On the optimal use of correlated information in contractual design under limited liability*

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Abstract

Riordan and Sappington (JET, 1988) show that in an agency relationship in which the agent's type is correlated with a public *ex post* signal, the principal may attain first best (full surplus extraction and efficient output levels) if the agent is faced with a lottery such that each type is rewarded for one signal realization and punished equally for all the others. Gary-Bobo and Spiegel (RAND, 2006) show that this kind of lottery is most likely to be locally incentive compatible when the agent is protected by limited liability. In this paper we investigate how the principal should construct the lottery to attain not only local but also global incentive compatibility. We first assess that the main issue with global incentive compatibility rests with intermediate types being potentially attractive reports to both lower and higher types. We then show that a lottery including three levels of profit (rather than only two) is optimal in that it is most likely to be globally incentive compatible under limited liability, if local incentive constraints are strictly satisfied. We identify conditions under which first best is implemented. In a setting with three types and three signals we also pin down the optimal distortions when those conditions are violated. In particular, when the first-best allocation is locally but not globally incentive compatible, output distortions are induced but all surplus is retained from the agent.

Keywords: Incentive compatibility; Limited liability; Correlated signals; Conditional probability; Full-rank condition

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1 Introduction

There is now notable work on contractual design in agency problems with correlated information. The pioneering studies, which we owe to Myerson [10], Crémer and McLean [2] (henceforth, CM), McAfee and Reny [9] and Riordan and Sappington [11] (henceforth, RS), identify necessary and sufficient conditions for full surplus extraction in settings in which the agent is not protected by limited liability. When such conditions are satisfied, the principal designs a payment scheme including a lottery related to the distribution of an external signal to be realized and observed *ex post* and correlated with the private information of the agent. All surplus is extracted from the agent by embedding in the lottery both rewards and punishments associated with the various possible signal realizations. However, a serious drawback of these mechanisms is that the punishments may be too high for the mechanisms to be viable when the agent is protected by limited liability.

Demougin and Garvie [3] and Gary-Bobo and Spiegel [5] (henceforth, GBS) investigate optimal screening under limited liability in the presence of correlated information. Demougin and Garvie [3] only consider the case in which the signal is binary. GBS show that this is without loss of generality when the principal is only concerned with local incentive-compatibility, in addition to limited liability. In that case, indeed, the principal is better off if she offers a lottery that admits only two levels of profit, a reward and a punishment. If more than two signals are available, then the reward is associated with only one signal and equal punishments are associated with all the other signals. However, it is not obvious that this is still the best strategy in environments in which global incentive compatibility is not implied by local incentive-compatibility. Hitherto the literature has not completely clarified which exact lottery the principal should adopt when incentive compatibility may be difficult to attain not only locally but also globally and the agent is protected by limited liability. Here is the contribution of our study.

A lottery yielding one reward and equal punishments to the agent was first proposed by RS. In addition to providing necessary and sufficient conditions for first-best implementation in the absence of limited liability, as already mentioned, they highlight that the principal can use such a lottery if the agent's cost function is less concave in type than the conditional likelihood function of the reward signal. GBS focus on situations in which the cost function is strictly convex in type and the conditional likelihood function of the reward signal is concave, hence the sufficient conditions identified by RS are satisfied. For the purpose of our study, we impose restrictions neither on the curvature of the cost function nor on that of the conditional likelihood function of the signal to which the highest profit is associated. In so doing, we allow for the total cost function to be concave in type, as may well be the case, for instance, if the agent has an affine cost function such that the fixed cost is inversely related to the privately known marginal cost. Moreover, we assume that there exist two signals (rather than only one, as in GBS) which, taken together with any of the other available signals, satisfy the monotonic likelihood property. Although we reinforce the assumption made by GBS in this respect, we nonetheless require the monotonic likelihood property, which is familiar in mechanism design, to hold only in a "partial" sense. That is, we only require it to hold for any triplet of signals which includes the extreme two, rather than for all signals in the feasible set. With this approach we can search for the best lottery that the principal could employ to implement first best under limited liability. Our results will depend on how the shape of the cost function compares with that of the conditional likelihood function, as in RS, but the family of cost functions such that full surplus extraction is at hand is likely to be richer than in the one-reward lottery scheme. More specific results are summarized hereafter.[1]

Overview of the results

We first show that the main difficulty with global incentive compatibility is rooted in the way in which the lotteries targeted to the intermediate types should be designed for those types to represent attractive reports neither to lower types nor to higher types. This is better understood if it is considered that the compensation to the agent blends together a cost reimbursement, which is a fixed payment related to the cost of production, and a lottery, which assigns rewards and punishments depending on the realization of the signal. On the one hand, by over-stating information, lower types gain on the cost reimbursement but lose in terms of lottery; on the other, by under-stating information, higher types gain in terms of lottery but lose on the cost reimbursement. This double circumstance constrains the principal in the design of the lotteries for the intermediate types.[2]

Second, when limited liability constraints are not too tight and local incentive compatibility can be attained with a lottery other than the one of GBS, the lottery which is most likely to be globally incentive compatible at the first-best allocation includes three distinct levels of profit for each type. With this structure of the lottery, the principal can more easily discourage over-statement by lower types, yet, without making under-statement significantly more attractive to higher types. This facilitates the principal's task of impeding that intermediate types be conveniently announced by any other type. Once the optimal lottery is characterized, a cut-off level of liability is determined, which dictates whether or not first best is implementable. This cut-off value depends on how the shape of the cost function with respect to type compares with the shape of the likelihood function of the reward signal. Put it differently, the exact family of cost functions for which first best is viable is determined, given the level of liability. For instance, first best is at reach if the agent's cost function is concave in type, rather than being convex as in GBS, but the degree of concavity is not too pronounced relative to that of the likelihood function of the reward signal.

[1] In GBS the exogenous signal is taken to affect the cost of production, rather than being a purely informative signal about that cost, as is usually assumed by the literature. We do not follow the approach of GBS to avoid introducing complications which are unnecessary to the purpose of our study.

[2] From the proofs of Corollary 1.4 and 1.5 in RS it emerges that first best is implementable once sufficient conditions are introduced, under which there is no conflict between incentive constraints. However, in subsequent studies it has not been clarified why such a conflict may arise and how it can be eliminated under limited liability.

Our third finding concerns situations in which the agent's liability is too low for the principal to be able to induce truthtelling without distortions, and hence for the first-best allocation to be effected. We show that the structure of the optimal lottery in this second-best scenario does not differ from that figured out in the first-best setting. An important aspect is that the level of liability which separates the regime under which local incentive compatibility is attained from that under which it is not, is also the level of liability which separates situations in which the optimal lottery includes three levels of profit from those in which, as in GBS, it includes only two levels of profit. Remarkably, in the former situations it is optimal to induce distortions in the volume of output to satisfy both upward and downward incentive constraints, whereas all surplus is retained from the agent.

Related literature

Our paper is first related to Myerson [10], CM and McAfee and Reny [9], who consider an environment in which a seller/principal auctions out an object to a number of potential buyers/agents whose preferences (types) are privately known and correlated. In that environment, the signals correlated with the type of each agent are generated endogenously by the reports collected by the principal from the other agents. From those studies we know that the principal retains all surplus in a Bayesian framework for *any utility function* of each agent, if and only if the vector of conditional probabilities of the type of any agent is linearly independent of the vector of conditional probabilities of the types of the other agents. Whereas this result is very appealing in contractual design, it nonetheless exhibits the aforementioned limit that it may induce very low compensations, in which case it would be difficult to attain in practice.

A second line of research to which this paper is related is pioneered by RS. They consider situations in which the principal deals with only one agent whose private information is correlated with some signal which is realized and publicly observed *ex post*. These are situations in which the signals are exogenous to the contractual relationship. However, provided that the external signals play the same role as the private information held by other agents, RS obtain a similar result to that derived by the first line of research. In addition, RS show that, for some *specific cost functions* of the agent, full surplus extraction is at hand in spite of the signals being less numerous than the possible types.[3] More precisely, whether or not the outcome is attainable depends on the relationship between the characteristics of the cost function of the agent and the properties of the likelihood functions of the signals. The most "parsimonious" lottery the principal can design in this context includes only two levels of profit. GBS show that the incentive scheme proposed by RS is most likely to satisfy the limited liability constraints because the punishments are spread equally among all signals but one. With our investigation we evidence that this is not necessarily the best lottery the principal can use because there

[3]In CM the types of the agents (the potential buyers of the object sold by the principal) determine their utilities. In RS, as in our study, the agent exerts an activity delegated by the principal and his type determines his cost of production.

are circumstances under which it fails to motivate some types to release information. We then highlight how the lottery should be amended to circumvent these difficulty.

Our work is also related to the study of Demougin and Garvie [3], who first analyze contractual design with correlated information in situations in which the agent is protected by limited liability. In their model with a continuum of types and a binary signal, limited liability is represented in two alternative ways. First, the transfer from the principal to the agent cannot be negative, meaning that the principal has no power to tax the agent under any conditions. A similar form of limited liability is also represented in the two-type two-signal model of Kessler *et al.* [6], who allow the transfer to be negative but not unbounded. Second, in Demougin and Garvie [3] the agent can incur no deficits. This is tantamount to imposing *ex post* participation constraints and entails that the agent recovers the entire cost borne to perform the task for the principal, regardless of the signal realization. In line with GBS, we generalize the latter kind of limited liability assuming that the agent can only be exposed to bounded deficits. Essentially, we refer to situations in which the principal is concerned with preserving the agent's financial viability, though not ensuring reimbursement of the entire cost. This is common practice, for instance, in regulated industries, in which firms' financial distress is generally prevented to avoid activity interruptions. Unlike in Demougin and Garvie [3] and Kessler *et al.* [6], and similarly to GBS, we allow for more than two signals, which is a crucial ingredient of our investigation.

As is well known, limited liability can alternatively be regarded as an extreme form of risk aversion. With that interpretation, our study is also related to Eso [4], who explores full surplus extraction in an agency problem with correlated information and risk aversion on the agent's side. Specifically, the author considers an auction in which the auctioneer/principal faces two potential buyers/agents, both risk averse. Their privately known valuations of the object offered for sale are correlated and can take only two values. By contrast, we develop the analysis considering a richer set of types. This extension enables us to capture the important circumstance that incentive compatibility is problematic essentially because intermediate types may potentially attract false reports by both lower types and higher types.

1.1 Outline

The reminder of the article is organized as follows. In section 2 we describe the model. In section 3 we present the first-best analysis. We first consider a discrete number of types and then allow for a continuum of types. In Section 4, we return to a discrete-type framework to investigate the second-best setting in which the level of liability is too low, or the cost function too concave in type to implement the first-best allocation. We conclude in section 5. Mathematical proofs are relegated to an appendix.

2 The model

A principal, P, delegates the production of a good (or service) to an agent. They are both risk neutral.

Consumption of q units of the good yields a gross utility of $S(q)$. The function $S(\cdot)$ is twice continuously differentiable with derivatives $S'(\cdot) > 0$ and $S''(\cdot) < 0$. Moreover, $S(0) = 0$ and the Inada's conditions are satisfied.

Production of q units of the good involves a cost of $C(q, \theta)$, where the "type" θ parametrizes the agent's productivity. A lower value of θ involves a lower total cost, for any given q, and will be referred to as a lower (more efficient) type. The function $C(\cdot, \cdot)$ is twice continuously differentiable in either argument with partial derivatives $(dC(q, \theta)/dq) \equiv C_q(q, \theta) > 0$ and $(dC(q, \theta)/d\theta) \equiv C_\theta(q, \theta) > 0$. Moreover, $(d^2C(q, \theta)/dqd\theta) = C_{q\theta}(q, \theta) > 0$, *i.e.*, less efficient types have higher marginal costs of production. As a compensation for the supply of q units of the good the agent receives a payment of z from P.

In the contracting stage, the agent knows his type whereas P is uninformed. It is commonly known that θ is drawn from the support $\Theta \equiv \left[\underline{\theta}, \overline{\theta}\right]$, where $\overline{\theta} > \underline{\theta} > 0$, with continuously differentiable density function $f(\theta)$ and cumulative distribution function $F(\theta)$. Alternatively, θ is known to take values in the discrete set $\Theta_T \equiv \{\theta_1, ..., \theta_T\}$, where T is the number of types, which have the natural ordering $\theta_1 < ... < \theta_T$. This alternative scenario will be considered in some parts of the analysis for expositional purposes. It will also be useful to present previous findings of the literature and develop comparisons. Notation will be adapted accordingly whenever necessary.

The agent's type is correlated with a random signal s, which is realized and publicly observed *ex post*, *i.e.*, after the contract is drawn up and the level of output is determined (or the output is produced). The realized signal (the "state of nature") is hard information, involving that a legally enforceable contract can be signed upon.[4] We take the signal to be drawn from the discrete support $N \equiv \{1, .., n\}$, where $n \geq 2$. The probability that signal s is realized conditional on the type being θ is $p_s(\theta)$. We assume that $p_s(\theta) > 0$, $\forall s \in N$, and that the function $p_s(\cdot)$ is twice continuously differentiable for all values of θ, with first and second derivative respectively denoted as $(dp_s(\theta)/d\theta) \equiv p'_s(\theta)$ and $\left(d^2p_s(\theta)/d\theta^2\right) \equiv p''_s(\theta)$. We also make the following assumption.

Assumption 1 *The conditional probabilities of the signals satisfy the following property:*

$$\frac{p_1(\theta)}{p_1(\theta')} > \frac{p_2(\theta)}{p_2(\theta')}, \qquad \forall \theta > \theta' \qquad\qquad if \quad n = 2$$

$$\frac{p_1(\theta)}{p_1(\theta')} > \frac{p_s(\theta)}{p_s(\theta')} > \frac{p_n(\theta)}{p_n(\theta')}, \quad \forall \theta > \theta', \quad \forall s \neq 1, n \quad if \quad n \geq 3$$

[4] For instance, in regulatory settings, the agent is a regulated firm and the signal can be the behaviour or the market performance of another firm, operating either in the same sector or in an analogous sector placed in a neighboring economy, which conveys information about the cost of the regulated firm. In other contexts, the signal can be the outcome of an audit of the activity run by the agent.

This is the usual monotonic likelihood property. However, in the general case of $n \geq 3$, the property is only required to hold partially, *i.e.*, for any triplet of signals including 1 and n. One can interpret it as follows. There exists an "extreme" signal 1 such that, for any subset of signal realizations which contains signal 1 and at least one more signal $s \neq 1$, the probability of signal 1 being drawn is increasing in type. There also exists an "extreme" signal n such that, for any subset of signal realizations which contains signal n and at least one more signal $s \neq n$, the probability of signal n being drawn is decreasing in type.

Invoking the Revelation Principle, we can confine attention to contractual offers $\{q(\theta), \mathbf{z}(\theta)\}$, $\forall\theta$, in which $q(\theta)$ is the quantity an agent of type θ is required to produce and $\mathbf{z}(\theta) \equiv \{z_1(\theta), ..., z_n(\theta)\}$ is the vector of the transfers he is assigned in the different states. The quantity is not conditioned on the signal because it is chosen (or the output is produced) prior to the signal realization. Accordingly, the net surplus of P in state s is $S(q(\theta)) - z_s(\theta)$. Denote $\widetilde{\pi}_s(\theta' | \theta) \equiv z_s(\theta') - C(q(\theta'), \theta)$ the profit an agent of type θ obtains in state s when he announces θ' to P (or, alternatively, when he picks the contractual option $\{q(\theta'), \mathbf{z}(\theta')\}$ within the menu of allocations). This is also written as follows:

$$\widetilde{\pi}_s(\theta' | \theta) = \pi_s(\theta') + C(q(\theta'), \theta') - C(q(\theta'), \theta). \tag{1}$$

For convenience, we further let $\pi_s(\theta) = \widetilde{\pi}_s(\theta | \theta)$ and denote the lottery designed for an agent of type θ as $\boldsymbol{\pi}(\theta) \equiv \{\pi_1(\theta), ..., \pi_n(\theta)\}$. We will say that in state s he receives a *reward* if $\pi_s(\theta) > 0$ and incurs a *punishment* if $\pi_s(\theta) < 0$. It is useful to remark that (1) would be the same if $z_s(\theta)$ were to include a fixed component related to the type and a stochastic component conditional on the signal realization, as considered by Bose and Zhao [1]. Consistent with this, the programme of P presented below only depends on the profits rather than on the exact structure of the transfers assigned to the various types in the different states.

The relationship between P and the agent unfolds as follows. Before contracting takes place, nature draws θ and the agent learns its realization. P addresses the contractual offer to the agent. If the agent rejects the offer, then the parties obtain their reservation payoffs and the relationship ends. If the agent accepts the offer, then he makes a report about his type to P (or, alternatively, he picks an option within the contractual menu) and produces accordingly. Next, the signal is realized and the contractually specified transfer is paid.

2.1 The programme of the principal

Referring to the profit $\pi_s(\theta)$ rather than to the transfer $z_s(\theta)$ with a standard change of variable, the programme of P is formulated as follows:

$$\underset{\{q(\theta);\boldsymbol{\pi}(\theta),\forall\theta\}}{Max} \int_{\underline{\theta}}^{\overline{\theta}} \sum_{s=1}^{n} \left(S(q(\theta)) - C(q(\theta),\theta) - \pi_s(\theta)\right) p_s(\theta)\, dF(\theta)$$

subject to

$$\mathbb{E}_s[\pi_s(\theta)] \geq \sum_{s=1}^{n} \pi_s(\theta') p_s(\theta) + C(q(\theta'),\theta') - C(q(\theta'),\theta),\ \forall\theta,\theta' \tag{IC}$$

$$\mathbb{E}_s[\pi_s(\theta)] \geq 0,\ \forall\theta \tag{PC}$$

$$\pi_s(\theta) \geq -L,\ \forall\theta,\ \forall s. \tag{LL}$$

(IC) is the incentive compatibility constraint whereby an agent of type θ is unwilling to report $\theta' \neq \theta$ (or to pick the contractual option targeted to type θ'). (PC) is the participation constraint which ensures that the expected value of the lottery designed for type θ, namely $\mathbb{E}_s[\pi_s(\theta)] \equiv \sum_{s=1}^{n} \pi_s(\theta) p_s(\theta)$, is non-negative. Thus, the agent incurs no loss in expectation. (LL) is the limited liability constraint which ensures that the maximum deficit to which the agent is exposed does not exceed $L > 0$ in each possible state. Essentially, this form of limited liability represents situations in which the principal would like to avoid the agent becoming so financially distressed that the activity must be interrupted, at least as long as the agent does not attempt to conceal information. For instance, in regulated industries, in which this is common practice, L could be interpreted as an indicator of financial viability, beyond which the regulated firm would go bankrupt.

