Tournaments

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Abstract

We derive robust comparative statics for general rank-order tournaments with additive and multiplicative noise. For unimodal distributions of noise, we show that individual equilibrium effort is unimodal in the number of players when it is deterministic. For a stochastic number of players, the unimodality is preserved for changes in the number of players in the sense of first-order stochastic dominance under an additional log-supermodularity restriction. Aggregate equilibrium effort can be increasing, decreasing or nonmonotone in the number of players. The existing results for Tullock contests with stochastic participation follow as a special case. Equilibrium effort decreases as noise becomes more dispersed, in the sense of dispersive order or appropriately defined entropy.

Keywords: tournament, comparative statics, stochastic number of players, unimodality, log-supermodularity, entropy

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1 Introduction

Tournaments are environments in which participants compete for a valuable prize by spending effort or other resources. Examples include R&D races, rent-seeking, wars and conflicts, and tournaments in organizations where promotions or bonuses are based on the relative performance of workers. Starting with the seminal contributions of Tullock (1980) and Lazear and Rosen (1981) there is by now a substantial theoretical literature on tournaments using the respective models.[1] An important feature of these models distinguishing them from "perfectly discriminating" contests or all-pay auctions (e.g., Hillman and Riley, 1989; Baye, Kovenock and De Vries, 1996; Siegel, 2009) is the presence of uncertainty, or "noise," in the winner determination process.[2] Jia (2008) and Jia, Skaperdas and Vaidya (2013) provide a unified framework for the two prominent tournament models showing that the contest success function (CSF) of Tullock (1980) can be obtained as a special case of a Lazear-Rosen tournament.[3]

Yet, the existing analysis of *general* tournament models is quite scarce. For tractability reasons, most of the literature uses either the Tullock CSF (also known as the lottery contest) and its lottery-form generalizations satisfying the axioms of Skaperdas (1996), or the Lazear-Rosen tournament with two players.[4] Little, if anything, is known in general about the basic comparative statics of the rank-order tournament model. Specifically, it is unknown how the individual and aggregate equilibrium effort is affected by the number of players and the shape of the distribution of noise. Common wisdom suggests that as the number of players increases the individual probability of winning goes down and hence so does the marginal gain from increasing one's effort, leading to lower effort in equilibrium. This is indeed the case in the Tullock contest.[5] However, in a Lazear-Rosen tournament with a uniformly distributed noise the symmetric equilibrium effort is independent of the number of players. Since the two models have different underlying noise

[1] For a recent summary, see, e.g., Konrad (2009), Congleton, Hillman and Konrad (2008), Corchón (2007), Connelly et al. (2014).

[2] Throughout this paper, we focus exclusively on models of "imperfectly discriminating" contests with noise and use "tournament" as a unifying term for such models.

[3] While it has been known in the demand estimation literature for a long time that the logit model can be derived from the random utility model (McFadden, 1974), in the tournament literature the Tullock and the Lazear-Rosen tournament models have been treated as two completely unrelated models, with the exception of Jia, Skaperdas and Vaidya (2013).

[4] Notable exceptions are the papers analyzing optimal prize structures in tournaments with risk-averse players (Nalebuff and Stiglitz, 1983; Green and Stokey, 1983; Krishna and Morgan, 1998; Akerlof and Holden, 2012) and heterogeneity (Balafoutas et al., 2017). See also a survey of the earlier literature by McLaughlin (1988).

[5] See, for example, surveys by Nitzan (1994) and Corchón (2007).

distributions, this suggests that the shape of the distribution of noise plays an important role in equilibrium comparative statics. At the same time, aggregate equilibrium effort is increasing in the number of players in both cases. How universal are these results? Can individual equilibrium effort *increase* in the number of players or can it be nonmonotone? Can aggregate effort decrease in the number of players?

Similar unanswered questions exist about the effect of the distribution of noise. Intuitively, as noise becomes more dispersed, the marginal gain from increasing one's effort declines and hence equilibrium effort should go down. Indeed, when the distribution of noise is uniform with support $[-a, a]$, the equilibrium effort is proportional to $\frac{1}{2a}$, confirming the intuition. Consider, however, the distribution of noise with pdf $f(t) = \frac{|t|}{a^2}$ on the same support. Even though its variance is higher than that of the uniform distribution and, more generally, it is dominated by the uniform distribution in the sense of second-order stochastic dominance (SOSD), this distribution leads to a higher equilibrium effort than the uniform distribution in a two-player tournament. The reason is, as we show, that this distribution has a lower *entropy*, and it is the Rényi entropy, and not the variance or SOSD ordering, that determines the effect of noise on the equilibrium effort.

In this paper, we start by analyzing the comparative statics of a general Lazear-Rosen tournament model.[6] We show that, in general, there is nothing robust about the comparative statics. Individual equilibrium effort can be increasing, aggregate effort can be decreasing, and both can be nonmonotone in the number of players. We show that the *unimodality* of the distribution of noise allows for at least some degree of universality, namely, the unimodality of equilibrium effort in the number of players, and provide a general characterization of the comparative statics for unimodal noise distributions. In the absence of unimodality any universality is lost.

We then turn to the analysis of general tournaments with a stochastic number of players. Indeed, in many situations the number of competitors is unknown to the tournament participants at the time they decide how much to invest in competition. This would be the case, for example, in coding contests where an unknown and potentially very large number of coders submit their solutions; in hiring tournaments where a job seeker does not know how many others she is up against; or in promotion tournaments where an employee may not know how many of her colleagues the management is considering for

[6]We use the formulation with additive noise. Models with multiplicative noise, such as the Tullock contest, are transformed into an appropriately defined tournament with additive noise and hence their comparative statics follow as a special case of a more general theory, see Section 2.2 (cf. also Jia, Skaperdas and Vaidya, 2013).

a senior position. Following the tradition of the literature on auctions with a stochastic number of bidders (e.g., McAfee and McMillan, 1987; Harstad, Kagel and Levin, 1990; Levin and Ozdenoren, 2004), we assume an arbitrary distribution of the number of players and explore the effects on equilibrium effort of changes in the parameters of the distribution leading to first-order stochastic dominance (FOSD); that is, we explore the effects of a stochastic increase in the number of players.

Similar to the deterministic participation case, we show that the unimodality of the distribution of noise plays a key role in robust comparative statics. We show that the preservation of unimodality under uncertainty requires an additional log-supermodularity condition imposed on the distribution of the number of players. This condition follows from similar arguments to those identified by Athey (2002) for the preservation of single-crossing under uncertainty. We also explore the effects of noise dispersion and show that they are governed by an appropriate entropy defined through a combination of the distribution of noise and the tournament size distribution.

Contests with a stochastic number of players and endogenous entry have been studied previously using the lottery contest model of Tullock (1980) and its generalizations (Münster, 2006; Myerson and Wärneryd, 2006; Lim and Matros, 2009; Fu and Lu, 2010; Fu, Jiao and Lu, 2011). Münster (2006) explores the effect of risk-aversion. He shows that when participation probability is sufficiently low equilibrium effort increases in the number of potential players, both under risk-neutrality and risk-aversion. Overall, effort is lower under risk-aversion (as compared to risk-neutrality) when participation probability is low, but higher when it is high. For an arbitrary distribution of group size with expectation μ, Myerson and Wärneryd (2006) compare aggregate equilibrium contest expenditure when the number of players is uncertain to the case when the number of players is equal to μ with certainty. They show that aggregate expenditure is strictly lower in the former case if it is guaranteed that the contest has at least one participant. Lim and Matros (2009) show that, for the binomial distribution of contest size, the equilibrium effort is nonmonotone and single-peaked in the participation probability when the number of potential players $n > 2$. They also show that, as long as the participation probability is not too high, effort is nonmonotone in the number of potential contestants. Fu, Jiao and Lu (2011) study the effect of disclosure of the number of participating players on aggregate effort. They show that disclosure or nondisclosure may be optimal depending on the properties of the "impact function" in the generalized lottery-form CSF; in the special case of lottery CSF of Tullock (1980), the principal is indifferent between disclosure and nondisclosure. Finally, Fu and Lu (2010) study endogenous entry and the optimal allocation of winner's prize and participation subsidy/fee. There is no contest size uncertainty in their model, however, because entry occurs sequentially and each player observes the number of prior entrants. Fu and Lu (2010) find that the optimal contract extracts all surplus from the contestants and restricts participation to two active players. More generally, our paper is related to the literature on games with population uncertainty, including auctions[7] and Poisson games.[8]

The rest of the paper is organized as follows. In Section 2, we set up the tournament model with additive noise and show how the case of multiplicative noise is reduced to it as well. In Section 3 we provide general results on the preservation of unimodality under uncertainty that we use in the following sections. In Section 4, we focus on tournaments with deterministic participation and present the comparative statics with respect to the number of players. In Section 5, we move on to the analysis of the model with stochastic participation, and Section 6 concludes. Proofs that are missing in the main text are contained in Appendix A.

2 Model setup

2.1 Additive noise

There are $k \geq 2$ identical, risk-neutral players indexed by $i \in \mathcal{K} = \{1, \ldots, k\}$. All players $i \in \mathcal{K}$ simultaneously and independently choose efforts $e_i \geq 0$. The cost of effort e_i to player i is $c(e_i)$, where $c(\cdot)$ is strictly increasing, strictly convex, and twice differentiable on $(0, c^{-1}(1)]$, with $c(0) = 0$. Efforts e_i are perturbed by random additive shocks u_i to generate the players' output levels $y_i = e_i + u_i$. Shocks u_i are i.i.d. with cumulative distribution function (cdf) F and probability density function (pdf) f defined on interval support U. When necessary, we will use u_l and u_h to denote, respectively, the lower and upper bounds of U, which may be finite or infinite.[9] We assume that f is atomless, continuous and piece-wise differentiable in the interior of U, and has an inverse quantile density $m(z)$ (defined below) that is continuous and piece-wise differentiable on $(0, 1)$

[7]For a theoretical analysis of auctions with a stochastic number of bidders see, e.g., McAfee and McMillan (1987), Harstad, Kagel and Levin (1990) and Levin and Ozdenoren (2004). For a theoretical analysis of endogenous entry in auctions see, e.g., Levin and Smith (1994) and Pevnitskaya (2004).

[8]See, e.g., Myerson (1998, 2000); Makris (2008, 2009); De Sinopoli and Pimienta (2009); Mohlin, Östling and Wang (2015); Kahana and Klunover (2015, 2016).

[9]In this type of models, it is typically assumed that the shocks are zero-mean. While this assumption can be made without loss of generality, it is not necessary because the probability of winning is determined by differences in shocks.

and integrable on $[0,1]$. The winner of the tournament is the player whose output is the highest.[10] The winner receives a prize normalized to one, whereas all other players receive zero.[11]

For a given vector of efforts $(e_1,\dots,e_k)$, the probability of player $i \in \mathcal{K}$ winning the tournament is given by

$$\begin{aligned}\Pr(y_i > y_j \,\forall j \in \mathcal{K}\setminus\{i\}) &= \Pr(e_i + u_i > e_j + u_j \,\forall j \in \mathcal{K}\setminus\{i\}) \\ &= \int_U \left[\prod_{j\in\mathcal{K}\setminus\{i\}} F(e_i - e_j + t)\right] dF(t). \qquad (1)\end{aligned}$$

Consider a symmetric pure strategy Nash equilibrium in which all players choose effort $e^* > 0$. Using (1), the expected payoff of player $i \in \mathcal{K}$ from some deviation effort e_i is

$$\pi_i(e_i, e^*) = \int_U F(e_i - e^* + t)^{k-1} dF(t) - c(e_i). \qquad (2)$$

The first-order condition for payoff maximization evaluated at $e_i = e^*$, $\left.\frac{\partial \pi_i(e_i,e^*)}{\partial e_i}\right|_{e_i=e^*} = 0$, gives

$$b_k = c'(e^*), \qquad b_k = (k-1)\int_U F(t)^{k-2} f(t) dF(t). \qquad (3)$$

Let $F^{-1}(z) = \inf\{t \in U : F(t) \geq z\}$ denote the quantile function of the distribution of noise. It is convenient to introduce an unnormalized density function $m(z) = f(F^{-1}(z))$, known as the inverse quantile density function (Parzen, 1979). Using the probability integral transformation $z = F(t)$, rewrite b_k in Eq. (3) as

$$b_k = (k-1)\int_0^1 z^{k-2} m(z) dz. \qquad (4)$$

Note that $c'(e^*)$ is a strictly increasing function; therefore, if Eq. (3) has a solution, it is positive and unique for $k \geq 2$. In what follows we assume that such a solution, e_k^*, exists, and that it is a symmetric pure strategy equilibrium, i.e., $e_i = e_k^*$ is the global maximum of function $\pi_i(e_i, e_k^*)$ given by (2).[12]

[10] Ties are broken randomly but, under the assumption of atomless f, occur with probability zero.

[11] A more general setting could involve up to n distinct prizes; however, in this paper we are not concerned with optimal contract design, and use the simplest "winner-take-all" prize structure.

[12] Equilibrium existence and comparative statics are two separate issues, and here we focus on the latter, leaving the discussion of equilibrium existence (and uniqueness) outside the scope of this paper. In the Lazear-Rosen tournament model, these are still open questions. It is generally understood that

2.2 Multiplicative noise

Via simple transformations of the distribution of noise and the cost of effort, the model above accommodates tournaments with multiplicative noise where player i's output is given by $y_i = e_i u_i$ and u_i are i.i.d. with a nonnegative support. The probability of player i winning the tournament of k players can then be written as

$$\Pr(e_i u_i > e_j u_j \ \forall j \in \mathcal{K} \setminus \{i\}) = \Pr(x_i + v_i > x_j + v_j \ \forall j \in \mathcal{K} \setminus \{i\}),$$

where $x_i = \ln e_i$ and $v_i = \ln u_i$. Defining $\hat{F}(v) = F(\exp(v))$ as the cdf of the transformed shocks v_i and $\hat{c}(x) = c(\exp(x))$ as the cost function for the transformed effort x, this model is reduced to the tournament model with additive noise, and all the results above go through.

Specifically, the first-order condition (3) for the transformed equilibrium effort, $x_k^* = \ln e_k^*$, is $\hat{b}_k = \hat{c}'(x_k^*)$, where $\hat{b}_k$ is based on distribution $\hat{F}$. Interestingly,

$$\hat{c}'(x) = c'(\exp(x)) \exp(x) = c'(e)e;$$

therefore, the first-order condition for the original equilibrium effort is $\hat{b}_k = c'(e_k^*)e_k^*$. This leads to the following proposition.

Proposition 1 *The symmetric equilibrium effort in a tournament with multiplicative noise is the same as in the tournament with additive noise distributed with cdf* $\hat{F}(v) = F(\exp(v))$ *and the cost of effort* $c_{\mathrm{m}}(e) = \int_0^e c'(x)x dx$.

