

# Growth-inequality nexus in a simple capital accumulation model \*

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Very preliminary. Do not quote.

## Abstract

In this paper we address the joint distribution and growth processes by combining the inherent conservative property of distributions, highlighted by the mean-field game literature, and simple capital accumulation dynamics of benchmark economic growth theory. Given an initial unequal distribution of capital, and assuming a deterministic setting, we show that there are three main types of evolutions: asymptotic equality but no long run growth, asymptotic growth and a stationary distribution featuring inequality, or growth together with increasing inequality. The last type of evolution is Pareto optimal if capital accumulation depends linearly on the capital stock. Introducing a multiplicative random capital redistribution process, we show that we always get an increase in inequality although it can occur together with growth (if noise is relatively low) or within a non-growth context (when noise is very high).

KEYWORDS: growth, inequality, PDE's, optimal control of PDE's

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# 1 Introduction

Although there were optimistic empirical work recording both economic growth and a long-run decrease in inequality ([Kuznets \(1955\)](#)) most of the evidence taking data after the WWII is not particularly supportive on the existence of a Kuznets curve (see [Barro \(2000\)](#), [Forbes \(2000\)](#) or [Anand and Segal \(2008\)](#)). Although the evidence is mixed (and very sensitive to methods used and data sampled) a short run positive relationship between growth and inequality is maybe the most robust result.

If we zoom out by including wider historical periods, the per-capita income ratio between the richest and poorest nations grew by a factor of 7 after the early 1800's (see [Maddison \(2007\)](#)). It may have grown at a factor in the thousands if we consider individuals. The acceleration of both growth and inequality generated by the first industrial revolution allowed for a consensus among economic historians on the existence of a "Great Divergence" (see [Pomeranz \(2000\)](#)), that started around 1800 and has progressively increased afterwards.

Those two types of evidence are not contradictory. They mean that there are countervailing forces acting in the short-to-medium run concurring to the reduction of inequality. First, globalzation and the flow of capital and ideas between countries may generate convergence forces. Particularly between successive "Industrial Revolutions" , after an initial outbreak of inequality, there are homogeinizing forces at work through international flows of capital and ideas (see [Milanovic \(2016\)](#)). Second, when the social fabric becomes fragile after heavy shocks generated by natural disasters or diseases, or man-made disasters, as wars and revolutions, it usually follows a period of reduction of inequality. [Scheidel \(2017\)](#) sees violence as the "Great Leveller".

We cannot discard those leveller effects as endogenous consequences of inequality itself: inequality between nations by widening relative prices creates an incentive for geographical relocation of capital, and inequality may jeopardize implicit or explicit contracts within nations, materialized in institutions safeguarding property rights, breeding political crises that can lead to massive relocation of capital ownership among social groups.

Does inequality causes growth or growth causes inequality ? When performing empirical research based on relatively short samples of data it is fundamental to have the causation working in the right direction. However, when we consider the long term, in a historical sense, the answer is they must be coupled processes: there is a growth-inequality nexus.

This paper addresses the growth-inequality (GI) nexus based on two simple ideas: first, as in simple growth models, we assume that the dynamics of income is generated by the dynamics

of capital accumulation, and, second, the distribution of capital is driven by a conservative process, because total mass should always sum to one at all times.

Whether we have an ad-hoc dynamics of capital accumulation (as in [Solow \(1956\)](#)), a centralized planner (as in the [Ramsey \(1928\)](#) model) or a decentralized market economy (as in mean-field game models), the capital accumulation process can be different, but the existence of an underlying conservative process cannot. This conservation property also holds whether we have a deterministic or a stochastic environment. This conservation property is at the core of the mean-field games approach (initiated by [Lasry and Lions \(2007\)](#) see [Gomes and Saúde \(2014\)](#) for a survey). From that conservation law three fundamental properties of the dynamics of the distribution are simultaneously determined: growth, ergodicity and inequality.

Lets us assume that the economy has one initial uneven distribution of capital, and therefore of income per capita. Economic growth can only exist if a significant part of the distribution of capital is stretched towards including in its support higher levels of the capital stock. However, the conservative law is still exerting its powerful effect. It only allows for two possibilities: either the distribution is stationary or it should work in the opposite sense of the capital accumulation by fixing an equal mass to lower levels of income. In the first case the process is ergodic and inequality would not be eliminated but would remain constant, but in the second it is non-ergodic and inequality will become asymptotically infinite. We will prove that those cases can occur with constant returns,  $AK$ , production functions.

Equality can only be reached in the long run if the process of potential stretching of the distribution is avoided. This case can only occur if long-run growth does not exist: the economy will converge to a finite, equality distributed capital stock. This is only possible if the production technology displays decreasing returns to scale as in the [Solow \(1956\)](#) model. In this case the process displays both ergodicity and equality but not economic growth.

Assuming that growth is possible, i.e, there is a constant returns technology, having an ergodic process requires a very specific redistribution policy. The inequality-amplifying mechanism is related to the existence of a positive local correlation between the savings rate and the capital stock. The only way to allow for growth and stationary inequality is to disconnect income, which is positively related to the capital stock, and savings.

Would it be optimal to follow an stationary ergodic distribution policy if we assume a simple utilitarian social welfare functional ? Using a distributional extension of the [Ramsey \(1928\)](#) model, in its endogenous growth [Rebelo \(1991\)](#) version, we will see that it is not. Although it would be optimal to introduce some re-distribution of capital it should not be extended to

the point at which it could reduce the asymptotic rate of economic growth (a similar result has been reached by [Bourguignon \(1981\)](#) using a static model).

Would the introduction of a stochastic redistribution of capital (through stochastic savings ) changes anything ? As in the mean-field games literature, while for the deterministic case the conservation law is modelled by a first-order PDE, in the stochastic case it is modelled by a forward Kolmogorov equation (or Fokker-Planck) parabolic PDE equation. Differently from the first-order PDE which features a distribution moving, in time, in the forward direction over its support, the introduction of noise adds a diffusion mechanism which can move the distribution in both ways. Assuming constant returns to scale, there are two possible results depending on the relative magnitude of the drift and the volatility components: if volatility is low, the process will display long run growth, and a non-ergodic increase in inequality; but if volatility is high there will not be long-run growth and the process will be ergodic, however converging for a state with increasing inequality and not equality as in the deterministic case.

The optimal central planner's solution for the stochastic version is consistent with one of those two dynamics. Therefore, the growth process, not the inequality increase process, can be countered by the existence of random redistribution of income.

**Related literature.** There are other equivalent approaches dealing with the joint determination of inequality and growth by [Quah \(2002\)](#) and [Azariadis and Stachurski \(2005\)](#) using maps. Although the results are similar using PDE's allows for an analytical approach and to the derivation of qualitative results.

In this paper the GI nexus results in a first-order PDE for the deterministic case in a parabolic PDE for the stochastic case. Optimal GI nexus are modelled as optimal control problems of first order or parabolic PDE's, generating systems of forward-backward PDE's. The recent interest in using PDE for modelling distributional issues in macro models, and in particular for mean-field games, has been surveyed in [Achdou et al. \(2014\)](#).