The first part of our study will be devoted to investigating under what conditions and in which way P implements the first-best allocation. This is defined by the optimality condition:

$$S'(q(\theta)) = C_q(q(\theta),\theta),\ \forall\theta, \tag{2}$$

together with the surplus extraction constraint:

$$\mathbb{E}_s[\pi_s(\theta)] = 0,\ \forall\theta. \tag{3}$$

Throughout this section, to save on notation, $q(\theta)$ will indicate the first-best quantity for an agent of type θ and $\pi_s(\theta)$ the profit assigned for the production of that quantity in state s. We further denote $\boldsymbol{\Pi}(\theta)$ the set of lotteries $\boldsymbol{\pi}(\theta)$ the elements of which satisfy (3).

2.2 Previous findings

Before turning to the analysis, it is useful to recall the previous findings on first-best implementation in settings with correlated information which are relevant for our study.

RS Assume that θ_t takes values in the set Θ_T, $C(q, \theta_t)$ is convex in θ_t, and $\exists i \in N$ such that $p_i(\theta_t)$ is increasing and concave in θ_t. If $L \to \infty$, *i.e.* the agent can be exposed to unbounded losses, then $\mathbf{\Pi}(\theta_t)$ is not empty for any θ_t. After presenting this result in Corollary 1.4, RS show that the principal effects the first-best allocation by adopting the binary lottery $\boldsymbol{\pi}^i(\theta_t)$ $\forall \theta_t$, defined as follows for any $t > 1$:

$$\pi_i(\theta_t) = (C(q(\theta_t), \theta_t) - C(q(\theta_t), \theta_{t-1})) \frac{1 - p_i(\theta_t)}{p_i(\theta_t) - p_i(\theta_{t-1})} \tag{4}$$

$$\pi_s(\theta_t) = -(C(q(\theta_t), \theta_t) - C(q(\theta_t), \theta_{t-1})) \frac{p_i(\theta_t)}{p_i(\theta_t) - p_i(\theta_{t-1})}, \ \forall s \neq i. \tag{5}$$

In Corollary 1.5, RS further show that if $n = 2$, types are drawn from Θ_3 and $p_i(\theta_3) > p_i(\theta_2) > p_i(\theta_1)$, then the lottery $\boldsymbol{\pi}^i(\theta_t)$ belongs to $\Pi(\theta_t)$ if and only if:

$$\frac{C(q, \theta_2) - C(q, \theta_1)}{C(q, \theta_3) - C(q, \theta_2)} \leq \frac{p_i(\theta_2) - p_i(\theta_1)}{p_i(\theta_3) - p_i(\theta_2)}. \tag{6}$$

This is ensured if the cost function is less concave in type than the conditional probability of signal i. In general, (6) can be satisfied when types θ_1 and θ_2 have similar costs of producing output q, relative to types θ_2 and θ_3, and/or when types θ_2 and θ_3 have similar probabilities of drawing signal i, relative to types θ_1 and θ_2.

GBS Take $C(q, \theta)$ to be convex in θ and $p_i(\theta)$ to be increasing and concave in θ for some $i \in N$. Moreover, $i = \arg\max_{s \in N} \{p_s'(\theta)/p_s(\theta)\}$, $\forall \theta$. That is, among all possible signals and for all possible types, signal i is the one the probability of which displays the highest rate of change as type increases. Notice that under this assumption the condition that RS impose on signal i in Corollary 1.5 is satisfied as well. Then, among all lotteries belonging to $\mathbf{\Pi}(\theta)$, the lottery that is most likely to satisfy (LL) is defined as follows:

$$\pi_i(\theta) = C_\theta(q(\theta), \theta) \frac{1 - p_i(\theta)}{p_i'(\theta)} \tag{7}$$

$$\pi_s(\theta) = C_\theta(q(\theta), \theta) \frac{p_i(\theta)}{-p_i'(\theta)}, \ \forall s \neq i. \tag{8}$$

This is the counterpart of lottery $\boldsymbol{\pi}^i(\theta_t)$, as figured out by RS, in the case of a continuum of types.[5] Being based on (8), one deduces that the first-best allocation is implemented if and only if:

$$C_\theta(q(\theta), \theta) \frac{p_i(\theta)}{p_i'(\theta)} \leq L, \ \forall \theta. \tag{9}$$

CM Take $\theta_t \in \Theta_T$ and $L \to \infty$. As long as the vectors $\mathbf{p}(\theta_t) \equiv \{p_1(\theta_t), ..., p_n(\theta_t)\}$ are linearly independent across types, $\mathbf{\Pi}(\theta_t)$ is non-empty for all θ_t. This follows from Farkas' lemma, which

[5] With a slight abuse, we will use the notation $\boldsymbol{\pi}^i(\cdot)$ to indicate this lottery regardless of whether types are drawn from a discrete set or a continuum range.

implies that for all θ_t there exists a $n-$dimensional vector $\mathbf{h}(\theta_t) \equiv \{h_1(\theta_t), ..., h_n(\theta_t)\}$ such that the following two conditions hold:

$$\sum_{s=1}^{n} h_s(\theta_t) p_s(\theta_t) = 0, \ \forall \theta_t \in \Theta_T \tag{10}$$

$$\sum_{s=1}^{n} h_s(\theta_t) p_s(\theta_{t'}) < 0, \ \forall \theta_t, \theta_{t'} \in \Theta_T. \tag{11}$$

By setting $\pi_s(\theta_t) = \gamma_t h_s(\theta_t)$, $\forall s$, $\forall t$, and choosing the "scaling" parameter γ_t arbitrarily big, all surplus is extracted from type θ_t and no incentive to mimic θ_t is triggered for any other type. First best is beyond reach if there exists some type θ_t for which no vector $\mathbf{h}(\theta_t)$ can be found such that (10) and (11) are satisfied.[6]

In substance, RS highlight that, as long as the agent can be imposed unlimited punishments, first best is possibly at hand even when the set of informative signals includes only two elements. As is evident from the definition of $\boldsymbol{\pi}^i(\theta_t)$, the agent's gain only depends on whether signal i is realized, rather than any other signal, regardless of how rich the subset of other signals is. From GBS we further retain that any other lottery belonging to $\mathbf{\Pi}(\theta)$ includes an element the value of which is below that of (5), involving that it is less likely to satisfy (LL). Under Assumption 1, $i = 1$ in our framework. The best known result in agency problems with correlated information is perhaps that of CM, who show that the first-best outcome is attained if the vectors of conditional probabilities of the signals are linearly independent. Importantly, this result is obtained regardless of the properties of the cost function. By setting rewards and punishments arbitrarily high, any untruthful report can be made unattractive. However, high punishments are unfeasible when the agent is protected by limited liability. One then needs to consider the properties of the cost and the probability functions to ascertain whether there exists some lottery that implements first best under limited liability, consistent with the analysis developed by GBS.

Our goal is to extend the analysis beyond that of GBS and investigate whether first best is attainable when (6) and (9) are not jointly satisfied, and what lottery should be adopted in that case. Indeed, with Assumption 1 being verified, (9) is most likely to hold for signal $i = 1$ but the associated lottery $\boldsymbol{\pi}^1(\theta)$ may fail to comply with (6) as required by RS. Whereas the assumption that some signal displays the highest likelihood ratio for all types is similar to that introduced by GBS, the assumption that some other signal displays the lowest likelihood ratio, also embodied in our Assumption 1, is made for the purpose of our study. Overall, Assumption 1 entails that the full-rank condition of CM must be satisfied for the extreme types but not

[6] The "only if" proof of CM shows that if the vector $\mathbf{h}(\theta_T)$ does not exist, then it is impossible to ensure that θ_T is not an attractive report to any type $\theta_t < \theta_T$. Notice however that the full-rank condition is not necessary for all types. In particular, it does not need to hold for type θ_1. This paves the way for the results drawn in the study of RS, in which first-best implementation does not necessarily depend on the full-rank condition. Bose and Zhao [1] show that Proposition 1 in RS implies that first best might be effected when the full-rank condition is violated.

necessarily for the intermediate types.[7] In this respect, our analysis diverges from that of CM and comes closer to that of RS and GBS.

3 Three types and two or three signals

We begin by considering a simple setting with three possible types. We will highlight the characteristics of the optimal lotteries when two and three signals are available. Mathematical derivations are reported in Appendix B.

3.1 Two signals

Take $n = 2$. Consider any lottery $\boldsymbol{\pi}(\theta_t) \in \Pi(\theta_t)$, designed for the generic type $\theta_t \in \Theta_3$, such that $\pi_2(\theta_t) < 0 < \pi_1(\theta_t)$. As surplus extraction requires $\sum_{s \in N} p_s(\theta_t)\pi_s(\theta_t) = 0$, we can express $\pi_1(\theta_t)$ in terms of $\pi_2(\theta_t)$ as $\pi_1(\theta_t) = -\pi_2(\theta_t)p_2(\theta_t)/p_1(\theta_t)$. This expression is useful to formulate the expected value of the lottery which type $\theta_{t'}$ is faced with, if it pretends θ_t, in terms of $\pi_2(\theta_t)$ only. Specifically, that lottery grants a profit of $-\pi_2(\theta_t)p_2(\theta_t)/p_1(\theta_t)$ with probability $p_1(\theta_{t'})$ and a profit of $\pi_2(\theta_t)$ with probability $p_2(\theta_{t'})$ so that its expected value to type $\theta_{t'}$ is $\pi_2(\theta_t)p_2(\theta_t)\left(\frac{p_2(\theta_{t'})}{p_2(\theta_t)} - \frac{p_1(\theta_{t'})}{p_1(\theta_t)}\right)$. Because $\pi_2(\theta_t) < 0$, under Assumption 1, this expected value is negative if $\theta_{t'} < \theta_t$ and positive in the converse case. That is, the lottery designed for type θ_t penalizes a lower type $\theta_{t'}$, if it pretends θ_t, because, as compared to θ_t, type $\theta_{t'}$ is more likely to draw signal 2 and less likely to draw signal 1. Conversely, that lottery favours a higher type $\theta_{t'}$, if it pretends θ_t, because, as compared to θ_t, type $\theta_{t'}$ is now less likely to draw signal 2 and more likely to draw signal 1. In addition to the lottery, the payoff of type $\theta_{t'}$, if it reports θ_t, includes the difference between the (false) cost reimbursed by P to the agent and the (real) cost incurred by the agent to perform the task. Overall, the payoff of type $\theta_{t'}$, if it reports θ_t, is given by:

$$\pi_2(\theta_t)p_2(\theta_t)\left(\frac{p_2(\theta_{t'})}{p_2(\theta_t)} - \frac{p_1(\theta_{t'})}{p_1(\theta_t)}\right) + C(q(\theta_t), \theta_t) - C(q(\theta_t), \theta_{t'}).$$

This expression is suggestive of what may incentivize type $\theta_{t'}$ to report θ_t. If $\theta_t > \theta_{t'}$, then type $\theta_{t'}$ loses in terms of lottery by reporting θ_t, but gains in terms of cost reimbursement (since $C(q(\theta_t), \theta_t) > C(q(\theta_t), \theta_{t'})$). On the opposite, if $\theta_t < \theta_{t'}$, then type $\theta_{t'}$ loses in terms of cost reimbursement (since $C(q(\theta_t), \theta_t) < C(q(\theta_t), \theta_{t'})$) but gains in terms of lottery. Thus, for both lower and higher types, there are two opposite effects at work. Remark that such effects follow from type θ_t being rewarded in state 1 and punished in state 2. In the converse case, under Assumption 1, lower types would obviously want to announce θ_t because, by doing so, they would gain both in terms of lottery and cost reimbursement, which justifies our choice to

[7] In Appendix A we show that, as long as Assumption 1 holds, $\mathbf{p}(\theta_1)$ and $\mathbf{p}(\theta_T)$ do not lie in the convex hull generated by the probability vectors of the other types. Moreover, there exist vectors $\mathbf{p}(\theta_t)$, $t \neq 1, T$, which lie in the convex hull generated by the probability vectors of the other types and do not violate Assumption 1.

consider a lottery such that $\pi_2(\theta_t) < 0 < \pi_1(\theta_t)$ in the first place. Taking this all into account, one can identify what requirements the profits should verify for not attracting false reports. In particular, the profits of the three types in state 2 must be such that

$$\pi_2(\theta_1) \geq \frac{C(q(\theta_1), \theta_{t'}) - C(q(\theta_1), \theta_1)}{-p_2(\theta_1)\left(\frac{p_1(\theta_{t'})}{p_1(\theta_1)} - \frac{p_2(\theta_{t'})}{p_2(\theta_1)}\right)}, \ t' = 2, 3, \tag{12}$$

$$\frac{C(q(\theta_2), \theta_3) - C(q(\theta_2), \theta_2)}{-p_2(\theta_2)\left(\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_2(\theta_3)}{p_2(\theta_2)}\right)} \leq \pi_2(\theta_2) \leq \frac{C(q(\theta_2), \theta_2) - C(q(\theta_2), \theta_1)}{-p_2(\theta_2)\left(\frac{p_2(\theta_1)}{p_2(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}\right)} \tag{13}$$

and

$$\pi_2(\theta_3) \leq \frac{C(q(\theta_3), \theta_3) - C(q(\theta_3), \theta_{t'})}{-p_2(\theta_3)\left(\frac{p_2(\theta_{t'})}{p_2(\theta_3)} - \frac{p_1(\theta_{t'})}{p_1(\theta_3)}\right)}, \ t' = 1, 2, \tag{14}$$

respectively. Considering also limited liability, the condition to the right of (13) must be satisfied jointly with the constraint $\pi_2(\theta_2) \geq -L$. Moreover, (14) must hold jointly with the constraint $\pi_2(\theta_3) \geq -L$. With some manipulation and the use of $p_2(\cdot) = 1 - p_1(\cdot)$, the following conditions are found to be necessary:

$$(C(q(\theta_3), \theta_3) - C(q(\theta_3), \theta_{t'}))\frac{p_1(\theta_3)}{p_1(\theta_3) - p_1(\theta_{t'})} \leq L, \ t' = 1, 2 \tag{15}$$

$$(C(q(\theta_2), \theta_2) - C(q(\theta_2), \theta_1))\frac{p_1(\theta_2)}{p_1(\theta_2) - p_1(\theta_1)} \leq L. \tag{16}$$

These two conditions are the counterpart of (9) in a setting with three types and two signals. They ensure that there exists a lottery such that types which exaggerate private information obtain non-positive payoffs under limited liability.

Being based on (12) to (16), we can deduce how the lottery should look like for each type and what requirements it should satisfy for first-best implementation. Let us begin with the extreme types. (12) and (14) evidence that it is easy to design lotteries such that none of those types represents an attractive report to any other, as long as the necessary conditions hold. First, it would suffice to set $\pi_s(\theta_1) = 0$, $\forall s$, because, in that case, higher types would lose money by producing $q(\theta_1)$ but being reimbursed only $C(q(\theta_1), \theta_1)$ rather than their true cost. Second, lower types are least motivated to pretend θ_3 if type θ_3 is assigned the lowest possible profit $(-L)$ in state 2, which lower types are more likely to draw than type θ_3. Thus, P can set $\pi_2(\theta_3) = -L$ and, accordingly, $\pi_1(\theta_3) = L(1 - p_1(\theta_3))/p_1(\theta_3)$.

As far as the intermediate type is concerned, contractual design looks more problematic. It is not plain that either extreme type can be prevented from announcing θ_2, even if (16) holds. In particular, because $\pi_2(\theta_2)$ must be set to satisfy both of the conditions in (13), the following requirement adds up to (15) and (16):

$$\frac{C\left(q\left(\theta_2\right),\theta_2\right)-C\left(q\left(\theta_2\right),\theta_1\right)}{C\left(q\left(\theta_2\right),\theta_3\right)-C\left(q\left(\theta_2\right),\theta_2\right)}\leq\frac{\frac{p_1(\theta_2)-p_1(\theta_1)}{p_1(\theta_2)}-\frac{p_2(\theta_2)-p_2(\theta_1)}{p_2(\theta_2)}}{\frac{p_1(\theta_3)-p_1(\theta_2)}{p_1(\theta_2)}-\frac{p_2(\theta_3)-p_2(\theta_2)}{p_2(\theta_2)}}. \tag{17}$$

Let us interpret (17). Given that type θ_1 gains on the cost reimbursement and loses on the lottery if it claims θ_2, whereas the converse occurs for type θ_3, there exist values of $\pi_2\left(\theta_2\right)$ such that types θ_1 and θ_3 are both discouraged from claiming θ_2 if and only if the ratio between the gain to type θ_1 and the loss to type θ_3 in terms of cost reimbursement does not exceed the ratio between the loss to type θ_1 and the gain to type θ_3 in terms of lottery, as (17) shows. With a binary signal, (17) is rewritten as (6), which is just the condition in Corollary 1.5 of RS, where now $q=q\left(\theta_2\right)$ and $i=1$. In line with the interpretation of (6), the gain/loss ratio in terms of cost reimbursement does not exceed the loss/gain ratio in terms of lottery if and only if the cost is less concave (more convex) than the conditional probability of signal 1.[8] Provided that (17) holds and $\pi_2\left(\theta_2\right)$ is set to comply with (13), (3) can then be used to determine $\pi_1\left(\theta_2\right)$. For instance, taking $\pi_2\left(\theta_2\right)$ to be the higher between the lower bound to the range of feasible values identified in (13) and $-L$, namely

$$\pi_2\left(\theta_2\right)=\max\left\{\frac{C\left(q\left(\theta_2\right),\theta_3\right)-C\left(q\left(\theta_2\right),\theta_2\right)}{-\frac{p_1(\theta_3)-p_1(\theta_2)}{p_1(\theta_2)}};-L\right\},$$

$\pi_1\left(\theta_2\right)$ is determined as follows:

$$\pi_1\left(\theta_2\right)=\min\left\{\frac{C\left(q\left(\theta_2\right),\theta_3\right)-C\left(q\left(\theta_2\right),\theta_2\right)}{\frac{p_1(\theta_3)-p_1(\theta_2)}{p_1(\theta_2)}};L\right\}\frac{1-p_1\left(\theta_2\right)}{p_1\left(\theta_2\right)}.$$

Overall, first best is at reach only if (17) (or, equivalently, (6) for $q=q\left(\theta_2\right)$ and $i=1$) holds jointly with (15) and (16). As we demonstrate in Appendix B.1, (17) also implies that the extreme types are more attracted by adjacent than non-adjacent types. Intuitively, because the lotteries that types θ_1 and θ_3 are faced with, if they announce θ_2, are not too extreme when (17) holds, those types will prefer the claim θ_2 to the claim θ_3 and θ_1, respectively. The benefit of this is that both upward and downward incentive constraints must only be verified locally. Therefore, taken together with (15) and (16), (17) is also sufficient for first-best implementation.[9]

[8] We formulate the condition identified by RS as (6), rather than as the equivalent condition (17), because this is useful to prepare the reader to the subsequent analysis with more than two signals.

