School Choice: Nash Implementation of Stable Matchings through Rank-Priority Mechanisms*

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Abstract

We consider school choice problems (Abdulkadiroğlu and Sönmez, 2003) where students are assigned to public schools through a centralized assignment mechanism. We study the family of so-called rank-priority mechanisms, each of which is induced by an order of rank-priority pairs. Following the corresponding order of pairs, at each step a rank-priority mechanism considers a rank-priority pair and matches an available student to an unfilled school if the student and the school rank and prioritize each other in accordance with the rank-priority pair. The Boston or immediate acceptance mechanism is a particular rank-priority mechanism. Our first main result is a characterization of the subfamily of rank-priority mechanisms that Nash implement the set of stable (i.e., fair) matchings (Theorem 1). We show that our characterization also holds for "sub-implementation" and "sup-implementation" (Corollaries 3 and 4). Our second main result is a strong impossibility result: under incomplete information, no rank-priority mechanism implements the set of stable matchings (Theorem 2).

Keywords: school choice; rank-priority mechanisms; stability; Nash implementation.
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1 Introduction

An important application of mechanism design is school choice (Abdulkadiroğlu and Sönmez, 2003).[1] In a *school choice problem* a group of students has to be assigned to a number of schools. Each school has a limited number of seats and a priority ordering over all students. Priority may reflect certain criteria such as walking distance to the school or having a sibling attending the school, etc. Each student (or his parents) has a ranking of the schools and his outside option (e.g., attending a private school or being home-schooled). A solution to a school choice problem is a *matching* that assigns each student to a school or his outside option while respecting the schools' capacities.

In practice, a school choice problem does not occur a single time nor in a single geographical location. Therefore, it is useful to consider *mechanisms*, i.e., systematic rules that associate a matching with each possible school choice problem. Among the mechanisms that are widely used in school choice programs around the world[2] is the Boston mechanism (Abdulkadiroğlu and Sönmez, 2003), aka the immediate acceptance mechanism. Given the students' rankings over schools and the schools' priorities over students, the immediate acceptance mechanism assigns students to schools by sequentially considering the 1^{st} ranked schools, the 2^{nd} ranked schools, etc. More precisely, at each step r, each school accepts (up to its remaining capacity) the highest priority students among those that have ranked it r^{th}.

The immediate acceptance mechanism is a member of the family of so-called *rank-priority mechanisms*. Each rank-priority mechanism is associated with an order of all pairs that consist of a (student's) rank and a (school's) priority. Given the students' rankings over schools and the schools' priorities over students, a rank-priority mechanism assigns step-by-step students to schools following the order of rank-priority pairs.[3] More specifically, at each step a rank-priority pair (r, f) is considered. If a school is ranked r^{th} by some available student and if the student has priority f for the school, then the student is assigned to the school provided that the school still has empty seats.[4] A student remains unassigned if he cannot be assigned to a school at any step.

We are interested in a fairness property for matchings known in the literature as *stability*. A matching is *stable* if it satisfies three conditions. First, *individual rationality*: each student

[1] For instance, Abdulkadiroğlu et al. (2005a) and Abdulkadiroğlu et al. (2005b and 2006) report on the redesign of the public school system in New York City and Boston, respectively. See Pathak (2011) and Abdulkadiroğlu (2013) for surveys on mechanism design in school choice.

[2] See, for instance, Kojima and Ünver (2014).

[3] It can be easily checked that the immediate acceptance mechanism is a rank-priority mechanism based on the order $(1, 1), (1, 2), \cdots, (1, n), (2, 1), \cdots, (2, n), \cdots, (m, 1), \cdots, (m, n)$, where m and n are the number of schools and students, respectively.

[4] At each step, multiple students can be assigned, but at most one student to any school.

should find his assignment acceptable, i.e., at least as good as his outside option. Second, *non-wastefulness*: if a student prefers a school to his assignment, then the school should not have an empty seat. Third, *no justified envy*: if a student prefers some school to his assignment and if the more preferred school has exhausted its capacity, then all seats at that school are occupied by higher priority students. Roth (1991, Section III) studies certain rank-priority mechanisms in the context of the assignment of medical residents to hospitals in different regions of the UK. More specifically, he shows that the UK rank-priority mechanisms are not stable (Roth, 1991, Proposition 4) and that agents have incentives to misrepresent their rankings.[5] Abdulkadiroğlu and Sönmez (2003, Section I.A) observe similar issues in the case of the immediate acceptance mechanism.

The fact that rank-priority mechanisms can be unstable and vulnerable to misrepresentation of rankings is not necessarily an insuperable problem. Indeed, if the students involved have the right incentives, it is possible that strategic interaction leads to a matching that is stable with respect to the true rankings (and priorities). More specifically, considering a complete information environment, we aim to determine which rank-priority mechanisms Nash implement the set of stable matchings.[6] In other words, which rank-priority mechanisms induce a game for which the set of Nash equilibrium outcomes coincides with the set of stable matchings? A partial answer to this question is obtained from Ergin and Sönmez (2006, Theorem 4): they show that all "monotonic" rank-priority mechanisms, and hence in particular the immediate acceptance mechanism, implement the set of stable matchings. Our first main result (Theorem 1) gives a complete answer: we characterize the family of rank-priority mechanisms that implement the set of stable matchings. Our necessary and sufficient condition is that the "top" of the order of rank-priority pairs be "quasi-monotonic." Loosely speaking, the top of an order satisfies quasi-monotonocity if the next priority appears only after the precedent priority has appeared with a sufficiently small rank.[7] One might suspect that by demanding only "sub-implementation" or "sup-implementation" (rather than "full implementation") one would obtain a larger family of rank-priority mechanisms than the family of quasi-monotonic mechanisms. However, for any non-quasi-monotonic mechanism we exhibit a school choice problem such that the set of equilibrium outcomes

[5] Ehlers (2008) studies strategies that can be used by the students when faced with these rank-priority mechanisms.

[6] The assumption of complete information and the study of Nash equilibria is far from unusual in the school choice literature. Some recent papers that take this approach are Ergin and Sönmez (2006), Pathak and Sönmez (2008), Haeringer and Klijn (2009), Bando (2014), Dur and Morrill (2016), Dur et al. (2016a,b), among others.

[7] We refer to Section 3 for the formal definition and examples of quasi-monotonic rank-priority mechanisms.

is non-empty, the set of stable matchings is a singleton, and yet neither of the two sets is a subset of the other (Proposition 2). So, our result also holds for "sub-implementation" and "sup-implementation": a rank-priority mechanism sub/sup-implements the set of stable matchings if and only if it is quasi-monotonic (Corollary 3/Corollary 4).

A natural question is whether our result still holds when the assumption of complete information is relaxed. Ergin and Sönmez (2006, Section 8) consider an incomplete information environment where students do know the priorities and the capacities of the schools but not the realizations of the other students' types. They show that the immediate acceptance mechanism may induce Bayesian Nash equilibria with unstable matchings in its support. Our second main result (Theorem 2) is a strong impossibility result: all rank-priority mechanisms exhibit the same feature as the immediate acceptance mechanism.

The remainder of the paper is organized as follows. In Section 2, we describe the school choice problem and rank-priority mechanisms. In Sections 3 and 4, we present our results for complete and incomplete information settings, respectively. Section 5 concludes.

2 Model

Let $\boldsymbol{I} = \{i_1, \ldots, i_n\}$ be the set of **students** and $\boldsymbol{S} = \{s_1, \ldots, s_m\}$ be the set of **schools**. We assume that $n \geq 2$ and $m \geq 1$. The sets I and S are kept fixed throughout.

Each student $i \in I$ has a complete, transitive, and strict **preference relation** $\boldsymbol{P_i}$ over the schools and "being unmatched" (e.g., attending a private school or being home-schooled), which is denoted by $\emptyset$. For each pair $s, s' \in S \cup \{\emptyset\}$, we write $s\, P_i\, s'$ if i prefers s to s', and $s\, R_i\, s'$ if i finds s as desirable as s', i.e., $s\, P_i\, s'$ or $s = s'$. A school $s \in S$ is acceptable (for P_i) if $s\, P_i\, \emptyset$. Given that only acceptable schools will be relevant, we often write a preference relation as an ordered list of acceptable schools (and $\emptyset$ to indicate the end of the list). Preference relation P_i can also be encoded through a function $r_i : S \to \{1, \ldots, m, \infty\}$ by setting $r_i(s) \equiv k$ if s is the k^{th} highest ranked acceptable school for P_i. (So, if $r_i(s) = 1$ then s is student i's most preferred acceptable school.) Otherwise, $r_i(s) \equiv \infty$. We refer to $r_i(s)$ as the **rank** of s in P_i. We will use P_i and r_i interchangeably. Let $P \equiv (P_i)_{i \in I}$ be the preference profile. For each $i \in I$, $P_{-i} \equiv (P_j)_{j \neq i}$.

Each school $s \in S$ has a **capacity** $\boldsymbol{q_s} \geq 1$ which is the (integer) number of seats it offers. Let $q = (q_{s_1}, \ldots, q_{s_m})$ be the capacity vector. Each school $s \in S$ has a complete, transitive, and strict **priority relation** $\boldsymbol{\succ_s}$ over the students. For each pair $i, i' \in I$, we write $i \succ_s i'$ if i has higher priority than i' for s. A priority relation can also be encoded through a function $f_s : I \to \{1, \ldots, n\}$ by setting $f_s(i) \equiv k$ if i is the k^{th} highest priority student for school s. (So, a small value of $f_s(\cdot)$ indicates a high priority for school s. E.g., if $f_s(i) = 1$ then i has the highest priority for s.) The integer $f_s(i)$ is the **priority** of i for s. We will use $\succ_s$ and f_s interchangeably. Let $\succ \equiv (\succ_s)_{s\in S}$ be the profile of priority relations.

A **problem** is a list $\boldsymbol{(P, \succ, q)}$ or, when no confusion is possible, $\boldsymbol{P}$ for short. Let $\mathcal{P}$ be the class of all problems.

A **matching** $\boldsymbol{\mu}$ for problem $P \in \mathcal{P}$ is a function $\mu : I \cup S \to 2^I \cup S$ such that (1) each student is assigned to one school or is unassigned, i.e., for each $i \in I$, $\mu(i) \in S \cup \{\emptyset\}$; (2) each school is assigned to a set of students that does not exceed its capacity, i.e., for each $s \in S$, $\mu(s) \in 2^I$ and $|\mu(s)| \leq q_s$; and (3) assignments are consistent, i.e., for each $i \in I$ and $s \in S$, $\mu(i) = s$ if and only if $i \in \mu(s)$. We call $\mu(i)$ the match of student i and if $\mu(i) = s \in S$, we say that student i is assigned to school s. Let $\boldsymbol{\mathcal{M}(P)}$ denote the **set of matchings** for problem $P \in \mathcal{P}$.