There is a recent literature on spatial Ramsey models (see [Brito \(2004\)](#), [Brock and Xepapadeas \(2008\)](#) and [Boucekkine et al. \(2009\)](#)) which also feature optimal control of parabolic PDE's. However, those models address the spatial distribution of capital and the existence of spatial agglomeration of production. Although the most common asymptotic state tend to display infinite variance, as in this paper, the interpretation is very different. In spatial-Ramsey models the independent variable is space an asymptotic infinite variance is associated to convergence in output along space. However, in this paper, because the independent variable is the capital stock owned by different ranks of the population an infinite variance means asymptotic

extreme inequality.

In section 2 we present the GI nexus for simple deterministic growth models and in section 3 we present the optimal distribution for a centralized economy. In section 4 we present the GI nexus for simple stochastic growth model and in 5 we study the optimal distribution. 6 contains some final remarks.

## 2 The GI nexus in simple growth models

Let  $k_i(t)$  be the capital stock at time  $t \in [0, \infty)$  of agents of type  $i \in \mathcal{I}$ , where  $\mathcal{I}$  is a continuum. There is a order relationship such that  $k_i(t) \leq k_j(t)$  if  $i \leq j$  for  $j \neq i \in \mathcal{I}$ .

Capital for agents of type  $i$  have, at every time time  $t$ , accumulates at rate  $\mu_i(k)$

$$dk_i(t) = \mu(k_i(t))dt.$$

This equation is a budget constraint such that  $\mu(k_i(t)) = y(k_i(t)) - c(k_i(t)) - \delta k_i(t)$ , where  $y$ ,  $c$  and  $\delta$  denote income, consumption and a capital depreciation rate, respectively.

The density of agents having capital stock  $k_i$ , at time  $t$  is denoted by  $n(k_i(t), t) \in \mathcal{C}_c^\infty(\mathbb{R}_+^2)$ , and satisfies the constraint

$$\int_{\mathcal{I}} n(k_i, t) di = 1, \text{ for every } t \in [0, \infty).$$

Because the indexing set  $\mathcal{I}$  has a order relationship we can see the capital stock of agents  $i$  as a mapping  $k_i : \mathcal{I} \mapsto \mathcal{K} \subseteq \mathbb{R}_+$ . Therefore, we can equivalently set the budget constrain and the density as satisfying

$$dk = \mu(k)dt, \tag{1}$$

and

$$\int_{\mathcal{K}} n(t, k) dk = 1, \text{ for every } t \in [0, \infty). \tag{2}$$

**Proposition 1.** *Assume that the density of agents verifies equation (2) and that the accumulation equation (1) holds. Then the distribution satisfies the first-order PDE <sup>1</sup>*

$$n_t(t, k) + (\mu(k)n(t, k))_k = 0, (t, k) \in \mathcal{K} \times [0, \infty) \tag{3}$$

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<sup>1</sup>We denote the partial derivatives by an index. For instance  $n_t(t, k) = \frac{\partial n}{\partial t}(t, k)$  and  $n_k(t, k) = \frac{\partial n}{\partial k}(t, k)$ . For functions we denote  $(\mu(k)n(t, k))_k = \frac{\partial}{\partial k}(\mu(k)n(t, k))$ . See the proof in the Appendix.

Equation (3) features a conservation law for the density. It basically implies that the distribution of capital among agents has always the density mass equal to one. Then we can compute the average per capita capital stock at time  $t$

$$\bar{k}(t) = \int_{\mathcal{K}} n(t, k) k dk \quad (4)$$

and its dispersion is given by the standard deviation, at time  $t$ ,

$$\sigma(t) = \sigma(K(t)) = \left( \int_{\mathcal{K}} n(t, k) (k - \bar{k}(t))^2 dk \right)^{\frac{1}{2}}. \quad (5)$$

There are two properties of the dynamics of the density we focus in this paper: the *long-run growth* properties and the *distributional* properties. We say there is long run growth if per capita capital is asymptotically unbounded, i.e.,  $\lim_{t \rightarrow \infty} \bar{k}(t) = +\infty$ . We say the distribution is stationary if  $\sigma(t) = \sigma^*$  constant for all  $t \in \mathbb{R}_+$  and it is ergodic if  $\lim_{t \rightarrow \infty} \sigma(t) = \sigma^*$ . There is equality if  $\sigma(t) = 0$  and inequality if  $\sigma(t) > 0$ . If the distribution is ergodic we can have equality or inequality in the long run if  $\sigma^* = 0$  or  $\sigma^* > 0$ , respectively.

**Assumption 1.** *In order to allow for long run growth we introduce the assumption that  $\mathcal{K} = \mathbb{R}_{++}$ . We also assume that the initial distribution,  $n(0, k) = \phi(k)$ , is a  $\mathcal{C}_c^\infty(\mathbb{R}_+)$  function satisfying, additionally the following properties*

$$\int_0^\infty \phi(k) dk = 1, \quad \lim_{k \rightarrow 0^+} \phi(k) = \lim_{k \rightarrow \infty} \phi(k) = 0, \quad (6)$$

and always displays inequality,

$$\int_0^\infty \phi(k) (k - \bar{k}(0))^2 dk > 0$$

where the initial aggregate per capita capital stock is

$$\bar{k}(0) = \int_0^\infty \phi(k) k dk > 0.$$

The solution for generic non-linear first order PDE's can display several types of behavior (see (Dafermos, 2000, p.13) : (1) blow-up if the the solution becomes infinite when a certain level for  $k$  is reached in finite time, (2) globally bounded solutions while  $k$  goes to infinite in infinite time; (3) progressive concentration along time tending asymptotically to a degenerate distribution concentrated at a finite value for  $k$ ,  $k^*$ , in infinite time; (4) shock-waves such that,

after a surface  $\Gamma(t, k)$  is reached, the solution becomes non-smooth and multivalued; or (5) rarefaction waves such that the distribution becomes increasingly dispersed.

A first-order PDE as equation (3), whose coefficients are independent from the endogenous variable  $n$ , can only have the first three types of behavior. We first need to distinguish cases for which  $\mu(k)$  is always positive, or can have any sign.

If we assume that function  $\mu(k)$  is continuous and differentiable, a standard method for solving first-order PDE is the method of characteristics. Characteristic lines can be seen as functions  $k(t)$  satisfying

$$\frac{dk(t)}{dt} = \mu(k(t))$$

such that the density  $n(t, k) = n(t, k(t))$  is constant. Given the initial distribution  $\phi(k)$  and the conservation law encoded in equation (6) the evolution of the stock of capital  $k_i(t)$  is described by an associated characteristic line.

Semi-linear first-order PDE's, as equation (3), can only have three types of characteristic lines: blow-up, parallel or converging to a point in infinite time. In our model only the last two can occur.