[9] Even if the lotteries are such that the agent does not lose more than L in equilibrium, he might still incur a greater loss if he were to choose an out-of-equilibrium report. The reason why this might occur is that the limited liability constraints are required to hold as long as the agent does not conceal information.

3.2 Three signals

We now take $n = 3$. One natural possibility is that the optimal lottery to effect first best for each type is simply an extension of the lottery that type is faced with when the signal is binary. That is, each type is assigned just the same profit in state 3 as in state 2, consistent with the incentive scheme characterized by GBS. However, other options might well be preferable. We thus need to identify the structure of the optimal lotteries in this framework. Proceeding as above, we formulate the incentive constraints whereby θ_t is an attractive report neither to lower types nor to higher types as follows:

$$\pi_3(\theta_t) \leq \frac{C(q(\theta_t), \theta_t) - C(q(\theta_t), \theta_{t'}) + \pi_2(\theta_t) p_2(\theta_t)\left(\frac{p_2(\theta_{t'})}{p_2(\theta_t)} - \frac{p_1(\theta_{t'})}{p_1(\theta_t)}\right)}{-p_3(\theta_t)\left(\frac{p_3(\theta_{t'})}{p_3(\theta_t)} - \frac{p_1(\theta_{t'})}{p_1(\theta_t)}\right)}, \ \forall \theta_{t'} < \theta_t \quad (18)$$

$$\pi_3(\theta_t) \geq \frac{C(q(\theta_t), \theta_{t'}) - C(q(\theta_t), \theta_t) + \pi_2(\theta_t) p_2(\theta_t)\left(\frac{p_1(\theta_{t'})}{p_1(\theta_t)} - \frac{p_2(\theta_{t'})}{p_2(\theta_t)}\right)}{-p_3(\theta_t)\left(\frac{p_1(\theta_{t'})}{p_1(\theta_t)} - \frac{p_3(\theta_{t'})}{p_3(\theta_t)}\right)}, \ \forall \theta_{t'} > \theta_t. \quad (19)$$

Because the signal can take three values, once the expected payoff is set equal to zero for each type the incentive constraints are expressed in terms of two profits, rather than only one as is the case with a binary signal. This is relevant to the determination of the necessary conditions under limited liability. Actually, searching for optimal lotteries along the lines of the analysis developed above, it will emerge that the necessary conditions are still (15) and (16).

Again we consider the extreme types first. θ_1 is not an attractive report for higher types as long as (19) holds for $t = 1$ and $t' = 2, 3$. As with a binary signal, this is the case if $\pi_s(\theta_1) = 0$, $\forall s$, and limited liability is clearly not an issue with this type. θ_3 is not an attractive report to lower types as long as (18) is satisfied for $t = 3$ and $t' = 1, 2$. To see what lottery type θ_3 should be faced with to that end, start from a situation in which $\pi_2(\theta_3) = \pi_3(\theta_3) < 0$, and hence $\pi_1(\theta_3) > 0$, and suppose that one of the two negative profits, say $\pi_2(\theta_3)$, is increased. Then, it might be necessary to reduce $\pi_3(\theta_3)$ to not violate (18). The reason why the adjustment is optimally made through a change in $\pi_3(\theta_3)$, rather than through a change in $\pi_1(\theta_3)$, is that signal 3 is less likely to be drawn by type θ_3 as compared to signal 1. Analogously, following an increase in $\pi_3(\theta_3)$ the adjustment is optimally made through a decrease in $\pi_2(\theta_3)$ because signal 2 is less likely to be drawn as compared to signal 1. However, in either case (LL) will be tightened since the decreased profit is already negative in the first place. Therefore, if there exists a lottery with $\pi_1(\theta_3) > 0$ which is incentive compatible under limited liability, then this feature is preserved if, in that lottery, $\pi_2(\theta_3) = \pi_3(\theta_3)$ and, in particular, if $\pi_2(\theta_3) = \pi_3(\theta_3) = -L$. Setting $\pi_2(\theta_3) = \pi_3(\theta_3)$ in (18), (14) is immediately retrieved jointly with the necessary condition (15). What changes here, with respect to the situation with a binary signal, is that there are now two signals, rather than only one, which are less likely to be drawn as compared to signal 1. Provided that the necessary conditions are satisfied when type θ_3 is exposed to the maximum deficit in the least likely state $(s = 3)$, they must also be satisfied when type θ_3

is exposed to the maximum deficit in the less likely between the two remaining states $(s=2)$. Hence, (15) can be satisfied by inflicting the highest punishment (L) to the least efficient type in two states of nature, rather than only one. That is, P can focus on a lottery in which $\pi_2(\theta_3)=\pi_3(\theta_3)=-L$ and, accordingly, $\pi_1(\theta_3)=L(1-p_1(\theta_3))/p_1(\theta_3)$.

We next turn to the design of the lottery for the intermediate type θ_2. We first consider the potential conflict between the incentive constraints whereby the higher and the lower type are unwilling to announce θ_2. The fact that the payment to the agent can be conditioned on three signals, rather than only two, provides P with an additional instrument to lessen this conflict. The necessary condition (17) is replaced by

$$\begin{aligned}&\frac{C(q(\theta_2),\theta_2)-C(q(\theta_2),\theta_1)}{\frac{p_3(\theta_1)}{p_3(\theta_2)}-\frac{p_1(\theta_1)}{p_1(\theta_2)}}-\frac{C(q(\theta_2),\theta_3)-C(q(\theta_2),\theta_2)}{\frac{p_1(\theta_3)}{p_1(\theta_2)}-\frac{p_3(\theta_3)}{p_3(\theta_2)}} \\ \leq\ &\pi_2(\theta_2)\,p_2(\theta_2)\left(\frac{\frac{p_1(\theta_3)}{p_1(\theta_2)}-\frac{p_2(\theta_3)}{p_2(\theta_2)}}{\frac{p_1(\theta_3)}{p_1(\theta_2)}-\frac{p_3(\theta_3)}{p_3(\theta_2)}}-\frac{\frac{p_2(\theta_1)}{p_2(\theta_2)}-\frac{p_1(\theta_1)}{p_1(\theta_2)}}{\frac{p_3(\theta_1)}{p_3(\theta_2)}-\frac{p_1(\theta_1)}{p_1(\theta_2)}}\right).\end{aligned} \tag{20}$$

Assuming that the difference in brackets in the right-hand side is negative, (20) is weakest when $\pi_2(\theta_2)$ is decreased to the minimum: $\pi_2(\theta_2)=-L$. Supposing that the equality $\pi_3(\theta_2)=\pi_2(\theta_2)$ is imposed, the necessary condition is again (17), and it can then be impossible to decrease $\pi_2(\theta_2)$ to $-L$. To see this, replace $\pi_2(\theta_2)=-L$ in (19) and rearrange to obtain

$$\pi_3(\theta_2)\geq\frac{C(q(\theta_2),\theta_3)-C(q(\theta_2),\theta_2)-L\frac{p_1(\theta_3)-p_1(\theta_2)}{p_1(\theta_2)}}{-p_3(\theta_2)\left(\frac{p_1(\theta_3)}{p_1(\theta_2)}-\frac{p_3(\theta_3)}{p_3(\theta_2)}\right)}-L. \tag{21}$$

This shows that, if L is sufficiently high for (15) (or (16)) to hold strictly, then it must be the case that $\pi_3(\theta_2)>-L$. Therefore, the conflict between incentives is weakest when the profit in state 2 is different from the profit in state 3. Turning back to type θ_2, the condition under which the term that multiplies $\pi_2(\theta_2)$ is negative in (20), and hence it is optimal to set $\pi_2(\theta_2)=-L$ (rather than $\pi_3(\theta_2)=-L$), is given by

$$\frac{\frac{p_1(\theta_2)-p_1(\theta_1)}{p_1(\theta_2)}-\frac{p_2(\theta_2)-p_2(\theta_1)}{p_2(\theta_2)}}{\frac{p_1(\theta_3)-p_1(\theta_2)}{p_1(\theta_2)}-\frac{p_2(\theta_3)-p_2(\theta_2)}{p_2(\theta_2)}}>\frac{\frac{p_1(\theta_2)-p_1(\theta_1)}{p_1(\theta_2)}-\frac{p_3(\theta_2)-p_3(\theta_1)}{p_3(\theta_2)}}{\frac{p_1(\theta_3)-p_1(\theta_2)}{p_1(\theta_2)}-\frac{p_3(\theta_3)-p_3(\theta_2)}{p_3(\theta_2)}}. \tag{22}$$

Notice that the left-hand side of (22) replicates the right-hand side of (17). Furthermore, the right-hand side of (22) is just the same as the left-hand side, except that the likelihood of signal 3 replaces the likelihood of signal 2. These observations are useful to interpret (22). If $\pi_2(\theta_2)$ is decreased, then by announcing θ_2 instead of telling the truth, type θ_1 loses and type θ_3 gains in terms of lottery. As long as the ratio between such loss and gain exceeds the ratio that would result from a decrease in $\pi_3(\theta_2)$ rather than in $\pi_2(\theta_2)$, the best strategy is to set $\pi_2(\theta_2)=-L$. Obviously, in the converse case, the best strategy would be to set $\pi_3(\theta_2)=-L$, instead. In any case, the lottery that is most likely to implement first best departs from that pinned down by GBS, which is such that $\pi_2(\theta_2) = \pi_3(\theta_2)$. For simplicity, here below we refer to the case in which (22) is satisfied, and hence it is optimal to set $\pi_2(\theta_2) = -L$.

Remark that, with $\pi_2(\theta_2) = -L$, P can easily assign a profit to type θ_2 in state 3 such that both (18) and (19) are satisfied. To see this, replace $\pi_2(\theta_2) = -L$ in (18) taken for $t = 2$ (and hence, $t' = 1$) and rearrange to obtain the following:

$$\pi_3(\theta_2) \leq -\frac{C(q(\theta_2), \theta_2) - C(q(\theta_2), \theta_1) - L\frac{p_1(\theta_2) - p_1(\theta_1)}{p_1(\theta_2)}}{p_3(\theta_2)\left(\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}\right)} - L. \tag{23}$$

Also recall that (19) taken for $t = 2$ (and hence, $t' = 3$) is rewritten as (21) when $\pi_2(\theta_2) = -L$. Joint inspection of (21) and (23) evidences that, under the necessary conditions (15) and (16), if there exists a range of feasible values of $\pi_3(\theta_2)$, then either it includes $-L$, or it lies entirely above $-L$. For instance, if P sets

$$\pi_3(\theta_2) = \frac{L\frac{p_1(\theta_3) - p_1(\theta_2)}{p_1(\theta_2)} - (C(q(\theta_2), \theta_3) - C(q(\theta_2), \theta_2))}{p_3(\theta_2)\left(\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}\right)} - L$$

and, accordingly,

$$\pi_1(\theta_2) = \frac{C(q(\theta_2), \theta_3) - C(q(\theta_2), \theta_2) - L\frac{p_3(\theta_3) - p_3(\theta_2)}{p_3(\theta_2)}}{p_1(\theta_2)\left(\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}\right)} - L,$$

provided $\pi_2(\theta_2) = -L$, then an incentive compatible lottery for the intermediate type is found, indeed. The following result can be thus stated.

Proposition 1 *Assume that $\theta_t \in \Theta_3$, $n = 3$ and (22) is satisfied. First best is implemented if and only if (15) and (16) hold jointly with*

$$\begin{aligned} & \frac{C(q(\theta_2), \theta_2) - C(q(\theta_2), \theta_1)}{\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}} - \frac{C(q(\theta_2), \theta_3) - C(q(\theta_2), \theta_2)}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}} \\ \leq \; & Lp_2(\theta_2)\left(\frac{\frac{p_2(\theta_1)}{p_2(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}}{\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}} - \frac{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_2(\theta_3)}{p_2(\theta_2)}}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}}\right). \end{aligned} \tag{24}$$

According to the proposition, implementation of the first-best allocation rests critically on (24). With this condition satisfied, P can find a profile of profits such that the incentives to lie upwards are eliminated jointly with the incentives to lie downwards, and hence incentive compatibility is attained in any reporting direction. Hence, to identify conditions for first-best implementation, it is necessary to ascertain for what features of the cost and probability functions and what magnitude of L (24) is satisfied. This approach will be followed to investigate the general case of a continuum of types here below.

4 A continuum of types and a finite number of signals

Take $\theta \in \Theta$ and $n \geq 3$. Considering that the expected payoff of type θ', if it reports θ, is given by

$$\mathbb{E}_s\left[\widetilde{\pi}_s\left(\theta\left|\theta'\right.\right)\right] = \sum_{s=1}^{n} \pi_s\left(\theta\right) p_s\left(\theta'\right) + C\left(q\left(\theta\right), \theta'\right) - C\left(q\left(\theta'\right), \theta'\right)$$

and that (3) must hold, we can state the local and the global incentive constraints as follows (mathematical derivations are found in Appendix B and C):

$$C_\theta\left(q\left(\theta\right), \theta\right) = \sum_{s=1}^{n} \pi_s\left(\theta\right) p'_s\left(\theta\right), \ \forall \theta \in \Theta \quad \text{(LIC)}$$

$$C(q(\theta), \theta) - C(q(\theta), \theta') \leq \sum_{s=1}^{n} \pi_s(\theta)(p_s(\theta) - p_s\left(\theta'\right)), \ \forall \theta', \theta \in \Theta. \quad \text{(GIC)}$$

These conditions ensure that θ is not an attractive report to any type $\theta' \neq \theta$ and that it will be chosen by type θ only. First suppose that θ' is in a neighborhood of θ. According to (LIC), any benefit from pretending θ to type θ' is eliminated if P designs profits for type θ such that the marginal change in the expected value of the lottery to type θ' ($\sum_{s=1}^{n} \pi_s\left(\theta\right) p'_s\left(\theta\right)$) is just as great as the marginal change in the cost reimbursement $\left(C_\theta\left(q\left(\theta\right), \theta\right)\right)$. Any deviation away from this rule would make the lie worth for some neighboring types: for higher types, if the marginal change in the expected value of the lottery is greater than the marginal change in the cost reimbursement; for lower types, if the converse occurs. Next suppose that θ' is not in a neighborhood of θ. According to (GIC), θ is not an attractive report to type θ' if the gain in cost reimbursement associated with that lie $(C(q(\theta), \theta) - C(q(\theta), \theta'))$ is lower than the loss on the lottery $(\sum_{s=1}^{n} \pi_s(\theta)(p_s((\theta) - p_s\left(\theta'\right)))$, when $\theta > \theta'$; and if the loss in cost reimbursement $(C(q(\theta), \theta') - C(q(\theta), \theta))$ exceeds the gain in the lottery $(\sum_{s=1}^{n} \pi_s(\theta)(p_s\left(\theta'\right) - p_s(\theta)))$, when $\theta < \theta'$ instead.

