Organizational Refinements of Nash Equilibrium*

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March 14, 2017

Abstract

Strong Nash equilibrium (see Aumann, 1959) and coalition-proof Nash equilibrium (see Bernheim et al., 1987) rely on the idea that players are allowed to form coalitions and to make joint deviations. They both consider a case in which any coalition can be formed. Be that as it may, there are many real life examples where some coalitions/subcoalitions cannot be formed. Furthermore, when all coalitions are formed, there may occur *conflicts of interest* such that a player is not able to choose an action that simultaneously meets the requirements of two coalitions that he/she is a member of. Stemming from these criticisms, we study an *organizational framework* where some coalitions/subcoalitions are not formed and where the coalitional structure are formulated in such a way that there remain no conflicts of interest. We define an *organization* as an ordered collection of partitions of the set of players in such a way that any partition is coarser than the partitions that precede it. For a given organization, we introduce the notion of *organizational Nash equilibrium.* We analyze the existence of equilibrium in a subclass of games with strategic complementarities and illustrate how the proposed notion refines the set of Nash equilibria in some examples of normal form games.

JEL Codes: C72
Keywords: Nash equilibrium, Refinements, Coalitional structure, Organizational structure, Games with strategic complementarities

*We would like to thank Daron Acemoglu, Mehmet Barlo, Tarık Kara, Emin Karagözoğlu, Özgür Kıbrıs, Semih Koray, Yusufcan Masatlıoğlu, Eiichi Miyagawa, Kemal Yıldız, as well as seminar participants at Bilkent University, Murat Sertel Workshop on Economic Design, and Conference on Economic Design. All remaining errors are ours.

1 Introduction

In the final scene of the western movie *The Good, the Bad and the Ugly*, there are three cowboys resolving their conflict via a truel. They all claim rights to some amount of money which will be collected by those who survive this truel. Clearly, this situation can be described as a strategic game. In the movie, it turns out that the Good cooperates with the Ugly whereas the Bad does not participate in any coalition. Now, knowing that only a single coalition is formed, would not the notions of *strong Nash equilibrium* (see Aumann, 1959) and *coalition-proof Nash equilibrium* (see Bernheim et al., 1987) be misleading? Perhaps more importantly, would not it be possible to find a new equilibrium notion that makes more precise predictions than *Nash equilibrium* (see Nash, 1951)? More generally, if it is *known* that there are players who cooperate with some of the other players but do not cooperate with some of them, then how come a notion which presumes that every player acts as his/her own or a notion which presumes that players participate in any combination of coalitions can make correct and/or precise predictions?

In non-cooperative game theory, there are numerous papers focusing on the ways of refining the set of Nash equilibria (see Aumann, 1959; Selten, 1965, 1975; Myerson, 1978; Kohlberg and Mertens, 1986; Bernheim et al., 1987, among others). Some of these equilibrium refinements allows players to form coalitions and to make joint deviations. Among these coalitional refinements, in this paper, we are mainly concerned with *strong Nash equilibrium* (SNE) and *coalition-proof Nash equilibrium* (CPNE).[1] Both of these equilibrium notions satisfy a certain type of coalitional stability. In particular, at a SNE, it should be the case that the members of any particular coalition prefer not to deviate collectively. As coalitions do not face too much restrictions in choosing their joint deviations, the set of SNE generally turns out to be empty. Stemming from this observation, Bernheim et al. (1987) propose the notion of CPNE according to which the members of a coalition cannot make binding commitments (i.e., agreements must be *self-enforcing*[2]). Accordingly, if no coalition is able to deviate from a strategy profile via self-enforcing contracts, then that strategy profile is said to be coalition-proof.

An important observation would be that both SNE and CPNE consider a case in which any coalition can be formed. However, in real life situations, we see that there are many instances at which some coalitions are not/cannot be formed. Moreover, even if a particular coalition is formed, this would not necessarily imply that all of its subcoalitions will be formed. Indeed, a game might have players that hate/dislike each other or that simply cannot communicate to form a coalition. Consider, for example,

[1] These are known to be the most prominent coalitional refinements of Nash equilibrium. For studies on these refinements, see Bernheim and Whinston (1987); Greenberg (1989); Dutta and Sen (1991); Konishi et al. (1997a,b, 1999) among others. In addition to these solution concepts, there are other refinements of Nash equilibrium which also utilize a coalitional structure: strong Berge equilibrium (Berge, 1957), the largest consistent set (Chwe, 1994), negotiation-proof Nash equilibrium (Xue, 2000), etc.

[2] A joint strategy profile of a coalition is *self-enforcing* if the coalition members do not desire further deviations.

two countries with a history of bad relations. These countries might not prefer to create a two-player coalition; or even if they meet at a global association, they might still refuse to form the two-player subcoalition.[3] Following the studies on *conference structure*s in the vein of Myerson (1980), consider also two academic scholars who have never met each other or anyone that could have connected them in a conference. Such scholars, even if they belong to the same society, cannot or choose not to collaborate. As for another example, we note that some coalitions cannot be formed because of some rules or regulations. For instance, the competition laws in many countries forbid firms that compete in the same market to cooperate. Yet, firms among which there is no competition are allowed to cooperate with each other. Along a similar line, in sports competitions, a player in a team is not permitted to form a coalition with a player of the opponent team whereas he/she is apparently cooperating with his/her teammates.

Another important observation on SNE and CPNE lies within the *actions* of players. Considering a coalition and its subcoalition, the notion of SNE allows both coalitions to determine joint strategy profiles in such a way that a member of the subcoalition cannot take an action that would simultaneously fulfill the interests of both coalitions (*vertical conflict of interest*). As a matter of fact, the notion of CPNE overcomes this conflict by restricting each coalition to respect the rationality of its subcoalitions/members. Be that as it may, since CPNE allows for coalitions that have non-empty intersection, a player participating in two coalitions may not be able to take an action that would simultaneously fulfill the interests of both coalitions (*horizontal conflict of interest*).

With the former observation in mind, is it really reasonable/feasible to control for all coalitions? If a coalition simply is not/cannot be formed, why would its members' *hypothetical* best actions be effective in the equilibrium behavior? Concerning the latter observation, can there be a specific structure that eliminates both vertical and horizantal conflicts of interests simultaneously? In this paper, motivated by the observations above and the associated questions, we aim to formulate two new equilibrium refinements: Our notions (i) resolve the problems of vertical and horizantal conflicts of interests, and (ii) prove to be more useful than the notions of SNE and CPNE (and even than Nash equilibrium) in cases where some coalitions are formed whereas some other coalitions are not formed.

Note that the former observation calls for a general coalitional structure that does not necessarily include some coalitions; whereas the latter observation calls for a specific framework, thereby restricting the set of coalitional structures to be studied. More precisely, in order to eliminate vertical conflicts of interest, every coalition should respect to the rationality of its subcoalitions/members (as it does in the case of CPNE). In addition to this, to be able to eliminate horizontal conflicts of interest, the coalitional structure should be formulated in such a way that for any pair of active (or formed)

[3] One may argue that they would if it will benefit them, but (i) forming a coalition does not necessarily make them better off (at the equilibrium) since there is strategic interaction with other players; and (ii) a possible reason why they would not participate in the same coalition is that it is somehow costly. For the latter, we must note that such a cost cannot be implemented into the payoff functions of the game.

coalitions, it is either the case that they are disjoint or that one coalition contains the other. This is what we refer to as *organizational structure*.

The intuition behind the *organizational framework* is as follows: In a non-cooperative game, players may prefer to form coalitions if they are allowed to. We assume that if a player is a member of a coalition, he/she cannot be a member of another coalition. Accordingly, the set of these coalitions turns out to be a partition of the player set. As coalitions may prefer to unite to form greater coalitions, in the next step we have another partition of the player set which will be coarser than the former partition. This recursively leads to a collection of partitions in which any partition is coarser than the partitions that precede it, i.e., to an *organization*. For an example, consider a university as a set of faculty members, each of whom belongs to one department. Moreover, each department belongs either to the school of social sciences or to the school of natural sciences. Similarly, a company (with its divisions, departments, units, and employees) can be considered to be another example.[4]

In this paper, we take the organizational structure as given.[5] Accordingly, for any organization, we define the notion of*organizational Nash equilibrium* (ONE) for which we utilize strict Pareto dominance to describe the preference relations of coalitions (Section 3). We provide a monotonicity property for the proposed notion in such a way that as we consider *greater* organizations, the equilibrium set is more refined; we analyze the existence of equilibrium for a subclass of games with strategic complementarities; and we study some examples of normal form games through which we understand how our organizational refinement works and how its predictions are different from those made by SNE and CPNE (Section 4). We conclude in Section 5.

