A general model of price competition with soft capacity constraints

Preliminary version

Marie-Laure Cabon-Dhersin*, Nicolas Drouhin†

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Abstract

We propose a general model of oligopoly with firms relying on a two factors production function. The factors are chosen sequentially. In a first stage, firms choose the level of the fixed factor. In a second stage, firms compete in price, and determine the level of variable factor necessary required to satisfy the whole demand. This setting generalizes the notion of capacity constraint. When the production function allows a certain degree of substitutability, the capacity constraint is "soft", implying a convex and smooth cost function in the second stage. We show that there exists a unique equilibrium prediction for the game, whatever the returns to scale. This equilibrium is characterized by a high level for price. We provide simulations, demonstrating non-standard results on the effects of the number of firms on the market price and welfare.

Key words: price competition, tacit collusion, convex cost, capacity constraint, limit pricing strategy, returns to scale.

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*Université de Rouen, CREAM, Adress: 3 avenue Pasteur, 76100 Rouen, France. (marie-laure.cabon-dhersin@univ-rouen.fr)

†Corresponding author at Ecole Normale Supérieure Paris-Saclay and CREST (UMR CNRS 9194), Ens Paris-Saclay 61 avenue du Président Wilson, 94230 Cachan, France. (drouhin@ens-paris-saclay.fr)

1 Introduction.

We study price competition between a variable number of capacity constrained firms. In our model, firms rely on a production function with two substitutable factors that are chosen sequentially. The first factor, is chosen in a first stage. In the second stage, the first factor is "fixed", and firms compete in price and adjust the level of the second "variable" factor to match their demand. In this setting, the level of fixed factor is a metaphor for capacity. Thus, in our model, firms are capacity constrained, but this constraint is "soft", because firms can always increase their production beyond their capacity optimal level, but at an increasing marginal production cost. Our results are general. They do not rely on a specific parametrical form of the production function, and returns to scale are not required to be constant or decreasing. The results are as follows. A continuum of subgame perfect Nash equilibria may exist that put into balance two opposite effects. In the second stage, the convexity of short run cost allows to sustain a high level of price as an equilibrium, like in Dastidar (1995) or in Cabon-Dhersin and Drouhin (2014). In the first stage, the level of fixed factor cannot be to small to avoid limit pricing strategies by competitors in the second stage. It is the convexity of the short-run cost function in the second stage that is determinant for the existence of the equilibrium. The fact, that this property can be achieved whatever the returns to scale, induces a very important and original property of our model: the existence of equilibrium of the whole game is disentangled from the nature of the returns to scale.

This paper bridges three lines of literature, the Bertrand-Dastidar convex cost approach of price competition, the Bertrand-Edgeworth constrained capacity approach of price competition, and the literature on capacities and limit pricing strategies.

In his seminal model of price competition, Joseph Bertrand (1883) considered an interaction between two firms that have identical linear cost functions and simultaneously set their prices. According to this model, even if the number of competing firms is small, price competition leads to a perfectly competitive outcome in a market for a homogeneous good. The unique equilibrium price equals the firm's (constant and common) marginal cost and the profit of each firm is equal to zero. This result is referred as the Bertrand Paradox. For a long time, following an initial insight of Edgeworth (1925) it has been believed that there was a serious equilibrium existence problem (in pure strategies) when considering decreasing returns to scale or, equivalently, considering a convex cost function. However, Dastidar (1995) proved that there exists a continuum of pure strategy Nash equilibria in price competition when costs are strictly convex. As usual in price competition, a firm undercutting its price will attract all the demand, but, because of the convexity of the cost function this move may not necessarily be profitable. Thus, at the equilibrium, price may be higher than the average cost and even higher than the marginal cost. Dastidar (2001) shows that, when the costs are sufficiently convex, the collusive outcome may even be an equilibrium. On the opposite, with strictly subadditive costs and symmetric firms, it can be shown that there exists no equilibrium in price competition (Dastidar, 2011b)[1]. The source of of subadditivity can be, either increasing returns to scale or the existence of fixed cost when variable unitary cost is constant or not too convex (Hoernig, 2007; Baye and Kovenock, 2008; Saporiti and Coloma, 2010). In this Bertrand-Dastidar approach of price competition, it is the convexity of the cost function that allows to solve the Bertrand Paradox. As mentioned in the introductory paragraph, in our model, the convexity of the short-run cost function in the second stage is due to the decreasing marginal productivity of the variable factor. Thus, we will share some properties of the Dastidarian framework, whatever the returns to scale.

As pointed out by Vives (1999), following Edgeworth (1925), there is a long tradition in Industrial Organization to solve the Bertrand Paradox considering the fact that firms are limited by their production capacities to match the demand. In the modern

[1]Dastidar (2011a) introduces asymmetric cost functions and proves that, in this case, when the monopoly break-even prices differ there exists an equilibrium even if costs are stricly subadditive.

literature, this argument has been put forward by Kreps and Scheinkman (1983). In a two-stage game, they obtain that quantity pre-commitment, in a first stage, and price competition, in a second stage, sustains the Cournot outcome under the condition that constrained capacities are not “too high”. As shown by Davidson and Deneckere (1986), this result is sensitive to the choice of a rationing rule for the residual demand (see Vives, 1999, p.124, for details). This result is build on "drastic" capacity constraints, that is, the marginal cost of production in excess of capacity is infinite. Our approach relies on the same type of two-stage game with a choice of capacity in a first stage. But, the softness of the capacity constraint induces a smoother cost function in the second stage. Some papers have tried to introduce some less-rigid capacity constraints (see Maggi, 1996; Boccard and Wauthy, 2000, 2004; Chowdhury, 2009, for example) directly in the cost function. Cabon-Dhersin and Drouhin (2014) provide rationale for those "soft" capacity constraints starting from the microeconomic production function and the sequential choice of the production factors. Burguet and Sákovics (2017) build on the same approach, but with a very different model of price competition in the second stage, in which firms are able to offer a different price to each consumer.