The first step is to identify profits of type θ such that (LIC) and (GIC) are satisfied. To that end, one can proceed similarly to the discrete-type analysis. First, being based on (3), express $\pi_1\left(\theta\right)$ in terms of the profits assigned in all states other than 1. Next, use the expression so obtained to reformulate (GIC) as a pair of conditions on $\pi_n\left(\theta\right)$, one written for any $\theta^- < \theta$ and the other for any $\theta^+ > \theta$, namely:[10]

$$\pi_n\left(\theta\right) p_n(\theta) \leq -\frac{\frac{C(q(\theta),\theta)-C\left(q(\theta),\theta^-\right)}{\theta-\theta^-}}{\frac{\frac{p_1(\theta)-p_1(\theta^-)}{\theta-\theta^-}}{p_1(\theta)} - \frac{\frac{p_n(\theta)-p_n(\theta^-)}{\theta-\theta^-}}{p_n(\theta)}} - \sum_{s\neq 1,n} \pi_s\left(\theta\right) p_s(\theta) \frac{\frac{\frac{p_1(\theta)-p_1(\theta^-)}{\theta-\theta^-}}{p_1(\theta)} - \frac{\frac{p_s(\theta)-p_s(\theta^-)}{\theta-\theta^-}}{p_s(\theta)}}{\frac{\frac{p_1(\theta)-p_1(\theta^-)}{\theta-\theta^-}}{p_1(\theta)} - \frac{\frac{p_n(\theta)-p_n(\theta^-)}{\theta-\theta^-}}{p_n(\theta)}} \quad (25)$$

[10] We let θ^- and θ^+ denote types respectively below and above θ, but not necessarily limit values around θ.

and

$$\pi_n(\theta)\, p_n(\theta) \geq -\frac{\frac{C\left(q(\theta),\theta^+\right)-C(q(\theta),\theta)}{\theta^+-\theta}}{\frac{\frac{p_1(\theta^+)-p_1(\theta)}{\theta^+-\theta}}{p_1(\theta)}-\frac{\frac{p_n(\theta^+)-p_n(\theta)}{\theta^+-\theta}}{p_n(\theta)}} - \sum_{s\neq 1,n} \pi_s(\theta)\, p_s(\theta) \frac{\frac{\frac{p_1(\theta^+)-p_1(\theta)}{\theta^+-\theta}}{p_1(\theta)}-\frac{\frac{p_s(\theta^+)-p_s(\theta)}{\theta^+-\theta}}{p_s(\theta)}}{\frac{\frac{p_1(\theta^+)-p_1(\theta)}{\theta^+-\theta}}{p_1(\theta)}-\frac{\frac{p_n(\theta^+)-p_n(\theta)}{\theta^+-\theta}}{p_n(\theta)}}. \tag{26}$$

Taking the limits for $\theta^- \to \theta$ and $\theta^+ \to \theta$ yields the following new formulation of (LIC):

$$\pi_n(\theta) = \frac{C_\theta(q(\theta),\theta)) + \sum_{s\neq 1,n} \pi_s(\theta) p_s(\theta) \left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_s'(\theta)}{p_s(\theta)}\right)}{-p_n(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}\right)}, \ \forall\theta \in \Theta. \tag{27}$$

This tells that, once P chooses the profits to be assigned to type θ in states 2 to $n-1$, she must set the profit in state n according to (27) to be able to prevent all neighboring types from reporting θ.

Once it is assessed that $\pi_n(\theta)$ must be chosen according to (27) and $\pi_1(\theta)$ such that (3) holds, it must be figured out how the profits should be set in states 2 to $n-1$ for (25) to (27) to be satisfied, taking into account that the limited liability constraint must hold in all states of nature. Considering that (27) is an alternative formulation of the local incentive constraint of type θ, whereas (25) and (26) is the global incentive constraint, as stated for types below and above θ, we will analyze the potential conflict between local incentive compatibility and limited liability separately from the potential conflict between global incentive compatibility and limited liability. Overall, the analysis will lead us to characterize the lottery which is most likely to satisfy the constraints altogether.

We first look at the potential conflict between local incentive compatibility and limited liability. Inspection of (27) highlights that, if any change is induced in $\pi_s(\theta)$, for some $s \neq 1, n$, then this change must be matched with an *opposite* variation in $\pi_n(\theta)$, and vice versa. Furthermore, an adjustment in $\pi_1(\theta)$ will be necessary to keep (3) satisfied. Therefore, switching from a lottery belonging to $\mathbf{\Pi}(\theta)$ to a new lottery also belonging to $\mathbf{\Pi}(\theta)$ requires at least three profits being varied. The following lemma is useful to understand how the profits will change in states 1 and n under Assumption 1 and, being based on that, to identify the lottery which is most likely to satisfy (LL) among those belonging to $\mathbf{\Pi}(\theta)$.

Lemma 1 *Take $n \geq 3$, $\boldsymbol{\pi}(\theta) \in \mathbf{\Pi}(\theta)$ $\forall\theta \in \Theta$, and any triplet of signals $\{i, j, k\} \in N$ such that:*

$$\frac{p_i'(\theta)}{p_i(\theta)} > \frac{p_j'(\theta)}{p_j(\theta)} > \frac{p_k'(\theta)}{p_k(\theta)}, \ \forall\theta \in \Theta. \tag{28}$$

For any given value of $\pi_s(\theta) \in \boldsymbol{\pi}(\theta)$, $\forall s \notin \{i, j, k\}$, if a change is induced in $\pi_i(\theta)$, then the new lottery belongs to $\mathbf{\Pi}(\theta)$ only if changes are also induced in $\pi_j(\theta)$ and $\pi_k(\theta)$, in opposite directions.

Under Assumption 1, if $\pi_1(\theta)$ is varied, then a change is also induced in $\pi_n(\theta)$ together with an opposite change in $\pi_s(\theta)$, for some $s \neq 1, n$. One can easily show that full surplus extraction is not attained unless at least one between $\pi_n(\theta)$ and $\pi_s(\theta)$ is a punishment, whereas $\pi_1(\theta)$ is a reward. Thus, being based on Lemma 1, one can conclude that the lottery which is most likely to satisfy (LL) is obtained by first setting $\pi_s(\theta) = \pi_n(\theta)$, $\forall s \neq 1, n$, in (27) and then checking that $\pi_n(\theta) \geq -L$. Not surprisingly, this lottery is tantamount to $\boldsymbol{\pi}^1(\theta)$, the one derived by GBS.[11] The intuition behind this result is understood by interpreting Lemma 1 with a similar reasoning to the discrete-type analysis. Begin by considering a lottery such that $\pi_j(\theta) > \pi_k(\theta)$. First decrease $\pi_j(\theta)$ and then increase $\pi_i(\theta)$ in such a way that the expected value of the lottery to type θ remains unchanged. Following these variations, a type θ^- slightly below θ becomes less motivated to pretend θ because it is less likely to draw signal i than signal j. On the opposite, a type θ^+ slightly above θ, which is more likely to draw signal i than signal j, becomes more prone to claim θ. To contain the attractiveness of report θ to type θ^+, the increase in $\pi_1(\theta)$ is limited by also inducing an increase in the profit in state k, which type θ^+ is less likely to draw. Provided that type θ^+ is also less likely to draw signal k than signal j, the decrease in $\pi_j(\theta)$ can be compensated by the increase in $\pi_i(\theta)$ and in $\pi_k(\theta)$ to such an extent that type θ^+ will be deterred from reporting θ. Overall, as long as the profit in state j does not fall below the profit in state k, (LL) is relaxed as $\pi_j(\theta)$ is decreased and $\pi_k(\theta)$ is increased, without tightening (27) and (3). A similar reasoning applies if we begin by considering a lottery such that, conversely, $\pi_j(\theta) < \pi_k(\theta)$, in which case the profit should be increased in state j and decreased in state k.

We now focus on the potential conflict between global incentive compatibility and limited liability. Recall that this conflict is ruled out in the problem analysed by GBS, due to the assumptions that the cost function is convex and the likelihood function of the reward signal is concave. Indeed, under those conditions, first best is attained by adopting the lottery $\boldsymbol{\pi}^1(\theta)$ $\forall\theta$, in line with Corollary 1.5 of RS.

The main question to our study is whether using the lottery $\boldsymbol{\pi}^1(\theta)$ is still the optimal strategy when the cost function is more concave than the probability function of the reward signal and (LL) is not binding under that lottery. More specifically, our aim is to understand whether and in which way P could take advantage of the liability slack to make the contract globally incentive compatible. To that end, the first step is to use the expression of $\pi_n(\theta)$ in (27) to reformulate (25) and (26) as presented here below.

Lemma 2 *Given (PC) and (LIC), (GIC) is satisfied if and only if the following two condi-*

[11] In their proof, GBS take any triplet of profits which includes the profit associated with signal 1 (*i.e.*, with the signal the conditional probability of which displays the property in our Assumption 1), and show that the other two profits should be equal to give the greatest chance of the limited liability constraints being satisfied. Lemma 1 emphasizes that this is due to property (28), which will be useful in our subsequent analysis.

tions are satisfied for any given $\theta \in \left(\underline{\theta}, \overline{\theta}\right)$:

$$C_\theta\left(q\left(\theta\right),\theta\right) \geq \left(\frac{p_1'\left(\theta\right)}{p_1(\theta)} - \frac{p_n'\left(\theta\right)}{p_n(\theta)}\right)\left[\frac{C\left(q\left(\theta\right),\theta\right) - C\left(q\left(\theta\right),\theta^-\right)}{\frac{p_n(\theta^-)}{p_n(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}}\right. \tag{29}$$
$$\left. + \sum_{s\neq 1,n}\pi_s\left(\theta\right)p_s(\theta)\left(\frac{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_s(\theta^-)}{p_s(\theta)}}{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_n(\theta^-)}{p_n(\theta)}} - \frac{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_S'(\theta)}{p_S(\theta)}}{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}}\right)\right], \ \forall\theta^- < \theta,$$

and

$$C_\theta\left(q\left(\theta\right),\theta\right) \leq \left(\frac{p_1'\left(\theta\right)}{p_1(\theta)} - \frac{p_n'\left(\theta\right)}{p_n(\theta)}\right)\left[\frac{C\left(q\left(\theta\right),\theta^+\right) - C\left(q\left(\theta\right),\theta\right)}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}}\right. \tag{30}$$
$$\left. + \sum_{s\neq 1,n}\pi_s\left(\theta\right)p_s(\theta)\left(\frac{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}} - \frac{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_s'(\theta)}{p_s(\theta)}}{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}}\right)\right], \ \forall\theta^+ > \theta.$$

Similarly to the discrete-type case, and for the reasons there explained, there is a potential conflict between (29) and (30). To avoid rise of the conflict, it is necessary to have the following condition satisfied for each possible triplet $\left\{\theta^-, \theta, \theta^+\right\} \in \Theta$:

$$\frac{C\left(q\left(\theta\right),\theta\right) - C\left(q\left(\theta\right),\theta^-\right)}{\frac{p_n(\theta^-)}{p_n(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}} - \frac{C\left(q\left(\theta\right),\theta^+\right) - C\left(q\left(\theta\right),\theta\right)}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}} \tag{31}$$
$$\leq \sum_{s\neq 1,n}\pi_s\left(\theta\right)p_s(\theta)\left(\frac{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}} - \frac{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_s(\theta^-)}{p_s(\theta)}}{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_n(\theta^-)}{p_n(\theta)}}\right).$$

Therefore, one needs first to check whether, for each possible report θ, there exists a lottery such that (31) holds without violating (LL). Once this is ascertained, one further needs to verify that such a lottery satisfies (29) and (30). As this is required for all possible pairs $\left\{\theta^-, \theta^+\right\}$, the analysis may look complex overall. The problem is tractable, in fact, thanks to the following result.

Lemma 3 (31) *is necessary and sufficient for* (29) *and* (30) *to hold.*

Once it is established that it suffices to check (31) to verify (29) and (30), it is possible to pin down the optimal incentive scheme according to the properties of the cost and the likelihood functions. To that end, it is useful to define:

$$\rho_s\left(\theta', \theta\right) \equiv \frac{p_s\left(\theta'\right) + \left(\theta - \theta'\right)p_s'\left(\theta'\right)}{p_s(\theta)}, \ \forall\theta' \neq \theta \in \Theta, \ \forall s \in N,$$

The magnitude of $\rho_s\left(\cdot,\cdot\right)$ is a measure of the curvature of the probability function of signal s. Indeed, $\rho_s\left(\theta', \theta\right) = 1$ if $p_s\left(\cdot\right)$ is linear, $\rho_s\left(\theta', \theta\right) < 1$ if $p_s\left(\cdot\right)$ is strictly convex, and $\rho_s\left(\theta', \theta\right) > 1$

if $p_s(\cdot)$ is strictly concave. Therefore, $\rho_s(\theta',\theta)$ can be used to assess how much the likelihood of signal s diverges for type θ' as compared to the likelihood of signal s for type θ. The more that $\rho_s(\theta',\theta)$ diverges from 1, the higher that the degree of convexity/concavity of $p_s(\cdot)$ is $\forall\theta'\neq\theta$, and the more that the likelihood of type θ' to draw signal s diverges from the likelihood of type θ. It can be shown that if

$$\frac{\rho_s(\theta',\theta)-\rho_1(\theta',\theta)}{\left|\frac{p_1(\theta')}{p_1(\theta)}-\frac{p_s(\theta')}{p_s(\theta)}\right|}<\frac{\rho_n(\theta',\theta)-\rho_1(\theta',\theta)}{\left|\frac{p_1(\theta')}{p_1(\theta)}-\frac{p_n(\theta')}{p_n(\theta)}\right|}, \tag{32}$$

then the term in brackets in the right-hand side of (31) is negative $\forall\theta$ such that $\theta^-\leq\theta\leq\theta^+$, with at least one of these inequalities holding strictly (the proof is found in Appendix G). Assuming that this is true, the lottery which is most likely to implement first best, denoted $\boldsymbol{\pi}^*(\theta)$, includes the following list of profits $\forall\theta\in\left(\underline{\theta},\overline{\theta}\right)$:

$$\begin{aligned}
\pi_1^*(\theta) &= \frac{C_\theta(q(\theta),\theta)-L\frac{p_n'(\theta)}{p_n(\theta)}}{p_1(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)}-\frac{p_n'(\theta)}{p_n(\theta)}\right)}-L && (33)\\
\pi_n^*(\theta) &= \frac{L\frac{p_1'(\theta)}{p_1(\theta)}-C_\theta(q(\theta),\theta)}{p_n(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)}-\frac{p_n'(\theta)}{p_n(\theta)}\right)}-L && (34)\\
\pi_s^*(\theta) &= -L,\ \forall s\neq 1,n. && (35)
\end{aligned}$$

Proposition 2 *Assume that $\theta\in\Theta$, $n\geq 3$ and (32) holds. First best is implemented if and only if either:*

$$\frac{C(q(\theta),\theta)-C\left(q(\theta),\theta^-\right)}{C\left(q(\theta),\theta^+\right)-C(q(\theta),\theta)}\leq\frac{p_1(\theta)-p_1(\theta^-)}{p_1(\theta^+)-p_1(\theta)},\ \forall\theta,\theta^-,\theta^+\in\Theta,\ \theta^-<\theta<\theta^+ \tag{36}$$

and

$$L\geq\left(C(q(\theta),\theta)-C\left(q(\theta),\theta^-\right)\right)\frac{p_1(\theta)}{p_1(\theta)-p_1(\theta^-)},\ \forall\theta^-,\theta\in\Theta,\ \theta^-<\theta \tag{37}$$

or (36) *is violated and:*

$$L\geq\frac{\frac{C(q(\theta),\theta)-C(q(\theta),\theta^-)}{\frac{p_n(\theta^-)}{p_n(\theta)}-\frac{p_1(\theta^-)}{p_1(\theta)}}-\frac{C(q(\theta),\theta^+)-C(q(\theta),\theta)}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}}}{-\sum_{s\neq1,n}p_s(\theta)\left(\frac{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}}-\frac{\frac{p_1(\theta^-)}{p_1(\theta)}-\frac{p_s(\theta^-)}{p_s(\theta)}}{\frac{p_1(\theta^-)}{p_1(\theta)}-\frac{p_n(\theta^-)}{p_n(\theta)}}\right)},\ \forall\theta,\theta^-,\theta^+\in\Theta,\ \theta^-<\theta<\theta^+. \tag{38}$$

This proposition extends Proposition 2 of GBS, according to which (37) is required under the assumption that the cost is convex in type, to the case in which the cost is possibly concave in type, as captured by (36) in line with Corollary 1.5 of RS, and, more importantly, to the case in which (36) does not hold jointly with (37) but first-best is still implemented.[12] This

[12]Notice that as θ^- approaches θ (37) reduces to (9) for $i=1$, which is the exact formulation in GBS. We present the condition as in (37) because this alternative formulation helps us stress that the necessity of the

result is useful in that it draws a single condition to be satisfied for first-best implementation when (36) does not hold, a condition which depends on how liable the agent is and on the properties of the cost and the likelihood functions. To interpret the result, it is first necessary to recall that it was obtained by identifying the lottery $\boldsymbol{\pi}^*(\theta)$ as being most likely to yield the first-best outcome. It is then useful to go through the following corollaries.

Corollary 1 $\pi_1^*(\theta) > \pi_1^1(\theta)$, $\pi_n^*(\theta) > \pi_n^1(\theta)$ *and* $\pi_s^*(\theta) < \pi_s^1(\theta)$, $\forall s \neq 1, n$, $\forall \theta \in (\underline{\theta}, \overline{\theta})$.

This corollary evidences in which way $\boldsymbol{\pi}^*(\theta)$ departs from the lottery pinned down by GBS. When the cost and the probability functions display the properties stated in Proposition 2, P should rely on Lemma 1 and proceed as follows. Starting from $\boldsymbol{\pi}^1(\theta)$, P should raise the profits in state 1 and n and decrease them in all other states. According to Lemma 1, P gains flexibility when switching from $\boldsymbol{\pi}^1(\theta)$ to a new lottery in which the profit in state 1 is raised and opposite changes are induced in the other profits. As explained in the discrete-type case, it is convenient to increase the profit of type θ in state 1 and decrease it in some state $s \neq 1$ because type θ^- is then led to bear a greater loss when reporting θ. This is because $\frac{p_1'(\theta)}{p_1(\theta)} > \frac{p_s'(\theta)}{p_s(\theta)}$, $\forall s \neq 1$, involving that type θ^- will obtain less with a signal it is more likely to draw and more with a signal it is less likely to draw. This process can be replicated for signal 1 and other $n-2$ signals with which profits higher than $-L$ are initially associated. On the other hand, for one signal realization the profit must be increased in order to weaken the incentive of type θ^- to exaggerate information. The remaining question is thus for which signal realization, beside 1, the profit should be increased and for which ones it should be decreased instead. Corollary 1 identifies those signals.

Corollary 2 (29) *is relaxed and* (30) *is tightened when* $\boldsymbol{\pi}^*(\theta)$ *replaces* $\boldsymbol{\pi}^1(\theta)$.

