed persons (25 years and over) who worked on an average Saturday, Sunday, and holiday periods–by education levels

Education	2008	2012
Less than high school diploma	6.82	6.30
High School graduate no college	6.44	6.76
Some college or associate degree	5.88	6.08
Bachelors degree or higher	4.31	4.21

Table 4: Employed persons (25 years and over) who worked on an average Saturday, Sunday, and holiday periods–by income level

Income	2008	2012
$0 –$530	6.70	7.26
$531 –$830	6.10	7.02
$831 –$1290	6.37	5.52
$1291 and higher	4.10	3.60

Tables 3 and 4 have been extracted from the American Time Use Survey 2012 report.

Appendix B   Calculations

B.1   Solving the Optimization Problem

Step 1. Incentive of the individuals

The total differential of $u(n) = U(c(n), x(n), z(n), n) = U(z(n) - T(z(n)), x(n), z(n), n)$ with respect to n reads:

$$\frac{du(n)}{n} = U_c \left[\frac{dz(n)}{dn} - \frac{dT}{dz} \frac{dz}{dn} \right] + U_x \frac{dx}{dn} + U_z \frac{dz}{dn} + U_n$$

$$= \left[U_c(1 - T'(z)) + U_z \right] \frac{dz(n)}{dn} + U_x \frac{dx}{dn} + U_n.$$

The two first conditions in (2) implies that $[U_c(1 - T'(z)) + U_z] = \frac{U_x}{n}$, which yields:

$$\frac{du(n)}{dn} = U_x \left[\frac{dx}{dn} + \frac{1}{n} \frac{dz(n)}{dn} \right] + U_n.$$

When the constraint is ineffective $U_x = 0$, then $\frac{du(n)}{dn} = U_n$. When the constraint is effective, $1 - x - \frac{z}{n} = 0$ yields $\frac{dx}{dn} + \frac{1}{n} \frac{dz}{dn} = \frac{z}{n^2}$. Thus, $\frac{du(n)}{dn} = U_x \frac{z}{n^2} + U_n$.

Step 2. Government's problem

Here we only provide the calculation for the case when $n < \tilde{n}$, such that the individual's incentive constraint is $\frac{du(n)}{dn} = U_x \frac{z}{n^2} + U_n$. When $n \geq \tilde{n}$, $\frac{du(n)}{dn} = U_n$. In latter case, the detailed calculations are documented in Wane (2001). The Hamiltonian can be written as

$$\mathcal{H}(c(n), z(n), u(n), \mu(n); \gamma) = \left[(u(n) - \beta P(c(n), c^*)) + \gamma \left(z(n) - c(n) \right) \right] f(n)$$

$$+ \mu(n) \left(U_x \frac{z}{n^2} + U_n \right),$$

in which, the choice variable are $c(n)$ and $z(n)$; the state variable is $u(n)$. Applying the Pontryagin maximum principle, we have two conditions $\frac{\partial H}{\partial z} = 0$ and $\frac{\partial H}{\partial u} = -\mu'(n)$.

The first order condition with respect to $z(n)$ reads

$$\frac{\partial \mathcal{H}}{\partial z(n)} = \left[-\beta P_c \frac{\partial c(n)}{\partial z(n)} + \gamma (1 - \frac{\partial c(n)}{\partial z(n)}) \right] f(n)$$

$$+ \mu(n) \left[\left(U_{xc} \frac{\partial c(n)}{\partial z(n)} + U_{xx} \frac{\partial x(n)}{\partial z(n)} + U_{xz} \right) \frac{z}{n^2} + U_x \frac{1}{n^2} + U_{nc} \frac{\partial c(n)}{\partial z(n)} + U_{nx} \frac{\partial x(n)}{\partial z(n)} + U_{nz} \right] = 0.$$