## 2.1 Parallel characteristics

Parallel characteristics exist when the accumulating function  $\mu(k)$  has the same sign for every  $k \in \mathbb{R}_+$ . In this case the PDE (3) has one unique classical solution:

**Proposition 2.** *Assume the initial distribution is  $n(k, 0) = \phi(k)$ . Then if  $\mu(k) > 0$  for all  $k \in \mathbb{R}_{++}$  equation (3) has one unique classic (smooth) solution*

$$n(t, k) = \phi\left(t - \int_0^k \frac{d\ell}{\mu(\ell)}\right) \frac{1}{\mu(k)} \quad (7)$$

If  $\phi(k) = \delta(k - \bar{k})$ , with the initial capital distribution evenly distributed at a level  $\bar{k}$ , we can the solution of equation (3) is

$$n(t, k) = e^{-\mu t} \delta(ke^{-\mu t} - \bar{k})$$

implying that the mean and the standard deviation are

$$\bar{k}(t) = \bar{k}e^{\beta t}$$

and  $\sigma(k(t)) = 0$  for all  $t \geq 0$ . In this case there is growth and permanent equality.

From now on we assume that there is initial heterogeneity, that is  $\sigma(0) > 0$ .

Assume the technology of production displays constant returns to scale,  $y_i(t) = Ak_i(t)$ , that there is depreciation of capital and that consumption is a linear function of the stock of capital,  $c_i(t) = ck_i(t)$ . Further assuming that agents have homogeneous technology and preferences, yielding

$$dk(t) = \mu k(t) = (A - \delta - c)k(t)$$

and that  $\mu > 0$ . The solution for equation (3) given an initial distribution  $n(x, 0) = \phi(k)$  is now

$$n(t, k) = e^{-\mu t} \phi(k e^{-\mu t}), \quad (t, k) \in \mathbb{R}_{++}^2 \quad (8)$$

We assume an initial distribution

$$\phi(k) = \frac{2}{\sqrt{\pi}(1 + \operatorname{erf}(k_0))} e^{-(k-k_0)^2}, \quad (9)$$

where  $\operatorname{erf}(k_0) = \frac{2}{\sqrt{\pi}} \int_0^{k_0} e^{-x^2} dx$ . The average capital stock and the standard deviations are

$$\bar{k}(0) = k_0 + \xi(k_0) \approx k_0$$

and

$$\sigma(0) = \left[ \frac{1}{2} - \xi(k_0) \bar{k}(0) \right]^{\frac{1}{2}} \approx \frac{1}{\sqrt{2}}$$

where  $\xi(k_0) \equiv \frac{e^{-k_0^2}}{\sqrt{\pi}(1 + \operatorname{erf}(k_0))}$ .  
equation (8) becomes

$$n(t, k) = 2e^{-\mu t} \xi(k_0) e^{-(k e^{-\mu t} - k_0)^2 + k_0^2}, \quad (t, k) \in \mathbb{R}_{++}^2.$$

The average capital per capita is

$$\bar{k}(t) = e^{\mu t} (k_0 + \xi(k_0))$$

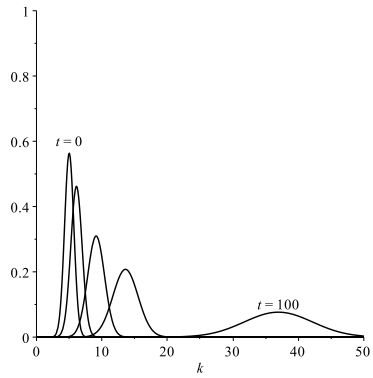
and the standard deviation is

$$\sigma(k(t)) = \frac{e^{\mu t}}{\sqrt{2}} [1 - 2\xi(k_0)(1 + \xi(k_0))]^{\frac{1}{2}}, \quad t \geq 0.$$

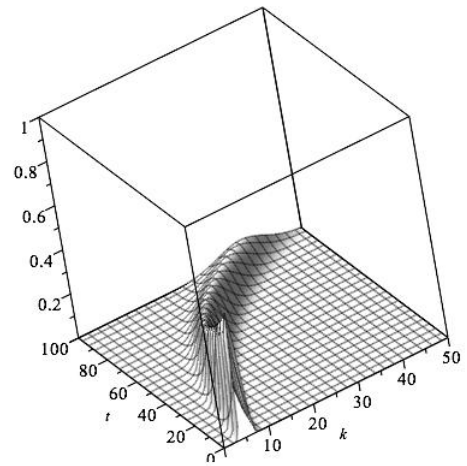
For values of  $k_0 > 2$  a very tight approximation of the average and the variance for  $k$  results

$$k(t) \approx k_0 e^{\beta t}, \quad \sigma(k(t)) \approx \frac{e^{\beta t}}{\sqrt{2}}.$$

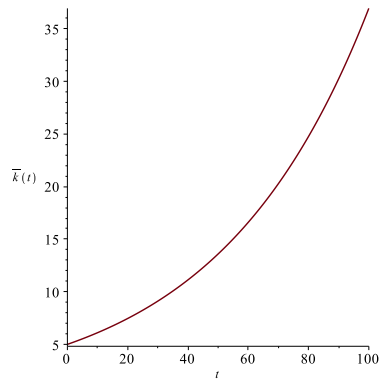




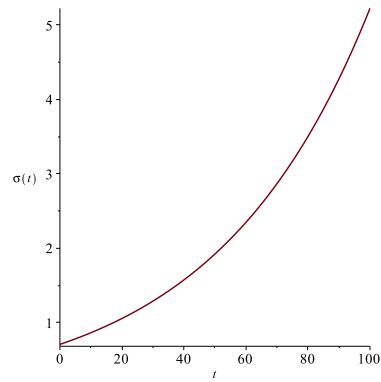
(a) Density for several increasing dates



(b) Density plot



(c) Average capital intensity



(d) Standard deviation

Figure 1: Linear accumulation function for  $\mu > 0$  and an initial normal distribution

In Figure 1 we represent the solution for several increasing times, a 3d plot and the average and the standard deviation of per capita capital.

If  $\mu > 0$  there is both growth and a progressively higher dispersion of income:

$$\lim_{t \rightarrow \infty} \bar{k}(t) = \lim_{t \rightarrow \infty} \sigma(t) = \infty.$$

This is natural because an initial heterogeneity is amplified because we assumed all agents face an equal rate of growth of the capital stock. Those assumptions imply that the characteristics are  $k_i(t) = k_i(0)e^{\mu t}$ .