Tullock contests

As an illustration, consider contests with the CSF of Tullock (1980). The probability of player i winning the contest of size k is given by $\frac{e_i^r}{\sum_{j=1}^k e_j^r}$, where $r > 0$ is a parameter measuring the level of noise (the "discriminatory power" of the contest) such that a lower r corresponds to higher noise. The cost of effort is linear, $c(e) = e$. Following Jia (2008), this probability of winning can be written as $\Pr(e_i u_i > e_j u_j \ \forall j \in \mathcal{K} \setminus \{i\})$ where $u_j > 0$ are i.i.d. with the Generalized Inverse Exponential (or inverse Weibull) distribution with cdf $F(u) = \exp(-u^{-r})$.

the symmetric pure strategy equilibrium exists if the variance of shocks u_i is sufficiently large and/or the effort cost function $c(\cdot)$ is sufficiently convex, cf. Nalebuff and Stiglitz (1983). Note that the second-order condition and the requirement that zero effort is not a best response are not sufficient for e_k^* to be a symmetric equilibrium because function $\pi_i(e_i, e_k^*)$ may have multiple local maxima in e_i. For completeness, we provide the second-order condition in Appendix A.

That is, the Tullock contest can be represented as a tournament with multiplicative noise. We can now use Proposition 1 to transform it into a tournament with additive noise. The transformed shocks $v_i = \ln u_i$ have the Generalized Type-I Extreme Value (or Gumbel) distribution with cdf $\hat{F}(v) = F(\exp(v)) = \exp[-\exp(-rv)]$ and pdf $\hat{f}(v) = r\exp[-rv - \exp(-rv)]$ (see Jia, Skaperdas and Vaidya, 2013). This pdf is unimodal, with a maximum at zero, and skewed to the right. The transformed cost of effort is $c_{\mathrm{m}}(e) = \int_0^e x dx = \frac{e^2}{2}$. The first-order condition then takes the form $\hat{b}_k = e_k^*$, where $\hat{b}_k$ is given by Eq. (4) with $m(z) = \hat{f}(\hat{F}^{-1}(z)) = -rz\ln z$,

$$\hat{b}_k = -r(k-1)\int_0^1 z^{k-2}\ln z dz = \frac{r(k-1)}{k^2}, \tag{5}$$

which is the equilibrium effort in the Tullock contest.

This approach can be further generalized to cover contests with a CSF of the form $\frac{h(e_i)}{\sum_{j=1}^k h(e_j)}$, where $h(\cdot)$ is a strictly increasing "impact function," and a possibly nonlinear cost of effort $c(e_i)$. By introducing effective efforts $x_i = h(e_i)$ and costs $C(x_i) = c(h^{-1}(x_i))$, such models are reduced to the Tullock contest with $r = 1$, and the results above apply. Specifically, Proposition 1 implies that the symmetric equilibrium level of effective effort, x^*, satisfies the equation $\frac{k-1}{k^2} = C'(x^*)x^*$, where $C'(x) = \frac{c(h^{-1}(x))}{h'(h^{-1}(x))}$. Substituting back $x^* = h(e_k^*)$, obtain for the equilibrium effort $\frac{k-1}{k^2} = \frac{c'(e_k^*)h(e_k^*)}{h'(e_k^*)}$.

3 Preservation of unimodality under uncertainty

In what follows, we explore the comparative statics of individual and aggregate equilibrium effort in tournaments with respect to the number of players, k. First, in Section 4, we assume that k is deterministically given; then, in Section 5, we allow k to be a realization of a nonnegative integer random variable with some probability mass function (pmf). In the latter case, we explore the comparative statics with respect to changes in the parameters of the pmf leading to first-order stochastic dominance (FOSD).

In both cases, we show that robust comparative statics can be obtained for *unimodal* distributions of noise $f(t)$. These comparative statics amount to preservation of unimodality under uncertainty. Indeed, note that coefficients b_k, Eq. (4), which determine the comparative statics in the case of deterministic group size, can be written in the from $b_k = \int_0^1 m(z)dz^{k-1}$, i.e., as expectations of inverse quantile density $m(z)$ with respect to an FOSD-ordered family of cdfs $F_{(k-1)}(z) = z^{k-1}$. Our first lemma in this section provides a necessary and sufficient condition for such expectations, generally of the form $\gamma(\theta) = \int_0^1 a(z)dH(z,\theta)$, where cdfs $H(z,\theta)$ are FOSD-ordered in θ, to be unimodal in θ for all unimodal functions $a(z)$. When we turn to the case of stochastic group size, equilibrium effort will be determined by discrete expectations of the form $\chi(\theta) = \sum_{k=1}^{n} x_k y_k(\theta)$, where $x = \{x_k\}_{k=1}^n$ is some sequence and $y(\theta) = \{y_k(\theta)\}_{k=1}^n$ is an FOSD-ordered family of pmfs. The second lemma in this section establishes a necessary and sufficient condition for such expectations to be unimodal in θ for all unimodal sequences x. We start with some definitions. All missing proofs are in Appendix A.

Definition 1 *A function (or sequence) $\phi : S \to \mathbb{R}$, where $S \subseteq \mathbb{R}$, is unimodal if there exists a $\hat{t} \in S$ such that $\phi(t)$ is nondecreasing for $t \leq \hat{t}$ and nonincreasing for $t \geq \hat{t}$. A function (or sequence) is interior unimodal if it is unimodal and nonmonotone.*

Definition 2 *A function $\psi : S_1 \times S_2 \to \mathbb{R}$, where $S_1, S_2 \subseteq \mathbb{R}$, is log-supermodular if for all $t_1, t_1' \in S_1$, $t_2, t_2' \in S_2$, such that $t_1' > t_1$ and $t_2' > t_2$,*

$$\psi(t_1, t_2')\psi(t_1', t_2) \leq \psi(t_1, t_2)\psi(t_1', t_2').$$

In other words, for all $t_2' > t_2$ the ratio $r(t_1, t_2, t_2') = \frac{\psi(t_1,t_2')}{\psi(t_1,t_2)}$ is nondecreasing in t_1.

Consider integrals of the form $\gamma(\theta) = \int_0^1 a(z)dH(z,\theta)$, where $a(z) : [0,1] \to \mathbb{R}$ is an integrable, continuous and piece-wise differentiable function and $H(z,\theta)$ is a cdf of a random variable $Z|\theta$ defined on $[0,1]$ and parameterized by $\theta \in \Theta \subseteq \mathbb{R}$.[13] We assume that an increase in θ leads to an upward probabilistic shift, in the FOSD sense, of $Z|\theta$; that is, $H(z,\theta)$ is nonincreasing in θ for all $z \in [0,1]$ and $\theta \in \Theta$. Let $H_\theta(z,\theta) \leq 0$ denote the derivative of $H(z,\theta)$ with respect to θ if θ is a continuous parameter (in which case we assume that $H(z,\theta)$ is differentiable) or the first difference, $H(z,\theta+d) - H(z,\theta)$, if θ is a discrete index with step size $d > 0$.

Lemma 1 *$\gamma(\theta)$ is unimodal for all unimodal functions $a(z)$ if and only if $|H_\theta(z,\theta)|$ is log-supermodular; that is, the ratio $r(z,\theta,\theta') = \frac{H_\theta(z,\theta')}{H_\theta(z,\theta)}$ is nondecreasing in z for any $\theta' > \theta$.*

Consider now sums of the form $\chi(\theta) = \sum_{k=1}^{n} x_k y_k(\theta)$, where x is a nonnegative sequence and $y(\theta) = (y_1(\theta), \ldots, y_n(\theta))$ is a pmf parameterized by $\theta \in \Theta \subseteq \mathbb{R}$. We will use

[13] Variables $Z|\theta$ do not have to have the same support; rather, we assume that $[0,1]$ includes all of their supports, and $H(0,\theta) = 1 - H(1,\theta) = 0$ for all $\theta \in \Theta$.

$Y_k(\theta) = \sum_{l=1}^{k} y_l(\theta)$ to denote the corresponding cumulative mass function (cmf), with $Y_n(\theta) = 1$. The upper bound of the sum, $n \geq 2$, can be finite or infinite and applies uniformly for all values of θ.[14] We assume that an increase in θ shifts the distribution $y(\theta)$ upward in the FOSD sense. Let $Y_k'(\theta) \leq 0$ denote the derivative or the first difference of the cmf with respect to θ.

Lemma 2 *$\chi(\theta)$ is unimodal for all unimodal sequences x if and only if $|Y_k'(\theta)|$ is log-supermodular; that is, the ratio $r(k,\theta,\theta') = \frac{Y_k'(\theta')}{Y_k'(\theta)}$ is nondecreasing in k for any $\theta' > \theta$.*

In some cases, the log-supermodularity condition of Lemma 2 may be difficult to check directly because there is no closed-form expression for the cmf $Y_k(\theta)$. The following lemma shows that a similar ratio condition can instead be checked for the probability-generating function (pgf) of distribution $y(\theta)$, defined as $G(z,\theta) = \sum_{k=1}^{n} y_k(\theta) z^{k-1}$. Probabilities $y_k(\theta)$ can be recovered from it as $y_k(\theta) = \frac{1}{(k-1)!} G^{(k-1)}(0,\theta)$. Moreover, the pgf can be related to the cmf $Y(\theta)$ as

$$\sum_{k=1}^{n} Y_k(\theta) z^{k-1} = \frac{G(z,\theta) - z^{n-1}}{1-z}. \tag{6}$$

It follows from Eq. (6) that $G(z,\theta)$ is nonincreasing in θ whenever $Y_k(\theta)$ is nonincreasing in θ for all k; that is, $G(z,\theta)$ behaves as an FOSD-ordered family of cdfs (except that $G(0,\theta) = y_1(\theta)$, which is, generally, nonzero). Let $G_\theta(z,\theta) \leq 0$ denote, similar to $H_\theta(z,\theta)$ in Lemma 1, either the derivative or the first difference of $G(z,\theta)$ with respect to θ.

Lemma 3 *$|G_\theta(z,\theta)|$ is log-supermodular if and only if $|Y_k'(\theta)|$ is log-supermodular; that is, the ratio $R(z,\theta,\theta') = \frac{G_\theta(z,\theta')}{G_\theta(z,\theta)}$ is nondecreasing in z for any $\theta' > \theta$ if and only if the ratio $r(k,\theta,\theta')$ in Lemma 2 is nondecreasing in k for any $\theta' > \theta$.*

The nondecreasing ratio conditions in Lemmas 1, 2 and 3 are well-known in the literature on comparative statics under uncertainty (Athey, 2002). They are also known as total positivity of order 2 (Karlin, 1968), and increasing likelihood ratio properties when applied to parameterized probability density functions (see, e.g., Shaked and Shanthikumar, 2007). Our results are most closely related to those of Athey (2002) on the comparative statics of expectations of the form $\gamma(\theta) = \int_0^1 a(z) dH(z,\theta)$ for single-crossing

[14] This is not to say that $y(\theta)$ have the same support for all $\theta \in \Theta$; rather, $n = \sup_{\theta \in \Theta} n(\theta)$, where $n(\theta)$ is the upper bound of the support of $y(\theta)$. The definitions of $y(\theta)$ are extended to the uniform support so that $y_k(\theta) = 0$ and $Y_k(\theta) = 1$ for $k > n(\theta)$.

functions $a(z)$. Lemma 1 is a straightforward corollary of these results applied to unimodal functions, i.e., functions with a single-crossing derivative. Indeed, assuming $a(1)$ is finite (which is the case for interior unimodal functions) and integrating by parts, $\gamma(\theta) = a(1) - \int_0^1 a'(z)H(z,\theta)dz$, where $a'(z)$ is single-crossing and hence, following Athey (2002), $\gamma'(\theta) = \int_0^1 a'(z)|H_\theta(z,\theta)|dz$ is single-crossing, i.e., $\gamma(\theta)$ is unimodal, if $|H_\theta(z,\theta)|$ is log-supermodular. Lemma 2 is a discrete version of Lemma 1 and follows similarly via "summation by parts." Lemma 3, however, is less straightforward; the equivalence of log-supermodality of a discrete cdf and the corresponding pgf is a new result with potentially broader applications.

4 Tournaments with deterministic group size

4.1 Individual equilibrium effort

Because the marginal cost function $c'(\cdot)$ is strictly increasing, the dependence of symmetric equilibrium effort e_k^* on k is determined entirely by coefficients b_k, Eq. (4), which can be interpreted as the marginal benefit of effort in equilibrium. Note that b_k is independent of k when the distribution of noise is uniform. The following lemma shows that the uniform distribution is, in fact, *the only* one for which it is the case.

Lemma 4 *Coefficients b_k are independent of k for $k \geq 2$ if and only if F is a uniform distribution.*

Generally, the properties of coefficients b_k are determined by the shape of the distribution of noise. One interpretation of coefficients b_k follows from writing them in the form $b_k = \int_0^1 m(z)dz^{k-1} = \mathrm{E}(m(Z_{(k-1)}))$, where $Z_{(k-1)}$ is the maximum of $k-1$ i.i.d. uniform random variables in $[0,1]$. From the ordering of variables $Z_{(k-1)}$ by first-order stochastic dominance, it follows immediately that if $f(t)$ is nonincreasing (nondecreasing) then b_k is nonincreasing (nondecreasing) in k for $k \geq 2$. Indeed, $m(z)$ has the same monotonicity as $f(t)$, and for a higher k the weights in the expectation $\mathrm{E}(m(Z_{(k-1)}))$ shift to the right. The nontrivial case emerges when $f(t)$ is nonmonotone.

Before turning to the main results describing the behavior of b_k for all k when $f(t)$ is unimodal, we present large-k asymptotic results for an arbitrary $f(t)$. As discussed above, as k increases, b_k is determined by increasingly higher order statistics $Z_{(k-1)}$ whose probability density is concentrated near $z = 1$; hence, the asymptotic behavior of b_k is determined by the shape of $m(z)$ near $z = 1$, which corresponds to the upper tail of pdf $f(t)$. Specifically, a nonincreasing (nondecreasing) upper tail of f will lead to a nonincreasing (nondecreasing) b_k for large k. The following proposition states the result formally.

Proposition 2 *Define* $\hat{z} = \inf\{z' \in [0,1] : m(z) \text{ is monotone on } (z',1)\}$. *If* $m(z)$ *is nonincreasing (nondecreasing) and nonconstant on* $(\hat{z},1)$*, then there exists a large enough* $\hat{k}$ *such that* b_k *is decreasing (increasing) for all* $k > \hat{k}$.

Point $\hat{z}$ defined in Proposition 2 determines the location of the "last" peak or dip of $m(z)$. If pdf f is monotone (and nonconstant), $\hat{z} = 0$ and b_k is either decreasing or increasing for all $k \geq 2$. If f is nonmonotone, b_k is asymptotically decreasing or increasing depending on whether the last turning point of f is a peak or a dip.

Unimodal distributions are an important class, for which universal global properties of coefficients b_k can be established. The most general result follows directly from Lemma 1: b_k is unimodal whenever $f(t)$ (and hence $m(z)$) is unimodal. Indeed, defining $H(z,k) = z^{k-1}$, it is easy to see that $|H_k(z,k)| = z^{k-1}(1-z)$ is log-supermodular.