This result formalizes the impossibility of lessening the global incentives both to overstate and to understate information by switching from one lottery to another in $\Pi(\theta)$. However, provided that (32) holds, when replacing $\boldsymbol{\pi}^1(\theta)$ with $\boldsymbol{\pi}^*(\theta)$ the positive effect of type θ^- becoming less eager to claim θ prevails on the negative effect of type θ^+ becoming more eager to do so. Indeed, under (32), one has:

$$\frac{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}} < \frac{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_s(\theta^-)}{p_s(\theta)}}{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_n(\theta^-)}{p_n(\theta)}}, \ \forall \theta^-, \theta, \theta^+ \in \Theta : \theta \in (\theta^-, \theta^+), \tag{39}$$

which is the counterpart of (22) in a setting with more than three types. Under (39), it is easier to lessen the conflict between the incentive constraints "from below" and "from above" if the profits of type θ are decreased to $-L$ in all states but 1 and n, rather than in all states but n only. Remarkably, when (GIC) is not a concern as in the setting considered by GBS, it suffices to refer to the rate of change of the conditional probability to determine the lottery

condition only results from the incentives of lower types to exaggerate information.

that is most likely to eliminate the tension between local incentive compatibility and limited liability. However, this is no longer the only requirement to be met in terms of probabilities as it comes to the incentive scheme that makes the tension between (GIC) and (LL) weakest. The curvature of the function $p(\cdot)$ becomes important as well because the potential gains and losses from the different lies depend on how the probabilities of the signals vary with type. The next corollary lists the necessary and sufficient conditions for (32) to hold, and hence for (31) to be weakest.

Corollary 3 *For* (32) *to hold* $\forall s \neq 1, n$:

it is necessary that $\rho_s(\theta', \theta) < \max\{\rho_1(\theta', \theta), \rho_n(\theta', \theta)\}$ *and sufficient that either*

$$\rho_s(\theta', \theta) < \rho_1(\theta', \theta) < \rho_n(\theta', \theta)$$

or

$$\rho_n(\theta', \theta) < \rho_s(\theta', \theta) < \rho_1(\theta', \theta);$$

it is necessary and sufficient that $\rho_n(\theta', \theta) - \rho_s(\theta', \theta)$ *is "sufficiently large" when*

$$\rho_1(\theta', \theta) < \rho_s(\theta', \theta) < \rho_n(\theta', \theta),$$

and that $\rho_n(\theta', \theta) - \rho_s(\theta', \theta)$ *is "sufficiently small" when*

$$\rho_s(\theta', \theta) < \rho_n(\theta', \theta) < \rho_1(\theta', \theta).$$

Intuitively, because any decrease in $\pi_s(\theta)$, where $s \neq 1, n$, is compensated with an increase in both $\pi_1(\theta)$ and $\pi_n(\theta)$ (recall Lemma 1), the lottery $\boldsymbol{\pi}^*(\theta)$ cannot be employed unless at least one between $p_1(\cdot)$ and $p_n(\cdot)$ is less convex/more concave than the conditional probability of any other signal. If this is not the case, then incentives to understate information are too strong for (31) to be weakened through the adoption of $\boldsymbol{\pi}^*(\theta)$. Specifically, (30) is tightened more than (29) is relaxed (recall Corollary 2). The remaining conditions listed in Corollary 3 are sufficient conditions on the degree of concavity/convexity of the likelihood functions for (32) to hold.

In substance, as long as (LL) does not bind in $\boldsymbol{\pi}^1(\theta)$ at least for some θ, the gain that P obtains by moving away from that lottery in such a way as to take advantage of the slack of (LL), resides in that global incentive compatibility is reconciled with limited liability for a wider family of cost functions. That is, first best is at hand in a richer variety of contractual relationships. To see this, rewrite (38) as follows:

$$\begin{aligned}
&\frac{C\left(q\left(\theta\right),\theta\right)-C\left(q\left(\theta\right),\theta^{-}\right)}{C\left(q\left(\theta\right),\theta^{+}\right)-C\left(q\left(\theta\right),\theta\right)} \qquad (40)\\
\leq\ &\frac{p_1(\theta)-p_1\left(\theta^{-}\right)}{p_1\left(\theta^{+}\right)-p_1(\theta)}+\left(\frac{\frac{p_n(\theta^{-})}{p_n(\theta)}-\frac{p_1(\theta^{-})}{p_1(\theta)}}{\frac{p_1(\theta^{+})}{p_1(\theta)}-\frac{p_n(\theta^{+})}{p_n(\theta)}}-\frac{\frac{p_1(\theta)-p_1(\theta^{-})}{p_1(\theta)}}{\frac{p_1(\theta^{+})-p_1(\theta)}{p_1(\theta)}}\right)\\
&-L\frac{\frac{p_n(\theta^{-})}{p_n(\theta)}-\frac{p_1(\theta^{-})}{p_1(\theta)}}{C\left(q\left(\theta\right),\theta^{+}\right)-C\left(q\left(\theta\right),\theta\right)}\sum_{s\neq 1,n}p_s(\theta)\left(\frac{\frac{p_1(\theta^{+})}{p_1(\theta)}-\frac{p_s(\theta^{+})}{p_s(\theta)}}{\frac{p_1(\theta^{+})}{p_1(\theta)}-\frac{p_n(\theta^{+})}{p_n(\theta)}}-\frac{\frac{p_1(\theta^{-})}{p_1(\theta)}-\frac{p_s(\theta^{-})}{p_s(\theta)}}{\frac{p_1(\theta^{-})}{p_1(\theta)}-\frac{p_n(\theta^{-})}{p_n(\theta)}}\right),
\end{aligned}$$

and observe that the last two terms in the right-hand side of (40), which are both positive, do not appear in the right-hand side of (17).

Corollary 4 (38) *is weaker than* (17).

This involves that the restrictions on the cost function are weaker than the sufficient condition identified by RS. Hence, in situations in which the conditional probabilities satisfy the assumptions previously made, P attains incentive compatibility under milder conditions by switching from $\boldsymbol{\pi}^1(\theta)$ to $\boldsymbol{\pi}^*(\theta)$, $\forall\theta\in\left(\underline{\theta},\overline{\theta}\right)$. In fact, $\boldsymbol{\pi}^*(\theta)$ is the lottery such that the restrictions on the cost function are weakest. Furthermore, this outcome is achieved only if the extent of the liability is higher than required by GBS.

Corollary 5 (38) *implies* (37) *if and only if* (36) *is violated.*

There is a simple conclusion to be drawn from this result. P can shift from $\boldsymbol{\pi}^1(\theta)$ to $\boldsymbol{\pi}^*(\theta)$ as long as (37) is slack, and she can take advantage of that slackness to relax the incentive compatibility constraints.

5 A second-best analysis with discrete types

There are multiple possible departures from first best. One departure occurs when (36) is satisfied but (37) is not. That is the case GBS consider in their second-best analysis. In that case, local incentive compatibility cannot be attained without violating (LL) unless P deviates from the first-best allocation. When it is (36) to be violated instead, one possibility is (32) not holding in Proposition 2 for at least one of the signals 1 and n, selected according to Assumption 1. However, intuition suggests that the lottery which is most likely to attain first best will then have similar characteristics to $\boldsymbol{\pi}^*(\theta)$, except that a pair of signals other than $\{1,n\}$ will be selected to satisfy (PC) and (LIC), involving that (38) will be tighter. A more interesting possibility to consider is that (38) does not hold, thus ruling out the second option in Proposition 2, which otherwise applies when (36) is violated. This is the case we now turn to explore. Notice that because (38) implies (37) (Corollary 5), our investigation will also include the case considered by GBS.

Resting on our first-best analysis, two questions arise naturally with regards to the case in which (36) is violated and L is not high enough for (38) to hold. First, one would like to know whether also in this setting the optimal lottery departs from that of GBS. Second, one wonders whether there is any type to be conceded an information rent at the second-best optimum.

To reply these questions, we hereafter develop the second-best analysis considering again a setting with three types and three signals. Our motivation for this focus is that, whereas standard solution methods are unlikely to be applicable in more complex settings with a continuum of types, in the first-best analysis we saw that the economic forces at work are neatly highlighted in the simple three-type framework.[13] Hence, we see no reason why the results we will derive with three types should not carry over with a continuum of types, once the technical complications are taken into account.

Formally, take $\theta_t \in \Theta_3$ and $n = 3$. In this setting, one should refer to (15) and (16) rather than to (37). Moreover, (38) specifies as follows:

$$L \geq \frac{\frac{C(q(\theta_2),\theta_2)-C(q(\theta_2),\theta_1)}{\frac{p_3(\theta_1)}{p_3(\theta_2)}-\frac{p_1(\theta_1)}{p_1(\theta_2)}} - \frac{C(q(\theta_2),\theta_3)-C(q(\theta_2),\theta_2)}{\frac{p_1(\theta_3)}{p_1(\theta_2)}-\frac{p_3(\theta_3)}{p_3(\theta_2)}}}{-p_2(\theta_2)\left(\frac{\frac{p_1(\theta_3)}{p_1(\theta_2)}-\frac{p_2(\theta_3)}{p_2(\theta_2)}}{\frac{p_1(\theta_3)}{p_1(\theta_2)}-\frac{p_3(\theta_3)}{p_3(\theta_2)}} - \frac{\frac{p_1(\theta_1)}{p_1(\theta_2)}-\frac{p_2(\theta_1)}{p_2(\theta_2)}}{\frac{p_1(\theta_1)}{p_1(\theta_2)}-\frac{p_3(\theta_1)}{p_3(\theta_2)}}\right)} \tag{41}$$

To keep notation parsimonious, we continue to denote $q(\theta_t)$ the quantity recommended from a generic type θ_t in the second-best setting, with the understanding that it no longer refers to the first-best production level. Accordingly, $\pi_s(\theta_t)$ will denote the profit assigned for the production of that quantity in state s.

Assume that, as long as P insists on the first-best allocation, the incentive compatibility constraints whereby the extreme types θ_1 and θ_3 are unwilling to claim θ_2 cannot be satisfied at once. Then, the issue is whether any of these types should be conceded an information rent to be motivated to tell the truth, and whether and how it is possible to extract all surplus from type θ_2. For simplicity, we let the expected value of the lottery be $R(\theta_t) = \sum_{s \in N} \pi_s(\theta_t) p_s(\theta_t)$, $\forall \theta_t$. This can be used to derive the following expression of the profit accruing to type θ_t in state 1 :

$$\pi_1(\theta_t) = \frac{R(\theta_t)}{p_1(\theta_t)} - \pi_2(\theta_t)\frac{p_2(\theta_t)}{p_1(\theta_t)} - \pi_3(\theta_2)\frac{p_3(\theta_t)}{p_1(\theta_t)}. \tag{42}$$

Considered for $t = 2$, (42) further leads to the following formulation of the incentive constraints whereby θ_2 is not an attractive report to types θ_1 and θ_3 :

$$\pi_3(\theta_2)p_3(\theta_2) \leq \frac{R(\theta_1) - R(\theta_2)\frac{p_1(\theta_1)}{p_1(\theta_2)} - (C(q(\theta_2),\theta_2) - C(q(\theta_2),\theta_1))}{\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}} - \pi_2(\theta_2)p_2(\theta_2)\frac{\frac{p_2(\theta_1)}{p_2(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}}{\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}} \tag{43}$$

[13] Technically speaking, the first-order approach may not be applicable because the contractual allocation is not necessarily differentiable.

and

$$\pi_3(\theta_2)\,p_3(\theta_2) \;\geq\; \frac{R(\theta_2)\frac{p_1(\theta_3)}{p_1(\theta_2)} - R(\theta_3) - \left(C(q(\theta_2),\theta_3) - C(q(\theta_2),\theta_2)\right)}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}} \qquad (44)$$
$$-\pi_2(\theta_2)\,p_2(\theta_2)\frac{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_2(\theta_3)}{p_2(\theta_2)}}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}}.$$

The other two adjacent incentive constraints to be considered are those whereby type θ_2 is unwilling to announce θ_1 and θ_3. Using (42) for $t = 1$ and $t = 3$ respectively, they are written as

$$R(\theta_2) \;\geq\; R(\theta_1)\frac{p_1(\theta_2)}{p_1(\theta_1)} + C(q(\theta_1),\theta_1) - C(q(\theta_1),\theta_2) \qquad (45)$$
$$-\sum_{s\neq 1}\pi_s(\theta_1)\,p_s(\theta_1)\left(\frac{p_1(\theta_2)}{p_1(\theta_1)} - \frac{p_s(\theta_2)}{p_s(\theta_1)}\right)$$

and

$$R(\theta_2) \;\geq\; R(\theta_3)\frac{p_1(\theta_2)}{p_1(\theta_3)} + C(q(\theta_3),\theta_3) - C(q(\theta_3),\theta_2) \qquad (46)$$
$$-\sum_{s\neq 1}\pi_s(\theta_3)\,p_s(\theta_3)\left(\frac{p_1(\theta_2)}{p_1(\theta_3)} - \frac{p_s(\theta_2)}{p_s(\theta_3)}\right).$$

Taking all these constraints into account, we state the programme of P as follows:

$$\underset{\{q(\theta_t);\boldsymbol{\pi}(\theta_t),\forall\theta_t\}}{Max}\;\sum_{\theta_t\in\Theta_3}\left(S(q(\theta_t)) - C(q(\theta_t),\theta_t) - \sum_{s\in N}\pi_s(\theta_t)\,p_s(\theta_t)\right)f(\theta_t)$$

subject to

(43) – (46), (PC), (LL),

and we derive our first result in the second-best setting (see Appendix M for the proof).

Lemma 4 *At optimum,* (43) *and* (44) *are both binding and* $\pi_2(\theta_2) = -L$.

The reason why (43) and (44) are both binding is that, given the rents designed for the three types, an increase in $\pi_3(\theta_2)$ (as associated with a decrease in $\pi_1(\theta_2)$) strengthens the incentive of type θ_1 to claim θ_2 to the same extent that it lessens the incentive of type θ_3 to claim θ_2. In other words, the marginal cost of tightening (43) is equal to the marginal benefit of relaxing (44). Moreover, the reason why the profit designed for type θ_2 is optimally decreased to $-L$ in state 2 is that, as in the first-best setting, this makes it least likely that (43) conflicts with (44).

It follows that the second-best lottery is structured either as $\boldsymbol{\pi}^1(\theta)$ or as $\boldsymbol{\pi}^*(\theta)$, depending on whether or not (LL) is binding for the intermediate type in state 3 (this does not entail that quantities will be set at their first-best levels though). Interestingly, the value of L that separates the regime in which $\pi_3(\theta_2) = -L$ (and hence, the lottery is structured as $\boldsymbol{\pi}^1(\theta)$) from that in which $\pi_3(\theta_2) > -L$ (and hence, the lottery is structured as $\boldsymbol{\pi}^*(\theta)$) is also the value that separates the regime in which types θ_1 and θ_2 are assigned an information rent from that in which they are not.

Conditions (15) and (16) are violated We already mentioned that when the limited liability constraints bind in the programme of P the optimal lottery is structured as $\boldsymbol{\pi}^1(\theta)$. Then, P concedes the information rents reported here below.

Proposition 3 *Assume that* $\theta_t \in \Theta_3$ *and* $n = 3$. *Suppose that* (15) *and* (16) *are violated at the first-best quantities. The following information rents are conceded:*

$$
\begin{aligned}
R(\theta_1) &= \left(C(q(\theta_3),\theta_3) - C(q(\theta_3),\theta_2) - L\frac{p_1(\theta_3) - p_1(\theta_2)}{p_1(\theta_3)}\right)\frac{p_1(\theta_1)}{p_1(\theta_2)} \\
&\quad + \left(C(q(\theta_1),\theta_2) - C(q(\theta_1),\theta_1) - L\frac{p_1(\theta_2) - p_1(\theta_1)}{p_1(\theta_1)}\right)\frac{p_1(\theta_1)}{p_1(\theta_2)},
\end{aligned}
\tag{47}
$$

$$
R(\theta_2) = C(q(\theta_3),\theta_3) - C(q(\theta_3),\theta_2) - L\frac{p_1(\theta_3) - p_1(\theta_2)}{p_1(\theta_3)} \tag{48}
$$

and $R(\theta_3) = 0$.

The expressions in (47) and (48) evidence that rents are given up exactly because, otherwise, it would be impossible to satisfy (LIC) without violating (LL) ((15) and (16) are violated at the first-best allocation). Not surprisingly, such expressions are tantamount to those obtained by GBS in their second-best analysis. Given that the contractual solution in this case is known from their study, we do not insist on it and turn to consider situations in which (15) and (16) are satisfied, instead, at the first-best allocation.

Conditions (15) and (16) hold The novel aspect to our second-best analysis is that when (15) and (16) hold, and hence it is not an issue to have the local incentive constraints satisfied under limited liability, P does not need to decrease $\pi_3(\theta_2)$ to the minimum of $-L$ to retain all surplus. A simple way to see this is to verify that both (47) and (48) are negative when these conditions are satisfied. Thus, by setting $\pi_3(\theta_2)$ strictly above $-L$, P can lessen the conflict between (43) and (44) without tightening (45) and (46), which eliminates the necessity to concede information rents. However, P needs the availability of an instrument to ensure that (43) and (44) hold at once. This instrument will be the quantity of type θ_2, which (43) and (44) depend upon. Specifically, P will adjust $q(\theta_2)$ until (41) is satisfied as an equality. This requires lowering the difference:

$$
\frac{C(q(\theta_2),\theta_2) - C(q(\theta_2),\theta_1)}{\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}} - \frac{C(q(\theta_2),\theta_3) - C(q(\theta_2),\theta_2)}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}}
$$

below its first-best value. Hence, with $C_{q\theta}(q,\theta) > 0$, $q(\theta_2)$ will be distorted upwards at the second-best optimum.