Given that the second order derivatives of U_x with respect to c, x, and z are null, U_{nx} and U_{nc} are null. Then, we obtain

$$\left[-\beta P_c \frac{\partial c(n)}{\partial z(n)} + \gamma(1 - \frac{\partial c(n)}{\partial z(n)})\right] f(n) + \mu(n) \left[U_{nz} + U_x \frac{1}{n^2}\right] = 0. \tag{B.1}$$

Using $c = z - T(z)$ and $U_c(1 - T'(z)) + U_z = \frac{U_x}{n}$, we have

$$
\begin{aligned}
\frac{\partial c}{\partial z} &= \frac{U_x}{nU_c} - \frac{U_z}{U_c} \\
&= \frac{\varepsilon}{n} + s. \tag{B.2}
\end{aligned}
$$

Combing (B.1) and (B.2) yields the first-order condition (9).

Step 3. Solution of $\mu(n)$ and γ

In order to write the co-state equation in terms of income, we use the condition $\partial H / \partial u = -\mu(n)'$:

$$
\begin{aligned}
\frac{\partial H}{\partial u} &= \left(1 + \gamma \frac{\partial z}{\partial u}\right) f(n) \\
&\quad + \mu(n) \left[\left(-U_{xx} \frac{\partial z}{\partial u} \frac{1}{n} + U_{xz} \frac{\partial z}{\partial u}\right) \frac{z}{n^2} + \frac{U_x}{n^2} \frac{\partial z}{\partial u} - \frac{U_{nx}}{n} \frac{\partial z}{\partial u} + U_{nz} \frac{\partial z}{\partial u}\right] \\
&= -\mu'(n).
\end{aligned}
$$

Then, assuming that the second derivatives, U_{xx}, U_{xz}, and U_{nx} are null, we can simplify this expression:

$$\frac{\partial H}{\partial u} = \left(1 + \gamma \frac{\partial z}{\partial u}\right) f(n) + \mu(n) \left(\frac{U_x}{n^2} \frac{\partial z}{\partial u} + U_{nz} \frac{\partial z}{\partial u}\right) = -\mu'(n).$$

Now we calculate $\partial z / \partial u$. First, we need to define an inverse function for $z(n)$. Given the binding constraint $x(n) = 1 - z(n)/n$, the variable $z(n)$ appears in two arguments of the utility function: $u(n) = U(c(n), x(n), z(n), n) = U\left(c(n), 1 - \frac{z(n)}{n}, z(n), n\right)$. This implies that the inverse function should look like this:

$$z(n) = \Gamma(u(n), c(n)).$$

Thus, we should take the inverse of all the $z-$components from the utility function. The total differential of $u(n)$ is

$$
\begin{aligned}
du &= U_c dc + U_x dx + U_z dz + U_n dn \\
&= U_c dc + \left(U_z - \frac{U_x}{n} \right) (\Gamma_u du + \Gamma_c dc) + U_n dn \\
&= \left(U_c + \left(U_z - \frac{U_x}{n} \right) \Gamma_c \right) dc + \left(U_z - \frac{U_x}{n} \right) \Gamma_u du + U_n dn \\
&= 0,
\end{aligned}
$$

which is equivalent to

$$
\left[1 - \left(U_z - \frac{U_x}{n} \right) \Gamma_u \right] du = \left(U_c + \left(U_z - \frac{U_x}{n} \right) \Gamma_c \right) dc + U_n dn.
$$

Then, we obtain: $\Gamma_u = \frac{1}{\left(U_z - \frac{U_x}{n} \right)}$ and $\Gamma_c = -\frac{U_c}{\left(U_z - \frac{U_x}{n} \right)}$. Replacing $\frac{\partial z}{\partial u}$ with Γ_u yields:

$$
\mu'(n) + \mu(n) \left[\frac{U_{nz} + U_x/n^2}{U_z - U_x/n} \right] + \left[1 + \frac{\gamma}{U_z - U_x/n} \right] f(n) = 0.
$$