Beyond the issue of price competition, strategic investment capacity decisions are also a very classical question in industrial organization with regard to entry deterrence (Spence, 1977; Dixit, 1980, among many others). In this kind of models, a choice of a capacity in excess in the first stage allows to drive away potential competitors. In our model, all the competitors are already operating the market. But a firm choosing a too low capacity level in the first stage, will be unable to match profitably competitors price in the second stage. A sufficiently high capacity level is thus required to avoid limit pricing strategy in the second stage.

Finally, we propose in this article a general model of price competition with "soft" capacity constraints which allows to examine the effects of the number of firms on the market price and welfare. Notably, we obtain that, when the number of firms is low, the equilibrium price may increase when new firms enter the market whatever the returns to scale. The welfare is not necessarily maximum when the price is minimum.

The papers is organized as follows. Section 2 characterizes rigourously the notion of "soft" capacity constraint, the complete model is solved in Section 3, and finally section 4 provides a "textbook example" to illustrate some interesting properties of the model and offers a general method for numerical simulations.

2 Characterisation of the soft capacity constraint

Firms produce an homogeneous good, and have all the same technology represented by a two factors production function. Factors will be chosen sequentially. We denote, z the quantity of the factor chosen in the first stage, subsequently referred as the 'fixed factor', and v, the quantity of the factor chosen in the second stage, i.e. 'the variable factor'. We denote y the level of production, and $f : \mathbb{R}_+^2 \to \mathbb{R}_+$. Thus, we have:

$$y = f(z, v) \tag{1}$$

We only assume that f is increasing in z and v, shows decreasing marginal factor productivity and is quasi-concave. Thus: $f_z > 0, f_v > 0, f_{zz} < 0$, $f_{vv} < 0$ and $-f_{zz}f_v^2 + 2f_{zv}f_vf_z - f_{vv}f_z^2 > 0$.

It is important to emphasize, that we make no general assumptions concerning the nature of the return to scale or the level of substitutability between the two factors of production.

When the factor z is fixed, the equation (1) defined the variable factor as an implicit function of z and y, $\hat{v}(y, z)$.

Lemma 1. *1) The function $\hat{v}$ is quasi-convex and fulfils:*

$$\hat{v}_y(y, z) = \frac{1}{f_v(z, v)} > 0 \tag{2}$$

$$\hat{v}_z(y,z) = -\frac{f_z(z,v)}{f_v(z,v)} < 0 \tag{3}$$

$$\hat{v}_{yy}(y,z) = -\frac{f_{vv}(z,v)\hat{v}_y(y,z)}{f_v(z,v)^2} > 0 \tag{4}$$

$$\hat{v}_{zz}(y,z) = \frac{-f_{zz}f_v^2 + 2f_{zv}f_vf_z - f_{vv}f_z^2}{f_v(z,v)^3} > 0 \tag{5}$$

$$\hat{v}_{yz}(y,z) = \hat{v}_{zy}(y,z) = -\frac{f_{vz}(z,v) + \hat{v}_z(y,z)f_{vv}(z,v)}{f_v(z,v)^2} < 0 \tag{6}$$

2) Moreover, if f is (strictly) concave then $\hat{v}$ is (strictly) convex.

Proof: By implicit differentiation of $\hat{v}$, we get Equations (2) to (6)

Then using the quasi-concavity of f, we get:

$$-\hat{v}_{zz}\hat{v}_y^2 + 2\hat{v}_{zy}\hat{v}_z\hat{v}_y - \hat{v}_{yy}\hat{v}_z^2 = f_{zz}f_v < 0 \tag{7}$$

That proves the quasi-concavity.

Moreover, it is easy to check that:

$$\begin{vmatrix} \hat{v}_{yy} & \hat{v}_{yz} \\ \hat{v}_{zy} & \hat{v}_{zz} \end{vmatrix} = \frac{1}{f_v^4} \begin{vmatrix} f_{zz} & f_{zv} \\ f_{vz} & f_{vv} \end{vmatrix}$$

If f is concave then this determinant is necessarily positive. The second order pure derivatives of $\hat{v}$ are also positive (cf. (4) and (5)), proving part 2) of the Lemma. □

Thus we are able to define the cost function as a function of (y,z). Denoting w_1,the price of factor z and, w_2, the price of the factor v, we have:

$$C(y,z) = \underbrace{w_1 z}_{FC(z)} + \underbrace{w_2\hat{v}(y,z)}_{VC(y,z)} \tag{8}$$

The level of the fixed factor corresponds to the choice of capacity. In this model, it is possible to match any incoming demand but at an increasing marginal cost. That is the reason why the capacity constraint is "soft". The sequential choice of production factor implies that the cost function is convex, whatever the returns to scale. Thus, when firms will compete in price in a second stage, our model will inherited the general properties of the Dastidarian framework. It is altogether important to notice that, as always the fixed cost depend on the level of the fixed factor, but also the variable cost. The choice of the capacity will have qualitative implications for the shape of the variable cost function. Most models starting from a arbitrary cost function usually miss this effect.