Now consider the case in which  $\mu = \bar{\mu}$  is positive and constant. Given an initial distribution  $\phi(k)$  the unique solution is

$$n(t, k) = \phi(k - \bar{\mu}t), \quad (t, k) \in \mathbb{R}_+^2$$

If we assume the same particular distribution as in equation (9), we get the density

$$n(t, k) = \frac{2}{\sqrt{\pi}(1 + \operatorname{erf}(k_0))} e^{-(k - k_0 - \mu t)^2}$$

and the average en standard deviation for per capita capital

$$\bar{k}(t) = \frac{1}{\sqrt{\pi}(1 + \operatorname{erf}(k_0))} \left( \sqrt{\pi}(\mu t + k_0)(1 + \operatorname{erf}(\mu t + k_0)) + e^{-(\mu t + k_0)^2} \right)$$

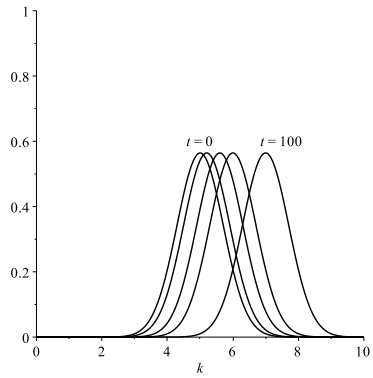
and

$$\begin{aligned} \sigma(t) = & \left( \frac{2}{\sqrt{\pi}(1 + \operatorname{erf}(k_0))} \right)^{\frac{1}{2}} \left\{ \frac{\sqrt{\pi}}{2}(1 + \operatorname{erf}(k_0)) \left( (\mu t + k_0 - \bar{k}(t))^2 + \frac{1}{2} \right) + \right. \\ & \left. + \left( \frac{\mu t + k_0}{2} - \bar{k}(t) \right) e^{-(\mu t + k_0)^2} \right\}^{\frac{1}{2}}. \end{aligned}$$

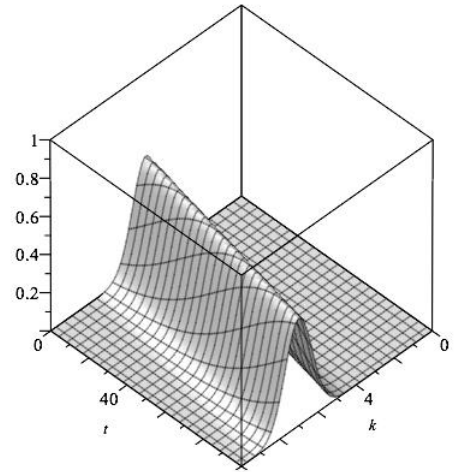
Figure 2 depicts 2d- and 3d- trajectories and the time paths for the mean and the standard deviation capital per capita. We can see that the average and the standard deviations can be approximate by

$$\bar{k}(t) \approx k_0 + \mu t, \quad \sigma(t) = \frac{1}{\sqrt{2}}, \quad t \geq 0$$

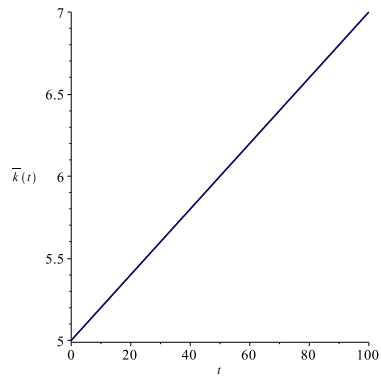
which means that we have long-run growth and a stationary distribution. Although initial inequality is not eliminated it does not increase either. In this case the characteristics for capital accumulation are linear  $k_i(t) = k_i(0) + \mu t$ .



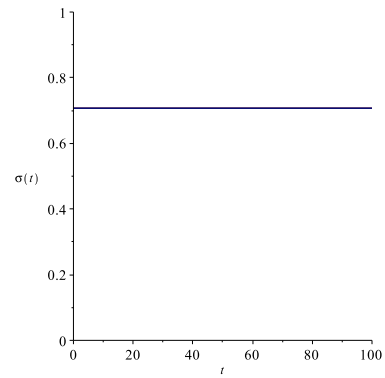
(a) Density for several increasing dates



(b) Density plot



(c) Average capital intensity



(d) Standard deviation

Figure 2: Constant accumulation function for  $\mu > 0$  and an initial normal distribution

## 2.2 Converging characteristics

Let the capital accumulation function to be as in the [Solow \(1956\)](#) model

$$\mu(k) = sAk^\alpha - \delta k, \quad (s, \alpha, \delta) \in (0, 1)^3, \quad A > 0$$

In this case the solution to equation (??) is

$$n(t, k) = e^{\delta k - sAk^\alpha} \phi \left( \left( \frac{sA}{\delta} + \left( k^{1-\alpha} - \frac{sA}{\delta} \right) e^{\delta(1-\alpha)t} \right)^{\frac{1}{1-\alpha}} \right) \quad (10)$$

Differently from the previous cases, the accumulation equation has a fixed point

$$k^* = \left( \frac{sA}{\delta} \right)^{\frac{1}{1-\alpha}},$$

which implies that the value of the accumulation function changes sign  $\mu(k) > 0$  if  $k \in (0, k^*)$  and  $\mu(k) < 0$  if  $k \in (k^*, \infty)$ . This implies that the characteristic lines

$$k_i(t) = \left( (k^*)^{1-\alpha} + (k_i(0)^{1-\alpha} - (k^*)^{1-\alpha}) e^{-(1-\alpha)\delta t} \right)^{\frac{1}{1-\alpha}},$$

will converge to point asymptotically  $k^*$ , in infinite time,<sup>2</sup> with a positive slope if  $0 < k < k^*$  or with a negative slope if  $k^* < k < \infty$ , or they are constant if  $k = k^*$ . I

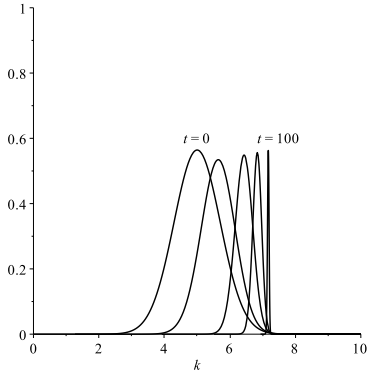
In this case the density is non-smooth locally at  $k^*$  and will converge to a degenerate distribution concentrated in  $k = k^*$ . This means that per average capital per capita  $\bar{k}(t)$  will also converge to  $k^*$ , i.e..  $\lim_{t \rightarrow \infty} \bar{k}(t) = k^*$  and the standard deviation will converge zero,  $\lim_{t \rightarrow \infty} \sigma(t) = 0$  (see [Figure 3](#)).

**Proposition 3.** *If there is one point  $k^*$  such that  $\mu(k^*) = 0$ , and  $\mu(k)$  is locally concave, then for any smooth initial distribution  $\phi(k)$  the density  $n(t, k)$  is a weak solution of [\(3\)](#) converging asymptotically to degenerate distribution concentrated in  $k = k^*$ .*

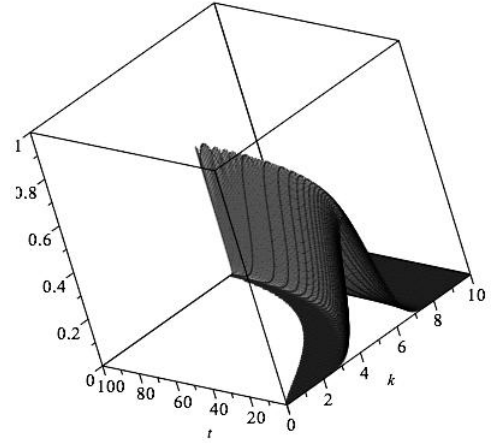
We can readily conclude that if the capital accumulation function has a fixed point and is locally concave then there will be asymptotic equality but there is no long-run growth. This is a case of  $\beta$ - and  $\sigma$ - convergence.