[4]We can provide more solid real life examples: In a doubles tennis match, there is a total of four players in two teams, and a player cannot participate in the same coalition with his/her opponent. For a larger organization, we can consider a football/soccer game in which there are twenty two players. Teammates playing in the same position form small coalitions; such as defenders, midfielders, strikers. Then these small coalitions join together to form the team. Clearly, no coalition includes players from both teams. As for another example, we can consider the market for GSM services. Since a consumer needs a mobile phone and a line to receive this service, there are phone producers and telecommunication companies operating in this market. Two telecommunication companies are forbidden to form a coalition, whereas a telecommunication company is allowed to form a coalition with a phone producer. In order for us to have an organization in such a scenario, we need a regularity condition that an agreement between a telecommunication company and a phone producer restricts one of them to make another agreement without the presence of the other. Considering such a regularity condition, a coalitional structure emerging from the transfer market seems like a better example: If a team signs a contract with a player, the player cannot sign another contract with another team. And if the team signs a contract with another player, this would not be without the presence of the former player. Finally, we recall the example mentioned at the beginning of the paper, which is the truel took place in the final scene of the western movie The Good, the Bad and the Ugly. There we see an organization since the only formed coalition is between the Good and the Ugly.

[5]Our approach may seem rather *ad-hoc*. However, it must be understood that we do not intend to impose any type of coalitional structures. We simply argue that (i) there are many real life examples in which the coalitional structure does not include some coalitions/subcoalitions and (ii) an important subset of such coalitional structures consists of organizations. We present a solution concept to be studied only for such coalitional structures.

2 Preliminaries

Let $\Gamma = (N, (X_i)_{i\in N}, (u_i)_{i\in N})$ be an $|N|$-player normal form game in which N denotes the finite set of players, X_i denotes the strategy set for player $i \in N$, and $u_i : \prod_{i\in N} X_i \to \mathbb{R}$ denotes the utility function for player $i \in N$. For any coalition $S \subset N$, let $X_S \equiv \prod_{i\in S} X_i$ denote the set of strategy profiles for the members of this coalition. For any $S \subset N$, set $X_{-S} = X_{N\setminus S}$. And further set $X_N = X$.

First, the following is the definition of Nash equilibrium.

Definition 2.1. Given a normal form game Γ, a strategy profile $x^* \in X$ is a **Nash equilibrium** if for every $i \in N$ and every $x'_i \in X_i$: $u_i(x^*) \geq u_i(x'_i, x^*_{-i})$.

The notion of strong Nash equilibrium (SNE) is a refinement of Nash equilibrium which indeed requires a *strong* notion of coalitional stability.

Definition 2.2. Given a normal form game Γ, a strategy profile $x^* \in X$ is a **strong Nash equilibrium** (SNE) if for no coalition $S \subset N$, there exists some $x'_S \in X_S$ such that for every $i \in S$: $u_i(x'_S, x^*_{-S}) > u_i(x^*)$.

Another well-known refinement of Nash equilibrium is the notion of coalition-proof Nash equilibrium (CPNE) proposed by Bernheim et al. (1987). Before proceeding to its definition, we first define a *reduced game*.

Definition 2.3. Given a normal form game Γ, a coalition $S \subset N$, and a strategy profile $x_{-S} \in X_{-S}$, the **reduced game** $\Gamma_S|_{x_{-S}} = (S, (X_i)_{i\in S}, (v_i)_{i\in S})$ is defined in such a way that for every $i \in S$, $v_i : X_S \to \mathbb{R}$ is given by $v_i(x'_S) = u_i(x'_S, x_{-S})$.

Bernheim et al. (1987) introduce *self-enforceability* so as to weaken the coalitional stability the notion of SNE requires. As a consequence, they obtain an equilibrium notion which is weaker than the notion of SNE. Because of this relation, the set of CPNE always includes the set of SNE.

Definition 2.4. Given a normal form game Γ,

(i) If Γ is a single-player game, then a strategy profile $x^* \in X$ is a **coalition-proof Nash equilibrium** (CPNE) if and only if x^* maximizes u_1.

(ii) Let $|N| > 1$ and assume that the set of CPNE has been defined for any game with less than $|N|$ players. Define a strategy profile $x^* \in X$ to be **self-enforcing** if for every $S \subsetneq N$: $x^*_S \in \text{CPNE}(\Gamma_S|_{x^*_{-S}})$. Then a strategy profile $x^* \in X$ is a **CPNE** if and only if it is self-enforcing and there is no other self-enforcing strategy profile $x \in X$ such that for every $i \in N$: $u_i(x) > u_i(x^*)$.

Despite both notions' plausible refinement structures, there are normal form games in which these refinements (i) cannot make any prediction or (ii) make undesirable predictions. For instance, the normal form game given in Table 1 has two Nash equilibria: (x_1, x_2, x_3) and (y_1, y_2, y_3); but it has no SNE. More precisely, the grand coalition N deviates from (x_1, x_2, x_3) to (y_1, y_2, y_3), whereas the coalition $\{1, 2\}$ deviates from

(y_1, y_2) to (x_1, x_2) given that Player 3 sticks to y_3. As a result, none of the Nash equilibria is coalitionally stable in the sense of SNE. Furthermore, this game has a unique CPNE: (x_1, x_2, x_3). In particular, we can see that the deviation by N from (x_1, x_2, x_3) to (y_1, y_2, y_3) is not self-enforcing, since the subcoalition $\{1, 2\}$ further deviates to (x_1, x_2). Since there is no other coalition that would like to deviate, the profile (x_1, x_2, x_3) turns out to be a CPNE. Also note that the unique CPNE is strictly Pareto dominated by the other Nash equilibrium, (y_1, y_2, y_3).

Table 1

x_3	x_2	y_2
y_1	0, 0, 0	0, 0, 0
x_1	1, 1, 1	0, 0, 0

y_3	x_2	y_2
y_1	0, 0, 0	2, 2, 2
x_1	3, 3, 0	0, 0, 0

We can also provide an example for which there exists no CPNE, hence no SNE.

Table 2

x_3	x_2	y_2
y_1	0, 0, 0	0, 0, 1
x_1	1, 1, 0	0, 0, 1

y_3	x_2	y_2
y_1	0, 1, 2	2, 0, 1
x_1	2, 0, 0	0, 2, 1

For instance, in the normal form game given in Table 2, there are two Nash equilibria: (x_1, x_2, x_3) and (y_1, y_2, x_3). The coalition $\{2, 3\}$ makes a self-enforcing deviation from (x_2, x_3) to (y_2, y_3) given that Player 1 sticks to x_1. In a similar manner, the same coalition deviates from (y_2, x_3) to (x_2, y_3) given that Player 1 sticks to y_1.

3 Organizational Refinements of Nash Equilibrium

In this section, we first introduce a notation relevant to our definition of organizational refinement. While doing that, we also provide equivalent definitions for the notions of SNE and CPNE. These equivalent definitions are consistent with the introduced notation which will make them apparently comparable to the notion proposed in this paper. Afterwards, we present the notion of organizational Nash equilibrium (ONE).

3.1 Criticisms of SNE and CPNE

For any strategy profile $x \in X$ and any set of strategy profiles $Y \subset X$, define $B_N(x, Y)$ and $B_N(Y)$ as follows:

$$B_N(x, Y) = B_N(Y) = \{y \in Y \mid \nexists z \in Y, \forall i \in N : u_i(y) < u_i(z)\}.$$

For any coalition $S \subset N$ with $|S| < |N|$, any strategy profile $x \in X$, and any set $Y_S \subset X_S$, define[6]

$$B_S(x, Y_S) = \{y \in X \mid y_S \in Y_S \text{ and } \nexists z_S \in Y_S, \forall i \in S : u_i(y_S, x_{-S}) < u_i(z_S, x_{-S})\}.$$

We refer to $B_S(x, Y_S)$ as the set of *rational* (or weakly Pareto optimal) responses of S to the strategy profile $x \in X$ within the set $Y_S \times X_{-S}$. For any strategy profile $x \in X$, define

$$B(x) = \bigcap_{S \subset N} B_S(x, X_S).$$

We now prove that a SNE is a *fixed point*[7] of this correspondence.

Proposition 3.1. *A strategy profile $x \in X$ is a strong Nash equilibrium if and only if $x \in B(x)$.*

Proof. Take any $x \in X$ such that $x \in B(x)$. Suppose that x is not a SNE. Then $\exists S \subset N$ such that $\exists z_S \in X_S, \forall i \in S : u_i(z_S, x_{-S}) > u_i(x_S, x_{-S})$. Then $x \notin B_S(x, X_S)$; a contradiction.

Conversely, take any $x \in \text{SNE}(\cdot)$. Suppose that $x \notin B(x)$. Then $\exists S \subset N$ such that $x \notin B_S(x, X_S)$; that is, $\exists z_S \in X_S, \forall i \in S : u_i(z_S, x_{-S}) > u_i(x_S, x_{-S})$; a contradiction. □

Unfortunately, without additional restrictions, the correspondence above is unlikely to be nonempty-valued due to two types of conflicts of interest, which we call *vertical* and *horizontal* conflicts of interest.