Proposition 4 *Assume that* $\theta_t \in \Theta_3$ *and* $n = 3$. *Suppose that, at the first-best quantities,* (15) *and* (16) *are satisfied but there is no lottery such that* (43) *and* (44) *hold. At optimum,* $q(\theta_2)$ *is distorted above the first-best level and all surplus is extracted from the agent.*

6 Conclusion

In a principal-agent model with correlated information and limited liability on the agent's side, we showed that focusing on the full-rank condition, the most common approach in the literature, is not necessarily the best approach. Provided that there exist at least three informative signals, the conditional probabilities of which satisfy the monotonic likelihood ratio property, it is enough to verify that the agent's liability is sufficiently high to ascertain whether first best is implementable, which is very useful in applications. Whereas Bose and Zhao [1] investigate first-best implementation when the full-rank condition does not hold, we proved that the possibility of attaining the first-best outcome under limited liability is not necessarily determined by the way in which the conditional probabilities of the signals depart from the full-rank condition. Moreover, the existence of an exact relationship between the extent of the liability and the admissible degree of concavity of the cost function (when this is not convex in type) involves that the set of technologies for which first best is at reach under limited liability is richer than that considered by GBS.

As a general view, our study contributes to shedding light on how to attain incentive compatibility in situations in which the principal faces more than two possible types of agent and there are more than two informative signals to be used in contractual design. Our findings point to the conclusion that it might be with loss of generality to restrict attention to the two-type case, or to a binary signal, when exploring principal-agent relationships with correlated information and limited liability.

References

[1] Bose, S., and J. Zhao (2007), "Optimal use of correlated information in mechanism design when full surplus extraction may be impossible, *Journal of Economic Theory*, 135, 357-381

[2] Crémer, J., and R. McLean (1988), "Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions," *Econometrica*, 56(6), 1247-1258

[3] Demougin, D., and D. Garvie (1991), "Contractual Design With Correlated Information Under Limited Liability," *RAND Journal of Economics*, 22, 477-489

[4] Eso, P. (2005), "An optimal auction with correlated values and risk aversion," *Journal of Economic Theory*, 125, 78-89

[5] Gary-Bobo, R., and Y. Spiegel (2006), "Optimal State-Contingent Regulation under Limited Liability," *RAND Journal of Economics*, 37, 431-448

[6] Kessler, A.S., C. Lulfesmann and P.W. Schmitz (2005), "Endogenous punishments in agency with verifiable ex post information," *International Economic Review*, 46(4), 1207-1231

[7] Lewis, T.R., and D.E.M. Sappington (1989), "Countervailing Incentives in Agency Problems", *Journal of Economic Theory*, 49, 294-313

[8] Maggi, G., and A. Rodriguez-Clare (1995), "On Countervailing Incentives," *Journal of Economic Theory*, 66, 238-263

[9] McAfee, P.R., and P.J. Reny (1992), "Correlated Information and Mechanism Design," *Econometrica*, 60(2), 395-421

[10] Myerson, R. "Optimal auction design," *Mathematics of Operations Research*, 6(1), 58-73

[11] Riordan, M., and D.E.M. Sappington (1988), "Optimal Contracts with Public Ex Post Information," *Journal of Economic Theory*, 45, 189-199

A Full-rank condition and Assumption 1

Suppose that the vector $\mathbf{p}(\theta_1)$ lies in the convex hull generated by the other probability vectors. Then, there exists a vector $(\lambda_2, ..., \lambda_T)$, where $\lambda_t \in [0,1], \forall t \in \{2, ..., T\}$, and $\sum_{t=2}^{T} \lambda_t = 1$, such that:

$$p_s(\theta_1) = \lambda_2 p_s(\theta_2) + ... + \lambda_T p_s(\theta_T), \ \forall s \in N.$$

Let us use this for $s = 1$ and $s \neq 1$ jointly with Assumption 1. We get:

$$\begin{aligned} p_1(\theta_1) &= \lambda_2 p_1(\theta_2) + ... + \lambda_T p_1(\theta_T) \\ &\Leftrightarrow \frac{p_1(\theta_1)}{p_1(\theta_2)} = \lambda_2 \frac{p_1(\theta_2)}{p_1(\theta_2)} + ... + \lambda_T \frac{p_1(\theta_T)}{p_1(\theta_2)} > \lambda_2 \frac{p_s(\theta_2)}{p_s(\theta_2)} + ... + \lambda_T \frac{p_s(\theta_T)}{p_s(\theta_2)} = \frac{p_s(\theta_1)}{p_s(\theta_2)}. \end{aligned}$$

The inequality $\frac{p_1(\theta_1)}{p_1(\theta_2)} > \frac{p_s(\theta_1)}{p_s(\theta_2)}$ contradicts Assumption 1. Similarly, suppose that there exists a vector $(\lambda_1, ..., \lambda_{T-1})$, where $\lambda_t \in [0,1], \forall t \in \{1, ..., T-1\}$, and $\sum_{t=1}^{T-1} \lambda_t = 1$, such that:

$$p_s(\theta_T) = \lambda_1 p_s(\theta_1) + ... + \lambda_{T-1} p_s(\theta_{T-1}), \ \forall s \in N.$$

Let us use this for $s = 1$ and $s \neq 1$ jointly with Assumption 1. We get:

$$\begin{aligned} p_1(\theta_T) &= \lambda_1 p_1(\theta_1) + ... + \lambda_{T-1} p_1(\theta_{T-1}) \\ &\Leftrightarrow \\ \frac{p_1(\theta_T)}{p_1(\theta_{T-1})} &= \lambda_1 \frac{p_1(\theta_1)}{p_1(\theta_{T-1})} + ... + \lambda_{T-1} \frac{p_1(\theta_{T-1})}{p_1(\theta_{T-1})} < \lambda_1 \frac{p_s(\theta_1)}{p_s(\theta_{T-1})} + ... + \lambda_{T-1} \frac{p_s(\theta_{T-1})}{p_s(\theta_{T-1})} = \frac{p_s(\theta_T)}{p_s(\theta_{T-1})}. \end{aligned}$$

The inequality $\frac{p_1(\theta_T)}{p_1(\theta_{T-1})} < \frac{p_s(\theta_T)}{p_s(\theta_{T-1})}$ contradicts Assumption 1.

Next take the vector $\mathbf{p}(\theta_t)$, where $t \notin \{1, T\}$, to lie in the convex hull generated by the probability vectors of the other types. This is equivalent to telling that there exists a vector

$(\lambda_1, ..., \lambda_{t-1}, \lambda_{t+1}, ..., \lambda_T)$, where $\lambda_t \in [0,1]$ $\forall t \in \{1, ..., t-1, t+1, ..., T\}$ and $\sum_{t' \neq t} \lambda_{t'} = 1$, such that:

$$\begin{aligned} p_s(\theta_t) &= \lambda_1 p_s(\theta_1) + ... + \lambda_{t-1} p_s(\theta_{t-1}) + \lambda_{t+1} p_s(\theta_{t+1}) + ... + \lambda_T p_s(\theta_T) \\ &\Leftrightarrow \\ \frac{p_s(\theta_t)}{p_s(\theta_{t+1})} &= \lambda_1 \frac{p_s(\theta_1)}{p_s(\theta_{t+1})} + ... + \lambda_{t-1} \frac{p_s(\theta_{t-1})}{p_s(\theta_{t+1})} + \lambda_{t+1} \frac{p_s(\theta_{t+1})}{p_s(\theta_{t+1})} + ... + \lambda_T \frac{p_s(\theta_T)}{p_s(\theta_{t+1})}. \end{aligned} \tag{49}$$

By taking $\mathbf{p}(\theta_t)$ such that

$$\begin{aligned} & \frac{p_{s'}(\theta_t)}{p_{s'}(\theta_{t+1})} \\ > & \frac{p_1(\theta_t)}{p_1(\theta_{t+1})} \\ > & \frac{p_{s'}(\theta_t)}{p_{s'}(\theta_{t+1})} + \lambda_1 \left(\frac{p_1(\theta_1)}{p_1(\theta_{t+1})} - \frac{p_{s'}(\theta_1)}{p_{s'}(\theta_{t+1})} \right) + ... + \lambda_{t-1} \left(\frac{p_1(\theta_{t-1})}{p_1(\theta_{t+1})} - \frac{p_{s'}(\theta_{t-1})}{p_{s'}(\theta_{t+1})} \right), \ \forall s' \neq 1, \end{aligned}$$

both Assumption 1 and (49) are satisfied. To see this, first use (49) for $s = 1$ to rewrite the second inequality here above as:

$$\begin{aligned} \lambda_{t+1} \frac{p_1(\theta_{t+1})}{p_1(\theta_{t+1})} + ... + \lambda_T \frac{p_1(\theta_T)}{p_1(\theta_{t+1})} &> \frac{p_{s'}(\theta_t)}{p_{s'}(\theta_{t+1})} + \lambda_1 \left(-\frac{p_{s'}(\theta_1)}{p_{s'}(\theta_{t+1})} \right) + ... + \lambda_{t-1} \left(-\frac{p_{s'}(\theta_{t-1})}{p_{s'}(\theta_{t+1})} \right) \\ &= \lambda_{t+1} \frac{p_{s'}(\theta_{t+1})}{p_{s'}(\theta_{t+1})} + ... + \lambda_T \frac{p_{s'}(\theta_T)}{p_{s'}(\theta_{t+1})} \end{aligned}$$

Then use (49) for s' to rewrite:

$$\lambda_{t+1} \frac{p_1(\theta_{t+1})}{p_1(\theta_{t+1})} + ... + \lambda_T \frac{p_1(\theta_T)}{p_1(\theta_{t+1})} > \lambda_{t+1} \frac{p_{s'}(\theta_{t+1})}{p_{s'}(\theta_{t+1})} + ... + \lambda_T \frac{p_{s'}(\theta_T)}{p_{s'}(\theta_{t+1})},$$

which is true by Assumption 1.

B Derivation of (GIC) with $\theta \in \Theta$ and $n \geq 3$

Using $\widetilde{\pi}_s(\theta | \theta') = z_s(\theta) - C(q(\theta), \theta')$ and $\pi_s(\theta) = \widetilde{\pi}_s(\theta | \theta)$, we have:

$$\mathbb{E}_s [\widetilde{\pi}_s(\theta | \theta')] = \sum_{s=1}^{n} \pi_s(\theta) p_s(\theta') + C(q(\theta), \theta) - C(q(\theta), \theta').$$

Because full surplus extraction requires $\sum_{s=1}^n \pi_s(\theta) p_s(\theta) = 0$, this is rewritten as (GIC). Further using $\sum_{s=1}^n \pi_s(\theta) p_s(\theta) = 0 \Leftrightarrow \pi_1(\theta) = -\sum_{s=2}^n \pi_s(\theta) \frac{p_s(\theta)}{p_1(\theta)}$, (GIC) is further rewritten as:

$$C(q(\theta), \theta) - C(q(\theta), \theta') \leq \sum_{s \neq 1, n} \pi_s(\theta) p_s(\theta) \left(\frac{p_1(\theta')}{p_1(\theta)} - \frac{p_s(\theta')}{p_s(\theta)} \right) + \pi_n(\theta) p_n(\theta) \left(\frac{p_1(\theta')}{p_1(\theta)} - \frac{p_n(\theta')}{p_n(\theta)} \right),$$

hence:

$$\pi_n(\theta)\,p_n(\theta)\left(\frac{p_1(\theta')}{p_1(\theta)}-\frac{p_n(\theta')}{p_n(\theta)}\right)\geq C(q(\theta),\theta)-C(q(\theta),\theta')-\sum_{s\neq 1,n}\pi_s(\theta)\,p_s(\theta)\left(\frac{p_1(\theta')}{p_1(\theta)}-\frac{p_s(\theta')}{p_s(\theta)}\right). \tag{50}$$

Recall that, by assumption, $\frac{p_1(\theta')}{p_1(\theta)}>\frac{p_n(\theta')}{p_n(\theta)}$ if and only if $\theta'>\theta$. Using this equivalence for $\theta^-<\theta$ and $\theta^+>\theta$, (50) is respectively rewritten as (25) and (26).

Here below we specify (GIC) with $\theta_t\in\Theta_3$ in the two cases of $n=2$ and $n=3$.

B.1 The case of $\theta_t\in\Theta_3$ and $n=2$

In this case, (25) and (26) specify as (12), (14) and (13). To check that the global incentive constraints are satisfied, we need to verify that (12) and (14) are respectively satisfied for $\theta_t=\theta_3$ and $\theta_t=\theta_1$, if they are for θ_2. This is the case when:

$$\frac{C(q(\theta_3),\theta_3)-C(q(\theta_3),\theta_1)}{C(q(\theta_3),\theta_3)-C(q(\theta_3),\theta_2)}\leq\frac{\frac{p_2(\theta_1)}{p_2(\theta_3)}-\frac{p_1(\theta_1)}{p_1(\theta_3)}}{\frac{p_2(\theta_2)}{p_2(\theta_3)}-\frac{p_1(\theta_2)}{p_1(\theta_3)}} \tag{51}$$

$$\frac{C(q(\theta_1),\theta_2)-C(q(\theta_1),\theta_1)}{C(q(\theta_1),\theta_3)-C(q(\theta_1),\theta_1)}\leq\frac{\frac{p_1(\theta_2)}{p_1(\theta_1)}-\frac{p_2(\theta_2)}{p_2(\theta_1)}}{\frac{p_1(\theta_3)}{p_1(\theta_1)}-\frac{p_2(\theta_3)}{p_2(\theta_1)}}. \tag{52}$$

Using $p_2(\cdot)=1-p_1(\cdot)$, these conditions are rewritten as:

$$\begin{aligned}\frac{p_1(\theta_3)-p_1(\theta_1)}{p_1(\theta_3)-p_1(\theta_2)}&\geq\frac{C(q(\theta_3),\theta_3)-C(q(\theta_3),\theta_1)}{C(q(\theta_3),\theta_3)-C(q(\theta_3),\theta_2)}\\ \frac{p_1(\theta_2)-p_1(\theta_1)}{p_1(\theta_3)-p_1(\theta_1)}&\geq\frac{C(q(\theta_1),\theta_2)-C(q(\theta_1),\theta_1)}{C(q(\theta_1),\theta_3)-C(q(\theta_1),\theta_1)}.\end{aligned}$$

Replacing $p_1(\theta_3)-p_1(\theta_1)$ with $p_1(\theta_3)-p_1(\theta_2)+p_1(\theta_2)-p_1(\theta_1)$ and $C(q(\theta_3),\theta_3)-C(q(\theta_3),\theta_1)$ with $C(q(\theta_3),\theta_3)-C(q(\theta c_3),\theta_2)+C(q(\theta_3),\theta_2)-C(q(\theta_3),\theta_1)$, the two conditions further become:

$$\frac{C(q(\theta_3),\theta_2)-C(q(\theta_3),\theta_1)}{C(q(\theta_3),\theta_3)-C(q(\theta_3),\theta_2)}\leq\frac{p_1(\theta_2)-p_1(\theta_1)}{p_1(\theta_3)-p_1(\theta_2)} \tag{53}$$

$$\frac{C(q(\theta_1),\theta_2)-C(q(\theta_1),\theta_1)}{C(q(\theta_1),\theta_3)-C(q(\theta_1),\theta_2)}\leq\frac{p_1(\theta_2)-p_1(\theta_1)}{p_1(\theta_3)-p_1(\theta_2)}, \tag{54}$$

which are equivalent to (6) for specified quantities $q(\cdot)$.

B.2 The case of $\theta_t\in\Theta_3$ and $n=3$

Write (25) and (26) for $n=3$, $\theta=\theta_2$ and, respectively, $\theta^-=\theta_1$ and $\theta^+=\theta_3$. $\exists\pi_3(\theta_2)$ which satisfies both (25) and (26) if and only if (20) is satisfied. Being based on the equality

$$\frac{p_s(\theta')\,p_1(\theta)-p_1(\theta')\,p_s(\theta)}{p_1(\theta)p_s(\theta)}=\frac{p_1(\theta)-p_1(\theta')}{p_1(\theta)}-\frac{p_s(\theta)-p_s(\theta')}{p_s(\theta)},$$

the term multiplied by $\pi_2(\theta_2)$ in (20) is negative if and only if (22) holds.

B.3 Proof of Proposition 1

In the text.

C Derivation of (LIC) and (27)

Recall $\widetilde{\pi}_s(\theta|\theta') = z_s(\theta) - C(q(\theta), \theta')$ and

$$\mathbb{E}_s[\widetilde{\pi}_s(\theta|\theta')] \equiv \sum_{s=1}^{n} (z_s(\theta) - C(q(\theta), \theta')) p_s(\theta'). \tag{55}$$

The first-order condition of the agent's problem, evaluated at $\theta' = \theta$, is given by:

$$\sum_{s=1}^{n} (z'_s(\theta) - C_q(q(\theta), \theta) q_\theta(\theta)) p_s(\theta) = 0. \tag{56}$$

From $t_s(\theta) = \pi_s(\theta) + C(q(\theta), \theta)$, we compute $z'_s(\theta) = \pi'_s(\theta) + C_q(q(\theta), \theta) q_\theta(\theta) + C_\theta(q(\theta), \theta)$, which we then replace into (56) to get:

$$C_\theta(q(\theta), \theta) = -\sum_{s=1}^{n} \pi'_s(\theta) p_s(\theta). \tag{57}$$

Because $\sum_{s=1}^{n} \pi_s(\theta) p_s(\theta) = 0$, $\forall\theta$, implies $-\sum_{s=1}^{n} \pi'_s(\theta) p_s(\theta) = \sum_{s=1}^{n} \pi_s(\theta) p'_s(\theta)$, $\forall\theta$, (57) is further rewritten as (LIC).

(GIC) is in the proof of Two and three signals.