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<sup>2</sup>This is different from the shock- or the rarefaction waves cases, which is much studied in the theory of first-order PDE's, where collision occurs at finite time.



(a) Density for several increasing dates



(b) Density plot

Figure 3: Solow accumulation function  $\mu(k) = sk^\alpha - \delta k$  and an initial normal distribution

## 2.3 Suming up

Summing up, while with a linear accumulation equation we have growth but inequality increases, in the case in which there is a fixed point and the accumulation function is concave we have asymptotic equality but there is no long-run growth. This conclusion seems natural if we depart from an unequal distribution and it just recasts well known results in a consistent growth-distributional framework.

For the existence of long-run growth there should be a section of the distribution with positive mass that should become unbounded. This fact together with the existence of mass conservation associated with the aggregate distribution, implies that the increase in inequality associated to long-run growth is a powerful force.

The only possibility to have ergodicity and growth, but still inequality, is if the accumulation is independent from the  $k$ . In this case, although the initial inequality is kept, it does not increase with time. This is only possible if the accumulation function is constant and independent from the capital stock. If we assume that income is a function of the capital stock this requires a consumption function which is constant and capital independent savings. Formally, if output is  $y(k) = (A - \delta)k$  then consumption should be of type  $c(k) = y(k) - \bar{\mu}$ .

The question naturally arises: would a Ramsey central planner allocate consumption in this way ?

### 3 The optimal GI nexus

We consider next the problem for a central planner who wants to maximize a social utility function by optimally choosing the change in the distribution of capital among agents. The central planner controls the allocation of consumption for agents identified by their capital stock. These assumptions yield the accumulation function is  $\mu(t, k) = Af(k) - c(t, k)$ .

We also assume a Bergson - Samuelson social welfare function in which the planner maximizes the present value of the average utility. As instantaneous average utility uses population distribution as a weighting scheme this implies that the utility is bounded, although it introduces the state variable in the utility functional.

The planner's problem is the following optimal control problem of a first-order PDE:

$$\max_{c(t,k)} \int_0^\infty \int_0^\infty u(c(t, k))n(t, k)dk e^{-\rho t} dt \quad (11)$$

subject to

$$n_t(t, k) + (\mu(k, c(t, k))n(t, k))_k = 0 \quad (12)$$

$$n(k, 0) = \phi(k) \quad (13)$$

where the population distribution satisfies

$$\int_0^\infty n(t, k)dk = 1, \quad \lim_{k \rightarrow \infty} = \lim_{k \rightarrow 0^+} = 0.$$

The first-order necessary conditions for an optimum are:

**Proposition 4.** *Let  $(c^*, n^*) \in \mathcal{C}_c^\infty(\mathbb{R}_{++}^2)$  be optimal distributions are let  $q \in \mathcal{PC}_c^\infty(\mathbb{R}_{++}^2)$  be a co-state variable. Then the following optimality conditions hold*

$$u'(c^*(t, k)) + \mu_c(k, c^*(t, k))n^*(t, k) = 0 \quad (t, k) \in \mathbb{R}_{++}^2 \quad (14)$$

$$u(c^*(t, k)) + q_t(t, k) + \mu(k, c^*(t, k))q_k(t, k) - \rho q(t, k) = 0, \quad (t, k) \in \mathbb{R}_{++}^2 \quad (15)$$

$$\lim_{t \rightarrow \infty} q(t, k)e^{-\rho t} = 0, \quad (t, k) \in \{t \rightarrow \infty\} \times \mathbb{R}_{++} \quad (16)$$

for admissible paths satisfying equations (12) and (13).

Our problem is called in the literature a mean-field optimal control problem or a optimal control of a conservation law <sup>3</sup>. The first-order conditions are a system of two first-order PDE:

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<sup>3</sup> See [Lasry and Lions \(2007\)](#) and [Bensoussan et al. \(2013\)](#).

the state variable is driven by a conservation law and the generalized Euler equation (12) and the co-state variable,  $q(t, k)$ , is driven by a Hamilton-Jacobi equation (14)<sup>4</sup>. The problem is constrained by the known initial distribution, in equation (13), and by a transversality condition (16).

The optimal consumption level is determined by equation (14) which states that the marginal utility of consumption should equal the average marginal cost measured by the marginal reduction in average savings.

In order to have an intuition on the behavior of the optimal distribution, and to allow for long-run growth, we introduce the following assumptions on preferences and technology:  $u(c) = \ln(c)$  and  $\mu(t, k) = Ak - c(t, k)$ . In this case, equations (14)-(15) imply that the optimality conditions become a recursive system,

$$q_t(t, k) + Akq_k(t, k) + \ln(q_k^{-1}(t, k)) - \rho q(t, k) - 1 = 0 \quad (17)$$

$$n_t(t, k) + ((Ak - q_k^{-1}(t, k)) n(t, k))_k = 0 \quad (18)$$

$$\lim_{t \rightarrow \infty} e^{-\rho t} q(t, k) = 0 \quad (19)$$

$$n(k, 0) = \phi(k) \quad (20)$$

A solution of the Euler equation (17), verifying the transversality constraint, is

$$\rho q(t, k) = \ln\left(\rho k e^{\frac{\gamma}{\rho}}\right) \quad (21)$$

where  $\gamma \equiv A - \rho$  is the optimal long run growth rate. The optimal consumption distribution is a function of the capital stock  $c^*(t, k) = \rho k$ . Then the budget constraint (18) is

$$n_t(t, k) + (\gamma k n(t, k))_k = 0$$

which has solution

$$n^*(t, k) = \phi(k e^{-\gamma t}) e^{-\gamma t}. \quad (22)$$

This has the behavior that we already studied in section 2 and depicted in Figure 1. The optimal policy is to redistribute because net savings are smaller than income  $\mu(k) = (A - \rho)k = \gamma k < Ak$ , although is designed in a way to allow for long-run growth. This means that this policy does not prevent inequality to increase along time.

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<sup>4</sup>See (Evans, 1998, ch 3).

## 4 The stochastic GI nexus

Can the introduction of uncertainty change those results ? That is, would keeping the assumption that there are constant returns to scale would a random redistribution of income introduce a break over this powerful force for increase in inequality ? Or would it reduce inequality or would, paradoxically, increase it ?

In this section we take the linear accumulation case to answering that question.

Let  $k_i = k \in \mathbb{R}_+$  be a the capital stock for agents  $i \in \mathcal{I}$  and again assume that the initial distribution is  $n(k, 0)$ . Now assume that the accumulation equation is given by the diffusion process

$$dK_i(t) = \mu(K(t))dt + \sigma(K(t))dW(t)$$

where  $W(\cdot)$  is a standard Brownian motion.