Recall that $b_1 = 0$ and $b_2 > 0$ in all cases; hence, for any $n \geq 2$ a unimodal sequence $\{b_k\}_{k=1}^n$ can either be nondecreasing or interior unimodal. The subsequence $\{b_k\}_{k=2}^n$, however, can also be nonincreasing. In what follows, we will mostly focus on the properties of the latter subsequence. Interesting special cases emerge when $f(t)$ is symmetric and/or $n = 3$.

Proposition 3 *(i) If* $f(t)$ *is interior unimodal then* $\{b_k\}_{k=2}^n$ *(and* $\{e_k^*\}_{k=2}^n$*) is unimodal.*
(ii) If $f(t)$ *is nonincreasing (and nonconstant) then* $\{b_k\}_{k=2}^n$ *(and* $\{e_k^*\}_{k=2}^n$*) is decreasing.*
(iii) If $f(t)$ *is nondecreasing (and nonconstant) then* $\{b_k\}_{k=2}^n$ *(and* $\{e_k^*\}_{k=2}^n$*) is increasing.*
(iv) For $n \geq 4$*, if* $f(t)$ *is interior unimodal and symmetric then* $b_2 = b_3$ *(and* $e_2^* = e_3^*$*), and* $\{b_k\}_{k=3}^n$ *(and* $\{e_k^*\}_{k=3}^n$*) is decreasing.*
(v) If $f(t)$ *is symmetric (not necessarily unimodal) then* $b_2 = b_3$ *(and* $e_2^* = e_3^*$*).*

Part (i) of Proposition 3 follows directly from Lemma 1, while parts (ii) and (iii) are straightforward special cases, as described above. Note that parts (ii) and (iii) only rely on the FOSD-ordering of cdfs $H(z,k) = z^{k-1}$, part (i) relies additionally on the log-supermodularity of $|H_k(z,k)|$, but none of the parts (i)-(iii) relies on the specific order-statistic structure of $H(z,k)$. In contrast, parts (iv) and (v) (proved in Appendix A) are more specialized and rely on that structure.

The unimodality of f is not necessary for the unimodality of b_k (and e_k^*), but it is a tight condition. That is, a non-unimodal distribution of noise can produce a non-unimodal

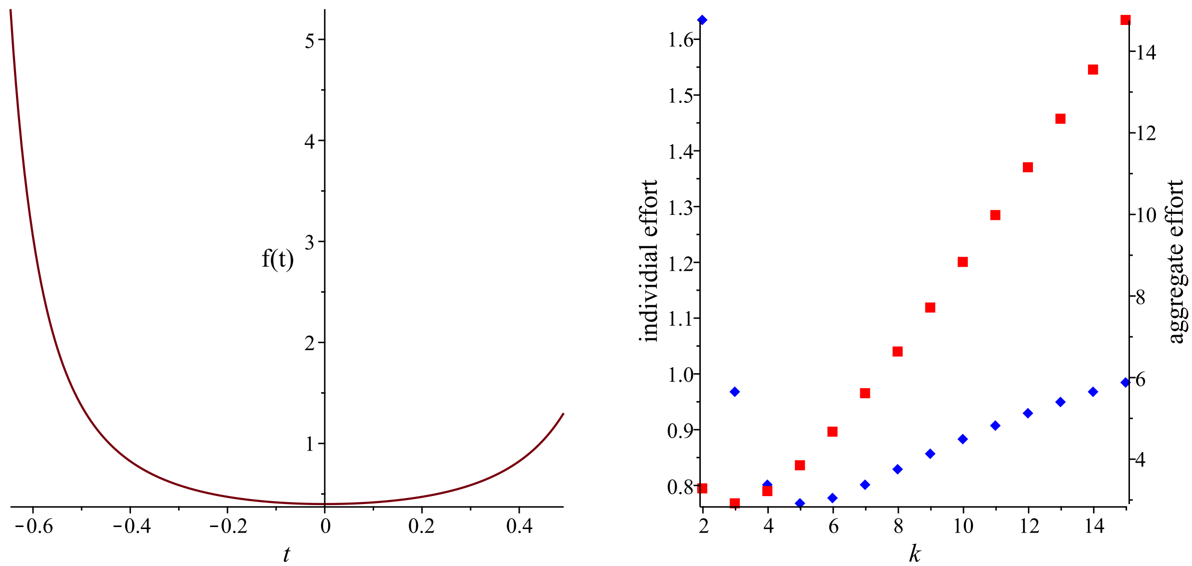

Figure 1: *Left*: The pdf $f(t)$ of a distribution with cdf $F(t) = 0.2\tan(2t) + 0.7$ defined on $[-0.646, 0.491]$. *Right*: Individual equilibrium effort e_k^* (blue diamonds, left scale) and aggregate equilibrium effort E_k^* (red squares, right scale) as a function of k for effort cost function $c(e) = \frac{1}{2}e^2$.

sequence $\{b_k\}$. This is illustrated in Figure 1 showing a bimodal pdf $f(t)$ (left) and the resulting bimodal sequence $\{e_k^*\}_{k=2}^n$ for $n = 15$ (right). At the same time, a non-unimodal $f(t)$ does not necessarily lead to a non-unimodal sequence $\{b_k\}$. For example, a bimodal distribution with pdf $f(t) = \frac{1}{2}[f_{N(-12,4)}(t) + f_{N(12,4)}(t)]$, where $f_{N(\mu,\sigma^2)}(t)$ is the pdf of the Normal distribution with mean μ and variance σ^2, generates a decreasing sequence $\{b_k\}_{k=2}^n$ for any $n \geq 3$. Thus, there is no "higher-order" universality of behavior of b_k for non-unimodal distributions.

Additionally, Proposition 3 allows us to characterize the behavior of b_k for *single-dipped* distributions such that $-f(t)$ is unimodal. Of interest is the case when $f(t)$ is single-dipped and nonmonotone (when f is monotone parts (ii) and (iii) of Proposition 3 apply).

Corollary 1 *(i) For $n \geq 3$, if $f(t)$ is single-dipped and nonmonotone then $\{b_k\}_{k=2}^n$ (and $\{e_k^*\}_{k=2}^n$) is single-dipped.*
(ii) For $n \geq 4$, if $f(t)$ is single-dipped, nonmonotone and symmetric then $b_2 = b_3$ (and $e_2^* = e_3^*$), and $\{b_k\}_{k=3}^n$ (and $\{e_k^*\}_{k=3}^n$) is increasing.

The example in Figure 1 illustrates part (i).

4.2 Aggregate equilibrium effort

Given the various possibilities for the dependence of individual equilibrium effort e_k^* on group size k, it is of interest to also explore how aggregate equilibrium effort $E_k^* = ke_k^*$ changes with the number of players. Considering a change from $k-1$ to k players, write the relative change in aggregate effort in the form

$$\delta E_k^* = \frac{E_k^* - E_{k-1}^*}{E_{k-1}^*} = \frac{k}{k-1}\frac{e_k^*}{e_{k-1}^*} - 1. \tag{7}$$

We will explore conditions for δE_k^* to be positive, i.e., for the aggregate effort to be increasing in k. As seen from (7), the number of players affects the aggregate equilibrium effort in two ways: The direct positive effect, represented by factor $\frac{k}{k-1} > 1$, and the indirect equilibrium effect, $\frac{e_k^*}{e_{k-1}^*}$, which can be less or greater than one. Obviously, aggregate effort will increase in k when $e_k^* \geq e_{k-1}^*$, i.e., whenever individual effort is nondecreasing in k. It is, however, also possible to have aggregate effort increasing in k when e_k^* is decreasing or nonmonotone. For example, in Tullock contests with linear costs individual effort $e_k^* = \frac{r(k-1)}{k^2}$ is decreasing but aggregate effort $E_k^* = \frac{r(k-1)}{k}$ is increasing in k.

It is difficult to proceed with the analysis of aggregate effort for a general cost function $c(e)$; therefore, we restrict attention to homogeneous cost functions of the form $c(e) = c_0 e^{\xi}$, $\xi > 1$. In this case Eq. (7) gives $\delta E_k^* = \frac{k}{k-1}\left(\frac{b_k}{b_{k-1}}\right)^{\frac{1}{\xi-1}} - 1$, which leads to the following proposition.

Proposition 4 *Suppose $c(e) = c_0 e^{\xi}$, $\xi > 1$. Then $E_k^* \geq E_{k-1}^*$ if and only if*

$$\frac{b_k}{b_{k-1}} \geq \left(\frac{k-1}{k}\right)^{\xi-1}. \tag{8}$$

One consequence of Proposition 4 is that for any $k \geq 3$ it is always possible to find a sufficiently high ξ such that $E_k^* \geq E_{k-1}^*$. The intuition is that a higher ξ makes the cost function more convex and hence, reduces the sensitivity of the equilibrium effort to its marginal benefit, i.e., b_k. Then, for a sufficiently high ξ the direct positive effect of a higher number of players dominates the indirect equilibrium effect. On the other hand, ξ can be arbitrarily close to 1 in which case the equilibrium effort becomes infinitely sensitive to b_k;[15] therefore, if $b_k < b_{k-1}$ for some k, it is always possible to find a $\xi > 1$

[15] As ξ gets closer to 1, it becomes more difficult to satisfy the second-order condition for payoff maximization at e_k^*, but for any given ξ it can always be satisfied for a sufficiently high c_0 and/or a sufficiently dispersed distribution of noise.

such that (8) does not hold and hence $E_k^* < E_{k-1}^*$.

For illustration, compare tournaments with group sizes $k = 2$ and 3. It follows from Proposition 3 that $b_3 \geq b_2$, and hence $E_3^* > E_2^*$, when $f(t)$ is symmetric or nondecreasing. However, if $f(t)$ is nonincreasing (and nonconstant), we have $b_3 < b_2$, in which case $E_3^* < E_2^*$ for $\xi < 1 + \frac{\ln\left(\frac{b_2}{b_3}\right)}{\ln\left(\frac{3}{2}\right)}$. For example, consider the distribution of noise with cdf $F(t) = t^\alpha$ and pdf $f(t) = \alpha t^{\alpha-1}$ on $[0,1]$, with $\alpha > \frac{1}{2}$.[16] This gives $m(z) = \alpha z^{\frac{\alpha-1}{\alpha}}$ and $b_k = \frac{\alpha^2(k-1)}{\alpha k-1}$; therefore, $\frac{b_3}{b_2} = \frac{2(2\alpha-1)}{3\alpha-1} < 1$ if and only if $\alpha < 1$, i.e., $f(t)$ is decreasing. For $\alpha = \frac{3}{4}$, we obtain $E_3^* < E_2^*$ for $\xi < 1 + \frac{\ln\left(\frac{5}{4}\right)}{\ln\left(\frac{3}{2}\right)} \approx 1.55$.

A natural question to ask is whether it can be established that E_k^* is unimodal for a unimodal $f(t)$. The answer is, in general, negative. Indeed, we can write $E_k^* = kc'^{-1}(b_k)$, where $c'^{-1}(\cdot)$ is the inverse marginal cost of effort. For a strictly convex $c(e)$, c'^{-1} is strictly increasing; therefore, $c'^{-1}(b_k)$ is unimodal for a unimodal $f(t)$. However, a product of a strictly increasing and unimodal functions is not necessarily unimodal. Additional restrictions on $f(t)$ and/or $c(e)$ are needed to ensure the unimodality of E_k^*. The following proposition provides further insights.

Proposition 5 *Suppose $c(e) = c_0 e^2$, $m(z)$ is twice differentiable, and $m(1)$ and $m'(1)$ are finite.*

(i) If $f(t)$ is log-concave, then $\{E_k^*\}_{k=2}^n$ is nondecreasing.

(ii) If $f(t)$ is log-convex and $f(u_h) = 0$, then $\{E_k^*\}_{k=2}^n$ is nonincreasing.

(ii) If $f(t)$ is first log-concave and then log-convex and $f(u_h) = 0$, then $\{E_k^*\}_{k=2}^n$ is unimodal.

The key property used in the proof of Proposition 5 is that the log-concavity (log-convexity) of $f(t)$ is equivalent to the concavity (convexity) of $m(z)$. Further, for a quadratic cost of effort $E_k^* \propto kb_k$ and, integrating (4) by parts twice, E_k^* can be expressed through an integral of $m''(z)$ (see the proof for details). Part (i) generalizes the results for the Tullock contest with linear costs. Indeed, as shown in Section 2.2, such a contest is equivalent to a tournament with a quadratic cost and Gumbel distribution of noise, which has a log-concave pdf. To understand part (ii), note that the log-convexity of $f(t)$ and condition $f(u_h) = 0$ imply that $f(t)$ is decreasing sufficiently fast. Then, not only does individual equilibrium efforts decrease (see Proposition 3(ii)) but the aggregate effort decreases too. For a simple example illustrating part (ii), consider the $F_{2,2}$-distribution whose pdf and cdf are $f(t) = \frac{1}{(1+t)^2}$ and $F(t) = \frac{t}{1+t}$ defined for $t \geq 0$. Then, $b_k = \frac{2}{k(k+1)}$

[16]The restriction $\alpha > \frac{1}{2}$ ensures that $m(z)$ is integrable on $[0, 1]$.

and the aggregate effort $E_k^* = \frac{2}{k+1}$ is strictly decreasing with the number of players.[17] Finally, for part (iii), the F-distribution and Beta distribution for some parameters, and the lognormal distribution are first log-concave and then log-convex (see Bagnoli and Bergstrom (2005) for details).

4.3 The effect of noise dispersion

Intuitively, when noise becomes more dispersed, the marginal gain from effort goes down and equilibrium effort should decrease. For example, when the distribution of noise is uniform on the interval $[-a, a]$, we have $b_k = \frac{1}{2a}$ for all $k \geq 2$; hence, as the variance of noise increases the equilibrium effort goes down. Similarly, in Tullock contests the dispersion of noise is determined by parameter r (see Section 2.2). As r goes down, noise becomes more dispersed and the equilibrium effort decreases.

Consider, however, a family of zero-mean, symmetrically distributed random variables $T|\alpha$, parameterized by $\alpha \geq 0$, with pdfs $f(t|\alpha) = \frac{\alpha+1}{2}|t|^\alpha$ defined on support $[-1, 1]$. An increase in α leads to a higher variance, $\mathrm{Var}(T|\alpha) = \frac{\alpha+1}{\alpha+3}$, and, more generally, shifts the distribution in terms of second-order stochastic dominance (SOSD). At the same time, $b_2 = \frac{(\alpha+1)^2}{2(2\alpha+1)}$ increases with α. In other words, an increase in noise leads to a higher equilibrium effort in a two-player tournament.