D Proof of Lemma 1

Suppose that some profit $\pi_i(\theta)$ is changed by ε. Accordingly, $\pi_j(\theta)$ is changed by ζ and $\pi_k(\theta)$ by δ such that (PC) is still saturated and the right-hand side of (LIC) does not vary. Dropping the argument θ everywhere for the sake of shortness, this requires:

$$\begin{aligned} \zeta p_j &= -\varepsilon p_i - \delta p_k \Leftrightarrow \zeta = -\varepsilon \frac{p_i}{p_j} - \delta \frac{p_k}{p_j} \\ \delta p'_k &= -\zeta p'_j - \varepsilon p'_i \Leftrightarrow \delta = -\zeta \frac{p'_j}{p'_k} - \varepsilon \frac{p'_i}{p'_k}. \end{aligned}$$

Replacing the expression of δ in that of ζ, we obtain:

$$\zeta = -\varepsilon \frac{p_i}{p_j} \frac{\frac{p'_i}{p_i} - \frac{p'_k}{p_k}}{\frac{p'_j}{p_j} - \frac{p'_k}{p_k}}. \tag{58}$$

Replacing (58) in the expression of δ, we further obtain:

$$\delta = \varepsilon \frac{p_i}{p_k} \frac{\frac{p_i'}{p_i} - \frac{p_j'}{p_j}}{\frac{p_j'}{p_j} - \frac{p_k'}{p_k}}. \tag{59}$$

Using (28) in (58) and (59), we deduce that $\mathrm{Si}\,gn\,(\zeta) \neq \mathrm{Si}\,gn\,(\delta)$.

E Proof of Lemma 2

Taking the expression of $\pi_n(\theta)\, p_n(\theta)$ from (27), pugging into (25) and making use of the inequalities $\frac{p_1'(\theta)}{p_1(\theta)} > \frac{p_n'(\theta)}{p_n(\theta)}$ and $\frac{p_n(\theta^-)}{p_n(\theta)} > \frac{p_1(\theta^-)}{p_1(\theta)}$ to rearrange, (25) is rewritten as (29). Similarly, (26) is rewritten as (30).

F Proof of Lemma 3

The necessity of (31) is obvious. To show sufficiency, we first let θ^+ tend to θ. Applying de L'Hopital's rule yields:

$$\frac{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}} = \frac{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_s'(\theta)}{p_s(\theta)}}{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}}.$$

Using this in (31), we obtain (29). Similarly, as θ^- tends to θ :

$$\frac{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_s(\theta^-)}{p_s(\theta)}}{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_n(\theta^-)}{p_n(\theta)}} = \frac{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_s'(\theta)}{p_s(\theta)}}{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}}.$$

Using this in (31), we obtain (30). Hence, (31) is sufficient as well.

G Proof of Proposition 2

G.1 Derivation of (38)

We see that

$$\frac{d}{d\theta^+}\left(\frac{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}}\right) < 0$$

if and only if

$$\frac{\frac{p_1'(\theta^+)}{p_1(\theta)} - \frac{p_s'(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}} < \frac{\frac{p_1'(\theta^+)}{p_1(\theta)} - \frac{p_n'(\theta^+)}{p_n(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}}. \tag{60}$$

Multiplying the numerator by $\theta^{+}-\theta$ in both sides, subtracting 1 from each side and manipulating further, (60) becomes:

$$\frac{\frac{p_s\left(\theta^{+}\right)-p_s'\left(\theta^{+}\right)\left(\theta^{+}-\theta\right)}{p_s(\theta)}-\frac{p_1\left(\theta^{+}\right)-p_1'\left(\theta^{+}\right)\left(\theta^{+}-\theta\right)}{p_1(\theta)}}{\frac{p_1\left(\theta^{+}\right)}{p_1(\theta)}-\frac{p_s\left(\theta^{+}\right)}{p_s(\theta)}}<\frac{\frac{p_n\left(\theta^{+}\right)-p_n'\left(\theta^{+}\right)\left(\theta^{+}-\theta\right)}{p_n(\theta)}-\frac{p_1\left(\theta^{+}\right)-p_1'\left(\theta^{+}\right)\left(\theta^{+}-\theta\right)}{p_1(\theta)}}{\frac{p_1\left(\theta^{+}\right)}{p_1(\theta)}-\frac{p_n\left(\theta^{+}\right)}{p_n(\theta)}}.$$

Using the definition of $\rho_s\left(\theta',\theta\right)$, this is rewritten as:

$$\frac{\rho_s\left(\theta^{+},\theta\right)-\rho_1\left(\theta^{+},\theta\right)}{\frac{p_1\left(\theta^{+}\right)}{p_1(\theta)}-\frac{p_s\left(\theta^{+}\right)}{p_s(\theta)}}<\frac{\rho_n\left(\theta^{+},\theta\right)-\rho_1\left(\theta^{+},\theta\right)}{\frac{p_1\left(\theta^{+}\right)}{p_1(\theta)}-\frac{p_n\left(\theta^{+}\right)}{p_n(\theta)}}, \tag{61}$$

which is satisfied by assumption.

We also see that:

$$\frac{d}{d\theta^{-}}\left(\frac{\frac{p_1\left(\theta^{-}\right)}{p_1(\theta)}-\frac{p_s\left(\theta^{-}\right)}{p_s(\theta)}}{\frac{p_1\left(\theta^{-}\right)}{p_1(\theta)}-\frac{p_n\left(\theta^{-}\right)}{p_n(\theta)}}\right)<0$$

if and only if

$$\frac{\frac{p_1'\left(\theta^{-}\right)}{p_1(\theta)}-\frac{p_s'\left(\theta^{-}\right)}{p_s(\theta)}}{\frac{p_s\left(\theta^{-}\right)}{p_s(\theta)}-\frac{p_1\left(\theta^{-}\right)}{p_1(\theta)}}>\frac{\frac{p_1'\left(\theta^{-}\right)}{p_1(\theta)}-\frac{p_n'\left(\theta^{-}\right)}{p_n(\theta)}}{\frac{p_n\left(\theta^{-}\right)}{p_n(\theta)}-\frac{p_1\left(\theta^{-}\right)}{p_1(\theta)}}. \tag{62}$$

Multiply both sides by $\left(\theta-\theta^{-}\right)$, subtract from either side and rearrange to obtain:

$$\frac{\frac{p_1\left(\theta^{-}\right)+p_1'\left(\theta^{-}\right)\left(\theta-\theta^{-}\right)}{p_1(\theta)}-\frac{p_s\left(\theta^{-}\right)+p_s'\left(\theta^{-}\right)\left(\theta-\theta^{-}\right)}{p_s(\theta)}}{\frac{p_s\left(\theta^{-}\right)}{p_s(\theta)}-\frac{p_1\left(\theta^{-}\right)}{p_1(\theta)}}>\frac{\frac{p_1\left(\theta^{-}\right)+p_1'\left(\theta^{-}\right)\left(\theta-\theta^{-}\right)}{p_1(\theta)}-\frac{p_n\left(\theta^{-}\right)+p_n'\left(\theta^{-}\right)\left(\theta-\theta^{-}\right)}{p_n(\theta)}}{\frac{p_n\left(\theta^{-}\right)}{p_n(\theta)}-\frac{p_1\left(\theta^{-}\right)}{p_1(\theta)}}.$$

Resting on the definition of ρ, this is rewritten as:

$$\frac{\rho_s\left(\theta^{-},\theta\right)-\rho_1\left(\theta^{-},\theta\right)}{\frac{p_s\left(\theta^{-}\right)}{p_s(\theta)}-\frac{p_1\left(\theta^{-}\right)}{p_1(\theta)}}<\frac{\rho_n\left(\theta^{-},\theta\right)-\rho_1\left(\theta^{-},\theta\right)}{\frac{p_n\left(\theta^{-}\right)}{p_n(\theta)}-\frac{p_1\left(\theta^{-}\right)}{p_1(\theta)}}, \tag{63}$$

which is satisfied by assumption.

Therefore, we have:

$$\frac{d}{d\theta^{+}}\left(\frac{\frac{p_1\left(\theta^{+}\right)}{p_1(\theta)}-\frac{p_s\left(\theta^{+}\right)}{p_s(\theta)}}{\frac{p_1\left(\theta^{+}\right)}{p_1(\theta)}-\frac{p_n\left(\theta^{+}\right)}{p_n(\theta)}}\right)<0 \text{ together with } \frac{d}{d\theta^{-}}\left(\frac{\frac{p_1\left(\theta^{-}\right)}{p_1(\theta)}-\frac{p_s\left(\theta^{-}\right)}{p_s(\theta)}}{\frac{p_1\left(\theta^{-}\right)}{p_1(\theta)}-\frac{p_n\left(\theta^{-}\right)}{p_n(\theta)}}\right)<0,$$

involving that the difference

$$\frac{\frac{p_1\left(\theta^{+}\right)}{p_1(\theta)}-\frac{p_s\left(\theta^{+}\right)}{p_s(\theta)}}{\frac{p_1\left(\theta^{+}\right)}{p_1(\theta)}-\frac{p_n\left(\theta^{+}\right)}{p_n(\theta)}}-\frac{\frac{p_1\left(\theta^{-}\right)}{p_1(\theta)}-\frac{p_s\left(\theta^{-}\right)}{p_s(\theta)}}{\frac{p_1\left(\theta^{-}\right)}{p_1(\theta)}-\frac{p_n\left(\theta^{-}\right)}{p_n(\theta)}}$$

is greatest as θ^- tends to θ and θ^+ tends to θ. For such values of θ^- and θ^+, the difference here above is found to be zero (by applying de L'Hopital's rule). Hence, for all pairs of types, the difference is non-positive. In definitive, for any given pair $\{\theta^-, \theta^+\}$ such that $\theta^- < \theta < \theta^+$, (31) is weakest if $\pi_s(\theta) = -L$, $\forall s \neq 1, n$. Substituting this value in (31) and rearranging yields (38).

G.2 Proof of (36) and (37)

Setting $\pi_s(\theta) = \pi_n(\theta)$ in (25), we see that $\pi_n(\theta) \geq -L$ if and only if (37) is satisfied. The fact that no other lottery satisfies (LL), if (LL) is not satisfied by $\boldsymbol{\pi}^1(\theta)$ (the lottery such that $\pi_s(\theta)$ is equal $\forall s \neq 1$), follows from Lemma 1.

Setting $\pi_s(\theta) = \pi_n(\theta)$ in (27) and then plugging the resulting expression of $\pi_n(\theta)$, we see that (25) and (26) are jointly satisfied if and only if (36) is satisfied.

H Proof of Corollary 1

Using $\pi_s(\theta) = -L$ in (27), $\pi_n(\theta)$ is rewritten as:

$$\pi_n(\theta) = \frac{L\sum_{s\neq 1,n} p_s(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_s'(\theta)}{p_s(\theta)}\right) - C_\theta(q(\theta), \theta)}{p_n(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}\right)}.$$

Replacing $\sum_{s\neq 1,n} p_s(\theta) = 1 - (p_1(\theta) + p_n(\theta))$ and $\sum_{s\neq 1,n} p_s'(\theta) = -(p_1'(\theta) + p_n'(\theta))$, $\pi_n(\theta)$ is further rewritten as (34).

Recalling that $\pi_1(\theta) = -\sum_{s=2}^n \pi_s(\theta)\frac{p_s(\theta)}{p_1(\theta)}$ because $\sum_{s=1}^n \pi_s(\theta) p_s(\theta) = 0$, and using $\pi_s(\theta) = -L$ and (34) in the expression of $\pi_1(\theta)$ we find:

$$\begin{aligned}
\pi_1(\theta) &= -\sum_{s\neq 1,n}^{n} \pi_s(\theta)\frac{p_s(\theta)}{p_1(\theta)} - \pi_n(\theta)\frac{p_n(\theta)}{p_1(\theta)} \\
&= \frac{L}{p_1(\theta)}\sum_{s\neq 1,n}^{n} p_s(\theta) - \left(\frac{L\frac{p_1'(\theta)}{p_1(\theta)} - C_\theta(q(\theta), \theta)}{p_n(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}\right)} - L\right)\frac{p_n(\theta)}{p_1(\theta)}
\end{aligned}$$

Replacing again $\sum_{s\neq 1,n} p_s(\theta) = 1 - (p_1(\theta) + p_n(\theta))$, $\pi_1(\theta)$ is further rewritten as (33).

We are left with checking that $\pi_1(\theta) \geq -L$ and $\pi_n(\theta) \geq -L$. The former is true because $p_n'(\theta) < 0$. The latter is implied by $\frac{p_1'(\theta)}{p_1(\theta)} > \frac{p_n'(\theta)}{p_n(\theta)}$ together with $C_\theta(q(\theta), \theta)\frac{p_1(\theta)}{p_1'(\theta)} \leq L$, which is implied by (37).

I Proof of Corollary 2

Recall that by applying de L'Hopital's rule one has:

$$\lim_{\theta^+\to\theta} \frac{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}} = \frac{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_s'(\theta)}{p_s(\theta)}}{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}}$$

and that:

$$\frac{d}{d\theta^+}\left(\frac{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}}\right)<0,\ \forall\theta^+>\theta$$

Hence, the term:

$$\sum_{s\neq 1,n}\pi_s(\theta)\,p_s(\theta)\left(\frac{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}}-\frac{\frac{p_1'(\theta)}{p_1(\theta)}-\frac{p_s'(\theta)}{p_s(\theta)}}{\frac{p_1'(\theta)}{p_1(\theta)}-\frac{p_n'(\theta)}{p_n(\theta)}}\right)$$

on the right-hand side of (30) is raised as $\pi_s(\theta)$ is decreased, so that (30) is relaxed. Also recall that:

$$\lim_{\theta^-\to\theta}\frac{\frac{p_1(\theta^-)}{p_1(\theta)}-\frac{p_s(\theta^-)}{p_s(\theta)}}{\frac{p_1(\theta^-)}{p_1(\theta)}-\frac{p_n(\theta^-)}{p_n(\theta)}}=\frac{\frac{p_1'(\theta)}{p_1(\theta)}-\frac{p_s'(\theta)}{p_s(\theta)}}{\frac{p_1'(\theta)}{p_1(\theta)}-\frac{p_n'(\theta)}{p_n(\theta)}},$$

and that:

$$\frac{d}{d\theta^-}\left(\frac{\frac{p_1(\theta^-)}{p_1(\theta)}-\frac{p_s(\theta^-)}{p_s(\theta)}}{\frac{p_1(\theta^-)}{p_1(\theta)}-\frac{p_n(\theta^-)}{p_n(\theta)}}\right)<0,\ \forall\theta^-<\theta.$$

Hence, also the term:

$$\sum_{s\neq 1,n}\pi_s(\theta)\,p_s(\theta)\left(\frac{\frac{p_1(\theta^-)}{p_1(\theta)}-\frac{p_s(\theta^-)}{p_s(\theta)}}{\frac{p_1(\theta^-)}{p_1(\theta)}-\frac{p_n(\theta^-)}{p_n(\theta)}}-\frac{\frac{p_1'(\theta)}{p_1(\theta)}-\frac{p_s'(\theta)}{p_s(\theta)}}{\frac{p_1'(\theta)}{p_1(\theta)}-\frac{p_n'(\theta)}{p_n(\theta)}}\right)$$

in the right-hand side of (29) is raised as $\pi_s(\theta)$ is decreased, so that (29) is tightened.

J Proof of Corollary 3

Condition (32) is satisfied if $\rho_s(\theta',\theta)<\rho_1(\theta',\theta)<\rho_n(\theta',\theta)$. We shall now consider cases in which one of these inequalities is violated.

First suppose that $\rho_s(\theta',\theta)>\rho_1(\theta',\theta)$ and $\rho_n(\theta',\theta)>\rho_1(\theta',\theta)$ for $\theta'\neq\theta$. Using these inequalities first for $\theta'=\theta^+$ and then for $\theta'=\theta^-$, we rewrite (32) as:

$$\frac{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_s(\theta^+)}{p_s(\theta)}}<\frac{\rho_n(\theta^+,\theta)-\rho_1(\theta^+,\theta)}{\rho_s(\theta^+,\theta)-\rho_1(\theta^+,\theta)}$$

and as:

$$\frac{\frac{p_n(\theta^-)}{p_n(\theta)}-\frac{p_1(\theta^-)}{p_1(\theta)}}{\frac{p_s(\theta^-)}{p_s(\theta)}-\frac{p_1(\theta^-)}{p_1(\theta)}}<\frac{\rho_n(\theta^-,\theta)-\rho_1(\theta^-,\theta)}{\rho_s(\theta^-,\theta)-\rho_1(\theta^-,\theta)}.$$

In either inequality, the left-hand side is greater than 1. It is thus necessary that $\rho_n(\theta',\theta)>\rho_s(\theta',\theta)$ and that the difference $\rho_n(\theta',\theta)-\rho_s(\theta',\theta)$ be sufficiently large.

Next suppose that $\rho_1(\theta',\theta)>\rho_n(\theta',\theta)$ whereas $\rho_s(\theta',\theta)<\rho_1(\theta',\theta)$. Using these inequalities first for $\theta' = \theta^+$ and then for $\theta' = \theta^-$, we rewrite (32) as:

$$\frac{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}} < \frac{\rho_1(\theta^+,\theta) - \rho_s(\theta^+,\theta)}{\rho_1(\theta^+,\theta) - \rho_n(\theta^+,\theta)}$$

and as:

$$\frac{\frac{p_n(\theta^-)}{p_n(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}}{\frac{p_s(\theta^-)}{p_s(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}} > \frac{\rho_1(\theta^-,\theta) - \rho_n(\theta^-,\theta)}{\rho_1(\theta^-,\theta) - \rho_s(\theta^-,\theta)}.$$

The left-hand side in the former condition is lower than 1; the left-hand side in the latter condition is above 1. For these two conditions to hold, it is sufficient that $\rho_s(\theta',\theta) > \rho_n(\theta',\theta)$. It is necessary that the difference $\rho_n(\theta',\theta) - \rho_s(\theta',\theta)$ be not too large.

We are left with the case in which $\rho_n(\theta',\theta) < \rho_1(\theta',\theta) < \rho_s(\theta',\theta)$. We see that (32) is violated.

K Proof of Corollary 4

Replacing $\pi_s(\theta) = -L$ in (31) and rearranging, (31) is rewritten as (40).