Finally, it is important to notice that the softness of the capacity constraint comes from the substitutability of the production factor. As noticed by Cabon-Dhersin and Drouhin (2014, p. 428) or Burguet and Sákovics (2017), if the case full complementarity of production factors (Leontief technology), our approach is equivalent with the usual "drastic" capacity constraint.

3 Equilibrium of the game

Firms will choose their level of fixed factor in a first stage, then compete in price in a second stage.

The demand of the whole market is continuous, twice differentiable and decreasing.

$D : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $D(p_{max}) = 0, D(0) = Q_{max}$.

The strategic variable for the firms in stage 2 is the price. We denote p_i the price of the firm i, $\vec{p} = (p_1, .., p_n)$ is the vector of price of all n firms on the market. We denote $p_L = \text{Min}\{p_1, .., p_n\}$, and we define the set $M = \{j \in \{1, .., n\} | p_j = p_L\}$. We denote $m = \text{Card}(M)$, the number of firms quoting the lowest price. Firms have to supply all the demand they face in stage 2 at the price p_i. The demand function of the firm i is defined as follows:

$$D_i(\vec{p}) = \begin{cases} 0 & \text{if } p_i > p_L \\ \\ \frac{D(p_i)}{m} & \text{if } p_i = p_L \end{cases}$$

We can now express the profit π_i for each firm i, when m firms operate the market (set the lowest price).

$$\pi_i(\vec{p}, z_i) = pD_i(\vec{p}) - w_1 z_i - w_2 \hat{v}\left(D_i(\vec{p}), z_i\right)$$

$$\pi_i(\vec{p}, z_i) = \begin{cases} -w_1 z_i & \text{if } p_i > p_L \\ p\frac{D(p)}{m} - w_1 z_i - w_2 \hat{v}\left(\frac{D(p)}{m}, z_i\right) \stackrel{\text{def}}{=} \hat{\pi}(p, z_i, m) & \text{if } p_i = p_L = p \end{cases}$$

The function $\hat{\pi}(p, z_i, m)$ represents the profit of the firm i when m firms (including firm i) quote the lowest price, p. It depends on the level of fixed factor invested in the first stage of the game. The assumption made before on the production function are sufficient to assure that $\hat{\pi}_{zz} < 0$. But, even if the profit is necessarily concave in y, it is not sufficient to assure the strict concavity of $\hat{\pi}$ according to p. For obtaining that result, we must assume that the demand function is not to concave, nor to convex:

$$-\frac{D'(p)^2}{m}\frac{\hat{v}_{yy}}{\hat{v}_y}\left(\frac{D(p)}{m}, z\right) < D''(p) < -2\frac{D'(p)}{p}$$

The left-hand size of the inequality corresponds to the sufficient condition for the short run cost to be convex in p and the right-hand size corresponds to the sufficient condition for the revenue function to be concave in p.[2]. Moreover, if we want this condition to hold whatever the number of the firms, and particularly when this number grows to infinity, then the left-hand side tends to zero, and the demand function should necessarily be convex (non-strictly).

To solve the equilibrium of the game in stage 2, it will be of first importance to test for a profitable deviation for the firm i, from an outcome in which m firms (including the firm i) quote the same price. For that purpose, we will define two thresholds. The first one, denoted $\bar{p}$, will be the maximum price for which a firm may not increase its profit by lowering its price. The second one, denoted $\hat{p}$ is the minimum price for which a firm may not increase its profit by increasing its price. In the traditional

[2] This assumption is very standard, even if, when starting directly from a cost function, it is hidden behind the general assumption of the concavity of the profit function according to p.

Bertrand competition model with constant average/marginal cost, those two thresholds are equals and correspond to the unique equilibrium of the game, corresponding to marginal cost pricing. In our more general setting with convex short-run cost function, as in Dastidar (1995), those two thresholds will never be equal, defining a price interval for each firm for which there is no profitable deviation.

Let us start by studying the possibility for a firm to increase its profit by undercutting its price. For that purpose we define, for $m \geqslant 2$, the function $\Omega(p,z,m) \stackrel{\text{def}}{=} \hat{\pi}(p,z,1) - \hat{\pi}(p,z,m)$. Ω can be interpreted as the incentive for a firm to lower its price when the market price is p. It means that when $\Omega \leq 0$ it is not profitable for the firm to undercut its price. On the opposite, when $\Omega > 0$ it profitable for the firm to do it.

Lemma 2. *For given z and m, there exists a unique threshold $\bar{p}(z,m) \in (0, p_{max})$ that solve $\Omega(p,z,m) = 0$*
For $p \leq \bar{p}$ we get $\Omega(p,z,m) \leq 0$ and for $p > \bar{p}$ we get $\Omega(p,z,m) > 0$

Proof: Let us start by expanding Ω and Ω_p

$$\Omega(p,z,m) = \left(\frac{m-1}{m}\right) D(p)p - w_2 \left(\hat{v}(D(p),z) - \hat{v}\left(\frac{D(p)}{m}, z\right)\right)$$

The derivative gives:

$$\begin{aligned} &\Omega_p(p,z,m) \\ &\quad = \left(\frac{m-1}{m}\right) D(p) + D'(p)\left[p - w_2\hat{v}_y(D(p),z) - \frac{1}{m}\left(p - w_2\hat{v}_y\left(\frac{D(p)}{m}, z\right)\right)\right] \end{aligned} \tag{9}$$

We are now going to prove the existence.