Next, we describe desirable properties of matchings. First, we are interested in a voluntary participation condition. A matching μ is **individually rational** for problem P if for each $i \in I$, $\mu(i)\ R_i\ i$. Second, a matching is non-wasteful if no student prefers a school with some empty seat to his match. Formally, a matching μ is **non-wasteful** for problem P if there is no student i and a school s such that $s\, P_i\, \mu(i)$ and $|\mu(s)| < q_s$. Finally, a student i is said to have justified envy if there is a school s such that i prefers s to his match, and i has higher priority at s than some student assigned to s. Formally, a student i has **justified envy** at μ for problem P if there is a school s and a student $j \in \mu(s)$ such that $s\, P_i\, \mu(i)$ and $i \succ_s j$. A matching μ is **stable** for P if it is *individually rational*, *non-wasteful*, and no student has *justified envy* for P. Let $\boldsymbol{\mathcal{S}(P)}$ denote the **set of stable matchings** for problem $P \in \mathcal{P}$. From Gale and Shapley (1962) it follows that for each $P \in \mathcal{P}$, $\mathcal{S}(P) \neq \emptyset$.

A mechanism φ is a function that selects for each problem a matching, i.e., for each $P \in \mathcal{P}$, $\varphi(P) \in \mathcal{M}(P)$. In this paper we focus on the family of rank-priority mechanisms which are defined next. Let $\pi : \{1, \ldots, m\} \times \{1, \ldots, n\} \to \{1, \ldots, m{\cdot}n\}$ be a bijection. Each element $\boldsymbol{(r, f)} \in \{1, \ldots, m\} \times \{1, \ldots, n\}$ is interpreted as a **rank-priority pair**, i.e., r is a rank and f is a priority. We often equivalently denote π by its induced order of rank-priority pairs, i.e., $(r^1, f^1), (r^2, f^2), \ldots, (r^{m\cdot n}, f^{m\cdot n})$ where for all k, $\pi(r^k, f^k) = k$. Thus, we will refer to $\boldsymbol{\pi}$ as an **order of rank-priority pairs**. Then, the **rank-priority mechanism** $\boldsymbol{\varphi^\pi}$ is defined as follows. Let Q be a profile of students' preferences. Set $\tilde{I} \equiv I$. For each $s \in S$, set $\tilde{q}_s \equiv q_s$. Matching $\varphi^\pi(Q)$ is obtained in $m{\cdot}n$ steps:

Step $k = 1, \ldots, m{\cdot}n$: As long as there are $i \in \tilde{I}$ and $s \in S$ such that

(c1) s has rank r^k in Q_i,

(c2) i has priority f^k for s, and

(c3) s still has some empty seat, i.e., $\tilde{q}_s > 0$,

assign student i to school s and set $\tilde{q}_s \equiv \tilde{q}_s - 1$ and $\tilde{I} \equiv \tilde{I} \backslash i$.

After step $m \cdot n$, the students in $\tilde{I}$ remain unmatched. Let $\varphi^{\pi}(Q)$ denote the thus induced matching. Note that at each step of the algorithm multiple students can be assigned (but at most one to each school). Let $\boldsymbol{\mathcal{F}}$ denote the **family of rank-priority mechanisms**.

Example 1. [A rank-priority mechanism]
Consider the school choice problem with $I = \{i_1, i_2, i_3, i_4\}$, $S = \{s_1, s_2, s_3\}$, $q = (2, 1, 1)$, and preferences P and priorities $\succ$ as given in Table 1. In each student's column, higher placed schools are more preferred (and unacceptable schools are omitted). In each school's column, higher placed students have higher priority. For instance, $f_{s_1}(i_2) = 1$.

Students				Schools		
P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
$\boldsymbol{s_1}$	$\boldsymbol{s_3}$	$\boxed{\boldsymbol{s_1}}$	$\boxed{s_3}$	$\boxed{i_2}$	$\boxed{i_1}$	i_1
$\boxed{s_2}$	s_2	s_2	$\boldsymbol{s_2}$	$\boldsymbol{i_1}$	$\boldsymbol{i_4}$	$\boldsymbol{i_2}$
$\emptyset$	$\boxed{s_1}$	$\emptyset$	s_1	$\boxed{\boldsymbol{i_3}}$	i_3	i_3
	$\emptyset$		$\emptyset$	i_4	i_2	$\boxed{i_4}$

Table 1: Preferences P and priorities $\succ$ in Example 1.

Consider the following order of rank-priority pairs π,

$$\pi : (2,1), (3,1), (3,2), (2,2), (1,2), (2,3), (1,3), (1,1), (2,4), (3,4), (3,3), (1,4). \tag{1}$$

To illustrate the algorithm above, we compute $\varphi^{\pi}(P)$. At step $1 = \pi(2,1)$, rank-priority pair $(2,1)$ is considered, i.e., rank 2 in each student's preference relation together with priority 1 in each school's priority relation. School s_2 has rank 2 in the preference relation of student i_1. In addition, student i_1 has priority 1 for school s_2. Moreover, school s_2 has still one empty seat. Hence, conditions (c1), (c2), and (c3) are satisfied for i_1 and s_2, and student i_1 is assigned to school s_2. There is no other student-school pair for which conditions (c1), (c2), and (c3) are met. Next, at step $2 = \pi(3,1)$, rank-priority pair $(3,1)$ is considered and student i_2 is assigned to school s_1. At step $3 = \pi(3,2)$, no student-school pair satisfies conditions (c1), (c2), and (c3), and hence no student is assigned. Similarly, at steps 4, 5, and 6, no student is assigned. At step $7 = \pi(1,3)$, student i_3 is assigned to school s_1. At steps 7–11, no student is assigned. Finally, at step $12 = \pi(1,4)$, student i_4 is assigned to school s_3. Hence, at problem P, the mechanism φ^{π} yields the "boxed" matching in Table 1:

$$\varphi^{\pi}(P) = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ s_2 & s_1 & s_1 & s_3 \end{pmatrix},$$

which is not stable, since the unique stable matching is

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ s_1 & s_3 & s_1 & s_2 \end{pmatrix}, \tag{2}$$

i.e., the boldfaced matching in Table 1. $\diamond$

We assume that priorities are determined by laws and that capacities are commonly known by the students.[8] Hence, students are the only strategic agents. A **strategy** is a preference relation. For each $i \in I$, let $\mathcal{P}_i$ denote the set of strategies. Let $\mathcal{P} \equiv \prod_{i\in I} \mathcal{P}_i$. Given a rank-priority mechanism φ^π, a **game** is a quadruple $\Gamma = (I, (\mathcal{P}_i)_{i\in I}, \varphi^\pi, P)$, or $\boldsymbol{\Gamma = (\varphi^\pi, P)}$ for short, where I is the set of players, $\mathcal{P}_i$ is the set strategies of player $i \in I$, φ^π is the outcome function, and the outcome is evaluated through the (true) preference relations P of the students.

Example 2. [A rank-priority mechanism, Example 1 cont'd]
Consider again the school choice problem and the order of rank-priority pairs π (see (1)) from Example 1. Suppose student i_2 reports the list[9] P'_{i_2} : s_1, s_3, $\emptyset$ instead of his true preference relation P_{i_2}, while the other students submit their true preference relations. At step $1 = \pi(2,1)$, student i_1 is again assigned to school s_2. However, this time no student is assigned at steps $2 = \pi(3,1)$ and $3 = \pi(3,2)$. At step $4 = \pi(2,2)$, student i_2 is assigned to school s_3. Then, it immediately follows that at problem $P' = (P'_{i_2}, P_{-i_2})$, the mechanism φ^π yields the matching

$$\varphi^\pi(P'_{i_2}, P_{-i_2}) = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ s_2 & s_3 & s_1 & s_1 \end{pmatrix}.$$

Since $\varphi^\pi_{i_2}(P'_{i_2}, P_{-i_2}) = s_3 \, P_{i_2} \, s_1 = \varphi^\pi_{i_2}(P)$, student i_2 has incentives to misrepresent his preferences to obtain a more preferred school. $\diamond$

Roth (1991) already observes the problem that Example 2 exhibits: rank-priority mechanisms are vulnerable to manipulation. For this reason, we will study the Nash equilibria of the games induced by rank-priority mechanisms. A strategy-profile $Q \in \mathcal{P}$ is a (Nash) equilibrium of the game (φ^π, P) if for each student i and for each Q'_i, $\varphi^\pi_i(Q_i, Q_{-i}) \, R_i \, \varphi^\pi_i(Q'_i, Q_{-i})$.

[8] In many school choice applications, students are prioritized at each school using some exogenous criteria, e.g., neighborhood or walk-zone priority (see Pathak, 2011 and Abdulkadiroğlu, 2013). Capacities are also often determined by laws. In particular, capacities cannot be manipulated (cf. Sönmez, 1997).

[9] P'_{i_2} says that school s_1 is the most preferred school, school s_3 is the second most preferred school, and school s_2 is not acceptable.

Let $\boldsymbol{\mathcal{E}(\varphi^{\pi}, P)}$ denote the **set of equilibria**. Let $\boldsymbol{\mathcal{O}(\varphi^{\pi}, P)}$ denote the **set of equilibrium outcomes**, i.e.,

$$\mathcal{O}(\varphi^{\pi}, P) = \{\mu \in \mathcal{M}(P) : \mu = \varphi^{\pi}(Q) \text{ and } Q \in \mathcal{E}(\varphi^{\pi}, P)\}.$$

Mechanism φ^{π} **(Nash) implements the set of stable matchings** if for each problem $P \in \mathcal{P}$, $\mathcal{O}(\varphi^{\pi}, P) = \mathcal{S}(P)$. Ergin and Sönmez (2006, Theorem 4) show that if φ^{π} is *monotonic*, then it implements the set of stable matchings. A rank-priority mechanism $\varphi^{\pi} \in \mathcal{F}$ is **monotonic** (Ergin and Sönmez, 2006) if

$$[(r, f) \neq (r', f'), r \leq r', \text{ and } f \leq f'] \implies \pi(r, f) < \pi(r', f'). \tag{3}$$

Since (3) is in fact a condition on π, we will often interchangeably refer to the *monotonicity* of π and φ^{π}. Let $\boldsymbol{\mathcal{F}^m}$ denote the **family of monotonic rank-priority mechanisms**. The **Boston** or **immediate acceptance mechanism** β (Abdulkadiroğlu and Sönmez, 2003) is a particular rank-priority mechanism where π lexicographically orders pairs (r, f), i.e., $\beta = \varphi^{\pi^{ia}}$ where $\pi^{ia} : (1,1), (1,2), \cdots, (1,n), (2,1), \cdots, (2,n), \cdots, (m,1), \cdots, (m,n)$. Note that the immediate acceptance mechanism is *monotonic*, i.e., $\beta \in \mathcal{F}^m$. In the next section we will see that *monotonicity* is not necessary for the implementation of the set of stable matchings.

3 Characterization

In this section, we introduce a weaker monotonicity property and prove that it characterizes the subfamily of rank-priority mechanisms that implement the set of stable matchings.