These examples show, perhaps surprisingly, that, in general, neither the variance nor SOSD ordering of noise distributions have a monotone effect on the equilibrium effort. To understand why this is the case, let u_1 and u_2 denote i.i.d. random variables with pdf f and recall that, from Eq. (3), $b_2 = \int_U f(t)^2 dt = f_{u_1-u_2}(0)$, where $f_{u_1-u_2}(\cdot)$ is the pdf of $u_1 - u_2$. In the example with variables $T|\alpha$ above, as α increases, the mass of the distribution is shifted away from the middle towards the edges of the support and, therefore, the density of $u_1 - u_2$ acquires a sharp peak at zero (and two additional, smaller peaks around -2 and $+2$) leading to an increase in b_2 even as the variance of $T|\alpha$ goes up.

For the rest of this section, we will use $b_k[f]$ and $e_k^*[f]$ to denote, respectively, the coefficient b_k and equilibrium effort e_k^* obtained from a noise distribution with pdf $f(t)$. Note that, from Eq. (3), b_2 can be written in the form $b_2[f] = \int_U f(t)^2 dt = \exp(-H[f])$, where $H[f]$ is the Rényi entropy of order 2, also known as "collision entropy" (Rényi,

[17] To illustrate the importance of the requirement $f(u_h) = 0$, consider again the example in Figure 1, where the pdf is log-convex but $f(u_h) > 0$. As the right panel shows, $E_3^* < E_4^* < E_2^* < E_5^*$ and E_k^* is monotonically increasing for $k \geq 5$.

1961).[18] Thus, in two-player tournaments equilibrium effort decreases in the *entropy* of the noise distribution. More generally, from Eq. (3),

$$b_k[f] = \frac{4(k-1)}{k^2} \int_U \left[\frac{k}{2} F(t)^{\frac{k}{2}-1} f(t)\right]^2 dt = \frac{4(k-1)}{k^2} b_2[f_{(k/2)}] = \frac{4(k-1)}{k^2} \exp(-H[f_{(k/2)}]), \tag{9}$$

where $f_{(k/2)}(t) = \frac{d}{dt} F(t)^{\frac{k}{2}}$ is the pdf corresponding to cdf $F_{(k/2)}(t) = F(t)^{\frac{k}{2}}$. Thus, coefficient b_k in a tournament of $k \geq 2$ players can be represented as an appropriately rescaled coefficient b_2 in a tournament of two symmetric "composite" players, each with the cdf of noise $F_{(k/2)}(t)$. The latter coefficient can then be expressed through the entropy of pdf $f_{(k/2)}$.

Proposition 6 *In a tournament of k players, equilibrium effort decreases in the Rényi entropy of order 2 of a distribution with pdf $f_{(k/2)}$.*

The representation (9) and Proposition 6 have a straightforward interpretation when k is even: Split the k players arbitrarily into two equal subgroups with $\frac{k}{2}$ players each. Then $F_{(k/2)}(t)$ is the cdf of noise of the two players whose shocks are the largest in each subgroup, and the player with a larger shock between the two subgroup "winners" will win the tournament. For an odd k, the two "composite players" can still be introduced, but they no longer have the same "human" analogues.

When support $[u_l, u_h]$ is finite, the entropy reaches its maximum for the uniform distribution. This leads to the following corollary.

Corollary 2 *Of all noise distributions with a finite support $[u_l, u_h]$, the distribution that minimizes the symmetric equilibrium effort in the tournament of $k \geq 2$ players has cdf $F_{\min}(t) = \left(\frac{t-u_l}{u_h-u_l}\right)^{\frac{2}{k}}$. The resulting minimized value of b_k is $b_k[f_{\min}] = \frac{4(k-1)}{k^2(u_h-u_l)}$.*

As seen from the corollary, the effort-minimizing noise distribution in a k-player tournament is uniform for $k = 2$, but for $k > 2$ it has a concave cdf and monotonically decreasing pdf, more so the larger the number of players k, such that $F_{\min}(t)^{\frac{k}{2}}$ is uniform.

An important sufficient condition that allows to rank entropy of different distributions and hence, equilibrium efforts is given by the *dispersive order*.[19]

[18] The general expression for the Rényi entropy of order α is $H_\alpha[f] = \frac{1}{1-\alpha} \ln \left(\int_U f(t)^\alpha dt\right)$.

[19] For recent applications of the dispersive order in the auction theory literature see, e.g., Ganuza and Penalva (2010) and Kirkegaard (2012).

Definition 3 *X is more dispersed than Y if for all $z, z' \in [0,1]$ such that $z' > z$*

$$F_X^{-1}(z') - F_X^{-1}(z) \geq F_Y^{-1}(z') - F_Y^{-1}(z).$$

and the inequality is strict in some open interval of z.

The definition is rather intuitive: X is more dispersed than Y if the distance between any two quantiles of X is at least as large as the distance between the same quantiles of Y. As discussed by Shaked and Shanthikumar (2007), whenever X is more dispersed than Y, $\text{Var}(X) \geq \text{Var}(Y)$; the converse, however, is not true. Similarly, the dispersive order for variables with equal means implies SOSD, but the converse is not true. Finally, whenever X is more dispersed than Y, it has a higher entropy. Moreover, the dispersive order is preserved for order statistics (Theorem 3.B.26 in Shaked and Shanthikumar, 2007), leading to the following result.

Lemma 5 *If X is more dispersed than Y then $H[f_{X(k/2)}] > H[f_{Y(k/2)}]$, and hence $e_k^*[f_X] < e_k^*[f_Y]$ for any $k \geq 2$.*

The proof of Lemma 5 is straightforward and based on Proposition 6 and the fact that X being more dispersed than Y is equivalent to $m_X(z) \leq m_Y(z)$ (see Appendix A).

An important special case which satisfies the dispersive order, allows for an explicit characterization of the equilibrium effort, and incorporates several important examples is when additional dispersion is generated by scaling: $X = sY$, where $s > 1$. A parameterized cdf $F(t,s)$ is said to have a scale parameter s if it satisfies $F(t,s) = F(\frac{t}{s}, 1)$. The corresponding scaled pdf is $f(t,s) = \frac{1}{s} f(\frac{t}{s}, 1)$. For example, the standard deviation of a zero-mean normal distribution, the length of the support of a uniform distribution, the expected value of an exponential distribution and the scale of the Gumbel distribution (and hence, the "discriminatory power" of the Tullock contest, see Section 2.2) are scale parameters. It is easy to see that an increase in s leads to a more dispersed distribution (Theorem 3.B.4 in Shaked and Shanthikumar, 2007) and hence to a lower equilibrium effort (Lemma 5). For an explicit characterization, note that if $[u_l, u_h]$ is the support of $f(t,1)$, then the support of $f(t,s)$ is $[su_l, su_h]$ and

$$b_k[f(t,s)] = (k-1)\int_{su_l}^{su_h} F(t,s)^{k-2} f(t,s)^2 dt = \frac{k-1}{s^2}\int_{su_l}^{su_h} F\left(\frac{t}{s},1\right)^{k-2} f\left(\frac{t}{s},1\right)^2 dt$$
$$= \frac{k-1}{s}\int_{u_l}^{u_h} F(u,1)^{k-2} f(u,1)^2 du = \frac{1}{s} b_k[f(t,1)].$$

Thus, individual and aggregate equilibrium efforts are decreasing in s.

In many cases of interest the dispersive order does not rank distributions. For example, a mean-preserving spread generated by adding an independent zero-mean random variable satisfies the dispersive order only under a special condition. In particular, suppose $X = Y + W$, where $\mathrm{E}(W) = 0$ and W is independent of Y. In this case X is more dispersed than Y for any W (and hence Lemma 5 applies) if and only if the pdf of Y is log-concave (Theorem 3.B.7 in Shaked and Shanthikumar, 2007).

Two (different) distributions cannot be ranked in the sense of dispersive order if they have the same finite support (Theorem 3.B.14. in Shaked and Shanthikumar, 2007). The following lemma may then help as it allows for ranking of some distributions directly in terms of the entropy.

Lemma 6 *Consider random variables X and Y defined on the same support $[u_l, u_h]$ (finite or infinite). If any of the following conditions holds then $H[f_X] \geq H[f_Y]$.*
(a) f_X and f_Y are nondecreasing and Y FOSD X;
(b) f_X and f_Y are nonincreasing and X FOSD Y;
(c) f_X and f_Y are interior unimodal and symmetric, and $(Y|Y \leq \mu)$ FOSD $(X|X \leq \mu)$, where $\mu = \mathrm{E}(X) = \mathrm{E}(Y)$.

Condition (a) in Lemma 6 is satisfied, for example, when f_X and f_Y are both nondecreasing and f_X crosses f_Y from above; that is, there exists a $\hat{t} \in [u_l, u_h]$ such that $f_X(t) \geq (\leq) f_Y(t)$ for $t \leq (\geq) \hat{t}$. Similarly, condition (b) is satisfied when f_X and f_Y are both nonincreasing and f_X crosses f_Y from below; and condition (c) is satisfied for symmetric unimodal f_X and f_Y when f_X crosses f_Y first from above and then from below. Of course, multiple crossings are also admissible as long as the FOSD relationships hold. Additionally, since a horizontal shift of the distribution of noise does not affect the equilibrium effort, what really matters in Lemma 6 is that the supports of X and Y are of the same size. The invariance to a horizontal shift also implies for part (c) that the means of X and Y can be different provided the support is infinite.

Note that first-order stochastic dominance is preserved by order statistics; therefore, if condition (a) in Lemma 6 is satisfied for f_X and f_Y, the same condition is satisfied for $f_{X(k/2)}$ and $f_{Y(k/2)}$ for any $k \geq 2$. This leads to the following result.

Corollary 3 *If f_X and f_Y satisfy condition (a) in Lemma 6 then $H[f_{X(k/2)}] \geq H[f_{Y(k/2)}]$ and hence $e_k^*[f_X] \leq e_k^*[f_Y]$ for any $k \geq 2$.*

5 Tournaments with stochastic group size

5.1 Model setup

Consider now a setting in which the number of players in the tournament, K, is a random variable taking nonnegative integer values. The maximal possible number of players $n \geq 2$ can be finite or infinite. Let $p = (p_0, p_1, \dots, p_n)$ denote the probability mass function (pmf) of K, where $p_k = \Pr(K = k)$ is the probability of having k players in the tournament, with $\sum_{k=0}^{n} p_k = 1$. The expected number of players $\bar{k} = \sum_{k=0}^{n} k p_k$ is finite. Operationally, it is convenient to think about a set of potential participants $\mathcal{N} = \{1, \dots, n\}$ from which a subset $\mathcal{K} \in 2^{\mathcal{N}}$ is randomly drawn such that $\Pr(|\mathcal{K}| = k) = p_k$, and subsets of the same cardinality $|\mathcal{K}|$ have the same probability of being drawn. Each player is informed if she is selected, but is not informed about the value of K.

Let S_i denote a random variable equal to 1 if player $i \in \mathcal{N}$ is selected for participation and zero otherwise, and let $\tilde{K} = (K|S_i = 1)$ denote the random number of players in the tournament from the perspective of a participating player. The distribution of $\tilde{K}$ is updated as (see, e.g., Harstad, Kagel and Levin, 1990)

$$\tilde{p}_k = \Pr(\tilde{K} = k) = \frac{p_k k}{\bar{k}}, \quad k = 1, \dots, n. \tag{10}$$

Equation (10) can be understood as follows (cf. Myerson and Wärneryd, 2006). Suppose n is finite (for an infinite n, a similar argument applies in the limit $n \to \infty$). For a given K, the probability for player i to be selected for participation is $\Pr(S_i = 1|K = k) = \frac{k}{n}$; thus,

$$\tilde{p}_k = \Pr(K = k|S_i = 1) = \frac{\Pr(S_i = 1|K = k) p_k}{\sum_{l=0}^{n} \Pr(S_i = 1|K = l) p_l} = \frac{\frac{k}{n} p_k}{\sum_{l=0}^{n} \frac{l}{n} p_l},$$

which gives (10).

Consider a symmetric pure strategy equilibrium in which all participating players choose effort $e^* > 0$. From Eq. (2), the expected payoff of a participating player i from some deviation effort e_i is

$$\pi_i(e_i, e^*) = \sum_{k=1}^{n} \tilde{p}_k \int_U F(e_i - e^* + t)^{k-1} dF(t) - c(e_i). \tag{11}$$

The first-order condition for payoff maximization evaluated at $e_i = e^*$, $\left.\frac{\partial \pi_i(e_i, e^*)}{\partial e_i}\right|_{e_i = e^*} = 0$, gives

$$B_p = c'(e^*), \quad B_p = \sum_{k=1}^{n} \tilde{p}_k (k-1) \int_U F(t)^{k-2} f(t) dF(t). \tag{12}$$

Changing the variable of integration to $z = F(t)$, obtain, similar to (4),

$$B_p = \sum_{k=1}^{n} \tilde{p}_k (k-1) \int_0^1 z^{k-2} m(z) dz = \int_0^1 m(z) d\tilde{G}(z). \tag{13}$$

Here, $\tilde{G}(z) = \sum_{k=1}^{n} \tilde{p}_k z^{k-1}$ denotes the probability-generating function (pgf) of distribution $\tilde{p}$.

Let e_p^* denote the unique positive solution of (12), assuming that it exists and it is a symmetric pure strategy equilibrium.[20] When p is degenerate at some k, Eq. (12) reduces to the deterministic group size case, Eq. (3). As before, since $c'(e^*)$ is strictly increasing in e^*, the comparative statics of equilibrium effort e_p^* with respect to parameters of distribution p are determined entirely by coefficients B_p.

Using Eqs. (13) and (10), and the definition of b_k, Eq. (3), coefficients B_p can also be written as

$$B_p = \sum_{k=1}^{n} \tilde{p}_k b_k = \mathrm{E}_{\tilde{p}}(b_K) = \frac{1}{\bar{k}} \sum_{k=2}^{n} p_k k b_k = \frac{1}{\bar{k}} \mathrm{E}_p(K b_K | K \geq 2) \mathrm{Pr}_p(K \geq 2). \tag{14}$$

Here, $\mathrm{E}_p(\cdot)$ and $\mathrm{Pr}_p(\cdot)$ denote expectation and probability with respect to distribution p. Note that the summation in (14) can start with $k = 2$ instead of $k = 1$ because $b_1 = 0$. Representation (14) shows, as expected, that only group sizes $k \geq 2$ contribute to the equilibrium effort.

5.2 Comparative statics for unimodal noise distributions

We are interested in the effects of changes in distribution p on coefficients B_p. In particular, we explore how B_p responds to a stochastic increase (in an appropriate sense) in the number of players in the tournament. To this end, consider a parameterized family of (updated) group size distributions $\{\tilde{p}(\theta)\}_{\theta \in \Theta}$, where $\Theta \subseteq \mathbb{R}$ is an interval of the real line or a set of consecutive discrete values. Let $\tilde{P}(\theta)$, $\tilde{G}(z, \theta)$ and $B_p(\theta)$ denote, respectively, the corresponding cmf, pgf and B_p.