For given z and m, $\Omega(0,z,m) = -w_2\left(\hat{v}(Q_{max},z) - \hat{v}\left(\frac{Q_{max}}{m}, z\right)\right) < 0$ (because $\hat{v}_y > 0$ and $Q_{max} > Q_{max}/m$). We also have $\Omega(p_{max},z,m) = 0$ with $\Omega_{p^-}(p_{max},z,m) = D'^-(p_{max})p < 0$ (with D'^- the left derivative of the demand function). Ω is continuous in p on the interval $[0, p_{max}]$. It starts being negative, and finish to converge to zero from above. It implies that there exists necessarily a $\bar{p}(z,m) \in (0, p_{max})$ that solve

$\Omega(p, z, m) = 0$

We now prove the uniqueness of $\bar{p}(z, m)$ on $(0, p_{max})$. Over this interval, we have $D(p) > D(p)/m > 0$. Moreover, from the strict convexity of $\hat{v}$, we have:

$$\hat{v}_y\left(\frac{D(p)}{m}, z\right) < \frac{\hat{v}(D(p), z) - \hat{v}\left(\frac{D(p)}{m}, z\right)}{D(p) - \frac{D(p)}{m}} < \hat{v}_y(D(p), z)$$

By definition of $\bar{p}$, we have $\frac{m-1}{m}D(\bar{p})\bar{p} = w_2\left(\hat{v}(D(\bar{p}), z) - \hat{v}\left(\frac{D(\bar{p})}{m}, z\right)\right)$ and thus

$$w_2\hat{v}_y\left(\frac{D(\bar{p})}{m}, z\right) < \bar{p} < w_2\hat{v}_y(D(\bar{p}), z) \tag{10}$$

Finally, considering Equation (9), it is now obvious that $\Omega_p(\bar{p}, z, m) > 0$. It means that, on the interval $(0, p_{max})$, Ω can only cut the x-axis from below. Being continuous, it can only happen once. □

$\bar{p}(z, m)$ is the highest price with no incentive to deviate when m firms operate the market. From Inequality (10), we can see that it corresponds with a strictly positive *mark-up*.

We can now study the possibility for a firm to increase profit by increasing its price. This case is much more simple, because in the second stage, the fixed cost, $w_1 z$, is sunk, and the firm has an interest to produce only if the variable part of the profit is positive. If it is not the case with the current price p, increasing the price will induce zero demand for the firm and thus zero production and lower loss.

Lemma 3. *For given z and $m \geq 1$, there exists a unique $\hat{p}(z, m)$ in the interval $(0, p_{max})$ such that: $\hat{\pi}(\hat{p}, z, m) = -w_1 z$*
Moreover, $\hat{\pi}(\hat{p}, z, m)$ decreases with m and $\hat{p}(z, m) < \bar{p}(z, m)$.

Proof: It is easy to check that $\hat{\pi}(0, z, m) < -w_1 z$, $\hat{\pi}(p_{max}, z, m) = -w_1 z$ and $\hat{\pi}_{p^-}(p_{max}, z, m) < 0$. Then, strict concavity of $\hat{\pi}$ in p implies existence and uniqueness of $\hat{p}$. Implicit differentiation of the the definition of $\hat{\pi}$ for a given z provides

$$\left.\frac{\mathrm{d}\hat{p}}{\mathrm{d}m}\right|_{dz=0} = \frac{1}{m}\frac{D(\hat{p})\left(\hat{p} - w_2\hat{v}_y\left(\frac{D(\hat{p})}{m}, z\right)\right)}{D(\hat{p}) + D'(\hat{p})(\hat{p} - w_2\hat{v}_y(\frac{D(\hat{p})}{m}, z))} < 0 \tag{11}$$

For $p < p_{max}$, we have $\frac{D(p)}{m} > D(p_{max}) = 0$. Then, using strict convexity of $\hat{v}$, we can write $\hat{v}\left(\frac{D(p)}{m}, z\right) - 0 < \left(\frac{D(p)}{m} - 0\right)\hat{v}_y\left(\frac{D(p)}{m}, z\right)$. By definition, $\hat{p}$ is such that $\hat{p}\frac{D(\hat{p})}{m} = w_2\hat{v}\left(\frac{D(\hat{p})}{m}, z\right)$ and then $\hat{p} < w_2\hat{v}_y\left(\frac{D(\hat{p})}{m}, z\right)$, that gives the sign of the implicit derivative and proves that $\hat{p}$ is decreasing in m. Thus, for $m \geq 2$, $\hat{p}(z,m) < \hat{p}(z,1)$ and $\hat{\pi}(\hat{p}(z,m), z, 1) < -w_1 z$. It follows that, for $m \geq 2$, $\Omega(\hat{p}(z,m), z, m) < 0$, implying $\hat{p}(z,m) < \bar{p}(z,m)$. $\square$

For a given z, $\hat{p}(z,m)$ is the minimum price compatible with a decision to produce in the second stage, when m firms operate the market.