Let π be an order of rank-priority pairs. For any priority $f \in \{1, \ldots, n\}$, we let $\pi(f)$ denote the first position in the order where priority f appears, i.e.,

$$\boldsymbol{\pi(f)} \equiv \min\{\, \pi(r, f) : r \in \{1, \ldots m\}\}.$$

Rank-priority mechanism φ^{π} is **quasi-monotonic** if for each priority $f \in \{1, \ldots, n-1\}$ there is a rank $r \in \{1, \ldots, m\}$ such that

$$(i)\ \pi(r, f) < \pi(f+1) \qquad \text{and} \qquad (ii)\ [r' < r \text{ and } f' < f] \implies \pi(r', f') \geq \pi(r, f). \tag{4}$$

Loosely speaking, *quasi-monotonocity* is satisfied if before any new priority $(f+1)$ appears (at step $\pi(f+1)$), the precedent priority (f) has already turned up with some rank r ($\pi(r, f) < \pi(f+1)$) such that no pair of strictly smaller rank r' and strictly smaller priority f' precedes it ($[r' < r$ and $f' < f] \Rightarrow \pi(r', f') \geq \pi(r, f)$).[10] Since (4) is a condition on

[10] We will provide some examples of rank-priority mechanisms to illustrate quasi-monotonicity in Example 3.

π, we will interchangeably refer to the *quasi-monotonicity* of π and φ^π. Note that *quasi-monotonicity* in fact only imposes restrictions on the rank-priority pairs that appear in π before position $\pi(n)$, i.e., the position in which priority n appears for the first time. For later convenience, we note that equivalently rank-priority mechanism φ^π is *quasi-monotonic* if for each priority $f \in \{1, \ldots, n-1\}$ and for each priority $f' \in \{1, \ldots, n-2\}$, $f' < f$, there is a rank $r_{f'} \in \{1, \ldots, m\}$ such that

$$(i)\ \pi(r_{f'}, f) < \pi(f+1) \qquad \text{and} \qquad (ii)\ r' < r_{f'} \Longrightarrow \pi(r', f') \geq \pi(r_{f'}, f). \tag{5}$$

Let $\boldsymbol{\mathcal{F}^q}$ denote the **family of quasi-monotonic rank-priority mechanisms**. The following lemma shows that *monotonicity* implies *quasi-monotonicity*.

Lemma 1. Monotonic *rank-priority mechanisms are* quasi-monotonic*, i.e.,* $\mathcal{F}^m \subseteq \mathcal{F}^q$.

Proof. Let φ^π be a *monotonic* rank-priority mechanism. Let $f \in \{1, \ldots, n-1\}$. Let $r = 1$. Then, by *monotonicity*,

$$\pi(r, f) = \pi(1, f) < \pi(1, f+1) = \pi(f+1),$$

which proves (i) in (4). Since there is no $r' < r$, (ii) in (4) is vacuously satisfied. Hence, φ^π is *quasi-monotonic*. □

We say that π (or equivalently, φ^π) satisfies **unit increments of priority (UIP)** if for each priority $f \in \{1, ..., n-1\}$,

$$\pi(f) < \pi(f+1).$$

UIP says that if we go through the order π, whenever a new priority appears it is exactly one unit larger than the maximal priority that we have encountered so far. Let $\boldsymbol{\mathcal{F}^u}$ denote the **family of rank-priority mechanisms that satisfy UIP**. The following result is immediate.

Lemma 2. Quasi-monotonicity *implies* unit increments of priority*, i.e.,* $\mathcal{F}^q \subseteq \mathcal{F}^u$.

Proof. Follows immediately from condition (i) in (4). □

Before we state and prove our main result, we provide some examples of rank-priority mechanisms to illustrate quasi-monotonicity.

Example 3. [Rank-priority mechanisms]
Consider the following orders of rank-priority pairs. A priority is in boldface whenever it appears for the first time in the order.

$$\begin{aligned}
\pi^{ia} \equiv \pi^1 \;&:\; (1,\mathbf{1}),(1,\mathbf{2}),\cdots,(1,\boldsymbol{n}),(2,1),\cdots,(2,n),\cdots,(m,1),\cdots,(m,n).\\
\pi^2 \;&:\; (1,\mathbf{1}),(2,1),\cdots,(m,1),(1,\mathbf{2}),\cdots,(m,2),\cdots,(1,\boldsymbol{n}),\cdots,(m,n).\\
\pi^3 \;&:\; \pi^3(r,f)<\pi^3(r',f') \iff r\cdot f<r'\cdot f' \text{ or } [r\cdot f=r'\cdot f' \text{ and } r<r'].\\
\pi^4 \;&:\; \pi^4(r,f)<\pi^4(r',f') \iff r\cdot f<r'\cdot f' \text{ or } [r\cdot f=r'\cdot f' \text{ and } f<f'].\\
\pi^5 \;&:\; (2,\mathbf{1}),(3,1),(3,\mathbf{2}),(2,2),(1,2),(2,\mathbf{3}),(1,3),(1,1),(2,\mathbf{4}),\cdots \text{ where } n=4.\\
\pi^6 \;&:\; (3,\mathbf{1}),(3,\mathbf{2}),(3,\mathbf{3}),(3,\mathbf{4}),\cdots \text{ where } n=4.\\
\pi^7 \;&:\; (4,\mathbf{1}),(3,1),(3,\mathbf{2}),(2,2),(1,1),(1,\mathbf{4}),\cdots.\\
\pi^8 \;&:\; (2,\mathbf{1}),(3,1),(3,\mathbf{2}),(1,1),(2,2),(1,\mathbf{3}),\cdots.
\end{aligned}$$

For $k=1,\ldots,4$, mechanism φ^{π^k} is *monotonic*. It is not difficult to check that for $k=5,6$, mechanism φ^{π^k} is *quasi-monotonic*, but not *monotonic*.[11] In the case of $k=5$, condition (4) is satisfied for priority $f=1$ (trivially), for priority $f=2$ (with rank 1 or 2, but not rank 3), and for priority $f=3$ (with rank 1, but not rank 2). Finally, mechanisms φ^{π^7} and φ^{π^8} are not *quasi-monotonic*. For φ^{π^7}, condition (i) in (4) is not satisfied (for $f=3$, $\pi(f)>\pi(f+1)$). For φ^{π^8}, condition (ii) in (4) is not satisfied.[12] To see this, let $f=2$ and note first that for (r,f) to satisfy (i), either $r=2$ or $r=3$. However, if $r=2$ then $(r',f')=(1,1)$ violates (ii), and if $r=3$ then $(r',f')=(2,1)$ violates (ii). $\diamond$

Our main result shows that *quasi-monotonicity* is a necessary and sufficient condition for the Nash implementation of the set of stable matchings.

Theorem 1. [Implementation: characterization]
A rank-priority mechanism $\varphi^\pi \in \mathcal{F}$ Nash implements the set of stable matchings if and only if it is quasi-monotonic, *i.e., $\varphi^\pi \in \mathcal{F}^q$.*

Theorem 1 immediately follows from Propositions 1 and 2, which are stated and proved below. We first provide an example that gives insights into how a *quasi-monotonic* rank-priority mechanism can implement the set of stable matchings (which is formalized in Proposition 1).

[11] So, $\mathcal{F}^q \not\subseteq \mathcal{F}^m$. One particular completion of π^5 is also discussed in Example 1.

[12] It is easy to complete π^8 so that φ^{π^8} satisfies UIP. So, $\mathcal{F}^u \not\subseteq \mathcal{F}^q$.

Example 4. [A rank-priority mechanism, Example 1 cont'd]
Consider again the school choice problem from Example 1 and (the unique) stable matching μ given in (2). Note that the order of rank-priority pairs π in (1) is *quasi-monotonic* (Example 3). We illustrate how μ can be obtained at an equilibrium of the game induced by φ^π. For each priority f with $f < n = 4$, let $r^*(f)$ be the rank that satisfies (4) such that $\pi(r^*(f), f) \leq \pi(r', f)$ for each $r' \in \{1, \ldots, m\}$ that satisfies (4). Then, $r^*(1) = 2$, $r^*(2) = 2$, and $r^*(3) = 1$. By convention we set $r^*(n) = r^*(4) \equiv 1$.

Let $i \in I$. The priority of student i for school $\mu(i)$ is $f_{\mu(i)}(i)$. Let Q_i be a list of exactly $r^*(f_{\mu(i)}(i))$ schools where school $\mu(i)$ is placed in (the last) position $r^*(f_{\mu(i)}(i))$. All other positions (i.e., $1, \ldots, r^*(f_{\mu(i)}(i)) - 1$) can be arbitrarily filled with other schools. In case of student i_1, the construction is as follows. Since at matching μ student i_1 is assigned to $\mu(i_1) = s_1$, it follows that $f_{\mu(i_1)}(i_1) = f_{s_1}(i_1) = 2$. Hence, $r^*(f_{\mu(i_1)}(i_1)) = r^*(2) = 2$. Then, a suitable strategy for student i_1 will be one in which precisely 2 schools are listed and where $\mu(i_1) = s_1$ appears in position 2. The first column in Table 2 gives (an example of) a suitable strategy. Using the same construction, strategies for the other students are obtained– see again Table 2. In particular, i_2 puts s_3 in position 2 of his list Q_{i_2}, i_3 puts s_1 in position 1 of hist list Q_{i_3}, and i_4 puts s_2 in position 2 of his list Q_{i_4}.

Q_{i_1}	Q_{i_2}	Q_{i_3}	Q_{i_4}
s_2	s_1	$\boldsymbol{s_1}$	s_3
$\boldsymbol{s_1}$	$\boldsymbol{s_3}$	$\emptyset$	$\boldsymbol{s_2}$
$\emptyset$	$\emptyset$		$\emptyset$

Table 2: Equilibrium Q in Example 4.

Next, we verify that $\varphi^\pi(Q) = \mu$. At steps 1, 2, and 3 no student is assigned. At step $4 = \pi(2, 2) = \pi(r^*(f_{\mu(i)}(i)), f_{\mu(i)}(i))$, each student $i \in \{i_1, i_2, i_4\}$ is assigned to school $\mu(i)$: students i_1, i_2, and i_4 obtain a seat at s_1, s_3, and s_2, respectively. Finally, at step $7 = \pi(1, 3)$ $= \pi(r^*(f_{\mu(i_3)}(i_3)), f_{\mu(i_3)}(i_3))$, student i_3 is assigned to school $s_1 = \mu(i_3)$. So, $\varphi^\pi(Q) = \mu$.

For a further clarification of the role of *quasi-monotonicity*, we informally explain why at Q each student i is *not* assigned to a school different from $\mu(i)$ *provided that at step* $\pi(r^*(f_{\mu(i)}(i)), f_{\mu(i)}(i))$ *school* $\mu(i)$ *still has some empty seat.*[13]

- First, a student i is not assigned to another school for which i has the same priority as for $\mu(i)$. For instance, student i_3 has priority 3 for all three schools. However, since i_3 put s_1 in the first position of Q_{i_3} and since the list Q_{i_3} consists of only 1 school,

[13] For a formal statement and proof we refer to Lemma 3.

student i_3 is assigned (to s_1) at step $7 = \pi(1,3) = \pi(r^*(3), 3)$. In particular, i_3 is not assigned to s_2 nor s_3.