[20] As in Section 2, we leave the issues of equilibrium existence and uniqueness outside the scope of this paper.

Suppose an increase in θ leads to a stochastic increase in the number of players in the sense of first-order stochastic dominance (FOSD); that is, assume that $\tilde{P}_k(\theta)$ is nonincreasing in θ for all $k = 1, 2, \ldots, n$. The simplest case that does not require any additional restrictions is when the sequence $\{b_k\}_{k=2}^n$ is nondecreasing (which implies that $\{b_k\}_{k=1}^n$ is nondecreasing because $b_1 = 0$). The following lemma and corollary follow immediately from (14) and Proposition 3.

Lemma 7 *Suppose an increase in θ leads to a stochastic increase in $\tilde{K}$ and $\{b_k\}_{k=2}^n$ is nondecreasing. Then $B_p(\theta)$ (and e_p^*) is nondecreasing in θ.*

Corollary 4 *Suppose an increase in θ leads to a stochastic increase in $\tilde{K}$ and $f(t)$ is nondecreasing. Then $B_p(\theta)$ (and e_p^*) is nondecreasing in θ.*

Note that a similar result cannot be established when $\{b_k\}_{k=2}^n$ is nonincreasing, because $b_1 = 0$ and hence $\{b_k\}_{k=1}^n$ would be nonmonotone, unless $p_1 = 0$ (for a more detailed discussions of results in the case when tournaments are restricted to have at least two participants, see Section 5.6); and when $\{b_k\}_{k=2}^n$ is interior unimodal, further restrictions are needed.

Let $\tilde{G}_\theta(z, \theta) \leq 0$ denote the derivative or the first difference of the pgf with respect to θ. Combined with Proposition 3, Lemmas 2 and 3 produce the following result.

Proposition 7 *Suppose an increase in θ leads to a stochastic increase in $\tilde{K}$ and*
(a) $f(t)$ is unimodal;
(b) $|\tilde{G}_\theta(z, \theta)|$ is log-supermodular; that is, the ratio $R(z, \theta, \theta') = \frac{\tilde{G}_\theta(z,\theta')}{\tilde{G}_\theta(z,\theta)}$ is nondecreasing in z for all $\theta' > \theta$.
Then $B_p(\theta)$ (and e_p^*) is unimodal in θ.

In the remainder of this section, we consider several examples of tournament size distributions that satisfy the log-supermodularity condition (b) of Proposition 7. The distributions we consider – the binomial, negative binomial, logarithmic and Poisson distributions – belong to a family known as power series distributions (PSD) characterized by pmfs of the form $p_k(\theta) = \frac{a_k \theta^k}{A(\theta)}$, where a_k are nonnegative numbers, $\theta \geq 0$ is a parameter, and $A(\theta) = \sum_{k=0}^\infty a_k \theta^k$ (where it is assumed that the sum exists) is the normalization function (Johnson, Kemp and Kotz, 2005). The pgf of PSD distributions is $G(z, \theta) = \frac{A(\theta z)}{A(\theta)}$. An important property of this family is that if pmf p belongs to it, so does the updated pmf $\tilde{p}$. Indeed, from (10),

$$\tilde{p}_k = \frac{k p_k}{\bar{k}} = \frac{k a_k \theta^k}{\sum_{k=1}^\infty k a_k \theta^k} = \frac{\tilde{a}_k \theta^k}{\tilde{A}(\theta)},$$

where $\tilde{a}_k = ka_k$ and $\tilde{A}(\theta) = \sum_{k=1}^{\infty} \tilde{a}_k \theta^k$; that is, $\tilde{p}_k$ also has the PSD form.

It can be shown that $G_\theta(z, \theta) \leq 0$ for any PSD distribution. Indeed,

$$G_\theta(z,\theta) = \frac{A'(\theta z)z}{A(\theta)} - \frac{A'(\theta)}{A(\theta)}\frac{A(\theta z)}{A(\theta)}$$
$$= \frac{\sum_{k=0}^{\infty} ka_k\theta^{k-1}z^k}{A(\theta)} - \frac{\sum_{k=0}^{\infty} ka_k\theta^{k-1}}{A(\theta)}\frac{\sum_{k=0}^{\infty} a_k\theta^k z^k}{A(\theta)}$$
$$= \frac{1}{\theta}\left(\mathrm{E}(Kz^K) - \mathrm{E}(K)\mathrm{E}(z^K)\right) = \frac{1}{\theta}\mathrm{Cov}(K, z^K) \leq 0.$$

Most importantly, PSD distributions satisfy the log-supermodularity condition of Proposition 7.

Proposition 8 *$|G_\theta(z,\theta)|$ is log-supermodular for PSD distributions.*

Proof of Proposition 8 Let $A_k(\theta) = \frac{1}{A(\theta)}\sum_{l=0}^{k} a_l\theta^l$ denote the cmf of a PSD distribution. We will prove that $|A_k'(\theta)|$ is log-supermodular; the result then follows by Lemma 3. Note that

$$A_k'(\theta) = \frac{1}{A(\theta)^2}\sum_{l=0}^{k}\sum_{m\geq 0} a_l a_m \theta^{l+m-1}(l-m) = -\frac{1}{A(\theta)^2}\sum_{l=0}^{k}\sum_{m\geq k+1} a_l a_m \theta^{l+m-1}(m-l).$$

Consider some $\theta' > \theta$ and let $\beta = \frac{\theta'}{\theta} > 1$. For convenience, introduce the notation $\alpha_{lm} = a_l a_m \theta^{l+m-1}(m-l)$. The ratio $r(k,\theta,\theta')$ from Lemma 2 is $\frac{A_k'(\theta')}{A_k'(\theta)} = \frac{A(\theta)^2}{A(\theta')^2}\frac{N_k}{D_k}$, where

$$N_k = \sum_{l=0}^{k}\sum_{m\geq k+1} \beta^{l+m-1}\alpha_{lm}, \quad D_k = \sum_{l=0}^{k}\sum_{m\geq k+1}\alpha_{lm}.$$

We need to show that $\frac{N_k}{D_k}$ is nondecreasing in k, or, equivalently, that $N_{k+1}D_k - N_k D_{k+1} \geq 0$. Notice that N_{k+1} can be expressed through N_k as follows:

$$N_{k+1} = N_k - \sum_{l=0}^{k}\beta^{l+k}\alpha_{l,k+1} + \sum_{m\geq k+2}\beta^{m+k}\alpha_{k+1,m}.$$

Similarly,

$$D_{k+1} = D_k - \sum_{l=0}^{k}\alpha_{l,k+1} + \sum_{m\geq k+2}\alpha_{k+1,m};$$

therefore,

$$\begin{aligned}
N_{k+1}D_k - N_k D_{k+1} &= \left(N_k - \sum_{l=0}^{k} \beta^{l+k}\alpha_{l,k+1} + \sum_{m\geq k+2} \beta^{m+k}\alpha_{k+1,m} \right) D_k \\
&- N_k \left(D_k - \sum_{l=0}^{k} \alpha_{l,k+1} + \sum_{m\geq k+2} \alpha_{k+1,m} \right) \\
&= \sum_{l=0}^{k} \alpha_{l,k+1}(N_k - \beta^{l+k}D_k) + \sum_{m\geq k+2} \alpha_{k+1,m}(\beta^{m+k}D_k - N_k).
\end{aligned}$$

It can be shown that each of the two terms in the last line is nonnegative. We demonstrate it explicitly for the first term; for the second term, the derivation is similar.

$$\begin{aligned}
&\sum_{l=0}^{k} \alpha_{l,k+1}(N_k - \beta^{l+k}D_k) = \sum_{l=0}^{k} \sum_{m\geq k+1} \sum_{l'=0}^{k} \left(\beta^{l'+m-1}\alpha_{l'm}\alpha_{l,k+1} - \beta^{l+k}\alpha_{l'm}\alpha_{l,k+1} \right) \\
&= \sum_{l=0}^{k} \sum_{m\geq k+1} \sum_{l'=0}^{k} \left(\beta^{l+m-1}\alpha_{lm}\alpha_{l',k+1} - \beta^{l+k}\alpha_{l'm}\alpha_{l,k+1} \right) \\
&\geq \sum_{l=0}^{k} \sum_{m\geq k+1} \sum_{l'=0}^{k} \beta^{l+k} \left(\alpha_{lm}\alpha_{l',k+1} - \alpha_{l'm}\alpha_{l,k+1} \right) \\
&= \sum_{l=0}^{k} \sum_{m\geq k+1} \sum_{l'=0}^{k} \beta^{l+k} a_l a_m a_{l'} a_{k+1} \theta^{l+m-1+l'+k} \left[(m-l)(k+1-l') - (m-l')(k+1-l) \right] \\
&= \sum_{l=0}^{k} \sum_{m\geq k+1} \sum_{l'=0}^{k} \beta^{l+k} a_l a_m a_{l'} a_{k+1} \theta^{l+m-1+l'+k} (m-k-1)(l-l') \\
&= \sum_{m\geq k+1} \beta^{k} a_m a_{k+1} \theta^{m-1+k} (m-k-1) \sum_{l=0}^{k} \sum_{l'=0}^{k} \beta^{l} a_l a_{l'} \theta^{l+l'} (l-l').
\end{aligned}$$

The sum over l and l' can be rewritten as

$$\begin{aligned}
\sum_{l=0}^{k} \sum_{l'=0}^{k} \beta^{l} a_l a_{l'} \theta^{l+l'} (l-l') &= A_k(\theta)^2 A(\theta)^2 [\mathrm{E}(\beta^L L) - \mathrm{E}(\beta^L)\mathrm{E}(L)] \\
&= A_k(\theta)^2 A(\theta)^2 \mathrm{Cov}(\beta^L, L) \geq 0.
\end{aligned}$$

Here, L is understood as a random variable with support $0, 1, \ldots, k$ and pmf $\frac{a_l \theta^l}{A_k(\theta)A(\theta)}$. The covariance is nonnegative because $\beta > 1$. ■

5.2.1 Example: The binomial distribution of group size

Consider the binomial distribution of tournament size, with $K \sim \text{Binomial}(n, q)$, where $n \geq 2$ and $q \in [0, 1]$. The updated probability of group size k is

$$\tilde{p}_k = \frac{1}{nq}\binom{n}{k} q^k (1-q)^{n-k} k = \binom{n-1}{k-1} q^{k-1}(1-q)^{n-k};$$

that is, from the perspective of a participating player, the distribution of the number of *other players*, $\tilde{K} - 1$, is $\text{Binomial}(n-1, q)$. An increase in q leads to an FOSD shift in the number of participants. We will now use Proposition 7 to show that, assuming $f(t)$ is unimodal, $B_p(q)$ is unimodal as a function of q.

The pgf for the updated binomial distribution is

$$\tilde{G}(z, q) = \sum_{k=1}^{n} \binom{n-1}{k-1} q^{k-1}(1-q)^{n-k} z^{k-1} = (1 - q + qz)^{n-1}. \tag{15}$$

It follows immediately that $\tilde{G}_q = -(n-1)(1-z)(1-q+qz)^{n-2} \leq 0$. In order to show that $|\tilde{G}_q|$ is log-supermodular, write for $q' = q + \delta$,

$$R(z, q, q') = \frac{-(n-1)(1-z)(1-q-\delta+qz+\delta z)^{n-2}}{-(n-1)(1-z)(1-q+qz)^{n-2}} = \left(\frac{1-(q+\delta)(1-z)}{1-q(1-z)}\right)^{n-2}.$$

It is easy to see that $R(z, q, q')$ is nondecreasing in z for any $\delta > 0$. Thus, all the assumptions of Proposition 7 are satisfied and $B_p(q)$ is unimodal.

Consider now the effect of an increase in the maximal number of players, n, for a fixed q, which also leads to an FOSD shift in the number of players. It follows from (15) that

$$\tilde{G}_n(z, n) = \tilde{G}(z, n+1) - \tilde{G}(z, n) = -q(1-z)(1-q+qz)^{n-1}.$$

Let $n' = n + d$, where $d > 0$ is an integer. This gives

$$R(z, n, n') = \frac{-q(1-z)(1-q+qz)^{n+d-1}}{-q(1-z)(1-q+qz)^{n-1}} = (1-q+qz)^d,$$

which is nondecreasing in z; hence, by Proposition 7, assuming $f(t)$ is unimodal, $B_p(n)$ is unimodal as a function of n.

For illustration, consider the $\text{Laplace}(0, 1)$ distribution of noise, whose pdf is $f(t) = \frac{1}{2}\exp(-|t|)$ and cdf is $F(t) = \frac{1}{2}\exp(t)$ for $t \leq 0$ and $F(t) = 1 - \frac{1}{2}\exp(-t)$ for $t \geq 0$. For

$K \sim \text{Binomial}(n, q)$, this distribution allows for a closed-form B_p. From Eqs. (13) and (15),

$$B_p = (n-1)q \int_0^1 (1-q+qz)^{n-2} m(z)dz, \tag{16}$$

which gives, for the Laplace$(0, 1)$ distribution of noise,

$$B_p = \frac{(1-q)^n - 2\left(1-\frac{q}{2}\right)^n + 1}{nq}. \tag{17}$$

Coefficients $b_k = \frac{1}{k}\left(1 - \frac{1}{2^{k-1}}\right)$ are decreasing for $k \geq 3$, with $b_2 = b_3$. Indeed, since the Laplace distribution is symmetric and unimodal, Proposition 3(iv) applies. Proposition (7) also applies, and B_p (and hence e_p^*) is unimodal in q and n. Note that, as seen from Eq. (16), B_p is a polynomial in q; therefore, the unimodality implies that it is either monotonically increasing or has a unique interior maximum in q, as illustrated in the left panel of Figure 2.