The price interval $[\hat{p}(z,m), \bar{p}(z,m)]$ will be of first importance when solving for the equilibrium of the game in stage 2. Those prices will have to be compared with the *purely collusive* price when m firms operate the market, denoted p^*.

Lemma 4. *For given z and $m \geq 1$, there exists a unique $p^*(z,m)$ in the interval $(0, p_{max})$ such that: $p^*(z,m) \overset{def}{=} \arg\max_p\{\hat{\pi}(p,z,m)\}$*
Moreover, $p^*(z,m) > \hat{p}(z,m)$.

Proof: It is easy to verify that $\hat{\pi}_p(0,z,m) > 0$ and $\hat{\pi}_{p^-}(p_{max},z,m) < 0$. $\hat{\pi}_p$ is continuous, assuring the existence of an interior maximum for the program. The strict concavity of $\hat{\pi}$ according to p assures the uniqueness of the maximum. Because $\hat{\pi}_p(\hat{p},z,m) > 0$, we have $p^*(z,m) > \hat{p}(z,m)$ $\square$
As a shortcut, this price can be interpreted as the cartel price when both firms have chosen the same level of fixed factors in the first stage (when $m = 1$, it is the monopoly price). We will see that p^* may belong to the interval $[\hat{p}(z,m), \bar{p}(z,m)]$, but it is not a necessity.

Lemma 5. *$\forall m \in [1,n]$, $\hat{p}(z,m), \bar{p}(z,m)$ and $p^*(z,m)$ are strictly decreasing in z and*

m on their respective domain.

Proof: At $\hat{p}$, we have $\frac{D(\hat{p})}{m}\hat{p} - w_2\hat{v}(\frac{D(\hat{p})}{m}, z) = 0$.

The derivative of the above expression with respect to z is:

$$\left.\frac{\mathrm{d}\hat{p}}{\mathrm{d}z}\right|_{dm=0} = \frac{w_2\hat{v}_z(\frac{D(\hat{p})}{m}, z)}{\frac{D'(\hat{p})}{m}\hat{p} + \frac{D(\hat{p})}{m} - w_2\frac{D'(\hat{p})}{m}\hat{v}_y(\frac{D(\hat{p})}{m}, z)} < 0$$

From Equation (11), we have

$$\left.\frac{\mathrm{d}\hat{p}}{\mathrm{d}m}\right|_{dz=0} < 0$$

At $\bar{p}$, we have $\Omega(\bar{p}, z, m) = 0$. The derivatives of the above equality with respect to z and m are:

$$\left.\frac{\mathrm{d}\bar{p}}{\mathrm{d}z}\right|_{dm=0} = -\frac{\Omega_z(\bar{p}, z, m)}{\Omega_p(\bar{p}, z, m)} = \frac{w_2\left(\hat{v}_z(D(\bar{p}), z) - \hat{v}_z(\frac{D(\bar{p})}{m}, z)\right)}{\Omega_p(\bar{p}, z, m)}$$

which is < 0 since $\Omega_p(\bar{p}, z, m) > 0$ and $\hat{v}_z < 0, \hat{v}_{yz} < 0$.

$$\left.\frac{\mathrm{d}\bar{p}}{\mathrm{d}m}\right|_{dz=0} = -\frac{\Omega_m(\bar{p}, z, m)}{\Omega_p(\bar{p}, z, m)} = -\frac{\frac{D(\bar{p})}{m^2}(\bar{p} - w_2\hat{v}_y(\frac{D(\bar{p})}{m}, z))}{\Omega_p(\bar{p}, z, m)}$$

which is < 0 since $\Omega_p(\bar{p}, z, m) > 0$ and from Equation (10), $\bar{p} > w_2\hat{v}_y(\frac{D(\bar{p})}{m}, z)$.

Finally, we obtain,

$$\left.\frac{\mathrm{d}p^*}{\mathrm{d}z}\right|_{dm=0} = -\frac{\hat{\pi}_{pz}(p^*, z, m)}{\hat{\pi}_{pp}(p^*, z, m)} = w_2\frac{D'(p^*)}{m}\frac{\hat{v}_{yz}(\frac{D(p^*)}{m}, z)}{\hat{\pi}_{pp}(p^*, z, m)} < 0$$

and

$$\left.\frac{\mathrm{d}p^*}{\mathrm{d}m}\right|_{dz=0} = -\frac{\hat{\pi}_{pm}(p^*, z, m)}{\hat{\pi}_{pp}(p^*, z, m)} = -w_2\frac{D'(p^*)}{m^3}\frac{D(p^*)\hat{v}_{yy}(\frac{D(p^*)}{m}, z)}{\hat{\pi}_{pp}(p^*, z, m)} < 0$$

□

We have now all the material to characterize the equilibrium prediction for the whole game.