- Second, a student i is not assigned to a school for which i has a priority that is larger (i.e., worse) than his priority for $\mu(i)$. For instance, student i_2 has a larger priority for school s_2 (priority 4) than for school s_3 (priority 2). By condition (i) in (4), the pair $(r^*(2), 2)$ appears in π before the first pair that contains priority 3, which in turn appears before the first pair that contains priority 4. Hence, i_2 is not assigned to s_2.
- Third, a student i is not assigned to a school for which i has a priority that is smaller (i.e., better) than his priority for $\mu(i)$. For instance, student i_2 has a smaller priority for school s_1 (priority 1) than for school s_3 (priority 2). By condition (ii) in (4), any pair that consists of a smaller rank than $r^*(2)$ and a smaller priority than 2 appears after the pair $(r^*(2), 2)$. (We do not have to worry about ranks that are larger than $r^*(2)$, because the list Q_{i_2} consists of exactly $r^*(2)$ schools.) Hence, i_2 is not assigned to s_1.

Finally, it remains to show that Q is an equilibrium. Since students i_1, i_2, and i_3 are assigned to their most preferred schools, they do not have a profitable deviation. So, we only need to see that i_4 cannot deviate and obtain a seat at s_3. Since μ is stable and i_4 prefers s_3 to $\mu(i_4)$, student i_4 has a larger (i.e., worse) priority (namely, priority 4) than student i_2 who occupies the only seat at s_3 and who has priority 2. Since student i_2 is assigned to s_3 at step $4 = \pi(r^*(2), 2)$ and $\pi(r^*(2), 2) < \pi(r, 4)$ for each $r \in \{1, \ldots, m\}$ (by *quasi-monotonicity*), no deviation by i_4 will give him a seat at s_3. Hence, i_4 does not have a profitable deviation and Q is an equilibrium. $\diamond$

The following lemma formalizes the observations from Example 4 and will be key in the proof of Proposition 1.

Lemma 3. *Let π be* quasi-monotonic. *Let $f^* \in \{1, \ldots, n\}$. If $f^* = n$, let $r^* \equiv 1$. Otherwise, let $r^* \in \{1, \ldots, m\}$ satisfy (4) for $f = f^*$ such that $\pi(r^*, f^*) \leq \pi(r', f^*)$ for each $r' \in \{1, \ldots, m\}$ that satisfies (4) for $f = f^*$.*

Let $i^* \in I$ and $s^* \in S$. Let $\succ$ be a profile of priority relations such that student i^* has priority f^* for school s^*. Consider strategy

$$Q^*_{i^*} \equiv \cdots, \underbrace{s^*}_{\textit{at rank } r^*}, \emptyset. \tag{6}$$

Apply the rank-priority algorithm of φ^π to $Q^* = (Q^*_{i^*}, Q_{-i^*})$, where Q_{-i^*} is any strategy-profile of the other students. Then, student i^* remains unassigned until the end of step $\pi(r^*, f^*) - 1$ (and hence is assigned to school s^* at step $\pi(r^*, f^*)$ if at that point the school still has an empty seat).

Proof. If $f^* = n$, then the list $Q^*_{i^*}$ only contains school s^*, and hence student i^* is not assigned at any step different from step $\pi(1, n) = \pi(r^*, f^*)$.

Suppose $f^* \neq n$. Consider any step k with $1 \leq k \leq \pi(r^*, f^*) - 1$. We show that student i^* is *not* assigned to a school at step k. Let $(r', f') \equiv \pi^{-1}(k)$. Since r^* satisfies (4) for $f = f^*$, $\pi(r^*, f^*) < \pi(f^* + 1)$. Then, $\pi(r', f') = k < \pi(r^*, f^*) < \pi(f^* + 1)$. Then, by Lemma 2, $f' < f^* + 1$.

Claim. $r' \geq r^*$

Proof of Claim. Suppose $f' < f^*$. Then, by *quasi-monotonicity*, $r' \geq r^*$. Now suppose $f' = f^*$. Assume $r' < r^*$. Then, r' also satisfies (4) for $f = f^*$. But then $\pi(r', f^*) = \pi(r', f') < \pi(r^*, f^*)$ yields a contradiction with the choice (definition) of r^*. So, $r' \geq r^*$. $\square$

Since strategy $Q^*_{i^*}$ (a) does not consist of more than r^* schools and (b) lists school s^* (for which i^* has priority f^*) at rank r^*, i^* is not assigned to a school at step $k = \pi(r', f') < \pi(r^*, f^*)$. $\square$

We can now state and prove the propositions that imply Theorem 1.

Proposition 1. [Quasi-monotonic mechanisms: implementation]
If a rank-priority mechanism is quasi-monotonic*, then it Nash implements the set of stable matchings.*

Proof. Let φ^π be *quasi-monotonic*. It is convenient to first introduce some more notation. For any priority $f \in \{1, \ldots, n-1\}$, let $r^*(f) \in \{1, \ldots, m\}$ be the rank that satisfies (4) such that $\pi(r^*(f), f) \leq \pi(r, f)$ for each $r \in \{1, \ldots, m\}$ that satisfies (4). By convention we set $r^*(n) \equiv 1$. We show that φ^π implements the set of stable matchings, i.e., for each problem $P \in \mathcal{P}$, $\mathcal{O}(\varphi^\pi, P) = \mathcal{S}(P)$. Let $P \in \mathcal{P}$.

We first prove the inclusion $\mathcal{S}(P) \subseteq \mathcal{O}(\varphi^\pi, P)$. Let $\mu \in \mathcal{S}(P)$. For each $i \in I$ with $\mu(i) \neq \emptyset$, let $f(i) \equiv f_{\mu(i)}(i)$. For each $i \in I$, define a strategy

$$Q_i \equiv \begin{cases} \emptyset & \text{if } \mu(i) = \emptyset; \\ \cdots, \underbrace{\mu(i)}_{\text{at rank } r^*(f(i))}, \emptyset & \text{if } \mu(i) \neq \emptyset, \end{cases}$$

where $\cdots$ is a(ny) list of $r^*(f(i)) - 1 < m$ different schools in $S \backslash \{\mu(i)\}$.

Obviously, for each $i \in I$ with $\mu(i) = \emptyset$, $\varphi^\pi_i(Q) = \emptyset = \mu(i)$. Now let $i \in I$ with $\mu(i) \neq \emptyset$. From Lemma 3 it follows that student i is not assigned until step $\pi(r^*(f(i)), f(i))$. Then, since for each $s \in S$, $|\mu(s)| \leq q_s$, it follows that each student $i \in I$ with $\mu(i) \neq \emptyset$ is assigned to $\mu(i)$ at step $\pi(r^*(f(i)), f(i))$. Hence, $\varphi^\pi(Q) = \mu$.

Next, we show that Q is an equilibrium. Suppose that some student $i \in I$ has a deviation Q'_i such that $\varphi^{\pi}_i(Q') = s \; P_i \; \mu(i)$ where $Q' \equiv (Q'_i, Q_{-i})$. Then, under Q', student i is assigned to school $s \in S$ at a step $\pi(r, f_s(i)) \geq \pi(f_s(i))$ for some $r \in \{1, \ldots, m\}$.

Since $\mu \in \mathcal{S}(P)$ and $s \; P_i \; \mu(i)$, (a) $|\mu(s)| = q_s$ and (b) for each $j \in \mu(s)$, $f(j) = f_s(j) < f_s(i)$ (so, in particular, $f(j) \neq n$). In view of (a), let $j \in \mu(s)$ such that $\varphi^{\pi}_j(Q') \neq s$. Since $f(j) \neq n$, it follows from the definition of $r^*(f(j))$ that $\pi(r^*(f(j)), f(j)) < \pi(f(j)+1)$. From (b), $f(j) + 1 \leq f_s(i)$. Hence from UIP, $\pi(f(j) + 1) \leq \pi(f_s(i))$. Hence, $\pi(r^*(f(j)), f(j)) < \pi(f_s(i))$. So, $\pi(r^*(f(j)), f(j)) < \pi(r, f_s(i))$. Then, since under Q' student i is assigned to school s at step $\pi(r, f_s(i))$, there is still an empty seat at s at step $\pi(r^*(f(j)), f(j))$. By Lemma 3, under Q', student j is assigned to s which contradicts $\varphi^{\pi}_j(Q') \neq s$. Hence, there is no profitable deviation for any student. So, $Q \in \mathcal{E}(\varphi^{\pi}, P)$. Hence, $\mu \in \mathcal{O}(\varphi^{\pi}, P)$. So, $\mathcal{S}(P) \subseteq \mathcal{O}(\varphi^{\pi}, P)$.

Finally, we prove the inclusion $\mathcal{O}(\varphi^{\pi}, P) \subseteq \mathcal{S}(P)$. Let $Q \in \mathcal{E}(\varphi^{\pi}, P)$ and $\mu = \varphi^{\pi}(Q)$. Suppose $\mu \notin \mathcal{S}(P)$. We distinguish between two cases.

Case 1: There is $i^* \in I$ with $\emptyset \, P_{i^*} \, \mu(i^*)$.
Let student i^* report $Q'_{i^*} = \emptyset$. Then, for $Q' \equiv (Q'_{i^*}, Q_{-i^*})$, $\varphi^{\pi}_{i^*}(Q') = \emptyset$. Hence, Q'_{i^*} is a profitable deviation.

Case 2a: There are $i^* \in I$ and $s^* \in S$ with $s^* \, P_i \, \mu(i)$ and $|\mu(s^*)| < q_{s^*}$.
Case 2b: There are $i^*, j^* \in I$ and $s^* \in S$ with $s^* \, P_i \, \mu(i)$, $j^* \in \mu(s^*)$, and $f_{s^*}(i^*) < f_{s^*}(j^*)$.
Let $f^* \equiv f_{s^*}(i^*)$, $r^* \equiv r^*(f^*)$, $k^* \equiv \pi(r^*, f^*)$, and

$$Q^*_{i^*} \equiv \cdots, \underbrace{s^*}_{\text{at rank } r^*}, \emptyset,$$

where $\cdots$ is a(ny) list of $r^* - 1 < m$ different schools in $S \backslash \{s^*\}$. Using the following claim we will prove that $Q^*_{i^*}$ is a profitable deviation for student i^*.

Claim. *Consider the rank-priority algorithm of φ^{π} for Q and $Q^* \equiv (Q^*_{i^*}, Q_{-i^*})$. Then, at the* beginning *of each step k, $1 \leq k \leq k^*$,*
(1.) *For each student i with $i \neq i^*$, if i is already assigned under Q, then he is also already assigned under Q^*.*
(2.) *For each school s, there are at least as many unassigned seats under Q^* as under Q.*

Proof of Claim. We prove the Claim by induction. Since the rank-priority algorithm starts with each student unassigned, the Claim holds for $k = 1$. Suppose the Claim holds for some step k, $1 \leq k < k^*$. Let $(r, f) \equiv \pi^{-1}(k)$. We will show that it also holds for step $k + 1$.
(1.) Let i, $i \neq i^*$, be a student who is already assigned at the beginning of step $k + 1$ under Q. If i got assigned to a school at some step l with $l < k$ under Q, then, by part 1 of the induction assumption, he is also already assigned at some step $l + 1 < k + 1$ under Q^*.