L Proof of Corollary 5

Comparing (37) with (38), we see that (38) is tighter than (37) if and only if:

$$\frac{\frac{C(q(\theta),\theta)-C(q(\theta),\theta^-)}{\frac{p_n(\theta^-)}{p_n(\theta)}-\frac{p_1(\theta^-)}{p_1(\theta)}} - \frac{C(q(\theta),\theta^+)-C(q(\theta),\theta)}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}}}{-\sum_{s\neq 1,n} p_s(\theta)\left(\frac{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}} - \frac{\frac{p_1(\theta^-)}{p_1(\theta)}-\frac{p_s(\theta^-)}{p_s(\theta)}}{\frac{p_1(\theta^-)}{p_1(\theta)}-\frac{p_n(\theta^-)}{p_n(\theta)}}\right)} > \left(C(q(\theta),\theta) - C(q(\theta),\theta^-)\right)\frac{p_1(\theta)}{p_1(\theta)-p_1(\theta^-)}.$$

Let us group the terms including $\left(C(q(\theta),\theta) - C(q(\theta),\theta^-)\right)$ to rewrite:

$$\begin{aligned}
&\left(C(q(\theta),\theta) - C(q(\theta),\theta^-)\right)\left[\frac{1}{\frac{p_n(\theta^-)}{p_n(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}}\right.\\
&+\frac{p_1(\theta)}{p_1(\theta)-p_1(\theta^-)}\left(\left(\frac{\frac{p_1(\theta^+)}{p_1(\theta)}\sum_{s\neq 1,n} p_s(\theta) - \sum_{s\neq 1,n} p_s(\theta^+)}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}}\right)\right.\\
&\left.\left.-\frac{\frac{p_1(\theta^-)}{p_1(\theta)}\sum_{s\neq 1,n} p_s(\theta) - \sum_{s\neq 1,n} p_s(\theta^-)}{\frac{p_1(\theta^-)}{p_1(\theta)} - \frac{p_n(\theta^-)}{p_n(\theta)}}\right)\right]\\
>&\ \frac{C(q(\theta),\theta^+) - C(q(\theta),\theta)}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}}
\end{aligned}$$

Using $\sum_{s\neq 1,n} p_s(\cdot) = 1 - p_1(\cdot) - p_n(\cdot)$ and rearranging further yields:

$$\left(C\left(q\left(\theta\right),\theta\right) - C\left(q\left(\theta\right),\theta^-\right)\right)\frac{p_1(\theta)}{p_1(\theta)-p_1(\theta^-)}\left(p_n(\theta) + \frac{1-p_n(\theta)-\frac{p_1(\theta)}{p_1(\theta^+)}\left(1-p_n(\theta^+)\right)}{\frac{p_1(\theta)}{p_1(\theta^+)}\left(\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}\right)}\right)$$

$$> \frac{C\left(q\left(\theta\right),\theta^+\right) - C\left(q\left(\theta\right),\theta\right)}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}}. \tag{64}$$

We now take the expression in brackets in the left-hand side of (64) and factorize $p_n(\theta)$ to develop as follows:

$$\begin{aligned}
&p_n\left(\theta\right)\left(1+\frac{1-p_n(\theta)-\frac{p_1(\theta)}{p_1(\theta^+)}\left(1-p_n(\theta^+)\right)}{\frac{p_1(\theta)}{p_1(\theta^+)}\left(p_n\left(\theta\right)\frac{p_1(\theta^+)}{p_1(\theta)}-p_n(\theta^+)\right)}\right)\\
=\;& p_n(\theta)\frac{p_n(\theta)-p_1(\theta)\frac{p_n(\theta^+)}{p_1(\theta^+)}+1-p_n(\theta)-\frac{p_1(\theta)}{p_1(\theta^+)}\left(1-p_n(\theta^+)\right)}{p_n(\theta)-p_1(\theta)\frac{p_n(\theta^+)}{p_1(\theta^+)}}\\
=\;& \frac{p_1(\theta^+)-p_1(\theta)}{p_1(\theta^+)-p_1(\theta)\frac{p_n(\theta^+)}{p_n(\theta)}}.
\end{aligned}$$

Using this, we can now rewrite (64) as:

$$\left(C\left(q\left(\theta\right),\theta\right) - C\left(q\left(\theta\right),\theta^-\right)\right)\frac{\frac{p_1(\theta^+)-p_1(\theta)}{p_1(\theta)-p_1(\theta^-)}}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}} > \frac{C\left(q\left(\theta\right),\theta^+\right) - C\left(q\left(\theta\right),\theta\right)}{\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}}$$

or, equivalently, as:

$$\frac{C\left(q\left(\theta\right),\theta\right) - C\left(q\left(\theta\right),\theta^-\right)}{C\left(q\left(\theta\right),\theta^+\right) - C\left(q\left(\theta\right),\theta\right)} > \frac{p_1(\theta)-p_1(\theta^-)}{p_1(\theta^+)-p_1(\theta)},$$

which means that (17) is violated. Therefore, (38) implies (37) if and only if (17) is violated.

M Proof of Lemma 4, Proposition 3 and 4

Denote $\gamma_s\left(\theta_t\right)$ the multiplier associated with (LL) when signal is s and type is θ_t, $\zeta\left(\theta_t\right)$ that associated with (PC) when type is θ_t, λ that associated with (43), μ that associated with (44),

β that associated with (46), δ that associated with (45. The Lagrangian of the programme is:

$$\sum_{\theta_t\in\Theta_3}\left(S\left(q\left(\theta_t\right)\right)-C\left(q\left(\theta_t\right),\theta_t\right)-R\left(\theta_t\right)\right)f\left(\theta_t\right)+\sum_{\theta_t\in\Theta_3}\sum_{s\in N}\gamma_s\left(\theta_t\right)\left(\pi_s\left(\theta_t\right)+L\right)+\sum_{\theta_t\in\Theta_3}\zeta\left(\theta_t\right)R\left(\theta_t\right)$$
$$+\mu\left\{\frac{R\left(\theta_1\right)-R\left(\theta_2\right)\frac{p_1(\theta_1)}{p_1(\theta_2)}-\left(C\left(q\left(\theta_2\right),\theta_2\right)-C\left(q\left(\theta_2\right),\theta_1\right)\right)-\pi_2\left(\theta_2\right)p_2(\theta_2)\left(\frac{p_2(\theta_1)}{p_2(\theta_2)}-\frac{p_1(\theta_1)}{p_1(\theta_2)}\right)}{\frac{p_n(\theta_1)}{p_n(\theta_2)}-\frac{p_1(\theta_1)}{p_1(\theta_2)}}\right.$$
$$\left.-\pi_3\left(\theta_2\right)p_3(\theta_2)\right\}$$
$$+\lambda\left\{\pi_3\left(\theta_2\right)p_3(\theta_2)\right.$$
$$\left.-\frac{R\left(\theta_2\right)\frac{p_1(\theta_3)}{p_1(\theta_2)}-R\left(\theta_3\right)-\left(C\left(q\left(\theta_2\right),\theta_3\right)-C\left(q\left(\theta_2\right),\theta_2\right)\right)-\pi_2\left(\theta_2\right)p_2(\theta_2)\left(\frac{p_1(\theta_3)}{p_1(\theta_2)}-\frac{p_2(\theta_3)}{p_2(\theta_2)}\right)}{\frac{p_1(\theta_3)}{p_1(\theta_2)}-\frac{p_3(\theta_3)}{p_3(\theta_2)}}\right\}$$
$$+\beta\left\{R\left(\theta_2\right)\frac{p_1(\theta_3)}{p_1\left(\theta_2\right)}-R\left(\theta_3\right)-\left(C\left(q\left(\theta_3\right),\theta_3\right)-C\left(q\left(\theta_3\right),\theta_2\right)\right)\frac{p_1(\theta_3)}{p_1\left(\theta_2\right)}\right.$$
$$\left.-\frac{p_1(\theta_3)}{p_1\left(\theta_2\right)}\sum_{s\neq 1}\pi_s\left(\theta_3\right)p_s(\theta_3)\left(\frac{p_s\left(\theta_2\right)}{p_s(\theta_3)}-\frac{p_1\left(\theta_2\right)}{p_1\left(\theta_3\right)}\right)\right\}\frac{p_1\left(\theta_2\right)}{p_1(\theta_3)}$$
$$+\delta\left\{R\left(\theta_2\right)\frac{p_1(\theta_1)}{p_1\left(\theta_2\right)}-R\left(\theta_1\right)+\left(C\left(q\left(\theta_1\right),\theta_2\right)-C\left(q\left(\theta_1\right),\theta_1\right)\right)\frac{p_1(\theta_1)}{p_1\left(\theta_2\right)}\right.$$
$$\left.+\frac{p_1(\theta_1)}{p_1\left(\theta_2\right)}\sum_{s\neq 1}\pi_s\left(\theta_1\right)p_s(\theta_1)\left(\frac{p_1\left(\theta_2\right)}{p_1\left(\theta_1\right)}-\frac{p_s\left(\theta_2\right)}{p_s(\theta_1)}\right)\right\}\frac{p_1\left(\theta_2\right)}{p_1(\theta_1)}.$$

We now characterize the solution.

First suppose that $\zeta\left(\theta_1\right)=\zeta\left(\theta_2\right)=0$. The Lagrangian is linear in both $R\left(\theta_2\right)\frac{p_1(\theta_3)}{p_1(\theta_2)}-R\left(\theta_3\right)$ and $R\left(\theta_1\right)-R\left(\theta_2\right)\frac{p_1(\theta_1)}{p_1(\theta_2)}$, with coefficients:

$$\beta\frac{p_1\left(\theta_2\right)}{p_1(\theta_3)}-\frac{\lambda}{\frac{p_1(\theta_3)}{p_1(\theta_2)}-\frac{p_n(\theta_3)}{p_n(\theta_2)}}$$
$$\frac{\mu}{\frac{p_n(\theta_1)}{p_n(\theta_2)}-\frac{p_1(\theta_1)}{p_1(\theta_2)}}-\delta\frac{p_1\left(\theta_2\right)}{p_1(\theta_1)}.$$

Suppose that $\beta=0$. Then, the former coefficient is negative, and hence $R\left(\theta_2\right)\frac{p_1(\theta_3)}{p_1(\theta_2)}-R\left(\theta_3\right)$ should be decreased until the point where the constraint with β is binding. Then, $\beta>0$, in contradiction with the hypothesis that $\beta=0$. Suppose that $\delta=0$. Then, the latter coefficient is positive, and hence $R\left(\theta_1\right)-R\left(\theta_2\right)\frac{p_1(\theta_1)}{p_1(\theta_2)}$ should be increased until the point where $\delta>0$, which contradicts the hypothesis that $\delta=0$. We thus conclude that if $\zeta\left(\theta_1\right)=\zeta\left(\theta_2\right)=0$, then both $\beta>0$ and $\delta>0$. Next suppose that $\zeta\left(\theta_1\right)>0$ and $\zeta\left(\theta_2\right)>0$. It is immediate to see that $\beta=0$ and $\delta=0$.

We now turn to show that $\gamma_3\left(\theta_2\right)=0$ is equivalent to $\beta>0$ and $\delta>0$, and hence it is equivalent to $\zeta\left(\theta_1\right)>0$ and $\zeta\left(\theta_2\right)>0$.

Suppose that $\gamma_3\left(\theta_2\right)=0$. The Lagrangian is linear in $\pi_3(\theta_2)$ with coefficient $(\lambda-\mu)\,p_3(\theta_2)$. If $\lambda>\mu=0$, then the Lagrangian increases with $\pi_3\left(\theta_2\right)$. Hence, $\pi_3\left(\theta_2\right)$ should be raised until the point where $\mu>0$, in contradiction with the hypothesis that $\mu=0$. Analogous contradiction emerges if we suppose that $\mu>\lambda=0$. Provided that at second best it cannot be $\lambda=\mu=0$ (as (43) and (44) do not hold jointly at the first-best allocation), it must be the case that $\lambda > 0$ and $\mu > 0$. Suppose that $\lambda \neq \mu$. As the two constraints with these multipliers are binding, it must be the case that $(\lambda - \mu)\, \pi_3(\theta_2)\, p_3(\theta_2) = 0$. However, if $\lambda \neq \mu$, then the Lagrangian either increases or decreases with $\pi_3(\theta_2)$, involving that it should be $\pi_n(\theta_2) \neq 0$, in contradiction with the requirement that $(\lambda - \mu)\, \pi_3(\theta_2)\, p_3(\theta_2) = 0$. We conclude that $\lambda = \mu$.

The Lagrangian is linear in $\pi_2(\theta_2)$ with the following coefficient:

$$p_2(\theta_2)\left(\lambda\frac{\frac{p_1(\theta_3)}{p_1(\theta_2)}-\frac{p_2(\theta_3)}{p_2(\theta_2)}}{\frac{p_1(\theta_3)}{p_1(\theta_2)}-\frac{p_3(\theta_3)}{p_3(\theta_2)}}-\mu\frac{\frac{p_2(\theta_1)}{p_2(\theta_2)}-\frac{p_1(\theta_1)}{p_1(\theta_2)}}{\frac{p_3(\theta_1)}{p_3(\theta_2)}-\frac{p_1(\theta_1)}{p_1(\theta_2)}}\right)=\lambda p_2(\theta_2)\left(\frac{\frac{p_1(\theta_3)}{p_1(\theta_2)}-\frac{p_2(\theta_3)}{p_2(\theta_2)}}{\frac{p_1(\theta_3)}{p_1(\theta_2)}-\frac{p_3(\theta_3)}{p_3(\theta_2)}}-\frac{\frac{p_2(\theta_1)}{p_2(\theta_2)}-\frac{p_1(\theta_1)}{p_1(\theta_2)}}{\frac{p_3(\theta_1)}{p_3(\theta_2)}-\frac{p_1(\theta_1)}{p_1(\theta_2)}}\right).$$

Relying on (39), this is found to be negative, involving that $\gamma_2(\theta_2) > 0$ and $\pi_2(\theta_2) = -L$, which completes the proof of Lemma 4.

We now verify the hypothesis that $\gamma_3(\theta_2) = 0$. Being based on the binding constraints with λ and μ, we see that $\pi_3(\theta_2) > -L$ if and only if the two conditions

$$\begin{aligned} & -Lp_3(\theta_2) \\ \leq\ & \frac{R(\theta_2)\frac{p_1(\theta_3)}{p_1(\theta_2)} - R(\theta_3) - \left(C(q(\theta_2),\theta_3) - C(q(\theta_2),\theta_2)\right) - \pi_2(\theta_2)\, p_2(\theta_2)\left(\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_2(\theta_3)}{p_2(\theta_2)}\right)}{\frac{p_1(\theta_3)}{p_1(\theta_2)} - \frac{p_3(\theta_3)}{p_3(\theta_2)}} \end{aligned}$$

and

$$\begin{aligned} & -Lp_3(\theta_2) \\ \leq\ & \frac{R(\theta_1) - R(\theta_2)\frac{p_1(\theta_1)}{p_1(\theta_2)} - \left(C(q(\theta_2),\theta_2) - C(q(\theta_2),\theta_1)\right) - \pi_2(\theta_2)\, p_2(\theta_2)\left(\frac{p_2(\theta_1)}{p_2(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}\right)}{\frac{p_3(\theta_1)}{p_3(\theta_2)} - \frac{p_1(\theta_1)}{p_1(\theta_2)}} \end{aligned}$$

are both satisfied. With $\pi_2(\theta_2) = -L$ these conditions are the same as the constraints with β and δ. Hence, if $\gamma_3(\theta_2) = 0$ and so $\pi_3(\theta_2) > -L$, then the constraints with β and δ are slack, in which case $\beta = 0$ and $\delta = 0$, further involving that $\zeta(\theta_1) > 0$ and $\zeta(\theta_2) > 0$. If $\gamma_3(\theta_2) > 0$ and so $\pi_3(\theta_2) = -L$, then $\beta > 0$ and $\delta > 0$, in which case $\zeta(\theta_1) = \zeta(\theta_2) = 0$. There are thus two solutions.

The first solution applies when $\gamma_3(\theta_2) > 0$, $\beta > 0$, $\delta > 0$ and $\zeta(\theta_1) = \zeta(\theta_2) = 0$. From the constraints associated with β and δ, we find

$$\begin{aligned} R(\theta_2) &= R(\theta_3)\frac{p_1(\theta_2)}{p_1(\theta_3)} + C(q(\theta_3),\theta_3) - C(q(\theta_3),\theta_2) \\ &\quad + \sum_{s\neq 1}\pi_s(\theta_3)\, p_s(\theta_3)\left(\frac{p_s(\theta_2)}{p_s(\theta_3)} - \frac{p_1(\theta_2)}{p_1(\theta_3)}\right) \qquad (65) \\ R(\theta_1) &= R(\theta_2)\frac{p_1(\theta_1)}{p_1(\theta_2)} + \left(C(q(\theta_1),\theta_2) - C(q(\theta_1),\theta_1)\right)\frac{p_1(\theta_1)}{p_1(\theta_2)} \qquad (66) \\ &\quad + \frac{p_1(\theta_1)}{p_1(\theta_2)}\sum_{s\neq 1}\pi_s(\theta_1)\, p_s(\theta_1)\left(\frac{p_1(\theta_2)}{p_1(\theta_1)} - \frac{p_s(\theta_2)}{p_s(\theta_1)}\right). \end{aligned}$$

Replacing in the Lagrangian, we see that P should set $R(\theta_3) = 0$ together with $\pi_s(\theta_1) = \pi_s(\theta_3) = -L$, $\forall s \neq 1$. Replacing $R(\theta_3) = 0$ and $\pi_2(\theta_3) = -L$ in (65) yields (48). Replacing the obtained value of $R(\theta_2)$ and $\pi_2(\theta_1) = -L$ in (66) yields (47). This completes the proof of Proposition 3.

The second solution applies when $\gamma_3(\theta_2) = 0$, $\zeta(\theta_1) > 0$, $\zeta(\theta_2) > 0$ and $\beta = \delta = 0$. Then, $R(\theta_1) = R(\theta_2) = 0$. Replacing these values in (43) and (44), together with $R(\theta_3) = 0$ and $\pi_2(\theta_2) = -L$, we obtain the necessary condition (41). This condition is binding because $\lambda > 0$ and $\mu > 0$, which completes the proof of Proposition 4.