Table 3

D	4, 0	1, 1
C	2, 2	0, 4
	C	D

A vertical conflict of interest arises between a coalition and its subcoalitions. For an example, consider *the Prisoner's Dilemma* which is represented by the matrix given in Table 3. Take the grand coalition N and observe that the only strategy profile which is strictly Pareto dominated by another is the unique Nash equilibrium: (D, D). Thus $B_N(X)$ includes all strategy profiles except (D, D); i.e., at each of the three strategy profiles in $B_N(X)$, at least one player is required to cooperate against his/her individual rationality. Hence there always is a player that faces a conflict of interest between coalitional rationality and his/her own individual rationality. More precisely,

[6]Note that $B_S(x, Y_S)$ does not impose any restriction on the joint strategy profile for the non-members of the coalition S. More precisely, for any $y = (y_S, y_{-S}) \in B_S(x, Y_S)$ and any $y'_{-S} \in X_{-S}$, we have $(y_S, y'_{-S}) \in B_S(x, Y_S)$ as well.

[7]A fixed point x of a correspondence of $F : X \to X$ is defined to satisfy $x \in F(x)$.

for any $x \in B_N(X)$, we either have $x \notin B_1(x, X_1)$ or $x \notin B_2(x, X_1)$, or both. This surely implies that for any $x \in X$: $B(x) = \varnothing$. It is also worth noting that for games with more players, the same type of conflict may arise between a coalition of at least three players and its subcoalitions of multiple players.

It is possible to eliminate such vertical conflicts of interest. For example, in a two-player game, no vertical conflict of interest would arise if the grand coalition respects the rationality of each player by restricting itself to the set of individually rational strategy profiles (i.e., the set of Nash equilibria). More generally, no vertical conflict of interest arises if each coalition respects the rationality of its proper subcoalitions, i.e., restricts itself to the strategy profiles from which none of its subcoalitions has an incentive to deviate.

As discussed below, this idea is closely related to self-enforceability in CPNE. Here we formalize the idea as follows. For a set of strategy profiles $Y \subset X$, define

$$[Y]_S = \{y_S \in X_S \mid \exists y_{-S} \in X_{-S} : (y_S, y_{-S}) \in Y\}.$$

Take any $x \in X$. For any $i \in N$, define $R_i(x) = B_i(x, X_i)$. For any coalition $S \subset N$ with $|S| = 2$, define

$$E_S(x) = \Big\{y \in X \,\Big|\, (y_S, x_{-S}) \in \bigcap_{i \in S} R_i((y_S, x_{-S}))\Big\}.$$

The set $[E_S(x)]_S$ can be interpreted as the set of Nash equilibria of the reduced game played by the coalition S given that the actions of the other players are fixed to x_{-S}. Then define

$$R_S(x) = B_S\left(x, [E_S(x)]_S\right).$$

This is the set of *rational* responses of coalition S among the strategy profiles which its members can jointly reach and which are *rational* for all of its members. In other words, coalition S restricts itself to the strategy profiles *acceptable* to all of its members. For any coalition $S \subset N$ with $|S| = 3$, define

$$E_S(x) = \Big\{y \in X \,\Big|\, (y_S, x_{-S}) \in \bigcap_{C \subsetneq S} R_C((y_S, x_{-S}))\Big\}. \tag{3.1}$$

Similarly, the set $[E_S(x)]_S$ can be interpreted as the *equilibrium* set of the corresponding reduced game. In a similar way,

$$R_S(x) = B_S\left(x, [E_S(x)]_S\right). \tag{3.2}$$

This is the set of *rational* responses of coalition S among the strategy profiles which its members can jointly reach and which are *rational* for all of its subcoalitions. In other words, coalition S restricts itself to the strategy profiles *acceptable* to all of its subcoalitions. Using (3.1) and (3.2) inductively, define $E_S(\cdot)$ and $R_S(\cdot)$ for $S \subset N$ with $|S| = 4, 5, \ldots, |N|$. Finally, for any strategy profile $x \in X$, define

$$R(x) = \bigcap_{S \subset N} R_S(x).$$

We first prove the following lemma.

Lemma 3.1. *For any coalition $S \subset N$ with $|S| \geq 2$ and any strategy profile $x \in X$, we have*

$$\bigcap_{C \subsetneq S} R_C(x) = \bigcap_{\substack{C \subset S \\ |C| = |S|-1}} R_C(x).$$

Proof. It is trivial that the left-hand side is included in the right-hand side.

Conversely, take any

$$x \in \bigcap_{\substack{C \subset S \\ |C| = |S|-1}} R_C(x),$$

and suppose that

$$x \notin \bigcap_{C \subsetneq S} R_C(x).$$

Then $\exists C \subsetneq S$ such that $x \notin R_C(x)$. Note that there exists $C' \subset S$ such that $C \subset C'$ and $|C'| = |S| - 1$. Since the definition is recursive, it follows that $x \notin R_{C'}(x)$; a contradiction. □

As we now show, a *fixed point* of the correspondence above turns out to be a CPNE, which was formally defined in the previous section.

Proposition 3.2. *A strategy profile $x \in X$ is a coalition-proof Nash equilibrium if and only if $x \in R(x)$.*

Proof. We prove this result by induction. Clearly, the statement holds when $|N| = 1$. For some $k \in \mathbb{N}$, assume that it holds when $|N| \leq k - 1$ and consider the case where $|N| = k$.

Take any $x \in X$ such that $x \in R(x)$. And suppose that x is not a CPNE. Then either (i) x is not self-enforcing; or (ii) x is self-enforcing but there exists another self-enforcing strategy profile $z \in R(x)$ such that $\forall i \in N : u_i(z) > u_i(x)$. If (i) is the case, then $\exists S \subsetneq N$ such that $x_S \notin \text{CPNE}(\Gamma_S|_{x_{-S}})$. But then $x \notin R_S(x)$ by the induction hypothesis; a contradiction. On the other hand, if (ii) is the case, it must be that $x \notin R_N(X)$; a contradiction.

Conversely, take any $x \in \text{CPNE}(\cdot)$. And suppose that $x \notin R(x)$. Then either (i) $x \notin R_N(x)$; or (ii) $x \notin \bigcap_{S \subsetneq N} R_S(x)$. If (i) is the case, then there exists another strategy profile $z \in R(x)$ such that $\forall i \in N : u_i(z) > u_i(x)$. By the induction hypothesis, it must be that z is self-enforcing; a contradiction. On the other hand, if (ii) is the case, then

$$x \notin \bigcap_{\substack{S \subset N \\ |S| = |N|-1}} R_S(x)$$

by Lemma 3.1. Following a similar reasoning, it follows for some $S \subset N$ with $|S| = |N| - 1$ that x_S is not a CPNE of the corresponding reduced game. This means that x is not self-enforcing; a contradiction. □

Although the definition of CPNE eliminates vertical conflicts of interest, there may still be *horizontal* conflicts of interest. A horizontal conflict of interest cannot arise between a coalition and its subcoalitions, by definition. It arises between two coalitions with a non-empty intersection. And, such a problem arises as soon as there are three players. For an example, consider the following normal form game given in Table 4. There exist two coalitions $A = \{1, 2\}$ and $B = \{2, 3\}$ with the corresponding sets of rational responses: $R_A((y_1, y_2, x_3)) = \{(x_1, z_2, \cdot)\}$ and $R_B((y_1, y_2, x_3)) = \{(\cdot, x_2, y_3)\}$. Surely, these sets have an empty intersection. The problem in this example is that Player 2 belongs to both coalitions A and B, and each of these coalitions requires him/her to behave differently than the other coalition requires. In other words, he/she faces a horizontal conflict of interest as it is impossible for him/her to choose an action that simultaneously meets the requirements of both coalitions.

Table 4

x_3

	x_2	y_2	z_2
z_1	1, 5, 8	0, 2, 0	7, 2, 3
y_1	0, 0, 3	5, 1, 8	4, 0, 0
x_1	0, 4, 3	0, 7, 3	7, 7, 0

y_3

	x_2	y_2	z_2
z_1	2, 0, 4	0, 1, 0	0, 0, 0
y_1	0, 2, 9	1, 0, 4	0, 0, 0
x_1	2, 0, 9	0, 0, 0	6, 6, 2

Since a horizontal conflict of interest arises between coalitions with a non-empty intersection, an easy way to eliminate this type of conflicts of interest would be to restrict coalition formation in such a way that all active (or formed) coalitions are mutually disjoint. However, we do not need to impose such a restrictive requirement. This is because if an active coalition is a proper subset of another active coalition, then there would still be no horizontal conflict of interest between these coalitions.

To sum up, if (i) one uses a similar formulation to $R(\cdot)$ above, and if (ii) any pair of active coalitions has the property that either they are disjoint or one is a subset of the other, then there is neither vertical nor horizontal conflicts of interest. Accordingly, we say that a collection $\mathscr{S}$ of coalitions in N is *conflict-free* if for any $A, B \in \mathscr{S}$, we have one of the following three properties: *(i)* $A \cap B = \varnothing$; *(ii)* $A \subset B$; or *(iii)* $B \subset A$.