The dynamics of the distribution is now given by the forward Kolmogorov (also called Fokker-Planck) equation

$$n_t(t, k) + (\mu(k)n(t, k))_k - \frac{1}{2} (\sigma^2(k)n(t, k))_{kk} = 0 \quad (23)$$

together with the initial condition  $n(k, 0) = \phi(k)$  for  $\phi(\cdot) \in L^1(\mathbb{R}_+)$

**Example : Multiplicative noise** We assume that the accumulation equation is the linear stochastic differential equation

$$dk(t) = \mu k(t)dt + \sigma k(t)dW(t)$$

with  $\mu > 0$  and  $\sigma > 0$ . This equation may generate long run growth, because of the linearity in the drift part, but by making the random distribution mechanism it can work against the inequality-increasing dependence on accumulation on the (private) level of capital stock.

This dynamics of the distribution of population is driven by the forward-parabolic PDE

$$n_t(t, k) + (\mu k n(t, k))_k - \frac{1}{2} (\sigma^2 k^2 n(t, k))_{kk} = 0 \quad (24)$$

where again the initial distribution is given  $n(0, k) = \phi(k)$ .

In the appendix we prove that the solution to this initial-value problem is

$$n(t, x) = \int_0^\infty \phi(\xi) g \left( t, \ln \left( \frac{k}{\xi} \right) \right) \frac{1}{\xi} d\xi. \quad (25)$$



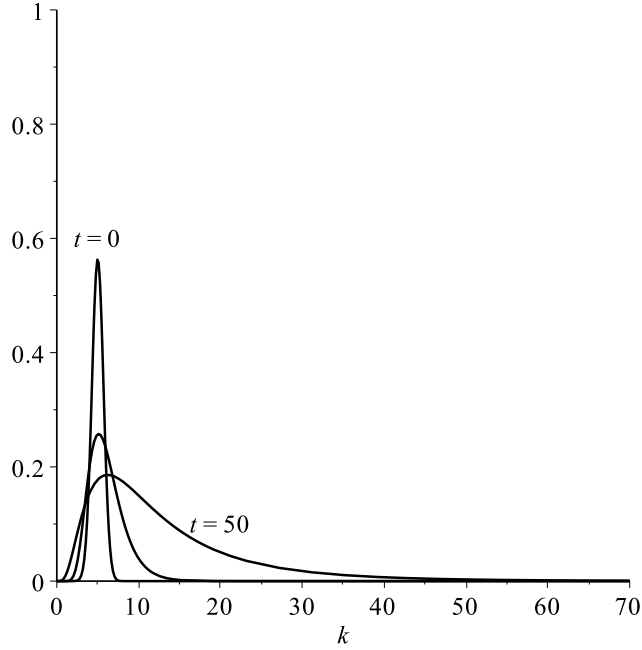


Figure 4: Stochastic accumulation function with multiplicative noise for the case  $\mu > \sigma^2$

where

$$g(t, y) = (2\pi\sigma^2t)^{-\frac{1}{2}} \exp \left[ (\mu - \sigma^2)t - \frac{(y - t(\mu - \frac{3}{2}\sigma^2))^2}{2\sigma^2t} \right], \quad x \in \mathbb{R}. \quad (26)$$

We show in the appendix that the real part of the characteristic exponent of this equation is

$$\text{Re}(\lambda(\omega)) \equiv \mu - \sigma^2 + 2(\pi\sigma\omega)^2$$

where  $\omega \in (-\infty, \infty)$  are frequencies. The drift component of the Gaussian kernel is  $e^{(\mu - \sigma^2)t}$ , and describes the time-behavior of the distribution along characteristic lines analogous to those of the deterministic model. Therefore, we readily conclude that: (1) if  $\mu > \sigma^2$  the drift component is dominant and the distribution will move forward along the domain of  $k$  in an analogous way as to the deterministic model; but (2) if  $\mu < \sigma^2$  the diffusion component is dominant and the distribution will not tend to move forward in time. Therefore, in both cases the standard deviation of the distribution will become asymptotically infinite, but while in the first case we have long run growth, in the second we will have not. This last case cannot occur in deterministic models.

Figure 4 shows trajectories for the case  $\mu > \sigma^2$ .

## 5 The optimal stochastic GI nexus

Assuming again a log utility function and the accumulation equation

$$dk_i(t) = (Ak_i(t) - c_i(t, k(t))) dt + \sigma k_i(t) dW(t)$$

the planners problem is

$$\max_{c(t,k)} \int_0^\infty \int_0^\infty u(c(t, k)) n(t, k) e^{-\rho t} dk dt \quad (27)$$

subject to

$$n_t(t, k) + (\mu(k, c(t, k)) n(t, k))_k - \frac{1}{2} (\sigma^2(k) n(t, k))_{kk} = 0 \quad (28)$$

$$n(k, 0) = \phi(k) \quad (29)$$

where the population distribution verifies the same properties as in the deterministic problem:  $\int_0^\infty n(t, k) dk = 1$  and  $n(0, k) = n(\infty, k) = 0$ .

**Proposition 5.** *Let  $(c^*, n^*) \in \mathcal{C}_c^\infty(\mathbb{R}_{++}^2)$  be optimal distributions are let  $q \in \mathcal{PC}_c^\infty(\mathbb{R}_{++}^2)$  be a co-state variable. Then the following optimality conditions hold*

$$u'(c^*(t, k)) + \mu_c(k, c^*(t, k)) n^*(t, k) = 0, \quad (t, k) \in \mathbb{R}_{++}^2 \quad (30)$$

$$u(c^*(t, k)) + q_t(t, k) + \mu(k, c^*(t, k)) q_k(t, k) - \rho q(t, k) + \frac{\sigma^2(k)}{2} q_{kk}(t, k) = 0, \quad (t, k) \in \mathbb{R}_{++}^2 \quad (31)$$

$$\lim_{t \rightarrow \infty} q(t, k) e^{-\rho t} = 0, \quad (t, k) \in \{t \rightarrow \infty\} \times \mathbb{R}_{++} \quad (32)$$

for admissible paths satisfying equations (28) and (29).

If we assume that the utility function is logarithmic then equations (30)-(31) also become a recursive PDE-system

$$q_t(t, k) + Ak q_k(t, k) + \ln(q_k^{-1}(t, k)) + \frac{1}{2} \sigma^2 k^2 q_{kk}(t, k) - \rho q(t, k) - 1 = 0 \quad (33)$$

$$n_t(t, k) + ((Ak - q_k^{-1}(t, k)) n(t, k))_k - \frac{\sigma^2}{2} (k^2 n(t, k))_{kk} = 0 \quad (34)$$

$$\lim_{t \rightarrow \infty} e^{-\rho t} q(t, k) = 0. \quad (35)$$

There is a time-independent solution for equation (33) which is the same solution as for the deterministic case, displayed in equation (21).