Now assume that i got assigned to a school, say $\bar{s}$, at step k under Q. Hence, student i has priority f for school $\bar{s}$ and student i's strategy Q_i lists $\bar{s}$ at rank r. We will prove that i is assigned to a school by the end of step k under Q^*. School $\bar{s}$ has at least one empty seat at the beginning of step k under Q. From part 2 of the induction assumption it follows that school $\bar{s}$ has at least one empty seat at the beginning of step k under Q^* as well. Suppose i is still unassigned at the beginning of step k under Q^*. Note that the rank-priority algorithm for Q^* considers at step k (student, school) pairs such that the student has priority f for the school and the student lists the school at rank r in Q^*. Since $i \neq i^*$, $Q^*_i = Q_i$, and hence student i is assigned to $\bar{s}$ at step k under Q^*. Hence, i is assigned to a school by the end of step k under Q^*.

(2.) Let $s \in S$. From part 2 of the induction assumption it follows that it is sufficient to show that if at step k under Q^* a student gets assigned to s, then at step k under Q the student also gets assigned to s or there are no seats left at s.

Let i be a student who gets assigned to s at step k under Q^*. Hence, student i has priority f for school s and student i's strategy Q^*_i lists s at rank r. Recall $k < k^* = \pi(r^*, f^*)$. From Lemma 3 it follows that under Q^* student i^* is not assigned until step $\pi(r^*, f^*)$. Hence, $i \neq i^*$. By assumption, i is still unassigned at the beginning of step k under Q^*. Then, from part 1 of the induction assumption it follows that i is still unassigned at the beginning of step k under Q as well. Then, since $i \neq i^*$, $Q_i = Q^*_i$, and hence student i is assigned to s at step k under Q if at that point s still has an empty seat. □

We complete the proof by showing that Q^*_i is a profitable deviation in both Case 2a and Case 2b. We first show that in both cases school s^* has at least one empty seat at the beginning of step $\pi(r^*, f^*)$ under Q.

In Case 2a, school s^* has at least one empty seat after applying the rank-priority algorithm to Q. Hence, school s^* has at least one empty seat at the beginning of step $\pi(r^*, f^*)$ under Q.

In Case 2b, student j^* is assigned to school s^* at a step $\pi(r', f_{s^*}(j^*))$ (where $r' \in \{1, \ldots, m\}$) under Q. Hence, school s^* has at least one empty seat at the beginning of step $\pi(r', f_{s^*}(j^*))$ under Q. Since $f^* = f_{s^*}(i^*) < f_{s^*}(j^*)$, it follows from the choice of r^* that $\pi(r^*, f^*) < \pi(r', f_{s^*}(j^*))$. Hence, school s^* has at least one empty seat at the beginning of step $\pi(r^*, f^*)$ under Q.

By part 2 of the Claim, school s^* has at least one empty seat at the beginning of step $k^* = \pi(r^*, f^*)$ under Q^* as well. Hence, by Lemma 3, i^* is assigned to s^* at step $\pi(r^*, f^*)$ under Q^*. So, $\varphi^\pi_{i^*}(Q^*) = s^*$. Hence, Q^*_i is a profitable deviation, which contradicts $Q \in \mathcal{E}(\varphi^\pi, P)$. Hence, $\mu \in \mathcal{S}(P)$. So, $\mathcal{O}(\varphi^\pi, P) \subseteq \mathcal{S}(P)$. □

Since any *monotonic* rank-priority mechanism is *quasi-monotonic* (Lemma 1) and the immediate acceptance mechanism is a *monotonic* rank-priority mechanism, we immediately obtain the following two corollaries to Proposition 1.

Corollary 1. [Ergin and Sönmez, 2006, Theorem 4]
Each monotonic *rank-priority mechanism Nash implements the set of stable matchings.*

Corollary 2. [Ergin and Sönmez, 2006, Theorem 1]
The immediate acceptance mechanism Nash implements the set of stable matchings.

In Section 5 we also show that Proposition 1 and its proof imply Theorem 2 in Dur et al. (2016b): any mechanism in the class considered by Dur et al. (2016b) Nash implements the set of stable matchings.

Next, we show that non-quasi-monotonic mechanisms do not Nash implement the set of stable matchings. In fact, we prove a stronger result: for any non-quasi-monotonic mechanism we construct a school choice problem for which (a) the unique stable matching cannot be obtained as an equilibrium outcome *and* (b) some equilibrium outcome is not stable. Propositions 1 and 2 prove Theorem 1.

Proposition 2. [Non-quasi-monotonic mechanisms: no implementation]
Let π *violate* quasi-monotonicity*. Then, there is a problem* P *with* $\mathcal{O}(\varphi^\pi, P) \neq \emptyset$*,* $|\mathcal{S}(P)| = 1$*,* $\mathcal{S}(P) \not\subseteq \mathcal{O}(\varphi^\pi, P)$*, and* $\mathcal{O}(\varphi^\pi, P) \not\subseteq \mathcal{S}(P)$*. In particular,* φ^π *does not Nash implement the set of stable matchings.*

Proof. It is convenient to first introduce some more notation. For any priority $f \in \{1, \ldots, n\}$, we let $r(f)$ denote the rank such that $(r(f), f)$ is the first pair in π in which priority f appears. In other words, $r(f) \in \{1, \ldots, m\}$ is such that for each $r \in \{1, \ldots, m\}$, $\pi(f) = \pi(r(f), f) \leq \pi(r, f)$.

In view of Lemma 2 it is sufficient to distinguish between the following two cases.

Case 1: φ^π violates UIP, i.e., $\varphi^\pi \notin \mathcal{F}^u$.
Then, there is a *smallest* priority $f \in \{1, \ldots, n-1\}$ with $\pi(f+1) < \pi(f)$. Thus, π takes the following form:

$$\pi: \quad \underbrace{\cdots, \cdots, \cdots, \cdots, \cdots}_{\substack{\text{only priorities}\\ 1,2,\ldots,f-1 \text{ appear}}}, \quad \underbrace{(r(f+1), f+1)}_{\substack{\text{first time}\\ \text{priority } f+1 \text{ appears}}}, \quad \cdots, \quad \underbrace{(r(f), f)}_{\substack{\text{first time}\\ \text{priority } f \text{ appears}}} \quad , \cdots. \tag{7}$$

Consider the school choice problem $(P, \succ, q)$ where preferences over schools P and priorities over students $\succ$ are given by the columns[14] in Table 3. Each school $s \in S$ has capacity

[14]So, all students find only s_1 acceptable, and all schools $s \in S$ have the same priority relation $i_1 \succ_s i_2 \succ_s \ldots \succ_s i_n$.

$q_s = f$. One easily verifies that $\mathcal{S}(P) = \{\mu\}$ where the unique stable matching μ is such that for each $k = 1, \ldots, f$, $\mu(i_k) = s_1$ and for each $k = f+1, \ldots, n$, $\mu(i_k) = \emptyset$.

Students' preferences	Schools' priorities
P_I	$\succ_S$
s_1	i_1
$\emptyset$	i_2
	$\vdots$
	i_n

Table 3: School choice problem in Case 1.

We first show $\mathcal{S}(P) \nsubseteq \mathcal{O}(\varphi^\pi, P)$. Suppose there is an equilibrium $Q \in \mathcal{E}(\varphi^\pi, P)$ such that $\varphi^\pi(Q) = \mu$. Since $\mu(i_f) = s_1$ and student i_f has priority f for school s_1, it follows from (7) that student i_f is assigned to school s_1 after step $\pi(r(f+1), f+1)$ under Q. Then, since $\varphi^\pi(Q) = \mu$, school s_1 has at least 1 empty seat at the beginning of step $\pi(r(f+1), f+1)$ under Q.

Consider any strategy of the form

$$Q'_{i_{f+1}} \equiv \cdots, \underbrace{s_1}_{\text{at rank } r(f+1)}, \emptyset,$$

for student i_{f+1}. Let $Q' \equiv (Q'_{i_{f+1}}, Q_{-i_{f+1}})$. Since student i_{f+1} has priority $f+1$ for all schools, i_{f+1} is not assigned to any school before step $\pi(r(f+1), f+1)$ under Q'. Since school s_1 has at least 1 empty seat at the beginning of step $\pi(r(f+1), f+1)$ under Q, it follows that school s_1 has at least 1 empty seat at the beginning of step $\pi(r(f+1), f+1)$ under Q' as well. Hence, student i_{f+1} is assigned to school s_1 at step $\pi(r(f+1), f+1)$ under Q'. Hence, $Q'_{i_{f+1}}$ is a profitable deviation for student i_{f+1}, contradicting $Q \in \mathcal{E}(\varphi^\pi, P)$. Hence, $\mu \in \mathcal{S}(P) \backslash \mathcal{O}(\varphi^\pi, P)$. So, $\mathcal{S}(P) \nsubseteq \mathcal{O}(\varphi^\pi, P)$.

Next, we show $\mathcal{O}(\varphi^\pi, P) \nsubseteq \mathcal{S}(P)$. Consider the strategy-profile Q in Table 4. Each student i_k with $k \in \{1, \ldots, f-1, f+1\}$ submits a list where school s_1 appears at rank $r(k)$. All other students submit the empty list.

Let $\mu' \equiv \varphi^\pi(Q)$. From (7) and the fact that each student i_k with $k \in \{1, \ldots, f-1, f+1\}$ has priority k for all schools, it follows that for each $k \in \{1, \ldots, f-1, f+1\}$, student i_k is assigned a seat at school s_1 at step $\pi(r(k), k)$ under Q. Since $\mu' \neq \mu$, μ' is not stable, i.e., $\mu' \notin S(P)$.

	Students' strategies	
	$Q_{i_k}, k \in \{1, \ldots, f-1, f+1\}$	$Q_{i_k}, k \in \{f, f+2, \ldots, n\}$
	$\vdots$	$\emptyset$
	$\vdots$	
$r(k) \rightarrow$	s_1	
	$\vdots$	

Table 4: Strategy-profile in CASE 1.

We show that $Q \in \mathcal{E}(\varphi^\pi, P)$. First, no student i_k with $k \in \{1, \ldots, f-1, f+1\}$ has a profitable deviation (since he gets his most preferred match). Second, no student i_k with $k \in \{f, f+2, \ldots, n\}$ can obtain a seat at his only acceptable school s_1 by means of some deviation Q'_{i_k}. To see this, let $Q' \equiv (Q'_{i_k}, Q_{-i_k})$. Since student i_k has priority k for all schools, i_k is not assigned to any school before step $\pi(r(k), k)$ under Q'. Since $\pi(r(k), k) > \pi(r(f+1), f+1)$ and school s_1 has no more empty seats after step $\pi(r(f+1), f+1)$ under Q, school s_1 has no more empty seats after step $\pi(r(f+1), f+1)$ under Q' either. So, student i_k does not obtain a seat at school s_1 under Q'. Hence, $Q \in \mathcal{E}(\varphi^\pi, P)$. Hence, $\mu' \in \mathcal{O}(\varphi^\pi, P) \backslash \mathcal{S}(P)$. So, $\mathcal{O}(\varphi^\pi, P) \nsubseteq \mathcal{S}(P)$.