Proposition 3.3. *Let $\mathscr{S}$ be a collection of coalitions in N and assume that it includes all singleton coalitions. Then $\mathscr{S}$ is conflict-free if and only if there exists a finite sequence $\mathcal{O} = \{P_0, P_1, \ldots, P_k\}$ of partitions of N with the following properties:*

(a) For any $S \in \mathscr{S}$, there exists $P \in \mathcal{O}$ with $S \in P$.

(b) For any $i \in \{0, \ldots, k-1\}$, P_i is finer than P_{i+1}.

Proof. Assume that $\mathscr{S}$ is conflict-free. We now construct a finite sequence $\mathcal{O}$ satisfying the properties above. First, let P_0 consist of all singleton coalitions. Let P_1 include all two-player coalitions in $\mathscr{S}$. For some $i \in N$, if $i \notin S$ for some $S \in P_1$, then let $\{i\}$ be included in P_1 as well. Then P_1 turns out to be a partition of N. Let P_2 include all

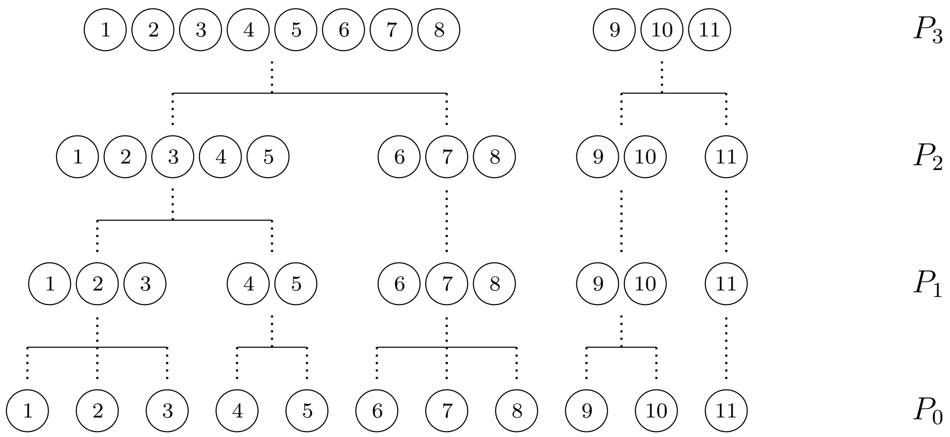

Figure 1: Example of an Organization

three-player coalitions in $\mathscr{S}$. For some $S \in P_1$, if $S \not\subset S'$ for some $S' \in P_2$, then let S be included in P_2 as well. Then P_2 turns out to be a partition of N. This process continues until there is no coalition remaining in $\mathscr{S}$. Accordingly, both (a) and (b) are satisfied by construction.

Conversely, take any two coalitions S and S' such that $S \in P$ and $S' \in P'$ for some $P, P' \in \mathcal{O}$. If $P = P'$, then $S \cap S' = \varnothing$. If $P \neq P'$, then one of them is finer than the other. Without loss of generality, assume that P is finer than P'. Then there exists $S'' \in P'$ such that $S \subset S''$. If $S' = S''$, then $S \subset S'$. If not, noting that $S' \cap S'' = \varnothing$, we have $S \cap S' = \varnothing$. Hence $\mathscr{S}$ is conflict-free. □

We define an *organization* $\mathcal{O} = \{P_0, P_1, \ldots, P_k\}$ of N as an ordered collection of partitions of N with properties (a) and (b) above, where $P_0 = \{\{1\}, \ldots, \{|N|\}\}$. Now, for a given organization $\mathcal{O} = \{P_0, P_1, \ldots, P_k\}$, let

$$\mathscr{S}^{\mathcal{O}} = \{S \subset N \mid \exists P \in \mathcal{O} \text{ such that } S \in P\}. \tag{3.3}$$

Given a partition $P \in \mathcal{O}$, let P_- be the coarsest partition in $\mathcal{O}$ that is finer than P and let P_+ be the finest partition in $\mathcal{O}$ that is coarser than P. To put it differently, P_- is the layer just below P and P_+ is that just above P.

Moreover, given a partition $P \in \mathcal{O}$ and a coalition $S \in P$, we define the *suborganization* $\mathcal{O}_{S\in P}$ as an ordered collection $\{P'_0, P'_1, \ldots, P'_-\}$ of partitions of S such that for each partition P'_t therein, we have $P'_t \subset P_t$. Finally, we define

$$\rho^{\mathcal{O}}(S, P) = \{C \in \mathscr{S}^{\mathcal{O}} \mid C \in P_- \text{ and } C \subset S\}, \tag{3.4}$$

to be the coarsest partition in $\mathcal{O}_{S\in P}$; which is indeed a partition of S.

For a concrete example, consider the organization illustrated in Figure 1. Let $\mathcal{O} = \{P_0, P_1, P_2, P_3\}$ where

$$\begin{aligned} P_0 &= \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}\}, \\ P_1 &= \{\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}, \{9, 10\}, \{11\}\}, \end{aligned}$$

$$
\begin{aligned}
P_2 &= \{\{1,2,3,4,5\},\{6,7,8\},\{9,10\},\{11\}\}, \text{ and} \\
P_3 &= \{\{1,2,3,4,5,6,7,8\},\{9,10,11\}\}.
\end{aligned}
$$

Moreover, if we consider $S = \{1,2,3,4,5,6,7,8\}$ and $P = P_3$, then the corresponding suborganization is $\mathcal{O}_{S \in P} = \{P'_0, P'_1, P'_-\}$ such that

$$
\begin{aligned}
P'_0 &= \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\}\}, \\
P'_1 &= \{\{1,2,3\},\{4,5\},\{6,7,8\}\}, \text{ and} \\
P'_- &= \{\{1,2,3,4,5\},\{6,7,8\}\}.
\end{aligned}
$$

3.2 Organizational Nash Equilibrium

In this subsection, we present a new refinement of Nash equilibrium for which there does not exist any vertical or horizontal conflict of interest. Consider a normal form game Γ and an organization $\mathcal{O} = \{P_0, P_1, \ldots, P_k\}$. Take any strategy profile $x \in X$. For any player $i \in N$, define $R_i^{\mathcal{O}}(x) = B_i(x, X_i)$. Then for any coalition $S \in P_1$, the sets $E_S^{\mathcal{O}}(x)$ and $R_S^{\mathcal{O}}(x)$ are defined as follows:

$$
\begin{aligned}
E_S^{\mathcal{O}}(x) &= \left\{ y \in X \;\middle|\; (y_S, x_{-S}) \in \bigcap_{\{i\} \in \rho^{\mathcal{O}}(S,P)} R_i^{\mathcal{O}}((y_S, x_{-S})) \right\} \\
R_S^{\mathcal{O}}(x) &= B_S\left(x, \left[E_S^{\mathcal{O}}(x)\right]_S\right)
\end{aligned}
$$

Moreover, for any coalition $S \in \mathscr{S}^{\mathcal{O}} \setminus P_1$, the sets $E_S^{\mathcal{O}}(x)$ and $R_S^{\mathcal{O}}(x)$ are inductively defined as follows:

$$
\begin{aligned}
E_S^{\mathcal{O}}(x) &= \left\{ y \in X \;\middle|\; (y_S, x_{-S}) \in \bigcap_{C \in \rho^{\mathcal{O}}(S,P)} R_C^{\mathcal{O}}((y_S, x_{-S})) \right\} \\
R_S^{\mathcal{O}}(x) &= B_S\left(x, \left[E_S^{\mathcal{O}}(x)\right]_S\right)
\end{aligned}
$$

Accordingly, for any $x \in X$, define

$$
R^{\mathcal{O}}(x) = \bigcap_{S \in \mathscr{S}^{\mathcal{O}}} R_S^{\mathcal{O}}(x).
$$

We call a strategy profile $x \in X$ satisfying $x \in R^{\mathcal{O}}(x)$ an *$\mathcal{O}$-organizational Nash equilibrium*, or simply an *organizational Nash equilibrium*. Let $\mathrm{ONE}^{\mathcal{O}}(\Gamma)$ be the set of $\mathcal{O}$-organizational Nash equilibria of Γ.

We first show that ONE is indeed a refinement of Nash equilibrium.

Proposition 3.4. *For any normal form game Γ and any organization $\mathcal{O}$,*

$$
\mathit{ONE}^{\mathcal{O}}(\Gamma) \subset \mathit{NE}(\Gamma)
$$

Proof. Omitted. □

It is worth noting that, given two organizations, the sets of equilibria may turn out to be very different. However, as we show in the following, the sets of ONE coincide for *equivalent* organizations.

Definition 3.1. Two organizations $\mathcal{O}$ and $\mathcal{O}'$ are **equivalent** if $\mathscr{S}^{\mathcal{O}} = \mathscr{S}^{\mathcal{O}'}$.