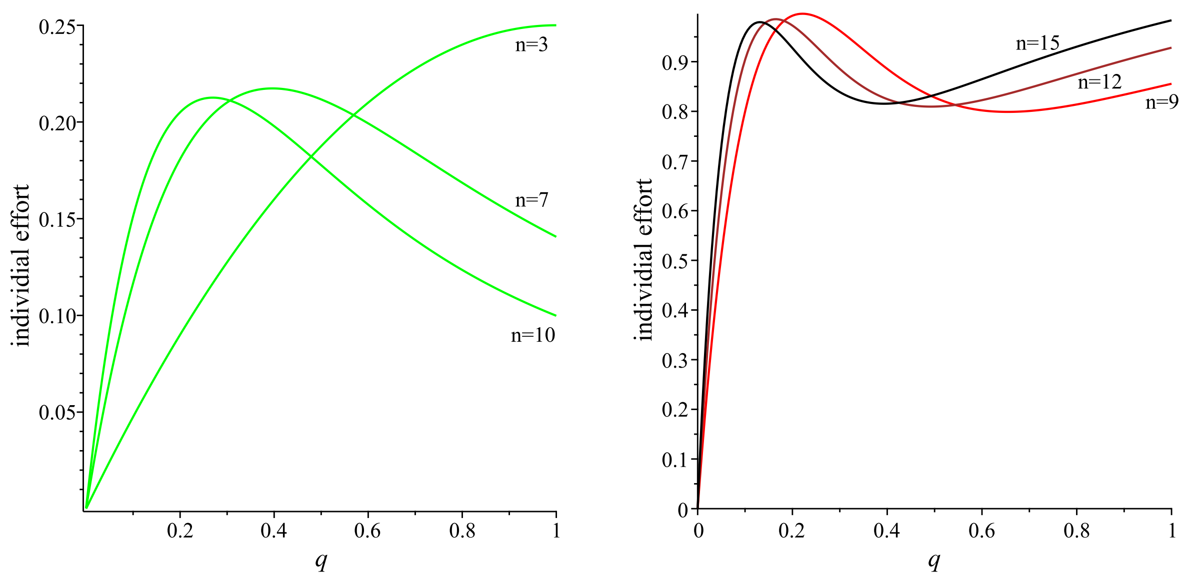

Figure 2: Individual effort as a function of q for different values of n for the binomial distribution of the number of players with parameters (n, q) and cost function $c(e) = \frac{1}{2}e^2$. *Left*: Noise is distributed according to the Laplace$(0, 1)$ distribution. *Right*: Noise is distributed according to a distribution with cdf $F(t) = 0.2\tan(2t) + 0.7$ on $[-0.646, 0.491]$ (see Figure 1).

We conclude this section by an example showing that, similar to the conditions of Proposition 3, the unimodality of $f(t)$ in Proposition 7 is a tight condition. Consider again the bimodal distribution shown in Figure 1, which produces a non-unimodal sequence $\{b_k\}$. This distribution generates a non-unimodal dependence of B_p (and e_p^*) on q shown in the right panel of Figure 2.[21]

5.2.2 Example: The negative binomial distribution of group size

Consider the negative binomial distribution of tournament size, with $K \sim \mathrm{NB}(m, q)$, where $m \geq 1$ and $q \in [0, 1]$. The geometric distribution is its special case, $\mathrm{NB}(1, q)$. The expected number of players is $\bar{k} = \frac{m(1-q)}{q}$ and hence, the updated probability of group size k is

$$\tilde{p}_k = \frac{q}{m(1-q)} \binom{m+k-1}{k} q^m (1-q)^k k = \binom{m+k-1}{k-1} q^{m+1} (1-q)^{k-1};$$

that is, from the perspective of a participating player, the distribution of the number of other players is $\mathrm{NB}(m+1, q)$. A *decrease* in q leads to an FOSD shift in the number of participants.

The pgf of the updated distribution is then

$$\tilde{G}(z, q) = \sum_{k=1}^{n} \binom{m+k-1}{k-1} q^{m+1} (1-q)^{k-1} z^{k-1} = \left(\frac{q}{1-q+qz} \right)^{m+1}.$$

Note that it is inversely related to its analogue for the binomial distribution (15). Since a lower q leads to a stochastic increase in the number of players, all the assumptions of Proposition 7 are satisfied and $B_p(q)$ is unimodal.

5.2.3 Example: The logarithmic distribution of group size

The logarithmic distribution of tournament size, $K \sim \mathrm{Logarithmic}(\theta)$, where $\theta \in (0, 1)$, has pmf $p_k = -\frac{\theta^k}{k \ln(1-\theta)}$ and expectation $\bar{k} = -\frac{\theta}{(1-\theta)\ln(1-\theta)}$. The updated probability of group size k is

$$\tilde{p}_k = (1-\theta)\theta^{k-1};$$

that is, $\tilde{K}$ has the geometric distribution with parameter $1-\theta$. Hence, it is covered by the negative binomial example above.

[21] Similar to Section 4.1, a bimodal distribution is not sufficient to generate a non-unimodal dependence of B_p on q. For example, the bimodal distribution with pdf $f(t) = \frac{1}{2}[f_{N(-12,4)}(t) + f_{N(12,4)}(t)]$ generates B_p which is strictly increasing in q for any n.

5.2.4 Example: The Poisson distribution of group size

Consider now the Poisson distribution of tournament size, with $k \sim \text{Poisson}(\lambda)$, where $\lambda > 0$. The updated probability of group size k is

$$\tilde{p}_k = \frac{1}{\lambda}\frac{\exp(-\lambda)\lambda^k}{k!}k = \frac{\exp(-\lambda)\lambda^{k-1}}{(k-1)!};$$

that is, similar to the binomial distribution, from the perspective of a participating player, the distribution of the number of other players, $K-1$, is Poisson(λ). An increase in λ leads to an FOSD shift in the number of participants. The pgf for the updated Poisson distribution is

$$\tilde{G}(z,q) = \sum_{k=1}^{\infty} \frac{\exp(-\lambda)\lambda^{k-1}}{(k-1)!} z^{k-1} = \exp(-\lambda + \lambda z).$$

Thus, $\tilde{G}_\lambda = -(1-z)\exp(-\lambda+\lambda z) \leq 0$. To check the log-supermodularity property, let $\lambda' = \lambda + \delta$ and write

$$R(z,\lambda,\lambda') = \frac{-(1-z)\exp(-\lambda-\delta+\lambda z+\delta z)}{-(1-z)\exp(-\lambda+\lambda z)} = \exp(-\delta+\delta z),$$

which is increasing in z. Thus, all the assumptions of Proposition 7 are satisfied and $B_p(\lambda)$ is unimodal.

5.2.5 Example: The uniform distribution of noise

When the distribution of noise is uniform, $b_k = b_2$ for any $k \geq 2$. Equation (13) then gives

$$B_p = b_2\left(\tilde{G}(1) - \tilde{G}(0)\right) = b_2\left(1 - \frac{p_1}{\bar{k}}\right), \tag{18}$$

leading to the following result.

Lemma 8 *Suppose F is a uniform distribution. Then $e_p^* \leq e_k^*$ for any $k \geq 2$, with equality if and only if $p_1 = 0$.*

Lemma 8 states that for a uniform distribution of noise the individual equilibrium effort of participating players in a tournament with stochastic group size cannot be higher than with deterministic group size, and is strictly lower if the probability for a player to be alone in the tournament is not zero. Indeed, if $p_1 = 0$, there are at least two players in the tournament (from the perspective of a player who has been selected), and the result follows because equilibrium effort is independent of tournament size for $k \geq 2$ when F is uniform (see Lemma 4).

5.3 The effect of noise dispersion

Similar to Section 4.3, suppose the distribution of group sizes, p, is fixed and consider the effect of changes in the dispersion of noise on the equilibrium effort. Throughout this section, we will use $B_p[f]$ and $e_p^*[f]$ to denote, respectively, coefficient B_p and the equilibrium effort e_p^* corresponding to the distribution of noise with pdf $f(t)$. Let $\tilde{g}(z) = \tilde{G}_z(z)$ denote the derivative of the pgf $\tilde{G}$ with respect to z. Changing the variable of integration to $z = F(t)$, rewrite (13) in the form

$$B_p[f] = \int_0^1 m(z)\tilde{g}(z)dz = \int_U \tilde{g}(F(t))f(t)^2 dt. \tag{19}$$

Consider a pdf $f_p(t)$ (with support U) defined as follows:

$$f_p(t) = \frac{1}{c_p} f(t)\sqrt{\tilde{g}(F(t))}, \quad c_p = \int_U f(t)\sqrt{\tilde{g}(F(t))}dt = \int_0^1 \sqrt{\tilde{g}(z)}dz, \tag{20}$$

where the normalization constant c_p is independent of f. Then Eq. (19) can be written in the form

$$B_p[f] = c_p^2 \int_U f_p(t)^2 dt = c_p^2 \exp(-H[f_p]), \tag{21}$$

where $H[\cdot]$ is the Rényi entropy. We arrive at the following results.

Proposition 9 *(i) In tournaments with stochastic participation, the equilibrium effort decreases in the Rényi entropy of a distribution with pdf f_p.*

(ii) Of all noise distributions with a finite support $[u_l, u_h]$, the equilibrium effort is minimized by the distribution such that $f_p(t) = \frac{1}{u_h - u_l}$; that is, cdf $F_{\min}$ satisfies the differential equation

$$F'(t) = \frac{c_p}{(u_h - u_l)\sqrt{\tilde{g}(F(t))}}. \tag{22}$$

The minimized value of B_p is $B_p[f_{\min}] = \frac{c_p^2}{u_h - u_l}$.

It is easy to see that the results for deterministic participation can be recovered as a special case for a degenerate p. The right-hand side of Eq. (22) decreases in t; hence, similar to the deterministic participation case, the effort-minimizing cdf is concave, with a monotonically decreasing pdf.

For illustration, consider $K \sim \text{Binomial}(n,q)$. From (15), $\tilde{g}(z) = (n-1)q(1-q+qz)^{n-2}$, $c_p = \sqrt{\frac{4(n-1)}{qn^2}}[1-(1-q)^{\frac{n}{2}}]$, and

$$f_p(t) = \frac{nqf(t)[1-q+qF(t)]^{\frac{n}{2}-1}}{2[1-(1-q)^{\frac{n}{2}}]}.$$

The equilibrium effort is minimized when $f_p(t)$ is uniform on $[u_l, u_h]$, and the minimized value of B_p is $B_p[f_{\min}] = \frac{4(n-1)[1-(1-q)^{\frac{n}{2}}]^2}{qn^2(u_h-u_l)}$.

Note that $\tilde{g}(z)$ is independent of the shape of the distribution of noise. Representation (19) then immediately implies that if X is more dispersed than Y then $B_p[f_X] \leq B_p[f_Y]$; thus, the dispersive order of noise distributions has the same effect on the equilibrium effort as in the deterministic participation case (cf. Lemma 5).

Lemma 9 *If X is more dispersed than Y then $e_p^*[f_X] \leq e_p^*[f_Y]$.*

5.4 A comparison between stochastic and deterministic participation

It may be of interest to compare expected aggregate effort in a tournament with stochastic participation, $E_p^* = \bar{k}e_p^*$, to aggregate effort in the tournament with deterministic participation of size $\bar{k}$, $E_{\bar{k}}^* = \bar{k}e_{\bar{k}}^*$. The results are summarized in the following proposition.

Proposition 10 *(a) Suppose $\bar{k} = \sum_{k=0}^{n} kp_k$ is integer. Suppose also that $p_0 = 0$ and for all $k \geq 1$ in the support of p kb_k is concave. Then $E_p^* \leq E_{\bar{k}}^*$; moreover, the inequality is strict if kb_k is strictly concave.*

(b) Suppose $\bar{k} \geq 2$ is integer. Suppose also that for all $k \geq 2$ in the support of p (i) kb_k is concave and (ii) b_k is nonincreasing. Then $E_p^* \leq E_{\bar{k}}^*$; moreover, the inequality is strict if kb_k is strictly concave or $p_1 > 0$.

The comparison between aggregate efforts E_p^* and $E_{\bar{k}}^*$ for a given $\bar{k}$ is equivalent to the comparison of individual efforts e_p^* and $e_{\bar{k}}^*$. The general intuition behind Proposition 10 is that B_p, which determines e_p^*, is proportional to the expectation of Kb_K, cf. Eq. (14), and the concavity of kb_k gives the result by Jensen's inequality. However, since this expectation is conditional and also divided by the expected number of players $\bar{k}$, additional qualifiers are needed. For part (a), note that $\bar{k} = \mathrm{E}_p(K)$ is the unconditional expectation of K while B_p is proportional to the expectation of Kb_K conditional on $K \geq 1$. By setting $p_0 = 0$, this conditional expectation becomes unconditional and Jensen's inequality gives the result. For part (b), as seen from (14), B_p can also be written as proportional to the expectation of Kb_K conditional on $K \geq 2$; while the expectation of K conditional on $K \geq 2$ is always (weakly) greater than the unconditional expectation of K. Then, the result is obtained using Jensen's inequality for conditional expectations (for concave kb_k) and the assumption that b_k is nonincreasing for $k \geq 2$. Part (a) of Proposition 10 generalizes the result of Myerson and Wärneryd (2006) who studied generalized Tullock contests with an arbitrary distribution of group size (subject to the restriction $p_0 = 0$). Part (b) generalizes the result of Lim and Matros (2009) who analyzed Tullock contests with $K \sim \text{Binomial}(n, q)$.

For examples of violations of the conditions of Proposition 10, when stochastic participation can lead to a higher expected aggregate effort, consider the binomial distribution of tournament size, $K \sim \text{Binomial}(n, q)$. Let q_{opt} denote the *optimal* participation probability, that is, the probability q that maximizes expected aggregate effort $E_p^* = \bar{k}e_p^*$ subject to the constraint $\bar{k} = nq$. The deterministic contest generates a higher aggregate effort if $q_{\text{opt}} = 1$. The binomial distribution violates the conditions of part (a) of Proposition 10 since $p_0 = (1-q)^n > 0$. Also, for the bimodal distribution of noise in Figure 1 both assumptions (i) and (ii) of part (b) do not hold. Then, $q_{\text{opt}} \approx 0.9$ for $\bar{k} = 3$ and $q_{\text{opt}} \to 0$ (that is, a tournament with $n \to \infty$ potential players, each with zero probability of participation, is optimal) for $\bar{k} \geq 4$. For the $F_{2,2}$-distribution of noise (see the end of Section 4.2) assumption (i) of part (b) is violated, and $q_{\text{opt}} \in (0, 1)$ for $3 \leq \bar{k} \leq 5$ while $q_{\text{opt}} \to 0$ for $\bar{k} \geq 6$.

5.5 Optimal disclosure of the number of players

Several authors investigated optimal disclosure policies under uncertainty, asking whether it makes sense for a principal whose goal is the maximization of aggregate effort, to disclose to players how many participants there are in the tournament. Lim and Matros (2009) show that in a standard Tullock contest with the binomial distribution of the number of players aggregate effort is independent of disclosure. Fu, Jiao and Lu (2011) generalize this result to lottery-form contests with CSFs of the form $\frac{h(e_i)}{\sum_{j=1}^{k} h(e_j)}$. They show that full disclosure (no disclosure) is optimal if $\frac{h(e)}{h'(e)}$ is strictly convex (concave), while the indifference is recovered when $\frac{h(e)}{h'(e)}$ is linear.[22] In this section, we generalize these results

[22]In asymmetric settings, the consequences of disclosure/nondisclosure become richer. For recent developments see, e.g., Denter, Morgan and Sisak (2014), Fu, Lu and Zhang (2016) and Zhang and Zhou (2016).

to arbitrary tournaments and arbitrary distributions of the number of players.