CASE 2: φ^π violates *quasi-monotonicity* but satisfies UIP, i.e. $\varphi^\pi \in \mathcal{F}^u \backslash \mathcal{F}^q$.
Since φ^π satisfies UIP, it follows that

$$\text{for each } \bar{f} \in \{1, \ldots, n-1\},\ \pi(r(\bar{f}), \bar{f}) < \pi(r(\bar{f}+1), \bar{f}+1). \tag{8}$$

Since φ^π violates *quasi-monotonicity*, it follows from (5) that there are two priorities $f \in \{1, \ldots, n-1\}$ and $f' \in \{1, \ldots, n-2\}$, $f' < f$, such that for each rank $\tilde{r} \in \{1, \ldots, m\}$ with $\pi(\tilde{r}, f) < \pi(f+1)$,

$$\text{there is a rank } \tilde{r}' < \tilde{r} \text{ with } \pi(\tilde{r}', f') < \pi(\tilde{r}, f). \tag{9}$$

It follows from (8) that there exists a *smallest* rank $r \in \{1, \ldots, m\}$ with $\pi(r, f) < \pi(f+1)$. Then, from (9), there is a rank $r' < r$, which implies that there are at least 2 schools, i.e., $m \geq 2$.

Consider a school choice problem $(P, \succ, q)$ where preferences over schools P and priorities over students $\succ$ are given by the columns in Table 5. School s_1 has capacity $q_{s_1} = 1$. Each school $s \neq s_1$ has capacity $q_s = n$. One easily verifies that $\mathcal{S}(P) = \{\mu\}$ where the unique stable matching μ is such that $\mu(i_1) = s_1$ and for each $i \neq i_1$, $\mu(i) = \emptyset$.

Students' preferences				Schools' priorities	
P_{i_1}	P_{i_2}	$P_{I\setminus\{i_1,i_2\}}$		$\succ_{s_1}$	$\succ_{S\setminus\{s_1\}}$
s_1	s_1	$\emptyset$		$\vdots$	$\vdots$
$\emptyset$	$\emptyset$		$f' \rightarrow$	$\vdots$	i_1
				$\vdots$	$\vdots$
			$f \rightarrow$	i_1	$\vdots$
			$f+1 \rightarrow$	i_2	i_2
				$\vdots$	$\vdots$

Table 5: School choice problem in CASE 2.

We first show $\mathcal{S}(P) \not\subseteq \mathcal{O}(\varphi^\pi, P)$. Suppose there is an equilibrium $Q \in \mathcal{E}(\varphi^\pi, P)$ such that $\varphi^\pi(Q) = \mu$. Then, under Q, student i_1 is assigned to school s_1 at some step $\pi(\bar{r}, f)$ where $\bar{r} \in \{1, \ldots, m\}$. Then, Q_{i_1} lists school s_1 at rank $\bar{r}$. Moreover, $\pi(\bar{r}, f) < \pi(r(f+1), f+1)$. To see this, we can use arguments similar to those in CASE 1. We include the arguments for the sake of clarity and completeness. Suppose that $\pi(\bar{r}, f) > \pi(r(f+1), f+1)$. Then, from $\varphi^\pi(Q) = \mu$ it follows that no student is assigned to s_1 before or at step $\pi(r(f+1), f+1)$ under Q. Consider any strategy of the form

$$Q'_{i_2} \equiv \cdots, \underbrace{s_1}_{\text{at rank } r(f+1)}, \emptyset,$$

for student i_2. Let $Q' \equiv (Q'_{i_2}, Q_{-i_2})$. Since student i_2 has priority $f+1$ for all schools, i_2 is not assigned to any school before step $\pi(r(f+1), f+1)$ under Q'. Since no student is assigned to s_1 before or at step $\pi(r(f+1), f+1)$ under Q, it follows that no student is assigned to s_1 before or at step $\pi(r(f+1), f+1)$ under Q' either. Hence, student i_2 is assigned to school s_1 at step $\pi(r(f+1), f+1)$ under Q'. Hence, Q'_{i_2} is a profitable deviation for student i_2, contradicting $Q \in \mathcal{E}(\varphi^\pi, P)$. So, $\pi(\bar{r}, f) < \pi(r(f+1), f+1) = \pi(f+1)$.

Suppose $\bar{r} \geq r$. Since $\pi(\bar{r}, f) < \pi(f+1)$, (9) implies that there is a rank $\bar{r}' < \bar{r}$ such that $\pi(\bar{r}', f') < \pi(\bar{r}, f)$. Since Q_{i_1} lists at least $\bar{r}$ schools, it lists some school, say $\bar{s}'$, at rank $\bar{r}'$. Obviously, $\bar{s}' \neq s_1$. Since i_1 has priority f' for school $\bar{s}'$ and $q_{\bar{s}'} = n$, it follows that student i_1 is assigned to some school before or at step $\pi(\bar{r}', f')$ under Q. Since $\pi(\bar{r}', f') < \pi(\bar{r}, f)$, this contradicts the fact that student i_1 is assigned to school s_1 at step $\pi(\bar{r}, f)$ under Q. Hence, $\bar{r} < r$.

We have shown that $\pi(\bar{r}, f) < \pi(f+1)$ and $\bar{r} < r$. However, this contradicts the minimality of r. So, $Q \notin \mathcal{E}(\varphi^\pi, P)$. Hence, $\mu \in \mathcal{S}(P) \setminus \mathcal{O}(\varphi^\pi, P)$. So, $\mathcal{S}(P) \not\subseteq \mathcal{O}(\varphi^\pi, P)$.

Next, we show $\mathcal{O}(\varphi^\pi, P) \nsubseteq \mathcal{S}(P)$. Consider the strategy-profile Q in Table 6. Student i_2 submits a list where school s_1 appears at rank $r(f+1)$. All other students submit the empty list.

	Students' strategies		
	Q_{i_1}	Q_{i_2}	$Q_{I\setminus\{i_1,i_2\}}$
	$\emptyset$	$\vdots$	$\emptyset$
		$\vdots$	
$r(f+1) \rightarrow$		s_1	
		$\vdots$	

Table 6: Strategy-profile in Case 2.

Let $\mu' \equiv \varphi^\pi(Q)$. Obviously, for each $i \neq i_2$, $\mu'(i) = \emptyset$ and $\mu(i_2) = s_1$. Since $\mu' \neq \mu$, μ' is not stable, i.e., $\mu' \notin S(P)$. We show that $Q \in \mathcal{E}(\varphi^\pi, P)$. First, none of the students $i_2, \ldots, i_n$ has a profitable deviation (since they get their most preferred match). Second, consider student i_1. The only possible improvement would be to get the seat at school s_1. Suppose Q'_{i_1} is such that $\varphi^\pi_{i_1}(Q') = s_1$ where $Q' \equiv (Q'_{i_1}, Q_{-i_1})$. Then, i_1 is assigned to s_1 before step $\pi(r(f+1), f+1)$ under Q'. (Otherwise i_2 would again grab the unique seat at s_1.) Since i_1 has priority f for s_1, i_1 is assigned to s_1 at some step $\pi(r,f) < \pi(r(f+1), f+1)$ where $r \in \{1, \ldots, m\}$. In particular, the list Q'_{i_1} consists of at least r schools and school s_1 appears at rank r. It follows from (9) that there exists a smallest rank $r' \in \{1, \ldots, m\}$ with $r' < r$ such that $\pi(r', f') < \pi(r, f)$. Since Q'_{i_1} lists a school at rank $r' < r$, say $s' \neq s_1$, and since student i_1 has priority f' for s' and $q_{s'} = n$, it follows that at step $\pi(r', f')$ student i_1 is assigned to s', which contradicts $\varphi^\pi_{i_1}(Q') = s_1 \neq s'$. Hence, i_1 does not have a profitable deviation. Hence, $Q \in \mathcal{E}(\varphi^\pi, P)$. Hence, $\mu' \in \mathcal{O}(\varphi^\pi, P) \setminus \mathcal{S}(P)$. So, $\mathcal{O}(\varphi^\pi, P) \nsubseteq \mathcal{S}(P)$. $\square$

Mechanism φ^π **(Nash) sub-implements the set of stable matchings** if for each problem $P \in \mathcal{P}$, $\mathcal{O}(\varphi^\pi, P) \subseteq \mathcal{S}(P)$. Similarly, mechanism φ^π **(Nash) sup-implements the set of stable matchings** if for each problem $P \in \mathcal{P}$, $\mathcal{S}(P) \subseteq \mathcal{O}(\varphi^\pi, P)$. Clearly, a mechanism implements the set of stable matchings if and only if it both sub-implements and sup-implements the set of stable matchings. As corollaries to Propositions 1 and 2 we obtain the following two results.

Corollary 3. [Sub-implementation: characterization]
A rank-priority mechanism $\varphi^\pi \in \mathcal{F}$ sub-implements the set of stable matchings if and only if it is quasi-monotonic, *i.e., $\varphi^\pi \in \mathcal{F}^q$.*

Corollary 4. [Sup-implementation: characterization]
A rank-priority mechanism $\varphi^\pi \in \mathcal{F}$ *sup-implements the set of stable matchings if and only if it is* quasi-monotonic, *i.e.,* $\varphi^\pi \in \mathcal{F}^q$*.*

4 Incomplete information

In the analysis of Section 3 we rely on the concept of Nash equilibrium. In particular, we assume complete information about preferences. A natural question is whether our result still holds when this assumption is relaxed. Ergin and Sönmez (2006, Section 8) consider an incomplete information environment where students do know the priorities and the capacities of the schools but not the realizations of the other students' types. They show that the immediate acceptance mechanism may induce a Bayesian Nash equilibrium with unstable matchings in its support. In this section, we prove a strong impossibility result: *all* rank-priority mechanisms exhibit the same feature as the immediate acceptance mechanism.

As before, let $I = \{i_1, \ldots, i_n\}$ and $S = \{s_1, \ldots, s_m\}$ denote the fixed set of students and schools, respectively. Furthermore, let $\succ = (\succ_s)_{s \in S}$ be the profile of priority relations and $q = (q_{s_1}, \ldots, q_{s_m})$ be the capacity vector. Each student $i \in I$ is now endowed with a **von Neumann-Morgenstern utility function** (or type) $\boldsymbol{u_i} : S \cup \{\emptyset\} \to \mathbb{R}$. We assume that for all $s, s' \in S \cup \{\emptyset\}$ with $s \neq s'$, $u_i(s) \neq u_i(s')$. Let $\mathcal{U}_i$ be the set of possible utility functions for student i. (For $i \neq j$, it is possible that $\mathcal{U}_i \neq \mathcal{U}_j$.) In our incomplete information setting, all students know the **probability distribution** $\boldsymbol{\mathbb{P}_i}$ over $\mathcal{U}_i$ where, without loss of generality, for each $u_i \in \mathcal{U}_i$, $\mathbb{P}_i(u_i) > 0$ and $\sum_{u_i \in \mathcal{U}_i} \mathbb{P}_i(u_i) = 1$, but only student i knows its realization. Let $\tilde{u}_i$ denote the random variable that determines student i's utility function. We assume that the collection $(\tilde{u}_i)_{i \in I}$ is independent. A **problem of incomplete information** is a list $\boldsymbol{(I, S, (\mathcal{U}_i)_{i \in I}, (\mathbb{P}_i)_{i \in I}, \succ, q)}$.