Proposition 1. *An outcome of the game* $(\vec{p}, \vec{z})$ *in which* n *firms operate the market at the same price* p^N *is a* Subgame Perfect Nash Equilibrium *(SPNE) if and only if the three following properties are simultaneously verified:*

1. ***Efficiency:*** *All* n *firms choose the same level of fixed factor* $z^N = z^*(p^N, n)$*, with* $z^*(p, n)$ *solution of the the program:*

$$(\mathcal{P}_1) \begin{cases} \max\limits_{z} \hat{\pi}(p, z, n) \\ s.t. \;\; p \leq \bar{p}(z, n) \end{cases}$$

2. ***Profitability:*** $\hat{\pi}(p^N, z^N, n) \geq 0$

3. ***Non existence of limit pricing strategy:***
$\hat{\pi}(p^N, z^N, n) \geq \hat{\pi}(\hat{p}(z^N, n), \underset{z}{\operatorname{argmax}}\, \hat{\pi}(\hat{p}(z^N, n), z, 1), 1)$

Proof: Let us assume that we are in a *SPNE* in which all n firms (indexed with i) operate the market at price p^N and have possibly variable level of fixed factor ranging between z_L, the lowest level and z_H the highest level. Because p^N is a Nash equilibrium in the second stage, each firm has no incentive to deviate in the second stage: $p^N \in \bigcap\limits_i [\hat{p}(z_i, n), \bar{p}(z_i, n)] = [\hat{p}(z_L, n), \bar{p}(z_H, n)] \neq \emptyset$. Three cases must be considered depending on the position of p^N in this interval.
Let us consider the possibility that $p^N = \hat{p}(z_L, n)$. But this case can be discarded because by definition $\hat{\pi}(\hat{p}(z_L, n), z_L, n) = -w_1 z_L < 0$. z_L cannot be an equilibrium strategy in the first stage (the firm will earn a strictly higher profit by playing $z = 0$). The second possible case is when p^N belongs to the interior of the interval, $p^N \in (\hat{p}(z_L, n), \bar{p}(z_H, n))$. For each firm i, we can compute the derivative $\hat{\pi}_z(p^N, z_i, n)$. If this derivative is negative, the firm i has an incentive to slightly decrease its level of fixed factor, z_i is not an equilibrium strategy in the first stage. Symmetrically when the derivative is positive, the firm has an incentive to slightly increase its level of fixed factor and z_i is not an equilibrium strategy in the first stage, either. Thus, to obtain a

SPNE in the second stage, it is necessary to have, for all i $\hat{\pi}_z(p^N, z_i, n) = 0$. Because $\hat{\pi}$ is strictly concave in z, $\hat{\pi}_z$ is strictly decreasing. It implies that $z_L = z_H$, all n firms should have the same level of fixed factor in a *SPNE* in which all those firms operate the market at the same price p^N.

The third (and last) possible case is when $p^N = \bar{p}(z_H, n)$. As in the preceding case, we can compute the derivative $\hat{\pi}_z(p^N, z_H, n)$. If this derivative is zero, then there is no incentive to deviate for the firm with the highest level of fixed factor. When this derivative is negative, firm H has an incentive to reduce its level of fixed factor in the first stage, it is incompatible with a SPNE. What happen when the derivative is strictly positive? In this case, from Lemma 5, we have, for any $\epsilon > 0$, $\bar{p}(z_H+\epsilon, n) < \bar{p}(z_H, n) = p^N$, by increasing its level of fixed factor in the first stage, the firm with the highest level will exclude itself from the possibility to sustain p^N as a Nash equilibria in the second stage. In this case, a necessary condition to have a *SPNE* is thus $\hat{\pi}_z(p^N, z_H, n) \geq 0$. Finally, is it possible in this case to have $z_L < z_H$? Because of the strict concavity of $\hat{\pi}$ in z, $\hat{\pi}_z(p^N, z_H, n) \geq 0$ and $z_L < z_H$ implies that $\hat{\pi}_z(p^N, z_L, n) > 0$. From Lemma 5, we also have $\bar{p}(z_L, n) > \bar{p}(z_H, n)$, the firm with a strictly lower z in the first period may increase its profit by increasing slightly its level of z without destabilizing the equilibrium at price p^N in the second stage. Thus, in this case, we cannot verify $z_L < z_H$ in a *SPNE*.

Cases 2 and 3 are mutually exclusive and cover all the possible *SPNE*. Thus, a *SPNE* necessary fulfils, for all i, $z_i = z_H = z_L = z^N$, and:

$$\left\{ \begin{array}{c} p^N < \bar{p}(z^N, n) \\ \text{and} \\ \hat{\pi}_z(p^N, z^N, n) = 0 \end{array} \right. \quad \text{OR} \quad \left\{ \begin{array}{c} p^N = \bar{p}(z^N, n) \\ \text{and} \\ \hat{\pi}_z(p^N, z^N, n) \geq 0 \end{array} \right.$$

It is easy to check that this last logical necessary condition is the same as the one required for z^N to be a solution of the program $(\mathcal{P}1)$ completing the proof of Part *1.* of Proposition 1. This efficiency condition is necessary to obtain a *SPNE* but it is not sufficient, because it only tests for the profitability of slight variation of the level of the fixed factor in the first stage. We shall also ruled out the profitability of substantial deviations in z in the first stage.

The first of this substantial deviation to be tested is a choice of $z = 0$ in the first stage and a price $p > p^N$ in the second. This move will induce zero profit for the firm. It is not profitable as long as $\hat{\pi}(p^N, z^N, n) \geq 0$. That prove part 2. of the proposition.