Without disclosure, the expected aggregate effort in the tournament is $E_p^* = \bar{k}e_p^* = \bar{k}c'^{-1}(B_p)$, where, from (14), $B_p = \mathrm{E}_{\tilde{p}}(b_K)$. With disclosure, the expected aggregate effort is $\mathrm{E}_p(Kc'^{-1}(b_K))$, which can be rewritten as

$$\mathrm{E}_p(Kc'^{-1}(b_K)) = \sum_{k=1}^{n} p_k k c'^{-1}(b_k) = \bar{k} \sum_{k=1}^{n} \tilde{p}_k c'^{-1}(b_k) = \bar{k}\mathrm{E}_{\tilde{p}}(c'^{-1}(b_K)).$$

Thus, comparing E_p^* and $\mathrm{E}_p(Kc'^{-1}(b_K))$ is equivalent to comparing $c'^{-1}(\mathrm{E}_{\tilde{p}}(b_K))$ and $\mathrm{E}_{\tilde{p}}(c'^{-1}(b_K))$.

It follows that the optimality of disclosure depends entirely on the concavity/convexity of c'^{-1}, and not on the nature of coefficients b_k. One special case is when b_k is constant in the support of $\tilde{p}$ (for example, noise is uniformly distributed and $p_1 = 0$); in this case the two expressions are equal. When b_k is not constant in the support of $\tilde{p}$, and c'^{-1} is concave (convex) and nonlinear for at least some distinct values of b_k, disclosure is not optimal (optimal). Note that the concavity (convexity) of c'^{-1} is equivalent to the convexity (concavity) of c', i.e., to the condition $c''' \geq (\leq) 0$.

Proposition 11 *Suppose b_k is non-constant for k in the support of $\tilde{p}$, and $c'(\cdot)$ is non-linear for at least some distinct values of b_k in the support of $\tilde{p}$. Then it is optimal to disclose (not disclose) the number of participants in the tournament if $c''' \leq (\geq) 0$.*

5.6 Tournaments with size $k \geq 2$

Proposition 7 on the unimodality of $B_p(\theta)$ in Section 5.2 is quite general, but it imposes a restriction on how θ may affect the distribution of tournament size, in the form of the log-supermodularity of $|\tilde{G}_\theta(z, \theta)|$. As we show in this section, the unimodality of $B_p(\theta)$ can also be obtained under an alternative set of restrictions on pmf p; namely, a requirement that $p_1 = 0$. In other words, in this section we consider tournaments in which, from the perspective of a participating player, the number of players is known to be at least two. Such tournaments are rather common in applications; indeed, it is common for organizers to have a provision that competition will be canceled if only one participant signs up.

We consider the effects of an upward probabilistic shift in the updated distribution of group size from $\tilde{p}$ to $\tilde{p}'$. When $\{b_k\}_{k=2}^n$ is nondecreasing, the result is straightforward and given by Lemma 7 and Corollary 4. Generally, when $\{b_k\}_{k=1}^n$ is nonmonotone, the effect of such a shift is ambiguous without additional restrictions on p and p'. Note that $p_1 = 0$

and $\tilde{p}'$ FOSD $\tilde{p}$ jointly imply that $p_1' = 0$. The following results then follow immediately from (14) and Proposition 3.

Lemma 10 *Suppose $\tilde{p}'$ FOSD $\tilde{p}$ and $p_1 = 0$. If $\{b_k\}_{k=2}^n$ is nonincreasing then $B_{p'} \leq B_p$ (and $e^*_{p'} \leq e^*_p$).*

Corollary 5 *Suppose $\tilde{p}'$ FOSD $\tilde{p}$ and $p_1 = 0$. Then,*
(i) if $f(t)$ is nonincreasing then $e^*_{p'} \leq e^*_p$;
(ii) for $n \geq 4$, if $f(t)$ is interior unimodal and symmetric then $e^*_{p'} \leq e^*_p$;
(iii) for $n = 3$, if $f(t)$ is symmetric then $e^*_{p'} = e^*_p$.

Parts (i) and (ii) follow from parts (ii) and (iv) of Proposition 3. Part (iii) follows from part (v) of Proposition 3.

Lemma 10 has one other interesting implication. When $\{b_k\}_{k=2}^n$ is nonincreasing, *the only way* e^*_p can be nonmonotone with respect to an upward probabilistic shift in $\tilde{p}$ is if $p_1 > 0$. Put differently, the possibility for a player to find herself alone in the tournament is the only mechanism through which the individual equilibrium effort can be nonmonotone in a parameter θ. One example is the Tullock contest, for which $b_k = \frac{r(k-1)}{k^2}$ decreases monotonically for $k \geq 2$, and Lim and Matros (2009) found that the individual equilibrium effort is nonmonotone in q for $K \sim \text{Binomial}(n, q)$. Lemma 10 shows that this nonmonotonicity is a consequence entirely of the fact that $p_1 = nq(1-q)^{n-1} > 0$. If the distribution of group size is replaced with a truncated binomial distribution such that $p_1 = 0$, the nonmonotonicity will go away. Of course, the nonmonotonicity can still arise even when $p_1 = 0$ if $\{b_k\}_{k=2}^n$ is nonmonotone; for example, if it is interior unimodal.

6 Conclusion

In this paper we derive robust comparative statics results for rank-order tournaments in which a player's effort is distorted by additive or multiplicative noise and the number of players is either deterministic or stochastic. The unimodality of the distribution of noise is critical for robust comparative statics, due to results on the preservation of unimodality under uncertainty. In the deterministic case, we show that the equilibrium effort is unimodal in the number of players when the distribution of noise is unimodal. In the stochastic case, the equilibrium effort is similarly unimodal in parameters shifting the distribution of the number of players in the sense of first-order stochastic dominance, albeit under an additional log-supermodularity restriction. The unimodality of the distribution of noise is a tight condition; we provide examples of non-unimodal noise distributions for which the comparative statics are no longer unimodal. We also show that, generally, there is no universality in the behavior of aggregate equilibrium effort.

The second dimension of our analysis is the effect of noise dispersion. We show that the equilibrium effort decreases in the appropriately defined Rényi entropy, as opposed to the often-cited variance or second-order stochastic dominance order. For the case of deterministic participation, it is the entropy of order statistics of the distribution of noise, while in the case of stochastic participation it is the entropy of a distribution that combines the distribution of noise with the distribution of tournament size. An important special case of entropy ordering that applies to both cases is the dispersive order of noise distributions.

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A Proofs

Second-order condition Differentiating the payoff function (2) twice with respect to e_i and setting $e_i = e^*$, obtain $\frac{\partial^2 \pi_i(e_i, e^*)}{\partial e_i^2}\Big|_{e_i=e^*} = \eta_k - c''(e^*)$, where

$$\eta_k = (k-1)\left[(k-2)\int_U F(t)^{k-3} f(t)^2 dF(t) + \int_U F(t)^{k-2} f'(t) dF(t)\right].$$

Integrating the second term by parts, obtain

$$\eta_k = \frac{k-1}{2}\left[(k-2)\int_U F(t)^{k-3} f(t)^2 dF(t) + f(u_h)^2 - f(u_l)^2 I_{k=2}\right],$$

where $I_{k=2}$ is an indicator equal to one if $k = 2$ and zero otherwise. Thus, when $k = 2$ and the distribution of noise is symmetric the second-order condition is always satisfied. Otherwise, the restriction $\eta_k - c''(e^*) < 0$ has to be imposed.

Proof of Lemma 1 (i) Sufficiency: When $a(z)$ is monotone, it follows immediately that $\gamma(\theta)$ is monotone. Suppose that $a(z)$ is interior unimodal; in this case, $a(1)$ is finite. Integrating by parts, obtain

$$\gamma(\theta) = a(1) - \int_0^1 a'(z) H(z, \theta) dz. \tag{23}$$

Let $\hat{z} \in (0, 1)$ denote a mode of $a(z)$. Differentiating, or taking the first difference, with respect to θ, and splitting the integral in (23), obtain

$$\gamma'(\theta) = -\int_0^{\hat{z}} a'(z) H_\theta(z,\theta) dz - \int_{\hat{z}}^1 a'(z) H_\theta(z,\theta) dz$$
$$= \int_0^{\hat{z}} a'(z) |H_\theta(z,\theta)| dz - \int_{\hat{z}}^1 |a'(z)| |H_\theta(z,\theta)| dz. \quad (24)$$

Suppose $\gamma'(\theta) \leq 0$ for some θ and consider a $\theta' > \theta$. Then (24) gives

$$\gamma'(\theta') = \int_0^{\hat{z}} a'(z) |H_\theta(z,\theta')| dz - \int_{\hat{z}}^1 |a'(z)| |H_\theta(z,\theta')| dz$$
$$= \int_0^{\hat{z}} a'(z) r(z,\theta,\theta') |H_\theta(z,\theta)| dz - \int_{\hat{z}}^1 |a'(z)| r(z,\theta,\theta') |H_\theta(z,\theta')| dz$$
$$\leq r(\hat{z},\theta,\theta') \int_0^{\hat{z}} a'(z) |H_\theta(z,\theta)| dz - r(\hat{z},\theta,\theta') \int_{\hat{z}}^1 |a'(z)| |H_\theta(z,\theta')| dz = r(\hat{z},\theta,\theta') \gamma'(\theta) \leq 0.$$

Here, the first inequality follows from the assumption that $r(z,\theta,\theta')$ is nondecreasing in z. Thus, we showed that $\gamma(\theta)$ is unimodal.

(ii) Necessity: Suppose that there exist $\theta' > \theta$ and a $z \in [0,1]$ such that $r(z,\theta,\theta')$ is decreasing in z. The proof consists in showing that a unimodal function $a(z)$ can then be constructed such that $\gamma(\theta)$ is not unimodal. By continuity, there exists an interval of positive length $[z_1, z_2]$ where $r(z,\theta,\theta')$ is strictly decreasing. First, define a unimodal function $a(z)$ such that it is nonzero only withing this interval. Furthermore, $a(z)$ can be defined in a way that $\gamma'(\theta) = 0$. For example, it can be defined as a piece-wise linear function such that $a'(z) = \int_{\hat{z}}^{z_2} |H_\theta(z,\theta)| dz$ for $z \in (z_1, \hat{z})$ and $|a'(z)| = \int_{z_1}^{\hat{z}} |H_\theta(z,\theta)| dz$ for $z \in (\hat{z}, z_2)$. In this case, it follows from (24) that $\gamma'(\theta) = 0$. Finally, we modify this $a(z)$ "slightly" to make $\gamma'(\theta)$ negative. For example, choose some $\epsilon > 0$ and set

$a'(z) = \int_{\hat{z}}^{z_2} |H_\theta(z,\theta)|dz - \epsilon$ for $z \in (z_1, \hat{z})$. Then

$$\begin{aligned}
\gamma'(\theta') &= \int_{z_1}^{\hat{z}} a'(z) r(z,\theta,\theta')|H_\theta(z,\theta)|dz - \int_{\hat{z}}^{z_2} |a'(z)| r(z,\theta,\theta')|H_\theta(z,\theta')|dz \\
&= r(z_1^*,\theta,\theta') \int_{z_1}^{\hat{z}} a'(z)|H_\theta(z,\theta)|dz - r(z_2^*,\theta,\theta') \int_{\hat{z}}^{z_2} |a'(z)||H_\theta(z,\theta')|dz \\
&= r(z_1^*,\theta,\theta') \left[\int_{\hat{z}}^{z_2} |H_\theta(z,\theta)|dz - \epsilon\right] \int_{z_1}^{\hat{z}} |H_\theta(z,\theta)|dz \\
&- r(z_2^*,\theta,\theta') \int_{\hat{z}}^{z_2} |H_\theta(z,\theta')|dz \int_{z_1}^{\hat{z}} |H_\theta(z,\theta)|dz \\
&= (r(z_1^*,\theta,\theta') - r(z_2^*,\theta,\theta')) \int_{z_1}^{\hat{z}} |H_\theta(z,\theta)|dz \int_{\hat{z}}^{z_2} |H_\theta(z,\theta')|dz \\
&- \epsilon r(z_1^*,\theta,\theta') \int_{\hat{z}}^{z_2} |H_\theta(z,\theta)|dz.
\end{aligned}$$

Here, $z_1^* \in (z_1, \hat{z})$ and $z_2^* \in (\hat{z}, z_2)$ exist due to the mean-value theorem for definite integrals. Note that $z_2^* > z_1^*$ and hence the first term in the last expression is positive, while the second term can be made arbitrarily small via the choice of ϵ; therefore, an $\epsilon > 0$ can be chosen such that $\gamma'(\theta') > 0$. Thus, $\gamma(\theta)$ is not unimodal. ■

Proof of Lemma 2 (i) Sufficiency: Rewrite $\chi(\theta)$ as follows:

$$\begin{aligned}
\chi(\theta) &= y_1(\theta)x_1 + y_2(\theta)x_2 + \ldots + y_{n-1}(\theta)x_{n-1} + y_n(\theta)x_n \\
&= Y_1(\theta)x_1 + (Y_2(\theta) - Y_1(\theta))x_2 + \ldots + (Y_{n-1}(\theta) - Y_{n-2}(\theta))x_{n-1} + (Y_n(\theta) - Y_{n-1}(\theta))x_n \\
&= x_n + Y_1(\theta)(x_1 - x_2) + Y_2(\theta)(x_2 - x_3) + \ldots + Y_{n-1}(\theta)(x_{n-1} - x_n) \\
&= x_n - \sum_{k=1}^{n-1} Y_k(\theta)\Delta x_{k+1},
\end{aligned}$$

where $\Delta x_{k+1} = x_{k+1} - x_k$. This "summation by parts" representation is similar to integration by parts and expresses the expectation $\chi(\theta)$ through the cmf $Y(\theta)$ and the first difference of x_k. Taking the derivative, or the difference, with respect to θ, obtain

$$\chi'(\theta) = -\sum_{k=1}^{n-1} Y_k'(\theta)\Delta x_{k+1} = \sum_{k=1}^{n-1} |Y_k'(\theta)|\Delta x_{k+1}.$$

Let $\hat{k}$ denote a mode of x such that $\Delta x_{k+1} \geq (\leq) 0$ for $k < (\geq) \hat{k}$. This gives

$$\chi'(\theta) = \sum_{k<\hat{k}} |Y_k'(\theta)| \Delta x_{k+1} - \sum_{k \geq \hat{k}} |Y_k'(\theta)||\Delta x_{k+1}|.$$

Suppose that $\chi'(\theta) \leq 0$ for some θ and consider a $\theta' > \theta$. Then

$$\begin{aligned}
\chi'(\theta') &= \sum_{k<\hat{k}} |Y_k'(\theta')| \Delta x_{k+1} - \sum_{k \geq \hat{k}} |Y_k'(\theta')||\Delta x_{k+1}| \\
&= \sum_{k<\hat{k}} |Y_k'(\theta)| r(k,\theta,\theta') \Delta x_{k+1} - \sum_{k \geq \hat{k}} |Y_k'(\theta)| r(k,\theta,\theta') |\Delta x_{k+1}| \\
&\leq r(\hat{k},\theta,\theta') \sum_{k<\hat{k}} |Y_k'(\theta)| \Delta x_{k+1} - r(\hat{k},\theta,\theta') \sum_{k \geq \hat{k}} |Y_k'(\theta||\Delta x_{k+1}| = r(\hat{k},\theta,\theta') \chi'(\theta) \leq 0.
\end{aligned}$$

Here, the first inequality follows from the assumption that $r(\hat{k},\theta,\theta')$ is nondecreasing in k.