Remark 3.1. Given a normal form game Γ and two equivalent organizations $\mathcal{O}, \mathcal{O}'$:

$$\text{ONE}^{\mathcal{O}}(\Gamma) = \text{ONE}^{\mathcal{O}'}(\Gamma).$$

Proof. Consider any coalition $S \in \mathscr{S}^{\mathcal{O}}$. Since $\mathcal{O}$ and $\mathcal{O}'$ are equivalent, $S \in \mathscr{S}^{\mathcal{O}'}$. Moreover for each subcoalition $C \subset S$, if $C \in \mathscr{S}^{\mathcal{O}}$, then $C \in \mathscr{S}^{\mathcal{O}'}$. This implies that in organizations $\mathcal{O}$ and $\mathcal{O}'$, the coalition S considers the rational responses of *the same* subcoalitions when making a joint decision. Then $R_S^{\mathcal{O}} = R_S^{\mathcal{O}'}$. Since S is arbitrarily chosen, it follows that $R^{\mathcal{O}} = R^{\mathcal{O}'}$. Hence the sets of ONE coincide. □

We now focus on the elimination of vertical and horizontal conflicts of interest. Although the former result in Proposition 3.5 is also valid for CPNE, the latter is only valid for ONE. The reason is that CPNE eliminates vertical conflicts of interest, but cannot eliminate horizontal conflicts of interest; whereas the notion of ONE is able to eliminate both types of conflict of interest.

Proposition 3.5. *For any normal form game Γ and any organization $\mathcal{O}$, there exists neither (i) vertical nor (ii) horizontal conflicts of interest within the analysis of ONE. Formally, (i) given two coalitions $S, S' \in \mathscr{S}^{\mathcal{O}}$ such that $S' \subsetneq S$, if $(y_S, \cdot) \in R_S^{\mathcal{O}}(x)$ for some $x \in X$, then $(y_{S'}, \cdot) \in R_{S'}^{\mathcal{O}}(y_S, x_{-S})$; and (ii) given two coalitions $S, S' \in \mathscr{S}^{\mathcal{O}}$ such that $S' \not\subset S$ and $S \not\subset S'$, for any $x \in X$: $R_S^{\mathcal{O}}(x) \cap R_{S'}^{\mathcal{O}}(x) \neq \varnothing$.*

Proof. For (i), consider any coalition $S \in P_1$ and set $S' = \{i\}$ for some member i of S. Take any $x \in X$ and any $(y_S, \cdot) \in R_S^{\mathcal{O}}(x)$. Noting that $R_S^{\mathcal{O}}(x) = B_S\left(x, \left[E_S^{\mathcal{O}}(x)\right]_S\right)$ by definition, we find that

$$(y_S, \cdot) \in \bigcap_{i \in S} R_i^{\mathcal{O}}((y_S, x_{-S})).$$

This implies that $(y_{S'}, \cdot) \in R_{S'}^{\mathcal{O}}(y_S, x_{-S})$.

Now, consider any coalition $S' \in P_2$. If $S' \subset S$ is a singleton, then the result similarly follows. If not, then $S' \in P_1$. Take any $x \in X$ and any $(y_S, \cdot) \in R_S^{\mathcal{O}}(x)$. Noting that $R_S^{\mathcal{O}}(x) = B_S\left(x, \left[E_S^{\mathcal{O}}(x)\right]_S\right)$ by definition, we find that

$$(y_S, \cdot) \in \bigcap_{\substack{C \subsetneq S \\ C \in P_0 \cup P_1}} R_C^{\mathcal{O}}((y_S, x_{-S})).$$

This implies that $(y_{S'}, \cdot) \in R_{S'}^{\mathcal{O}}(y_S, x_{-S})$.

The rest follows recursively.

As for (ii), consider two coalitions $S, S' \in \mathscr{S}^{\mathcal{O}}$ such that $S' \not\subset S$ and $S \not\subset S'$. By the definition of organizations, it must be that S and S' are disjoint. The proof concludes with an observation that each $R_S^{\mathcal{O}}$ concerns only the relevant part of strategy profiles for S; i.e., if $(y_S, y_{-S}) \in R_S^{\mathcal{O}}(x)$ then for every $y'_{-S} \in X_{-S}$: $(y_S, y'_{-S}) \in R_S^{\mathcal{O}}(x)$. □

According to the definition of SNE, any coalition of players can jointly deviate to any of their joint strategy profiles. On the other hand, our organizational refinement restricts the set of coalitions that can deviate and the set of strategy profiles that a particular coalition can deviate to. Accordingly, our notion of ONE turns out to be weaker than the notion of SNE.[8]

Proposition 3.6. *For any normal form game* Γ *and organization* $\mathcal{O} = \{P_0, P_1, \ldots, P_k\}$,

$$\mathit{SNE}(\Gamma) \subset \mathit{ONE}^{\mathcal{O}}(\Gamma).$$

Proof. Take any $x^* \in \text{SNE}(\Gamma)$. Suppose that x^* is not an $\mathcal{O}$-organizational Nash equilibrium of Γ. We then see that there should exist some partition(s) $P_t \in \mathcal{O}$ such that there is $S \in P_t$ satisfying $x^*_S \notin R^{\mathcal{O}}_S(x^*)$. We take the one with the smallest t and denote it by $\bar{P}$. The corresponding coalition is denoted by $\bar{S}$.

Then for every $S' \in \bar{P}_-$ with $S' \subset \bar{S}$:

$$x^*_{S'} \in \text{ONE}^{\mathcal{O}_{S' \in \bar{P}_-}}(\Gamma_{S'}|_{x^*_{-S'}}).$$

It must be that there is $y_{\bar{S}} \in X_{\bar{S}}$ such that

$$(i)\ \forall i \in \bar{S} : u_i(y_{\bar{S}}, x^*_{-\bar{S}}) > u_i(x^*) \text{ and}$$
$$(ii)\ \forall S' \in \bar{P}_- \text{ with } S' \subset \bar{S} : y_{S'} \in \text{ONE}^{\mathcal{O}_{S' \in \bar{P}_-}}(\Gamma_{S'}|_{x^*_{-S'}}).$$

We then conclude that $x^*_{\bar{S}}$ is not Pareto optimal for $\bar{S}$ given that the complementary coalition chooses $x^*_{-\bar{S}}$. This is a contradiction; which completes the proof that x^* is an $\mathcal{O}$-organizational Nash equilibrium. □

3.3 Illustrative Examples

By Proposition 3.6, we understand the relation between the predictions of ONE and SNE. In this section we consider two examples of normal form games which help us understand how ONE refines the set of Nash equilibria and how its predictions are different from those of CPNE. Insights gained from these equilibrium analyses will be further discussed in Concluding Remarks below.

We first recall the three-player normal form game given in Table 4 for which we have observed horizontal conflicts of interest. This game has three Nash equilibria: (z_1, x_2, x_3), (y_1, y_2, x_3), and (x_1, z_2, y_3). The coalition $\{1, 2\}$ deviates from the first and the second, the coalition $\{2, 3\}$ deviates from the second and the third, and the coalition $\{1, 3\}$ deviates from the first and the third. Further note that all of these deviations are self-enforcing. Therefore, this game possesses no SNE or CPNE.

In this example, we consider

$$\mathcal{O}_1 = \{P_0, \{\{1, 2\}, \{3\}\}, \{N\}\},$$
$$\mathcal{O}_2 = \{P_0, \{\{1\}, \{2, 3\}\}, \{N\}\},$$
$$\mathcal{O}_3 = \{P_0, \{\{1, 3\}, \{2\}\}, \{N\}\};$$

[8] This also implies that an ONE exists for every normal form game which possesses a SNE.

Table 4 [Revisited]

x_3

	x_2	y_2	z_2
z_1	1, 5, 8	0, 2, 0	7, 2, 3
y_1	0, 0, 3	5, 1, 8	4, 0, 0
x_1	0, 4, 3	0, 7, 3	7, 7, 0

y_3

	x_2	y_2	z_2
z_1	2, 0, 4	0, 1, 0	0, 0, 0
y_1	0, 2, 9	1, 0, 4	0, 0, 0
x_1	2, 0, 9	0, 0, 0	6, 6, 2

and we find the following sets of ONE:

$$\begin{aligned}\mathrm{ONE}^{\mathcal{O}_1}(\cdot) &= \{(x_1, z_2, y_3)\},\\ \mathrm{ONE}^{\mathcal{O}_2}(\cdot) &= \{(z_1, x_2, x_3)\},\\ \mathrm{ONE}^{\mathcal{O}_3}(\cdot) &= \{(y_1, y_2, x_3)\}.\end{aligned}$$

The arguments are as follows: In O_1, the only active two-player coalition is $\{1, 2\}$. This coalition deviates from (z_1, x_2, x_3) and (y_1, y_2, x_3), but not from (x_1, z_2, y_3). The other two-player coalitions cannot deviate from any of these strategy profiles, since they are not formed. Noting that every Nash equilibrium is Pareto optimal for the grand coalition, we conclude that (x_1, z_2, y_3) is the unique coalitionally stable outcome for this particular organization. As for O_2 and O_3, similar reasonings would apply.