The second of this substantial variation in z to be tested is when a firm decide to choose z sufficiently high to sustain a price sufficiently low in the second stage to "exclude" the competitors from the market. This is a limit pricing strategy. If all other firms plays z^N in a first stage, they will have an interest to sustain any p^N such that $p^N \geq \hat{p}(z^N, n)$ (remember that in the second stage the fixed cost is sunken and the criteria of decision is the positivity of the variable profit). Thus, limit price is $p^N \geq \hat{p}(z^N), n) - \epsilon$ with $\epsilon > 0$ and the smallest possible. The most profitable value of z to sustain such a price is $\underset{z}{\operatorname{argmax}}\, \hat{\pi}(\hat{p}(z^N, n), z, 1)$. If a firm deviates using this "limit pricing strategy", it will operate the market alone. This move is profitable only when part 3. of the proposition is not fulfilled. $\square$

Proposition 2. *The outcome in which all n firms choose the same level of fixed factor z^C in the first stage and quote the same price p^C in the second stage, with z^C and p^C solution of the program:*

$$(\mathcal{P}_2)\left\{\begin{array}{ll} & \max\limits_{z,p} \hat{\pi}(p, z, n) \\ s.t. & p \leq \bar{p}(z, n) \\ & \hat{\pi}(p, z, n) \geq 0 \\ & \hat{\pi}(p, z, n) \geq \hat{\pi}(\hat{p}(z, n), \underset{\tilde{z}}{\operatorname{argmax}}\, \hat{\pi}(\hat{p}(z, n), \tilde{z}, 1), 1) \end{array}\right.$$

is a Subgame Perfect Nash Equilibrium of the game. Moreover, $\hat{\pi}(z^C, p^C)$ is the Payoff Dominant Subgame Perfect Nash Equilibrium of the game.

Proof: It is obvious, thanks to the envelop theorem, that the solution of $\mathcal{P}_2$

fulfills the conditions of Proposition 1, proving that (z^C, p^C) is a *SPNE*. It follows that $\hat{\pi}(z^C, p^C)$ is *Payoff Dominant*, because all firms have the same technology. $\square$

In the remaining, we will consider (p^C, z^C) as the predictable outcome of the price competition game with soft capacity constraint. As pointed by Cabon-Dhersin and Drouhin (2014) the solution of program $(\mathcal{P}_2)$ is collusive by nature (i.e. it corresponds to a joint profit maximisation program). However, as proved in Proposition 2, this solution correspond to a Subgame perfect Nash equilibrium, meaning that it is "stable" in the sense of the Nash equilibrium, a result that is very unusual in a nonrepeated game.

However, the outcome is not fully collusive because the joint profit maximisation is constrained by two important conditions. For being an equilibrium in the second stage, it is required to have $\hat{\pi}(p, z, m) \geq \hat{\pi}(p, z, 1)$, and, for being an equilibrium in the first stage, it required to fulfil the **non-existence of limit pricing strategy**.

4 A textbook example

4.1 Generale procedure and parametrization

The model in this article is builded on very general assumptions: sequential choice of two substitutable factors, quasi-concavity of the production function, decreasing marginal factor productivity. Proposition (1) and (2) show that the equilibrium prediction of the whole game can be described as a solution of a maximisation program subject to three different inequality constraints. At the equilibrium, each of those constraints may be binding or slack. Thus, there will be a threshold for each of those constraints to be binding. We will name, $\tilde{p}(n) = \bar{p}(z^*(p, n), n)$ the threshold to exclude profitable deviation in the second stage. We will name p^0, the threshold for firms to earn a positive profit and p_L the threshold to avoid a choice of fixed factor in the first stage that can make limit pricing strategy profitable in the second stage. However, even if each of this constraint has simple economic meaning, it may be difficult to understand from scratch how they interact in the equilibrium. For that purpose, we find useful to solve numerically a simple parametrical example, a kind of "textbook case" assuming Cobb-Douglas production function and linear demand. We will show that, even if the assumptions are simple, they are sufficient to demonstrate the full richness of our theoretical framework. The production function:

$$f(z, v) = A\left(z^{1-\alpha}v^{\alpha}\right)^{\rho} \tag{12}$$

with $\rho > 0$, the scale elasticity of production, and, $\rho\,\alpha < 1$ and $\rho\,(1-\alpha) < 1$ because of the decreasing factor marginal productivity. Of course, with all generality, the Cobb-Douglas production function is quasi-concave. It will be concave when $\rho = 1$ (constant returns to scale) and strictly concave when $\rho < 1$ (decreasing returns to scale). Taking $y = f(z, v)$, it is easy to obtain by direct calculation the function $\hat{v}$:

$$\hat{v}(y, z) = \frac{y^{\frac{1}{\alpha\rho}}}{A^{\frac{1}{\alpha\rho}}z^{\frac{1-\alpha}{\alpha}}} \tag{13}$$

Thus, the function $\hat{\pi}$ can be written when n firms operate the market:

$$\hat{\pi}(p, z, n) = p\frac{D(p)}{n} - w_1 z - w_2\frac{y^{\frac{1}{\alpha\rho}}}{A^{\frac{1}{\alpha\rho}}z^{\frac{1-\alpha}{\alpha}}} \tag{14}$$

The demand function is assumed to be linear:

$$D(p) = b(p_{max} - p) \tag{15}$$

with $b > 0$.

In this section, we will take the Payoff dominant subgame perfect Nash Equilibrium of Proposition (2) (i.e. the solution of programme $(\mathcal{P}_2)$) as the predictable outcome of our general model of price competition with soft capacity constraint. But, because the "non-existence of limit pricing strategy condition" can be tricky to deal with directly, we will proceed sequentially.