It is straightforward that the most general form of solution is:

$$
\mu(n) = \int_n^{\widetilde{n}} \left(1 + \frac{\gamma}{U_z - U_x/p} \right) \exp \left(\int_n^p \frac{U_{nz} + U_x/m^2}{U_z - U_x/m} dm \right) f(p) dp.
$$

Note that $U_x(n) \neq 0$ when $n < \widetilde{n}$; thus the solution of $\mu(n)$ for low-skilled population should be

$$
\begin{aligned}
\mu(n) &= \int_n^{\widetilde{n}} \left(1 + \frac{\gamma}{U_z - U_x/p} \right) \exp \left(\int_n^p \frac{U_{nz} + U_x/m^2}{U_z - U_x/m} dm \right) f(p) dp \\
&+ \underbrace{\int_{\widetilde{n}}^{\overline{n}} \left(1 + \frac{\gamma}{U_z} \right) \exp \left(\int_{\widetilde{n}}^p \frac{U_{nz}}{U_z} dm \right) f(p) dp}_{\widetilde{\mu}}
\end{aligned}
$$

where the second term of the equation is a constant, that is, $\widetilde{\mu} \equiv \mu(\widetilde{n})$. When $n \geqslant \widetilde{n}$, the co-state variable becomes:

$$
\mu(n) = \int_n^{\overline{n}} \left(1 + \frac{\gamma}{U_z} \right) \exp \left(\int_n^p \frac{U_{nz}}{U_z} dm \right) f(p) dp.
$$

Therefore, besides the usual transversality conditions $\mu(\underline{n}) = \mu(\overline{n}) = 0$, we have an additional transition condition, $\mu(\widetilde{n}) = \widetilde{\mu}$. Next, we can obtain an explicit solution for γ using the initial condition, $\mu(\underline{n}) = 0$.

$$
\int_{\underline{n}}^{\widetilde{n}} \exp \left(\int_n^p \frac{U_{nz} + U_x/m^2}{U_z - U_x/m} dm \right) f(p) dp + \gamma \int_{\underline{n}}^{\widetilde{n}} \frac{1}{U_z - U_x/p} \exp \left(\int_n^p \frac{U_{nz} + U_x/m^2}{U_z - U_x/m} dm \right) f(p) dp
$$

$$
+ \int_{\widetilde{n}}^{\overline{n}} \exp \left(\int_{\widetilde{n}}^p \frac{U_{nz}}{U_z} dm \right) f(p) dp + \gamma \int_{\widetilde{n}}^{\overline{n}} \frac{1}{U_z} \exp \left(\int_{\widetilde{n}}^p \frac{U_{nz}}{U_z} dm \right) f(p) dp = 0.
$$

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The above equality yields (12).

B.2   Calculation under the Explicit Utility Function

Given the quasi-linear utility function defined in (17), the three marginal utilities are:

$$U_c = \frac{1}{c}, \quad U_z = -\frac{1}{n}, \quad \text{and } U_x = \begin{cases} \sigma & \text{for } c(n) < \widetilde{c}; \\ 0 & \text{for } c(n) \geqslant \widetilde{c}. \end{cases}$$

The second order derivatives check: $U_{nz} = 1/n^2$, $U_{nx} = 0$, $U_{nc} = 0$, and the two elasticity functions are given by:

$$s = \frac{c}{n}, \quad s_n = -\frac{c}{n^2}; \quad \text{and } \varepsilon = \begin{cases} \sigma c & \text{for } c(n) < \widetilde{c} \\ 0 & \text{for } c(n) \geqslant \widetilde{c} \end{cases}, \quad \varepsilon_n = 0.$$

Then, using the general formula in Section 3.1, we can obtain (18) and (19) for the explicit utility function. Using (11), the co-state variable is

$$\mu(n) = \int_n^{\widetilde{n}} (1 - \frac{\gamma p}{1+\sigma}) \left(\frac{n}{p}\right) f(p) dp + \int_{\widetilde{n}}^{\overline{n}} (1 - \gamma p) \left(\frac{\widetilde{n}}{p}\right) f(p) dp.$$