It is worth mentioning here that, as illustrated in the example above, it may be the case that each Nash equilibrium is supported by some organizational structure as the unique coalitionally stable outcome. However, this is not necessarily true for all normal form games, since the notions of ONE and CPNE coincide in two-player games:

Remark 3.2. In a two-player normal form game, since the only non-trivial organization includes all possible coalitions, ONE and CPNE coincide.

We now provide an example in which the non-empty sets of CPNE and ONE are disjoint. This implies that one set of equilibrium does not necessarily include the other. To do this, we refer to the normal form game given in Table 1:

Table 1 [Revisited]

x_3

	x_2	y_2
y_1	0, 0, 0	0, 0, 0
x_1	1, 1, 1	0, 0, 0

y_3

	x_2	y_2
y_1	0, 0, 0	2, 2, 2
x_1	3, 3, 0	0, 0, 0

Recall that there are two Nash equilibria: (x_1, x_2, x_3) and (y_1, y_2, y_3), and that there exists a unique CPNE: (x_1, x_2, x_3). The reason is that the coalition $\{1, 2\}$ makes a self-enforcing deviation from (y_1, y_2, y_3); whereas the deviation of the grand coalition from (x_1, x_2, x_3) to (y_1, y_2, y_3) will be blocked by the subcoalition $\{1, 2\}$, as they would further deviate from (y_1, y_2, y_3). On the other hand, if we analyze ONE of this game for the organization $\{P_0, \{\{1\}, \{2, 3\}\}, \{N\}\}$, we find that the unique ONE turns out to be (y_1, y_2, y_3). The reason is that since the coalition $\{1,2\}$ is now inactive, they would not deviate from (y_1, y_2, y_3) and they would not block the deviation of the grand coalition from (x_1, x_2, x_3) to (y_1, y_2, y_3). As a matter of fact, by a similar reasoning, also for the organizations $\{P_0, \{\{1,3\},\{2\}\},\{N\}\}$ and $\{P_0,\{N\}\}$, the unique ONE of this game would be (y_1, y_2, y_3). It is also worth mentioning that this is an example which highlights the importance and usefulness of our organizational refinement. Apparently, the unique ONE strictly Pareto dominates the unique CPNE.

4 The Results

4.1 Existence of Equilibrium

Although the formulation of our organizational refinement eliminates both types of conflicts of interest, a normal form game might not have an ONE for some organizations. For an example, consider the normal form game given in Table 5. This game has a unique Nash equilibrium: $x \equiv (x_1, x_2, x_3)$.

Table 5

x_3	x_2	y_2
y_1	0, 0, 0	2, 2, 0
x_1	1, 1, 1	0, 0, 0

y_3	x_2	y_2
y_1	0, 1, 1	1, 0, 1
x_1	1, 0, 0	0, 1, 1

Considering the organization $\mathcal{O} = \{P_0, \{\{1,2\},\{3\}\}\}$, we have $R_1^{\mathcal{O}}(x) = (x_1,\cdot,\cdot)$, $R_2^{\mathcal{O}}(x) = (\cdot, x_2, \cdot)$, and $R_{\{1,2\}}^{\mathcal{O}}(x) = (y_1, y_2, \cdot)$. Accordingly, $R^{\mathcal{O}}(x) = \varnothing$ which implies the non-existence of ONE.

The observation above leads to a natural question to find classes of normal form games for which an ONE exist. In this part of the paper, we prove the existence of our organizational refinement in a subclass of games with strategic complementarities (see Topkis, 1998; Amir, 2005; Vives, 2005, among others).[9] Below are the definitions that will be utilized throughout this subsection.

A set is a *lattice* if it contains the supremum and the infimum of every pair of its elements. A lattice is *complete* if each non-empty subset has a supremum and an infimum.[10] Moreover, a subset Y of a lattice X is a *subcomplete sublattice* of X if for every non-empty subset Y' of Y, the supremum of Y' and the infimum of Y' exist and are contained in Y.

[9] Games with strategic complemetarities are commonly utilized in the literature both for the existence of Nash equilibrium (see Zhou, 1994; Echenique, 2005; Calciano, 2007; Keskin et al., 2014, among others) and for the existence of the refinements of Nash equilibrium; such as minimally altruistic Nash equilibrium (see Karagozoglu et al., 2013), perfect equilibrium (see Carbonell-Nicolau and McLean, 2014), and strong Berge equilibrium (see Keskin and Saglam, 2014).

[10] Note that a complete lattice X is compact in its interval topology which is the topology generated by taking the closed intervals, $[y,z] = \{x \in X : y \leq x \leq z\}$ with $y, z \in X$ as a subbasis of closed sets (see Birkhoff (1967)).

Let X be a lattice and T be a partial order. A function $f : X \to \mathbb{R}$ is called *quasi-supermodular* if for all $x, y \in X$, $f(x) \geq f(x \wedge y)$ implies $f(x \vee y) \geq f(y)$ and $f(x) > f(x \wedge y)$ implies $f(x \vee y) \geq f(y)$. We say that a function $f : X \times T \to \mathbb{R}$ satisfies the *single crossing property* in (x, t) if for all $x, x' \in X$ and $t, t \in T$ with $x > x'$ and $t > t'$: $f(x, t') \geq f(x', t')$ implies $f(x, t) \geq f(x', t)$ and $f(x, t') > f(x', t')$ implies $f(x, t) > f(x', t)$.

The following definition of games with strategic complementarities is provided by Milgrom and Shannon (1994) and Milgrom and Roberts (1996).

Definition 4.1. A normal form game Γ is a **game with strategic complementarities** if for every $i \in N$: (i) X_i is a non-empty complete lattice; (ii) u_i is upper semi-continuous in x_i and continuous in x_{-i}; and (iii) u_i is quasi-supermodular in x_i and has the single crossing property in (x_i, x_{-i}).

As shown by Milgrom and Shannon (1994), in a game with strategic complementarities, the smallest and the largest serially undominated strategy profiles[11] exist and they are in fact the smallest and the largest Nash equilibria of the game, respectively (see their Theorem 12). Furthermore, as shown by Milgrom and Roberts (1996), an additional monotonicity assumption would suffice for the existence of CPNE in a subclass of games with strategic complementarities. In particular, these authors assume that each utility function u_i is non-decreasing/non-increasing in x_{-i} (see their Theorem 2). In the following, by weakening the monotonicity assumption, we prove the existence of our organizational refinement.

Proposition 4.1. *Consider a game with strategic complementarities Γ and an organization $\mathcal{O}$. Assume that for every $i \in N$ and every $S \in \mathscr{S}^{\mathcal{O}}$ that includes i: either (i) u_i is non-decreasing in x_{-S}, or (ii) u_i is non-increasing in x_{-S}. Then there exists an $\mathcal{O}$-organizational Nash equilibrium for this game.*

Proof. Assume that (i) is the case. Consider the largest Nash equilibrium of the game, denoted by x^*. As we know from Milgrom and Shannon (1994), x^* is also the largest serially undominated strategy profile in this game. Consider any coalition $S \in P_1$ and the corresponding reduced game $\Gamma_S|_{x^*_{-S}} = (S, (X_i)_{i \in S}, (v_i)_{i \in S})$. By definition, this reduced game is a game with strategic complementarities in which each v_i is non-decreasing in x_{-i}. We also know that x^*_S is a Nash equilibrium for $\Gamma_S|_{x^*_{-S}}$; which means that x^*_S would survive the iterated elimination of strictly dominated strategies in the reduced game. As a matter of fact, x^*_S turns out to be the largest serially undominated strategy profile in this game.[12] It then follows from Milgrom and Shannon (1994) that x^*_S is the largest Nash equilibrium for the reduced game. Applying Theorem A2 of Milgrom and Roberts (1996), for each subcoalition $C \subset S$, playing x^*_C is preferred to playing any other strategy profile in the reduced game. That theorem surely applies for the coalition S itself. Accordingly, for any Nash equilibrium y_S of the reduced

[11] A strategy profile is said to be *serially undominated* if it survives the iterated elimination of strictly dominated strategies.

[12] The reader is referred to the Appendix for the proof of this claim.

game $\Gamma_S|_{x^*_{-S}}$, we have for every $i \in S$: $u_i(y_S, x^*_{-S}) \le u_i(x^*_S)$. Therefore, x^*_S is a coalitional best response for S. Since S is arbitrarily chosen, for every coalition $S \in P_1$: $(x^*_S, \cdot) \in R^{\mathcal{O}}_S(x^*)$.