As before, we assume that students are the only strategic agents. For each $i \in I$, let $\mathcal{P}_i$ be the set of all complete, transitive, and strict preference relations over $S \cup \{\emptyset\}$. A **strategy** of student i is a function $\sigma_i : \mathcal{U}_i \to \mathcal{P}_i$. Let Σ_i denote the set of student i's strategies and let $\Sigma \equiv \prod_{i \in I} \Sigma_i$. Given a rank-priority mechanism φ^π, $\boldsymbol{\Gamma = (I, (\mathcal{U}_i)_{i \in I}, (\mathbb{P}_i)_{i \in I}, (\Sigma_i)_{i \in I}, \varphi^\pi)}$ is a **Bayesian game**.

A strategy-profile $\sigma = (\sigma_{i_1}, ..., \sigma_{i_n}) \in \Sigma$ is a **(Bayesian Nash) equilibrium** of Γ if for each student $i \in I$, σ_i assigns an optimal action to each $u_i \in \mathcal{U}_i$, i.e., maximizes student i's expected payoff given the other students' strategies. Formally, for each $i \in I$, each $u_i \in \mathcal{U}_i$, and each $P_i' \in \mathcal{P}_i$,

$$\mathbb{E}\left[u_i[\ \varphi_i^\pi(\sigma_i(u_i), (\sigma_j(\tilde{u}_j))_{j \neq i})\ ] \right] \geq \mathbb{E}\left[u_i[\ \varphi_i^\pi(P_i', (\sigma_j(\tilde{u}_j))_{j \neq i})\ ] \right], \tag{10}$$

where the expected payoff is computed with respect to the vector of random variables $(\sigma_j(\tilde{u}_j))_{j\neq i}$. Let $\boldsymbol{\mathcal{E}(\Gamma)}$ denote the **set of equilibria**. The **support** of a strategy-profile $\sigma \in \Sigma$ is the set of matchings that can be obtained with strictly positive probability, i.e.,

$$\Big\{ \mu \ : \ \text{ there is } (u_i)_{i\in I} \in \prod_{i\in I} \mathcal{U}_i \ \text{ s.t. } \varphi^\pi\big(\sigma_{i_1}(u_{i_1}),\ldots,\sigma_{i_n}(u_{i_n})\big) = \mu \Big\}.$$

Next, we show that for each rank-priority mechanism, there is a problem of incomplete information with a Bayesian Nash equilibrium such that its support contains an unstable matching.

Theorem 2. [Incomplete information: impossibility of "stable support"]
Let $m \geq 3$ and $n \geq 4$. For each rank-priority mechanism φ^π, there is a problem of incomplete information with a Bayesian Nash equilibrium such that its support contains an unstable matching, i.e., for each $\varphi^\pi \in \mathcal{F}$, there is $\sigma \in \mathcal{E}(\Gamma)$ such that for some $(u_i)_{i\in I} \in \prod_{i\in I}\mathcal{U}_i$,

$$\varphi^\pi\big(\sigma_{i_1}(u_{i_1}),\ldots,\sigma_{i_n}(u_{i_n})\big) \notin \mathcal{S}(P_{u_{i_1}},\ldots,P_{u_{i_n}}),$$

where for each $i \in I$, P_{u_i} is the preference relation over $S\cup\{\emptyset\}$ such that for all $s,s' \in S\cup\{\emptyset\}$, $s\,P_{u_i}\,s'$ if $u_i(s) > u_i(s')$.

Proof. Let φ^π violate *quasi-monotonicity*. Then, the statement follows immediately from Theorem 1. Let φ^π be *quasi-monotonic*. Assume there are $n = 4$ students and $m = 3$ schools.[15] Let $(I, S, (\mathcal{U}_i)_{i\in I}, (\mathbb{P}_i)_{i\in I}, \succ, q)$ be any school choice problem[16] of incomplete information with $I = \{1,2,3,4\}$, $S = \{a,b,c\}$, $\mathcal{U}_1 = \{u_1\}$, $\mathcal{U}_2 = \{u_2\}$, $\mathcal{U}_3 = \{u_3\}$, $\mathcal{U}_4 = \{u_4^\emptyset, u_4^a\}$, $\mathbb{P}_1(u_1) = 1$, $\mathbb{P}_2(u_2) = 1$, $\mathbb{P}_3(u_3) = 1$, and $\mathbb{P}_4(u_4^\emptyset) = \mathbb{P}_4(u_4^a) = \frac{1}{2}$. The utility functions u_1, u_2, u_3, u_4^a, and $u_4^\emptyset$ are given by the columns in Table 7. The only condition (apart from the partial specification of u_1) that we impose on the utility functions is that the induced preferences are those described by the corresponding columns[17] in Table 8. The profile of priority relations $\succ$ is also described in Table 8. Finally, school a has capacity 2 and schools b and c each have capacity 1.

Let $(r(f), f)$ be the first pair in π in which priority f appears. Let $(r^2(4), 4)$ and $(r^3(4), 4)$ be the pair in π in which priority 4 appears for the second and third time, respectively. Note that $\{r(4), r^2(4), r^3(4)\} = \{1,2,3\}$. We also observe that since $\varphi^\pi \in \mathcal{F}^q$, before step $\pi(r(2), 2)$ only pairs with priority 1 are considered. In particular, $\pi(r(1), 1) = 1$. We will use these facts in the remainder of the proof.

[15] A proof for the case with $m > 3$ or $n > 4$ can easily be obtained by introducing unacceptable schools.
[16] For the sake of clarity, we let integers and letters denote students and schools, respectively.
[17] Note that we simplify notation by writing P_1,P_2,P_3,$P_4^\emptyset$, and P_4^a instead of P_{u_1},P_{u_2},P_{u_3},$P_{u_4^\emptyset}$, and $P_{u_4^a}$.

	Students				
	u_1	u_2	u_3	$u_4^\emptyset$	u_4^a
a	3	$*$	$*$	$*$	$*$
b	2	$*$	$*$	$*$	$*$
c	$*$	$*$	$*$	$*$	$*$
$\emptyset$	0	$*$	$*$	$*$	$*$

Table 7: The utility functions in Theorem 2. Each $*$ can be arbitrarily chosen provided that each column induces the preferences in the corresponding column of Table 8.

Students					Schools		
P_1	P_2	P_3	$P_4^\emptyset$	P_4^a	$\succ_a$	$\succ_b$	$\succ_c$
a	a	a	$\emptyset$	a	2	1	2
b	$\emptyset$	b		$\emptyset$	4	3	4
$\emptyset$		c			1	2	1
		$\emptyset$			3	4	3

Table 8: Induced preferences and the priorities in Theorem 2.

Consider any strategy-profile $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ such that

- at $\sigma_1(u_1)$, b has rank $r(1)$,
- at $\sigma_2(u_2)$, a has rank $r(1)$,
- at $\sigma_3(u_3)$,

  case I: if $r(2) \neq r(4), r^2(4)$, then b has rank $r(2)$, a has rank $r(4)$, and c has rank $r^2(4)$,
  case II: if $r(2) = r(4)$, then b has rank $r(2)$, a has rank $r^2(4)$, and c has rank $r^3(4)$,
  case III: if $r(2) = r^2(4)$, then b has rank $r(2)$, a has rank $r(4)$, and c has rank $r^3(4)$, and
- $\sigma_4(u_4^\emptyset) = P_4^\emptyset$ and at $\sigma_4(u_4^a)$, a has rank $r(2)$.

We compute the support of σ. Since $\pi(r(1), 1) = 1$, at step 1, student 1 is assigned to school b and student 2 is assigned to school a (independently of the realization of student 4's utility function ($u_4^\emptyset$ or u_4^a)). Students 3 and 4 are not assigned to a school at step 1, because students 3 and 4 do not have priority 1 for any school. To determine the assignment of the latter two students, we consider the two possible realizations of student 4's utility function separately.

First, consider realization $u_4^\emptyset$. In this case, student 4 obviously remains unassigned. As a consequence, student 3 is assigned to school a at step $\pi(r(4), 4)$ or $\pi(r^2(4), 4)$. To see this, note that after step 1, only schools a and c have an empty seat. Moreover, student 3 has priority 4 for both schools a and c. In case I, since $\pi(r(4), 4) < \pi(r^2(4), 4)$, student 3 is assigned to school a at step $\pi(r(4), 4)$. In case II, since $\pi(r^2(4), 4) < \pi(r^3(4), 4)$, student 3 is assigned to school a at step $\pi(r^2(4), 4)$. In case III, since $\pi(r(4), 4) < \pi(r^3(4), 4)$, student 3 is assigned to school a at step $\pi(r(4), 4)$. So, under realization $u_4^\emptyset$, the resulting outcome is matching $\mu^\emptyset$ as depicted in Table 9.

	1	2	3	4
$\mu^{\emptyset}$	b	a	a	$\emptyset$
μ^{a}	b	a	c	a

Table 9: The support of σ in Theorem 2.

Second, consider realization u_4^a. Recall that before step $\pi(r(2), 2)$ only pairs with priority 1 are considered. Since student 2 is the only student with priority 1 for school 1 and since he has been assigned a seat at school a at step 1, at the beginning of step $\pi(r(2), 2)$ school a has still one empty seat. Moreover, since student 4 does not have priority 1 for any school, student 4 is not assigned to any school before step $\pi(r(2), 2)$. But then (by definition of $\sigma_4(u_4^a)$) student 4 is assigned to school a at step $\pi(r(2), 2)$. Consequently, student 3 is assigned to school c at step $\pi(r^2(4), 4)$ or $\pi(r^3(4), 4)$. So, under realization u_4^a, the resulting outcome is matching μ^a as depicted in Table 9.

Next, we show that σ is an equilibrium by checking that none of the four students has a profitable deviation, i.e., inequality (10) is satisfied for each student $i \in I$.

Since student 2 gets his most preferred match, student 2 does not have a profitable deviation. Since student 4 gets his most preferred match, either being unassigned or school a under each realization of his utility function, student 4 does not have a profitable deviation.