Given that $\mu(\underline{n}) = 0$, it follows

$$\underline{n} \int_{\underline{n}}^{\widetilde{n}} \frac{f(n)}{p} dp - \gamma \underline{n} \int_{\underline{n}}^{\widetilde{n}} \frac{f(n)}{1+\sigma} dp + \widetilde{n} \int_{\widetilde{n}}^{\overline{n}} \frac{f(p)}{p} dp - \gamma \widetilde{n} \int_{\widetilde{n}}^{\overline{n}} f(p) dp = 0.$$

Thus, we obtain:

$$\gamma = \frac{\displaystyle \underline{n} \int_{\underline{n}}^{\widetilde{n}} \frac{f(p)}{p} dp + \widetilde{n} \int_{\widetilde{n}}^{\overline{n}} \frac{f(p)}{p} dp}{\displaystyle \underline{n} \int_{\underline{n}}^{\widetilde{n}} \frac{f(p)}{1+\sigma} dp + \widetilde{n} \int_{\widetilde{n}}^{\overline{n}} f(p) dp}.$$

To calculate the labor supply, we distinguish between two cases. First, when $n < \widetilde{n}$, the household's utility concern is:

$$U_c \frac{\partial c}{\partial n} + U_z \frac{\partial z}{\partial n} = \frac{\partial z}{\partial n} \frac{U_x}{n}$$

$$\text{that is,} \quad z'(n) = \frac{\partial z}{\partial n} = \frac{U_c c'(n)}{U_x/n - U_z} = \frac{n U_c c'(n)}{1 + U_x}.$$

Then, we can define quantity v_1 as

$$v_1(n) \;=\; \int_{\underline{n}}^n z'(m)dm = \int_{\underline{n}}^n \frac{mU_c c'(m)}{1 + U_x} dm = \frac{1}{1+\sigma} \int_{\underline{n}}^n m \frac{c'(m)}{c(m)} dm$$

$$=\; \frac{1}{1+\sigma} \left(n \ln c(n) - \underline{n} \ln c(\underline{n}) - \int_{\underline{n}}^n \ln c(m)dm \right).$$

Second, when $n \geqslant \widetilde{n}$, the marginal utility of x is null. Thus, in this case the following integral is calculated separately:

$$v_2(n) \;=\; \int_{\underline{n}}^n z'(m)dm = \int_{\widetilde{n}}^n mU_c c'(m)dm$$

$$=\; v_1(\widetilde{n}) + n \ln c(n) - \widetilde{n} \ln c(\widetilde{n}) - \int_{\widetilde{n}}^n \ln c(m)dm.$$

Next, we determine the constant term K using the government's budget constraint with $\overline{R} = 0$:

$$\int_{\underline{n}}^{\overline{n}} [z(n) - c(n)] f(n)dn = 0$$

or equivalently $$\int_{\underline{n}}^{\overline{n}} \left[\int_{\underline{n}}^n z'(m)dm - K - c(n) \right] f(n)dn = 0.$$

Then,

$$K \;=\; \int_{\underline{n}}^{\overline{n}} \left[\int_{\underline{n}}^n z'(m)dm - c(n) \right] f(n)dn$$

$$=\; \int_{\underline{n}}^{\widetilde{n}} \left[\int_{\underline{n}}^n z'(m)dm - c(n) \right] f(n)dn + \int_{\widetilde{n}}^{\overline{n}} \left[\int_{\underline{n}}^n z'(m)dm - c(n) \right] f(n)dn$$

$$=\; \int_{\underline{n}}^{\widetilde{n}} [v_1(n) - c(n)] f(n)dn + \int_{\widetilde{n}}^{\overline{n}} [v_2(n) - c(n)] f(n)dn.$$

That completes the proof.