Now, consider any coalition $S' \in P_2$ and the corresponding reduced game $\Gamma_{S'}|_{x^*_{-S'}}$. Noting that $\mathcal{O}_{S' \in P_2}$ is the suborganization for this coalition and considering the arguments above, we know that $x^*_{S'}$ is an $\mathcal{O}_{S' \in P_2}$-organizational Nash equilibrium of this game. This is because for every member $i \in S'$: $(x^*_i, \cdot) \in R^{\mathcal{O}}_i(x^*)$ and for every active subcoalition $C' \subset S'$ in this suborganization: $(x^*_{C'}, \cdot) \in R^{\mathcal{O}}_{C'}(x^*)$. Applying Theorem A2 of Milgrom and Roberts (1996) once again, we conclude that for the coalition S', playing $x^*_{S'}$ is preferred to playing any other ONE in the reduced game $\Gamma_{S'}|_{x^*_{-S'}}$. Then it similarly follows that for every coalition $S' \in P_2$: $(x^*_{S'}, \cdot) \in R^{\mathcal{O}}_{S'}(x^*)$.

Finally, it recursively follows that for every coalition $S'' \in \mathscr{S}^{\mathcal{O}}$: $(x^*_{S''}, \cdot) \in R^{\mathcal{O}}_{S''}(x^*)$. Hence, $x^* \in R^{\mathcal{O}}(x^*)$, which in turn implies that $x^* \in \text{ONE}(\Gamma)$.

Arguments for (ii) similarly follow. □

In the following normal form game given in Table 6, we can demonstrate how the existence result works:

Table 6

x_3:

	x_2	y_2
y_1	3, 4, 0	5, 5, 0
x_1	4, 4, 1	5, 3, 3

y_3:

	x_2	y_2
y_1	0, 1, 1	2, 2, 2
x_1	1, 1, 0	1, 0, 2

This example is a game with strategic complementarities since each utility function u_i is quasi-supermodular in x_i and has the single crossing property in (x_i, x_{-i}). Furthermore, u_1 is non-decreasing in x_2 and u_2 is non-decreasing in x_1. Also note that u_3 is not monotone in x_1 or x_2, since (x_1, y_2, x_3) yields the highest utility for Player 3. Accordingly, this normal form game satisfies the conditions of our existence result for the organization $\mathcal{O}^* = \{P_0, \{\{1,2\},\{3\}\}\}$, but not for the other possible organizations. On top of that, it does not satisfy the conditions for the existence of CPNE provided by Milgrom and Roberts (1996). There are two Nash equilibria: (x_1, x_2, x_3) and (y_1, y_2, y_3). Now, analyzing the set of $\mathcal{O}^*$-organizational Nash equilibria, we see that the coalition $\{1,2\}$ deviates from (x_1, x_2, x_3) to (y_1, y_2) given that Player 3 sticks to x_3. Noting that they would not deviate from the other Nash equilibrium, we find that (y_1, y_2, y_3) is the unique ONE for this game. On the other hand, there exists no CPNE. The reason is that (i) the coalition $\{1,2\}$ still makes the aforementioned deviation and (ii) the coalition $\{1,3\}$ makes a self-enforcing deviation from (y_1, y_2, y_3) to (x_1, x_3) given that Player 2 sticks to y_2.

As for another example satisfying the conditions of Proposition 4.1, consider the four-player normal form game given in Table 7. Once again, u_1 is non-decreasing in x_2 and u_2 is non-decreasing in x_1. We can also see that neither u_3 nor u_4 is monotone in either of the other players' strategies. This game has two Nash equilibria: (x_1, x_2, x_3, x_4)

Table 7

		x_3		y_3	
		x_2	y_2	x_2	y_2
y_4	y_1	3, 3, 2, 0	3, 3, 0, 2	1, 0, 1, 1	2, 2, 1, 2
	x_1	3, 3, 2, 0	3, 3, 2, 0	0, 0, 1, 1	0, 1, 1, 1
x_4	y_1	2, 5, 1, 1	5, 5, 1, 0	4, 3, 0, 0	5, 5, 0, 0
	x_1	4, 4, 1, 1	5, 2, 1, 1	3, 3, 0, 0	3, 4, 0, 0

and (y_1, y_2, y_3, y_4). Considering the organization $\mathcal{O}^*$ above, we note that the coalition {1,2} deviates from (x_1, x_2, x_3, x_4) to (y_1, y_2) given that Players 3 and 4 stick to (x_3, x_4), and we find that (y_1, y_2, y_3, y_4) is the unique ONE for this game. Now, if we consider a greater organization by adding $P_2 = \{\{1, 2, 3\}, \{4\}\}$ into the existing organization $\mathcal{O}^*$, the unique ONE ceases to exist. The reason is that the coalition $\{1, 2, 3\}$ deviates from (y_1, y_2, y_3, y_4) to (x_1, x_2, x_3) given that Player 4 sticks to y_4. As we see in this particular example, adding a new coalition for which the utility functions of its members do not have a monotonicity relation as described in Proposition 4.1 *might* lead to the non-existence of equilibrium.

Another interesting note is that if we consider the organization $\{P_0, \{\{1, 2, 3\}, \{4\}\}\}$, then the weakly Pareto optimal Nash equilibrium (x_1, x_2, x_3, x_4) is realized as the unique ONE. Accordingly, one can claim that removing small coalitions from an organization or replacing small coalitions in an organization with larger coalitions *might* turn out to be socially beneficial.

4.2 A Monotonicity Property

We start with a nice result indicating that the introduced refinement structures follow in a monotonic fashion. In a normal form game, given two organizations $\mathcal{O}$ and $\mathcal{O}'$, we say that $\mathcal{O}$ is *greater* than $\mathcal{O}'$ if (*i*) $P \in \mathcal{O}'$ implies that $P \in \mathcal{O}$ and (*ii*) $\big[P' \in \mathcal{O}$ and $P' \notin \mathcal{O}'\big]$ implies that P' is coarser than the coarsest partition in $\mathcal{O}'$. We show that the equilibrium set is more refined for *greater* organizations.[13]

Proposition 4.2. *For any normal form game* Γ*, if an organization* $\mathcal{O}$ *is greater than another organization* $\mathcal{O}'$*, then*

$$ONE^{\mathcal{O}}(\Gamma) \subset ONE^{\mathcal{O}'}(\Gamma).$$

[13]It is important here that we do not compare any particular partitions when comparing organizations. Our definition of "being greater" indicates that an organization is greater than another if the former completely preserves the structure of the latter and additionally includes coarser partitions. Alternative definitions *may* yield different results.

Proof. Take any $x^* \in \mathrm{ONE}^{\mathcal{O}}(\Gamma)$. By definition, $x^* \in R^{\mathcal{O}}(x^*)$. This implies that for every $S \in \mathscr{S}^{\mathcal{O}}$: $(x^*_S, \cdot) \in R^{\mathcal{O}}_S(x^*)$. Since $\mathcal{O}$ is greater than $\mathcal{O}'$, we know that $\mathscr{S}^{\mathcal{O}'} \subset \mathscr{S}^{\mathcal{O}}$ and that for every $S \in \mathscr{S}^{\mathcal{O}} \setminus \mathscr{S}^{\mathcal{O}'}$: $\nexists S' \in \mathscr{S}^{\mathcal{O}'}$ such that $S \subset S'$. Accordingly, for every $S' \in \mathscr{S}^{\mathcal{O}'}$: $(x^*_{S'}, \cdot) \in R^{\mathcal{O}'}_{S'}(x^*)$. Therefore, $x^* \in R^{\mathcal{O}'}(x^*)$; i.e., $x^* \in \mathrm{ONE}^{\mathcal{O}'}(\Gamma)$. Then it follows that $\mathrm{ONE}^{\mathcal{O}}(\Gamma) \subset \mathrm{ONE}^{\mathcal{O}'}(\Gamma)$. □

This monotonicity property leads to the following observation.

Corollary 4.1. *Consider a normal form game Γ that possesses a Nash equilibrium. Take any increasing sequence of organizations $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_t$ such that $\mathcal{O}_1 = \{P_0\}$ and for any $i \in \{1, \ldots, t-1\}$, $\mathcal{O}_{i+1}$ is greater than $\mathcal{O}_i$. By Proposition 4.2, we know that for any $i \in \{1, \ldots, t-1\}$,*

$$\mathit{ONE}^{\mathcal{O}_{i+1}}(\Gamma) \subset \mathit{ONE}^{\mathcal{O}_i}(\Gamma).$$

In addition to that, this sequence has a maximum organization for which the set of ONE is non-empty.