Consider student $i = 1$. Since $\mathcal{U}_1 = \{u_1\}$, we only have to check inequality (10) for $u_i = u_1$. It follows immediately from Table 9 that

$$\mathbb{E}\left[\, u_1[\ \varphi_1^{\pi}(\sigma_1(u_1), (\sigma_j(\tilde{u}_j))_{j\neq 1})\ ]\right] = 2. \tag{11}$$

Suppose there is $P_1' \in \mathcal{P}_1$ such that

$$\mathbb{E}\left[\, u_1[\ \varphi_1^{\pi}(P_1', (\sigma_j(\tilde{u}_j))_{j\neq 1})\ ]\right] > \mathbb{E}\left[\, u_1[\ \varphi_1^{\pi}(\sigma_1(u_1), (\sigma_j(\tilde{u}_j))_{j\neq 1})\ ]\right]. \tag{12}$$

Since $\mathbb{P}_4(u_4^{\emptyset}) = \mathbb{P}_4(u_4^a) = \frac{1}{2}$,

$$\begin{aligned}\mathbb{E}\Big[u_1[\ \varphi_1^{\pi}(P_1', (\sigma_j(\tilde{u}_j))_{j\neq 1})\,]\Big] = \ & \frac{1}{2}u_1\big(\varphi_1^{\pi}(P_1', \sigma_2(u_2), \sigma_3(u_3), \sigma_4(u_4^{\emptyset}))\big) + \\ & \frac{1}{2}u_1\big(\varphi_1^{\pi}(P_1', \sigma_2(u_2), \sigma_3(u_3), \sigma_4(u_4^{a}))\big).\end{aligned} \tag{13}$$

Consider the rank-priority algorithm for π at $(P_1', \sigma_2(u_2), \sigma_3(u_3), \sigma_4(u_4^a))$. Since student 1 has priority 3 for school a and since before step $\pi(r(2), 2)$ only pairs with priority 1 are considered, student 1 cannot be assigned to school a before or at step $\pi(r(2), 2)$. However, by the end of step $\pi(r(2), 2)$, school a does no longer have empty seats: student 2 is assigned to a at step $\pi(r(1),1)$ and student 4 is assigned to a at step $\pi(r(2),2)$. Hence, $\varphi_1^\pi(P_1', \sigma_2(u_2), \sigma_3(u_3), \sigma_4(u_4^a))) \neq a$. But then, from (11), (12), (13), and the specification of u_1 in Table 7 it follows that P_1' is a ranking that includes a as an acceptable school and

$$\varphi_1^\pi(P_1', \sigma_2(u_2), \sigma_3(u_3), \sigma_4(u_4^\emptyset)) = a. \tag{14}$$

Then, together with the fact that before step $\pi(r(2),2)$ only pairs with priority 1 are considered, we have that for each $r \in \{1,2,3\}$ and each $f \in \{1,2,3,4\}$ with $\pi(r,f) < \pi(r(2),2)$, $f = 1$ and at P_1' school b does not have rank r (otherwise, student 1 would be assigned to school b, which contradicts (14)).[18]

Following the rank-priority algorithm for π at $(P_1', \sigma_2(u_2), \sigma_3(u_3), \sigma_4(u_4^\emptyset))$, we find that at step $\pi(r(1),1)$, student 2 is assigned to a; no further assignments take place until step $\pi(r(2),2)$; at step $\pi(r(2),2)$, student 3 is assigned to b; and student 1 is assigned to a at some step $\pi(r,3) > \pi(r(2),2)$ where $r \in \{1,2,3\}$.

Following the rank-priority algorithm for π at $(P_1', \sigma_2(u_2), \sigma_3(u_3), \sigma_4(u_4^a))$, we find again that at step $\pi(r(1),1)$, student 2 is assigned to a and that no further assignments take place until step $\pi(r(2),2)$. However, this time, at step $\pi(r(2),2)$, student 3 is assigned to b *and* student 4 is assigned to a. So, after step $\pi(r(2),2)$ schools a and b do no longer have empty seats. Hence, $\varphi_1^\pi(P_1', \sigma_2(u_2), \sigma_3(u_3), \sigma_4(u_4^a)) \in \{\emptyset, c\}$. Hence, from the specification of u_1 in Table 7,

$$u_1\big(\varphi_1^\pi(P_1', \sigma_2(u_2), \sigma_3(u_3), \sigma_4(u_4^a))\big) \leq 0. \tag{15}$$

Substituting (14) and (15) in (13) yields a contradiction with (11) and (12). We conclude that there is no $P_1' \in \mathcal{P}_1$ that satisfies (12), i.e., student 1 does not have a profitable deviation.

Finally, consider student $i = 3$. Since $\mathcal{U}_3 = \{u_3\}$, we only have to check inequality (10) for $u_i = u_3$. Since $\mathbb{P}_4(u_4^\emptyset) = \mathbb{P}_4(u_4^a) = \frac{1}{2}$, it follows immediately from Table 9 that

$$\mathbb{E}\Big[\, u_3[\ \varphi_3^\pi(\sigma_3(u_3), (\sigma_j(\tilde{u}_j))_{j\neq 3})\ ]\Big] = \frac{1}{2}u_3(a) + \frac{1}{2}u_3(c). \tag{16}$$

Suppose there is $P_3' \in \mathcal{P}_3$ such that

$$\mathbb{E}\Big[\, u_3[\ \varphi_3^\pi(P_3', (\sigma_j(\tilde{u}_j))_{j\neq 3})\ ]\Big] > \mathbb{E}\Big[\, u_3[\ \varphi_3^\pi(\sigma_3(u_3), (\sigma_j(\tilde{u}_j))_{j\neq 3})\ ]\Big]. \tag{17}$$

Note that

$$\begin{aligned}\mathbb{E}\Big[u_3[\ \varphi_3^\pi(P_3', (\sigma_j(\tilde{u}_j))_{j\neq 3})\,]\Big] = \;& \frac{1}{2}u_3\big(\varphi_3^\pi(\sigma_1(u_1), \sigma_2(u_2), P_3', \sigma_4(u_4^\emptyset))\big) + \\ & \frac{1}{2}u_3\big(\varphi_3^\pi(\sigma_1(u_1), \sigma_2(u_2), P_3', \sigma_4(u_4^a))\big)\end{aligned} \tag{18}$$

[18]In fact, at P_1', school b may not even be acceptable.

Equations (16), (17), and (18) yield

$$\begin{aligned} u_3\big(\varphi_3^\pi(\sigma_1(u_1), \sigma_2(u_2), P_3', \sigma_4(u_4^\emptyset))\big) \quad + \quad u_3\big(\varphi_3^\pi(\sigma_1(u_1), \sigma_2(u_2), P_3', \sigma_4(u_4^a))\big) \quad > \\ u_3(a) \quad + \quad u_3(c). \end{aligned} \tag{19}$$

Since at both $(\sigma_1(u_1), \sigma_2(u_2), P_3', \sigma_4(u_4^\emptyset))$ and $(\sigma_1(u_1), \sigma_2(u_2), P_3', \sigma_4(u_4^a))$, student 1 is assigned to the unique seat at b at step $1 = \pi(r(1), 1)$, student 3 cannot be assigned to school b, i.e.,

$$\varphi_3^\pi(\sigma_1(u_1), \sigma_2(u_2), P_3', \sigma_4(u_4^\emptyset)) \neq b \quad \text{and} \tag{20}$$

$$\varphi_3^\pi(\sigma_1(u_1), \sigma_2(u_2), P_3', \sigma_4(u_4^a)) \neq b. \tag{21}$$

Then, from (19), (20), (21), and the conditions imposed on u_3 by Table 8, it follows that

$$\varphi_3^\pi(\sigma_1(u_1), \sigma_2(u_2), P_3', \sigma_4(u_4^\emptyset)) = \varphi_3^\pi(\sigma_1(u_1), \sigma_2(u_2), P_3', \sigma_4(u_4^a)) = a. \tag{22}$$

Now consider the rank-priority algorithm for π at $(\sigma_1(u_1), \sigma_2(u_2), P_3', \sigma_4(u_4^a))$. Recall that before step $\pi(r(2), 2)$ only pairs with priority 1 are considered. At step $\pi(r(1), 1)$, student 1 is assigned to b and student 2 is assigned to a; no further assignments take place until step $\pi(r(2), 2)$; and at step $\pi(r(2), 2)$, student 4 is assigned to school a. Hence, $\varphi_3^\pi(\sigma_1(u_1), \sigma_2(u_2), P_3', \sigma_4(u_4^a)) \neq a$, which contradicts (22). We conclude that there is no $P_3' \in \mathcal{P}_3$ that satisfies (17), i.e., student 3 does not have a profitable deviation. Hence, σ is an equilibrium.

Finally, to complete the proof, notice that the support of σ contains the unstable matching $\mu^\emptyset$ (see Table 9): at $\mu^\emptyset$, student 1 has justified envy with respect to student 3 and school a. □

Notice that to tackle any rank-priority mechanism $\varphi^\pi \in \mathcal{F}^q$ in the proof of Theorem 2, we only use the fact that $\varphi^\pi \in \mathcal{F}^u$. Moreover, for all rank-priority mechanisms in $\mathcal{F}^q$ we use the same problem of incomplete information and the same strategy-profile to prove the statement. Therefore we obtain the following corollary.

Corollary 5.
Let $m \geq 3$ and $n \geq 4$. There is a problem of incomplete information and a strategy-profile σ such that for each rank-priority mechanism $\varphi^\pi \in \mathcal{F}^u$, σ is a Bayesian Nash equilibrium with an unstable matching in its support.

5 Concluding remarks

Since rank-priority mechanisms are vulnerable to preference manipulation our focus has been to establish *fairness* (by means of *stability*) in equilibrium. Our analysis has shown that a large class of rank-priority mechanisms may serve the goal of obtaining fairness in equilibrium. Indeed, from the point of view of implementation of the set of stable matchings, all *quasi-monotonic* rank-priority mechanisms are equivalent in the complete information framework (Theorem 1) as well as in an incomplete information framework (Theorem 2).

Even though we have restricted our attention to the family of rank-priority mechanisms, we can easily obtain Nash implementation of the set of stable matchings for a class of mechanisms that are not rank-priority mechanisms. More specifically, let φ be a mechanism that consists of the following two phases. In the first phase, students are matched to schools according to some *quasi-monotonic* mechanism φ^{π}, but only considering the rank-priority pairs that appear in π up to and including step $\pi(n)$. At the end of step $\pi(n)$, the second phase starts: unmatched students are matched to schools so that individual rationality and non-wastefulness are satisfied. Then, since *quasi-monotonicity* only imposes restrictions on the rank-priority pairs that appear in π before position $\pi(n)$, it is easy to see that the proof of Proposition 1 yields the Nash implementation of the set of stable matchings for φ. In particular, we obtain Nash implementation for the adaptive Boston or immediate acceptance with skips mechanism (Alcalde, 1996, Harless, 2015, and Dur, 2015).

Dur et al. (2016b) consider the class of mechanisms that (1) maximize the number of students matched to their reported first choices and (2) yield a matching in which no student forms a blocking pair with his first choice. They show that the set of students that receive their first choice under each of these mechanisms always coincides with the set of students that receive their first choice under the Boston or immediate acceptance mechanism (Dur et al., 2016b, Lemma 1). Hence, any mechanism in their class can be described as a "two-phase" mechanism where the first phase consists of the first $n = \pi^{ia}(n)$ rank-priority steps of the Boston mechanism. Then, from the observation in the previous paragraph, we obtain Theorem 2 in Dur et al. (2016b): any mechanism in the class considered by Dur et al. (2016b) Nash implements the set of stable matchings.

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