(ii) Necessity: Suppose that there exist $\theta' > \theta$ and k such that $r(k-1,\theta,\theta') > r(k,\theta,\theta')$. As in the proof of Lemma 1, we will show that it is possible to construct a unimodal sequence x such that $\chi(\theta)$ is not unimodal. Set $x_l = a$ for all $l \leq k-1$ and $x_l = b$ for all $l \geq k+1$; furthermore, set $x_k > \max\{a,b\}$. The resulting sequence x is interior unimodal with mode k and satisfies $\Delta x_k > 0$, $\Delta x_{k+1} < 0$, and $\Delta x_l = 0$ for all $l \neq k, k+1$. Then

$$\chi'(\theta) = |Y_{k-1}'(\theta)| \Delta x_k - |Y_k'(\theta)||\Delta x_{k+1}|.$$

Choosing a, x_k and b so that $\Delta x_k = |Y_k'(\theta)| - \epsilon$ for some $\epsilon > 0$ and $|\Delta x_{k+1}| = |Y_{k-1}'(\theta)|$, obtain $\chi'(\theta) = -\epsilon |Y_{k-1}'(\theta)| < 0$. However,

$$\begin{aligned}
\chi'(\theta') &= |Y_{k-1}'(\theta')| \Delta x_k - |Y_k'(\theta')||\Delta x_{k+1}| \\
&= r(k-1,\theta,\theta') |Y_{k-1}'(\theta)| (|Y_k'(\theta)| - \epsilon) - r(k,\theta,\theta') |Y_k'(\theta)||Y_{k-1}'(\theta)| \\
&= (r(k-1,\theta,\theta') - r(k,\theta,\theta')) |Y_k'(\theta)||Y_{k-1}'(\theta)| - \epsilon r(k-1,\theta,\theta') |Y_{k-1}'(\theta)|.
\end{aligned}$$

The first term on the last line is strictly positive, while the second term can be made arbitrarily small through the choice of ϵ; thus, an $\epsilon > 0$ can be chosen such that $\chi'(\theta') > 0$, i.e., $\chi(\theta)$ os not unimodal. ■

Proof of Lemma 3 (i) Sufficiency: By differentiating, or taking the first difference of, Eq. (6) with respect to θ, obtain

$$\sum_{k=1}^{n} Y_k'(\theta) z^{k-1} = \frac{G_\theta(z,\theta)}{1-z},$$

which gives, for some $\theta' > \theta$,

$$R(z,\theta,\theta') = \frac{|G_\theta(z,\theta')|}{|G_\theta(z,\theta)|} = \frac{\sum_{k=1}^{n} |Y_k'(\theta')| z^{k-1}}{\sum_{k=1}^{n} |Y_k'(\theta)| z^{k-1}} = \frac{\sum_{k=1}^{n} |Y_k'(\theta)| r(k,\theta,\theta') z^{k-1}}{\sum_{k=1}^{n} |Y_k'(\theta)| z^{k-1}}. \tag{25}$$

Define a pmf $\alpha_k(z) = \frac{|Y_k'(\theta)| z^{k-1}}{\sum_{l=1}^{n} |Y_l'(\theta)| z^{l-1}}$ and the corresponding cmf $A_k(z) = \sum_{l=1}^{k} \alpha_k(z)$. Then (25) can be written as an expectation $R(z,\theta,\theta') = \sum_{k=1}^{n} \alpha_k(z) r(k,\theta,\theta')$ of a nondecreasing random variable $r(K,\theta,\theta')$. This expectation is nondecreasing in z provided an increase in z leads to an FOSD increase in distribution $\alpha(z)$, i.e., if $A_k(z)$ is nonincreasing in z. The derivative of $A_k(z)$ is

$$A_k'(z) = \frac{d}{dz}\left(\frac{\sum_{l=1}^{k} |Y_l'(\theta)| z^{l-1}}{\sum_{l=1}^{n} |Y_l'(\theta)| z^{l-1}}\right) = \frac{1}{(\sum_{l=1}^{n} |Y_l'(\theta)| z^{l-1})^2} \sum_{l=1}^{k} \sum_{l'=1}^{n} |Y_l'(\theta)||Y_{l'}'(\theta)| z^{l+l'-3}(l-l')$$

$$= \frac{1}{(\sum_{l=1}^{n} |Y_l'(\theta)| z^{l-1})^2} \sum_{l=1}^{k} \sum_{l'=k+1}^{n} |Y_l'(\theta)||Y_{l'}'(\theta)| z^{l+l'-3}(l-l') \leq 0. \tag{26}$$

(ii) Necessity: Define $\Delta r_{l+1} = r(l+1,\theta,\theta') - r(l,\theta,\theta')$, and suppose that $\Delta r_{k+1} < 0$ for some k and $\theta' > \theta$. Using the same "summation by parts" transformation as at the start of the proof of Lemma 2, write

$$R(z,\theta,\theta') = r(n,\theta,\theta') - \sum_{l=1}^{n-1} A_l(z) \Delta r_{l+1},$$

which gives, differentiating with respect to z,

$$R_z(z,\theta,\theta') = \sum_{l=1}^{n-1} |A_l'(z)| \Delta r_{l+1}.$$

Choose $Y_l(\theta)$ so that $Y_l'(\theta) = 0$ for all $l \neq k, k+1$ and $Y_k'(\theta), Y_{k+1}'(\theta) < 0$. Equation (26) then gives

$$A_k'(z) = \frac{-|Y_k'(\theta)||Y_{k+1}'(\theta)| z^{2k-2}}{(|Y_k'(\theta)| z^{k-1} + |Y_{k+1}'(\theta)| z^k)^2} < 0$$

and $A'_l(z) = 0$ for all $l \neq k$; therefore, we obtain $R_z(z, \theta, \theta') = |A'_k(z)|\Delta r_{k+1} < 0$, which is a contradiction. ■

Proof of Lemma 4 Sufficiency is obvious: If F is a uniform distribution, $m(z)$ is a constant and $b_k = m(0)$ (for $k \geq 2$). Conversely, suppose $b_k = b_2$ for all $k \geq 2$. This implies $(k+1)m_k = b_2$ and hence $m_k = \frac{b_2}{k+1}$ for all $k = 0, 1, \ldots$. The moment-generating function of $m(z)$, defined as $\phi(t) = \mathrm{E}(\exp(tZ))$, can be written in the form of expansion over moments, $\phi(t) = \sum_{k=0}^{\infty} \frac{m_k}{k!} t^k$, which gives

$$\phi(t) = \sum_{k=0}^{\infty} \frac{b_2}{(k+1)!} t^k = \frac{b_2}{t}(\exp(t) - 1).$$

This is the moment-generating function of an (unnormalized) uniform distribution on $[0, 1]$, implying $m(z)$ is a constant and F is uniform. ■

Proof of Proposition 2 Recall that $b_k = \int_0^1 m(z) dz^{k-1}$; therefore, integrating by parts,

$$b_k - b_{k+1} = \int_0^1 m(z) d(z^{k-1} - z^k) = -\int_0^1 z^{k-1}(1-z)m'(z)dz.$$

Suppose $m(z)$ is nonincreasing and nonconstant on $(\hat{z}, 1)$ (the case of a nondecreasing and nonconstant $m(z)$ is proved similarly). Then

$$\begin{aligned}
b_k - b_{k+1} &= -\int_0^{\hat{z}} z^{k-1}(1-z)m'(z)dz + \int_{\hat{z}}^1 z^{k-1}(1-z)|m'(z)|dz \\
&\geq \int_{\hat{z}}^1 z^{k-1}(1-z)|m'(z)|dz - \int_0^{\hat{z}} z^{k-1}(1-z)|m'(z)|dz \\
&= M_1 \int_{\hat{z}}^1 z^{k-1}dz - M_2 \int_0^{\hat{z}} z^{k-1}dz,
\end{aligned}$$

where M_1 and M_2 are positive constants (independent of k), the existence of which follows from the mean-value theorem for definite integrals. Evaluating the integrals, further obtain

$$b_k - b_{k+1} \geq \frac{1}{k}[M_1(1 - \hat{z}^k) - M_2 \hat{z}^k] = \frac{1}{k}[M_1 - \hat{z}^k(M_1 + M_2)].$$

Since $\hat{z} < 1$, it is clear that the last expression becomes positive for a sufficiently large k. ■

Proof of Proposition 3 Define

$$\Delta b_{k+3} = b_{k+3} - b_{k+2} = \int_0^1 \left[(k+2)z^{k+1} - (k+1)z^k\right] m(z)dz, \quad k = 0, 1, \ldots, n-3. \quad (27)$$

Integrating by parts, obtain

$$\Delta b_{k+3} = \int_0^1 m(z)d(z^{k+2} - z^{k+1}) = \int_0^1 z^{k+1}(1-z)m'(z)dz. \quad (28)$$

For part (iv), the symmetry of $f(t)$ around its mean μ implies $f(t) = f(2\mu - t)$ and $F(t) = 1 - F(2\mu - t)$ for all $t \in U$. Letting $z = F(t) = 1 - F(2\mu - t)$, obtain $1 - z = F(2\mu - t)$, $F^{-1}(1-z) = 2\mu - t$ and $m(1-z) = f(F^{-1}(1-z)) = f(2\mu - t) = f(t) = f(F^{-1}(z)) = m(z)$. Thus, the symmetry of the distribution of noise implies $m(z) = m(1-z)$ and $m'(z) = -m'(1-z)$ for all $z \in [0, 1]$.

This gives, via a change of variable $z \to 1 - z$,

$$\Delta b_{k+3} = -\int_0^{\frac{1}{2}} z(1-z)[(1-z)^k - z^k]m'(z)dz,$$

which immediately implies that $\Delta b_3 = 0$ and $\Delta b_{k+3} < 0$ for $k > 0$.

For part (v), note that $b_2 = \int_0^1 m(z)dz$ and, if $m(z) = m(1-z)$ (which only requires symmetry but not unimodality of f),

$$b_3 = 2\int_0^1 zm(z)dz = 2\int_0^1 (1-z)m(1-z)dz = 2\int_0^1 (1-z)m(z)dz = 2b_2 - b_3,$$

which implies $b_2 = b_3$. ■

Proof of Proposition 5 Given the cost function, $E_k^* = \frac{1}{2c_0}kb_k$. Integrating by parts twice, obtain

$$\begin{aligned} E_k^* &\propto k(k-1)\int_0^1 z^{k-2}m(z)dz = k\left[m(1) - \int_0^1 m'(z)z^{k-1}dz\right] \\ &= km(1) - m'(1) + \int_0^1 m''(z)z^k dz, \quad k \geq 2, \end{aligned}$$

which gives

$$\Delta E_{k+1}^* = E_{k+1}^* - E_k^* \propto m(1) - \int_0^1 m''(z)(z^k - z^{k+1})dz.$$

Since $m(1) = 0$, noting that log-concavity (log-convexity) of $f(t)$ is equivalent to concavity (convexity) of $m(z)$, proves parts (i) and (ii).

For part (iii), note that if $f(t)$ is first log-concave and then log-convex, then $-m''(z)$ is single crossing and hence, $-m'(z)$ is unimodal. Since $z^k - z^{k+1}$ is log-supermodular, Lemma 1 implies the result. ■

Proof of Lemma 5 Definition 3 is equivalent to the requirement that $F_X^{-1}(z) - F_Y^{-1}(z)$ is nondecreasing in z. Differentiating with respect to z, obtain $\frac{1}{f_X(F_X^{-1}(z))} - \frac{1}{f_Y(F_Y^{-1}(z))} \geq 0$, or, using the definition of inverse quantile density, $m_X(z) \leq m_Y(z)$ (with a strict inequality in some open interval). Equation (4) then gives the result. ■

Proof of Lemma 6 For part (a), note that since f_X and f_Y are nondecreasing and Y FOSD X, for any nondecreasing function $u(t)$ we have $\int f_Y(t)u(t)dt \geq \int f_X(t)u(t)dt$. Using $u(t) = f_Y(t)$, obtain $\int f_Y(t)^2 dt \geq \int f_X(t)f_Y(t)dt$; using $u(t) = f_X(t)$, obtain $\int f_Y(t)f_X(t)dt \geq \int f_X(t)^2 dt$. Combining the two inequalities, obtain the result.

For part (b), similarly, note that X FOSD Y and hence for any nonincreasing function $u(t)$ we have $\int f_X(t)u(t)dt \leq \int f_Y(t)u(t)dt$. Using $u(t) = f_Y(t)$ and $u(t) = f_X(t)$ consecutively, obtain the result.

For part (c), note that due to symmetry $b_2[f_X] = 2\int_{u_l}^{\mu} f_X(t)^2$, and similarly for f_Y, where $\mu = \mathrm{E}(X) = \mathrm{E}(Y)$ is the middle of the interval $[u_l, u_h]$. Functions f_X and f_Y satisfy the conditions of part (a) on $[u_l, \mu]$, and the result follows. ■

Proof of Proposition 10 In order to compare $E_p^* = \bar{k}e_p^*$ to $E_{\bar{k}}^* = \bar{k}e_{\bar{k}}^*$, we need to compare e_p^* and $e_{\bar{k}}^*$, i.e., it is sufficient to compare B_p given by (14) and $b_{\bar{k}}$.

(a) Suppose $p_0 = 0$ and kb_k is concave for $k \geq 1$. Then

$$B_p = \frac{1}{\bar{k}}\sum_{k=1}^{n} p_k k b_k = \frac{1}{\bar{k}}\mathrm{E}_p(Kb_K) \leq \frac{1}{\bar{k}}\bar{k}b_{\bar{k}} = b_{\bar{k}},$$

where the inequality follows from Jensen's inequality, which will be strict if kb_k is strictly concave.

(b) From Jensen's inequality for conditional expectations, and assumptions (i) and (ii),

$$\mathrm{E}_p(Kb_K|K \geq 2) \leq \mathrm{E}_p(K|K \geq 2)b_{\mathrm{E}_p(K|K\geq 2)} \leq \mathrm{E}_p(K|K \geq 2)b_{\bar{k}}.$$

The first inequality will be strict if kb_k is strictly concave. Multiplying both sides by

$\Pr_p(K \geq 2)$,

$$\mathrm{E}_p(Kb_K|K \geq 2)\Pr_p(K \geq 2) \leq \mathrm{E}_p(K|K \geq 2)\Pr_p(K \geq 2)b_{\bar{k}},$$

or

$$\bar{k}B_p \leq \sum_{k=2}^{n} kp_k b_{\bar{k}} \leq \sum_{k=0}^{n} kp_k b_{\bar{k}} = \bar{k}b_{\bar{k}}.$$

The last inequality will be strict if $p_1 > 0$. Thus, we showed that $B_p \leq b_{\bar{k}}$, with strict inequality if kb_k is strictly concave or $p_1 > 0$. ■