Therefore, we obtain a PDE for the optimal dynamics of the distribution

$$n_t(t, k) + (\gamma kn(t, k))_k - \frac{\sigma^2}{2} (k^2 n(t, k))_{kk} = 0$$

whose solution is

$$n^*(t, x) = \int_0^\infty \phi(\xi) g\left(t, \ln\left(\frac{k}{\xi}\right)\right) \frac{1}{\xi} d\xi. \quad (36)$$

where

$$g(t, y) = (2\pi\sigma^2 t)^{-\frac{1}{2}} \exp\left[(\gamma - \sigma^2)t - \frac{(y - t(\gamma - \frac{3}{2}\sigma^2))^2}{2\sigma^2 t}\right], \quad x \in \mathbb{R}. \quad (37)$$

The optimal distribution has the same properties as those analysed in section 5: if  $\gamma > \sigma^2$  there will be long-run growth and if  $\gamma < \sigma^2$  there will not be long-run growth. In both cases there is increasing in inequality.

## 6 Final remarks

Although other types of distribution dynamics that have been shown can describe well the data, as twin-peaks (Quah (2002)), or "elephant-shaped" change in the distribution profile, or waves of equality-inequality following industrial revolutions (Milanovic (2016), or even Kuznets (1955)) we think are non inconsistent with the existence of a conservation law.

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## A Proofs

*Proof of Proposition 1.* Taking a time derivative of equation (2)

$$\frac{d}{dt} \left( \int_{\mathcal{K}} n(t, k) dk \right) = 0. \quad (38)$$

Expanding, and writing  $\mathcal{K} = (\underline{k}, \bar{k})$ , we get

$$\begin{aligned} \frac{d}{dt} \left( \int_{\mathcal{K}} n(t, k) dk \right) &= \int_{\mathcal{K}} \frac{\partial n}{\partial t}(t, k) dk + \int_{\underline{k}}^{\bar{k}} n(t, k) \frac{dk(t)}{dt} = \\ &= \int_{\mathcal{K}} \frac{\partial n}{\partial t}(t, k) dk + \int_{\underline{k}}^{\bar{k}} n(t, k) \mu(k) = \\ &= \int_{\mathcal{K}} \left( \frac{\partial n}{\partial t}(t, k) + \frac{\partial}{\partial k} (\mu(k)n(t, k)) \right) dk. \end{aligned}$$

Equation (38) holds if and only iff the distribution  $n(t, k)$  solves the first order PDE (3).  $\square$

*Proof of Proposition 2.* Equation (3) can be expanded to

$$\frac{\partial n}{\partial t}(t, k) + \mu(k) \frac{\partial n}{\partial k}(t, k) + s'(k)n(t, k) = 0$$

Let

$$u(t, k) = \mu(k)n(t, k).$$

But

$$\frac{\partial u}{\partial t}(t, k) + \mu(k) \frac{\partial u}{\partial k}(t, k) = \mu(k) \left( \frac{\partial n}{\partial t}(t, k) + \mu(k) \frac{\partial n}{\partial k}(t, k) + \mu'(k)n(t, k) \right).$$

If  $\mu(k) > 0$  for all  $k \in \mathbb{R}_+$  then (3) implies that  $u(t, k)$  follows the homogeneous first-order PDE

$$\frac{\partial u}{\partial t}(t, k) + \mu(k) \frac{\partial u}{\partial k}(t, k) = 0.$$

We can solve this equation by using the method of characteristics, i.e., curves satisfying  $k = k(t)$  such that the value of  $u(t, k)$  is constant. Define  $v(t) = u(t, k(t))$ . Then

$$\frac{dv}{dt}(t) = \frac{\partial u}{\partial t}(t, k(t)) + \frac{\partial u}{\partial k}(t, k(t)) \frac{dk}{dt}(t).$$

Therefore, along a characteristic the following ordinary differential equations should hold

$$\begin{cases} \frac{dk}{dt}(t) = \mu(k(t)) \\ \frac{dv}{dt}(t) = 0 \end{cases}$$

We solve the first equation by separation of variables and integrating

$$\int \frac{dk}{\mu(k)} = \int dt = t - \xi$$

where  $\xi$  is a constant of integration. Therefore, along a characteristic we have  $\xi = t - \int^k \frac{d\ell}{\mu(\ell)}$ , and the second equation yields  $v(t) = v(0) = u(\xi, 0) = \phi(\xi)$ , where  $\xi = k(0)$ .  $\square$

*Derivation of equation (10).* In order to apply the method of characteristics we set  $t = t(x)$ ,  $k = k(x)$  and  $z(x) = n(t(x), k(x))$  (see (Evans, 1998, section 3.2)). Taking derivatives as regards variable  $x$  we have the system

$$\begin{aligned} \frac{dt(x)}{dx} &= 1 \\ \frac{dk(x)}{dx} &= \mu(k(x)) = sAk(x)^\alpha - \delta k(x) \\ \frac{dz(x)}{dx} &= -\mu'(k(x))z(k(x)) \end{aligned}$$

Solving and taking the initial values  $(t(0), k(0), z(0)) = (0, \xi, \phi(\xi))$  we get the solutions

$$t(x) = x \tag{39}$$

$$k(x) = \left( \frac{sA}{\delta} + \left( \xi^{1-\alpha} - \frac{sA}{\delta} \right) e^{-\delta(1-\alpha)x} \right)^{\frac{1}{1-\alpha}} \tag{40}$$

$$z(x) = \phi(\xi) e^{-\int^k \mu'(k(l)) dl} \tag{41}$$

Transforming back to  $(t, k)$  and solving equation (40) for  $\xi$  we get

$$\xi = \left( \frac{sA}{\delta} + \left( k^{1-\alpha} - \frac{sA}{\delta} \right) e^{\delta(1-\alpha)t} \right)^{\frac{1}{1-\alpha}}$$

and because  $n(t, k) = \phi(\xi) e^{-\mu(k)}$ , we readily obtain equation (10).  $\square$

*Proof of Proposition 4.* Assume we know the optimal consumption distribution  $(c^*(t, k))_{(t,k) \in \mathbb{R}_+^2}$ . We denote  $u^*(t, k) = u(c^*(t, k))$  and  $\mu^*(t, k) = \mu(k, c^*(t, k))$ .

In the optimum the value of the program is

$$\begin{aligned} V(c^*, n^*) &= \int_0^\infty \int_0^\infty u^*(t, k) n(t, k) e^{-\rho t} dk dt = \\ &= \int_0^\infty \int_0^\infty u^*(t, k) n^*(t, k) e^{-\rho t} - \lambda(t, k) (n_t^*(t, k) + (\mu^*(t, k) n^*(t, k))_k) dk dt \end{aligned}$$

where  $\lambda(t, k)$  is the co-state variable. Using integration by parts we get

$$\begin{aligned} V(c^*, n^*) &= \int_0^\infty \int_0^\infty [u^*(t, k)e^{-\rho t} + (\lambda_t(t, k) + \mu^*(t, k)\lambda_k(t, k))] n^*(t, k) dk dt + \\ &\quad - \int_0^\infty \lambda(t, k)n^*(t, k) dk \Big|_{t=0}^\infty - \int_0^\infty \lambda(t, k)\mu^*(t, k)n^*(t, k) dt \Big|_{k=0}^\infty \end{aligned} \quad (42)$$

Next we introduce admissible perturbations in both state and control variables,  $n(t, k) = n^*(t, k) + \varepsilon\delta_n(t, k)$  and  $c(t, k) = c^*(t, k) + \varepsilon\delta_c(t, k)$ . The perturbed state variable should satisfy  $\int_0^\infty \delta_n(t, k) dk = 1$ ,  $\lim_{k \rightarrow \infty} \delta_n(t, k) = \lim_{k \rightarrow 0^+} \delta_n(t, k) = 0$  and  $\delta_n(0, k) = 0$ .