Example 4.1. Consider a five-player normal form game given in Table 8 for which there are three Nash equilibria: $(x_1, x_2, y_3, y_4, x_5)$, $(y_1, y_2, x_3, y_4, x_5)$, $(y_1, y_2, y_3, x_4, x_5)$. We now set

$$\begin{aligned} P_0 &= \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}, \\ P_1 &= \{\{1, 2\}, \{3\}, \{4\}, \{5\}\}, \\ P_2 &= \{\{1, 2, 3\}, \{4\}, \{5\}\}, \text{ and} \\ P_3 &= \{\{1, 2, 3, 4\}, \{5\}\}. \end{aligned}$$

Table 8

y_4, x_3:

	x_2	y_2
y_1	1, 0, 1, 0, 0	0, 1, 1, 0, 0
x_1	0, 1, 1, 0, 0	1, 0, 1, 0, 0

y_4, y_3:

	x_2	y_2
y_1	1, 0, 0, 1, 0	0, 1, 0, 1, 0
x_1	0, 1, 0, 1, 0	1, 0, 0, 1, 0

y_5 x_4, x_3:

	x_2	y_2
y_1	1, 0, 0, 1, 0	0, 1, 0, 1, 0
x_1	0, 1, 0, 1, 2	1, 0, 0, 1, 0

y_5 x_4, y_3:

	x_2	y_2
y_1	1, 0, 1, 0, 0	0, 1, 1, 0, 0
x_1	0, 1, 1, 0, 0	1, 0, 1, 0, 0

First consider $\mathcal{O}_1 = \{P_0\}$. Then the set of ONE surely coincides with the set of Nash equilibria. Then we consider $\mathcal{O}_2 = \{P_0, P_1\}$. We can see that the coalition $\{1, 2\}$ deviates to (x_1, x_2) given that Players 3, 4, and 5 stick to (x_3, y_4, x_5). Since there is no further deviation, $(y_1, y_2, x_3, y_4, x_5)$ is not coalitionally stable in the sense of ONE.

y_4, x_3:

	x_2	y_2
y_1	0, 0, 0, 1, 1	1, 1, 1, 1, 1
x_1	2, 2, 0, 0, 1	0, 0, 0, 1, 1

y_4, y_3:

	x_2	y_2
y_1	0, 0, 1, 0, 1	1, 1, 0, 1, 1
x_1	3, 3, 3, 3, 1	0, 0, 1, 0, 1

x_5, x_4, x_3:

	x_2	y_2
y_1	0, 0, 1, 0, 1	2, 2, 0, 0, 1
x_1	4, 4, 4, 4, 1	0, 0, 1, 0, 1

x_5, x_4, y_3:

	x_2	y_2
y_1	0, 0, 0, 1, 1	2, 2, 2, 2, 1
x_1	1, 1, 0, 0, 1	0, 0, 0, 1, 1

And since there is no other deviation, the remaining two Nash equilibria turn out to be ONE of this game:

$$(x_1, x_2, y_3, y_4, x_5), (y_1, y_2, y_3, x_4, x_5).$$

If we consider $\mathcal{O}_3 = \{P_0, P_1, P_2\}$, we can see that the coalition $\{1, 2, 3\}$ deviates to (x_1, x_2, x_3) given that Players 4 and 5 stick to (x_4, x_5). Since there is no further deviation, $(y_1, y_2, y_3, x_4, x_5)$ is not coalitionally stable in the sense of ONE. And since there is no other deviation, there exists a unique ONE:

$$(x_1, x_2, y_3, y_4, x_5).$$

Finally, when we consider $\mathcal{O}_4 = \{P_0, P_1, P_2, P_3\}$, the coalition $\{1, 2, 3, 4\}$ deviates to (x_1, x_2, x_3, x_4) given that Player 5 sticks to x_5. Since there is no further deviation, the set of ONE turns out to be empty.

It is also worth mentioning here that, considering all of the deviations above, the five-player normal form game has neither a SNE nor a CPNE.

The following observation also follows from Proposition 4.2.

Corollary 4.2. *Consider a normal form game* Γ *that possesses a Nash equilibrium. Take any organization* $\mathcal{O}$ *and consider the set of all organizations* $\mathcal{O}'$ *such that* $\mathcal{O}$ *is greater than or equal to* $\mathcal{O}'$*. In this set there exists a maximum organization for which the set of ONE is non-empty.*

In some normal form games, there may exist a "too big" organization which will fail to take an action (or, reach an equilibrium). In such cases, it might help to dissolve all of the coalitions in the final layer of the organization and play the game as a "smaller" organization. The corollary above indicates that the dissolving process stops at a unique point for which the normal form game possesses an ONE.

5 Concluding Remarks

We have studied cases in which some coalitions is not/cannot be formed. Taking the organizational structures as given, we have introduced a refinement of Nash equilibrium. We have showed the existence of equilibria in certain classes of games. Moreover, through remarks and examples, we have further analyzed how the set of Nash equilibria is refined by our notion.

Organizational refinements can lead to many interesting and fruitful questions. First, one can study the robustness of equilibrium. More precisely, some of Nash equilibria may remain to be an equilibrium for any given organization, whereas some of them may fail to be an equilibrium as soon as some organization is formed. Then one can consider the former to be the most robust Nash equilibrium, whereas the latter to be the least robust. In that sense, any two Nash equilibria can be compared in terms of robustness to organizational deviations. Such an analysis may also provide general insights for certain classes of games.

Second, one can study the endogenous formation of organizations. There can be several methods for this exercise. Either (i) players may have pre-defined preferences over the set of coalitions/organizations which *somehow* induce organizational structures; or (ii) as the set of equilibria is now known for any given organization, players may strategically form coalitions/organizations under the condition that each player would prefer the organizational structure that would yield *the best* set of equilibria. As an example, recall the game given in Table 4: Either of the three Nash equilibria can be captured by a certain organization. Among the three Nash equilibria, (x_1, z_2, y_3) is Pareto optimal for the coalition $\{1, 2\}$. And Players 1 and 2 are able to reach there by forming the two-player coalition, thereby blocking the formation of $\{1, 3\}$ and $\{2, 3\}$.

Third, one can analyze policy implications. Notice that the formation of coalitional/organizational structures does not have to be strategic (as described above). For instance, a *social planner* may be interested in forming a socially optimal organization. As an example, recall the game given in Table 1: Since there exist such an organization for which the unique ONE strictly Pareto dominates the unique CPNE, a social planner would prefer to forbid the formation of $\{1, 2\}$.

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Appendix

We first note that for a player $i \in N$, a strategy $x_i \in X_i$ is *strictly dominated* if there exists another strategy $x'_i \in X_i$ such that for every $x_{-i} \in X_{-i}$:

$$u_i(x'_i, x_{-i}) > u_i(x_i, x_{-i}).$$

The following lemma is used in the proof of Proposition 4.1.

Lemma 5.1. *In a normal form game Γ, assume that x^* is a Nash equilibrium which is the largest serially undominated strategy profile. Then for any coalition $S \subset N$, x^*_S turns out to be the largest serially undominated strategy profile of the reduced game $\Gamma_S|_{x^*_{-S}}$.*

Proof. First note that x^*_S is a Nash equilibrium of the reduced game $\Gamma_S|_{x^*_{-S}}$. Therefore, it is a serially undominated strategy profile.

We now describe a particular procedure of iterated elimination of strictly dominated strategies: We start with $\Gamma^0 \equiv \Gamma$. At stage 1, only Player 1's dominated strategies are eliminated and we obtain Γ^1. At stage 2, only Player 2's dominated strategies are eliminated and we obtain Γ^2. Following in a similar manner, we reach stage n. At this stage, only Player n's dominated strategies are eliminated and we obtain Γ^n. At stage $n+1$, only Player 1's dominated strategies are eliminated and we obtain Γ^{n+1}. More generally, for any $k \in \mathbb{N} \cup \{0\}$, only Player i's dominated strategies are eliminated in stage $i + kn$. The procedure continues until we reach Γ^∞.

Without loss of generality, assume that Player 1 is a member of S and that $x_1 \in X_1$ is strictly dominated by some $x'_1 \in X_1$. In the reduced game $\Gamma_S|_{x^*_{-S}}$, we have

$$u_i(x'_i, x_{S\setminus\{i\}}, x^*_{-i}) > u_i(x_i, x_{S\setminus\{i\}}, x^*_{-i}).$$

for every $x_{S\setminus\{i\}} \in X_{S\setminus\{i\}}$. Hence, x_i remains to be strictly dominated in the reduced game.

Now, we start with $\Gamma^0_S = \Gamma_S|_{x^*_{-S}}$. At stage 1, only Player 1's dominated strategies in Γ^0 are eliminated and we obtain Γ^1_S. Note that Player 1 may have additional dominated strategies in the reduced game; but we do not eliminate those at this stage. Also notice that Γ^1_S is a reduced game of Γ^1. At stage 2, if Player 2 is a member of S, then only Player 2's dominated strategies in Γ^1 are eliminated and we obtain Γ^2_S. Otherwise, we set $\Gamma^2_S = \Gamma^1_S$. In either case, Γ^2_S is a reduced game of Γ^2. Following in a similar manner, we eventually obtain a reduced game of Γ^∞ above.

Given a player $i \in S$, we know that each strategy $y_i \not\leq x^*_i$ is eliminated in some stage of the iterated elimination of strictly dominated strategies for Γ. Following the procedure above, the same should happen in the corresponding stage of the iterated elimination for $\Gamma_S|_{x^*_{-S}}$. Accordingly, x^*_S turns out to be the largest serially undominated strategy profile. □