If the program is optimum it verifies  $V(c^*, n^*) \geq V(c, n)$  for all admissible pairs  $(c, n)$ . A necessary condition for this is that

$$\delta V(c^*, n^*) = \lim_{\varepsilon \rightarrow 0} \frac{V(c^* + \varepsilon\delta_c, n^* + \varepsilon\delta_n)}{\varepsilon} = \frac{\partial V(c^*, n^*)}{\partial \varepsilon} = 0.$$

Defining the current-value co-state variable as  $q(t, k) = \lambda(t, k)e^{\rho t}$  the integral derivative is

$$\begin{aligned} \delta V(c^*, n^*) &= \int_0^\infty \int_0^\infty (u_c^*(t, k) + \mu_c^*(t, k)q_k(t, k)) e^{-\rho t} n^*(t, k) \delta_c(t, k) dk dt + \\ &\quad + \int_0^\infty \int_0^\infty [u_c^*(t, k) + (q_t(t, k) - \rho q(t, k) + \mu^*(t, k)q_k(t, k))] e^{-\rho t} \delta_n(t, k) dk dt + \\ &\quad - \int_0^\infty q(t, k) e^{-\rho t} \delta_n(t, k) dk \Big|_{t=0}^\infty + \\ &\quad - \int_0^\infty q(t, k) e^{-\rho t} (\mu_c^*(t, k)n^*(t, k)\delta_c(t, k) + \mu^*(t, k)\delta_n(t, k)) dt \Big|_{k=0}^\infty \end{aligned} \quad (43)$$

It is equal to zero if equation (14)-(16) hold.  $\square$

*Proof of equation (25)*. By the change in variable,  $x = x(k) \equiv \ln(k)$  we can transform equation (24) into a linear PDE with constant coefficients

$$u_t(t, x) = \frac{\sigma^2}{2} u_{xx}(t, x) - \left( \mu - \frac{3}{2}\sigma^2 \right) u_x(t, x) + (\sigma^2 - \mu) u(t, x), \quad (t, x) \in \mathbb{R}_+^2 \quad (44)$$

where  $n(t, k) = u(t, x(k)) = u(t, \ln x(k))$ . Change, for a while, the domain of  $x$  from  $\mathbb{R}_+$  to  $\mathbb{R}$  and introduce the Fourier transform of  $u(t, x)$ ,

$$U(t, \omega) = \mathcal{F}[u(t, x)](\omega) \equiv \int_{-\infty}^\infty u(t, x) e^{-2\pi i x \omega} dx$$

where  $i^2 = -1$ . Then, equation (44) can be transformed into

$$U_t(t, \omega) = -\lambda(\omega)U(t, \omega) \quad (45)$$



where

$$\lambda(\omega) = \sigma^2 - \mu - \frac{(2\pi\sigma\omega)^2}{2} - \left(\mu - \frac{3}{2}\sigma^2\right) 2\pi i\omega.$$

As equation (45) is a linear ODE for  $U(t, \cdot)$  it has the solution

$$U(t, \omega) = \Psi(\omega)G(t, \omega)$$

where  $\Psi(\omega)$  is an arbitrary function and  $G(t, \omega)$  is the Gaussian kernel

$$G(t, \omega) = \begin{cases} 1 & t = 0 \\ e^{-\lambda(\omega)t} & t > 0. \end{cases}$$

Taking inverse Fourier transforms,

$$u(t, x) = \mathcal{F}^{-1}[U(t, \omega)](x) \equiv \int_{-\infty}^{\infty} u(t, x) e^{2\pi i x \omega} d\omega$$

we know that  $u(t, x) = \mathcal{F}^{-1}[\Psi(\omega)G(t, \omega)](x) = \psi(x) * \tilde{g}(t, x)$ , that is

$$u(t, x) = \int_{-\infty}^{\infty} \psi(s)g(t, x-s)ds \quad (46)$$

where  $g(t, x) = \mathcal{F}^{-1}[G(t, \omega)](x) = \mathcal{F}^{-1}[e^{-\lambda(\omega)t}](x)$  and  $\psi(x) = \mathcal{F}^{-1}[\Psi(\omega)](x)$ . Then  $g(0, x) = \delta(x)$ , for  $t = 0$  and  $g(t, x)$  for  $t > 0$  is in equation (26).

Transforming back  $x = \ln(k)$  and using the initial condition  $n(0, k) = \phi(x)$  we finally get equation (25).  $\square$

*Proof of Proposition 5.* Using the same approach as the in the proof of Proposition 4, the value function is, at the optimum,  $V(c^*, n^*) = V_1^* + V_2^*$  where  $V_1^*$  is the same as in equation (42) and

$$\begin{aligned} V_2^*(c^*, n^*) &= \frac{1}{2} \int_0^{\infty} \int_0^{\infty} \lambda(t, k) (\sigma^2(k)n(t, k))_{kk} dkdt = \\ &= \frac{1}{2} \int_0^{\infty} \int_0^{\infty} \sigma^2(k)n(t, k)\lambda_{kk}(t, k)dkdt - \\ &\quad - \frac{1}{2} \int_0^{\infty} \lambda(t, k) (\sigma^2(k)n(t, k))_k - \sigma^2(k)n(t, k)\lambda_k(t, k)dt \Big|_{k=0}^{\infty} \end{aligned}$$

The integral derivative is now  $\delta V(c^*, n^*) = \delta V_1(c^*, n^*) + \delta V_2(c^*, n^*)$  where  $\delta V_1^*$  is the same as in equation (43) and

$$\begin{aligned} \delta V_2^*(c^*, n^*) &= \frac{1}{2} \int_0^{\infty} \int_0^{\infty} \sigma^2(k)\lambda_{kk}(t, k)\delta_n(t, k)dkdt - \\ &\quad - \frac{1}{2} \int_0^{\infty} \sigma^2(k) [\lambda(t, k) (1 + n_{kk}(t, k)) - \lambda_k(t, k)] \delta_n(t, k)dt \Big|_{k=0}^{\infty}. \end{aligned}$$

The admissibility condition for the perturbation  $\delta_n(t, k)$ , together with the definition of  $q(t, k) \equiv e^{-\rho t}\lambda(t, k)$  lead to the optimality conditions (30)-(32).  $\square$