Step 1. We solve the program $(\mathcal{P}_1)$ for a given number of firms, n, and a given price, $p \in (0, p_{max})$, and a given vector of parameters $(\alpha, \rho, A, b, w_1, w_2)$. We obtain $z^*(p,n)$, the efficient level of fixed factor to sustain the price p. We are thus able to calculate $\Pi(p,n) = \hat{\pi}(p, z^*(p,n), n)$.

Step 2. For a given n, we are able to repeat the process of step 1 for any $p \in (0, p_{max})$. So we are able to draw point by point $\Pi(p,n)$ as a function of p.

Step 3. For each point $(p, \Pi(p,n))$ calculated in Step 2., we are able to test the **profitability** condition $\Pi(p,n)) \geq 0$.

Step 4. For each point $(p, \Pi(p,n))$ calculated in Step 2., we are able to calculate $\hat{p}(z^*(p,n),n)$ and then test for the **non existence of limit pricing strategy**: $\hat{\pi}(p, z^*(p,n), n) \geq \hat{\pi}(\hat{p}(z^*(p,n),n), \operatorname*{argmax}_{\tilde{z}} \hat{\pi}(\hat{p}(z^*(p,n),n), \tilde{z}, 1), 1)$.

Step 5. For each point $(p, \Pi(p,n))$ calculated in Step 2., we can check if the **condition of non profitable deviation in stage 2** is binding or not.

4.2 Effect of the concavity of the variable cost and the fixed factor price

Let us take an numerical example.[3] Figure 1 represents the case of a duopoly (n=2), with constant returns to scale (ρ=1). We take $\alpha = .7$, $p_{max} = 10$ and normalize all the other parameters to 1. The lower graphic represents $z^*(p,n)$ the solution of the Program $(\mathcal{P}_1)$ *i.e.* the efficient level of capital taking into account the non profitable deviation in stage 2 constraint. For $p \leq \tilde{p}$, the constraint is slack (the price p is a p^*). Of course, the relation between z^* and p is decreasing, converging to zero as p tends

[3]The numerical computations are made using Wolfram Research Mathematica 11. Optimizations programs are numerically solved using the function NMaximize and the value of $\hat{p}(z^*(p,n),n)$ is solved using the function Findroot.

to p_{max}. Conversely, for $p \geq \tilde{p}$, the constraint is binding (the price p is a $\bar{p}$). We shall notice that a binding constraint implies a much lower level of z for a given price.

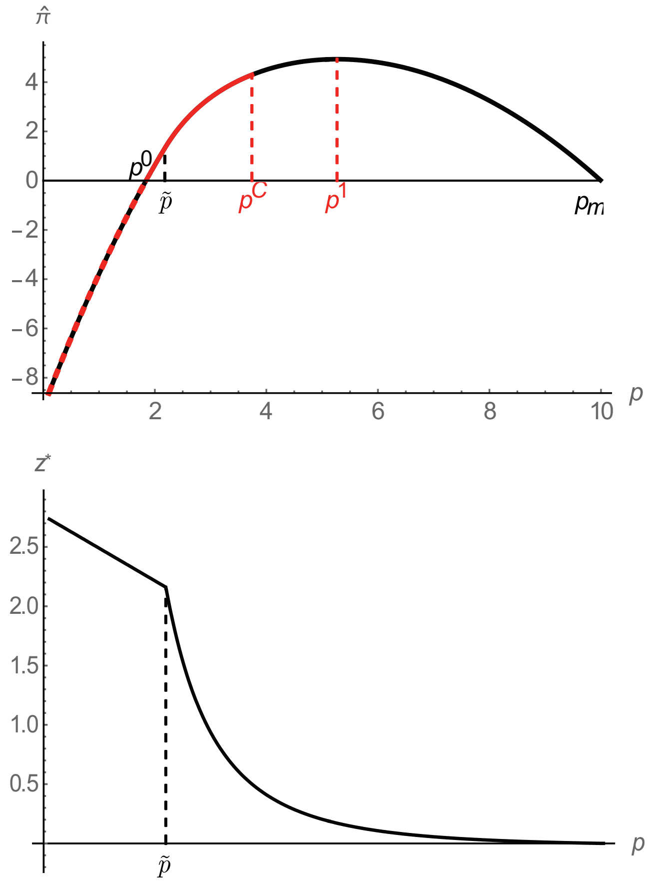


Figure 1: A first example ($\rho = 1$, $\alpha = .7$, $n = 2$, w_1=1, w_2=1).

The upper graph of Figure 1 shows the whole function $\Pi(p, n) = \hat{\pi}(p, z^*(p, n), n)$ for $p \in (0, p_{max})$. The left dotted part, for $p \in (0, p^0)$, corresponds to negative profits. Thus, this price interval cannot be a *SPNE* of the two-stage game. The right part of the curve (in black) does not fulfil the **non-existence of limit pricing strategy** (checked at Step 4 of our procedure), they cannot correspond either to a *SPNE*. Consequently, the remaining red part of the curve, corresponds to values of p for which both **profitability** and **non-existence of limit pricing strategy** conditions are fulfilled. It means that every couple (p, z) such that p belongs to the interval $[p^0, p^C]$ and $z = z^*(p, n)$ are *SPNE* of the two-stage game. It is easy to check that p^C corresponds to the "Payoff dominant" *SPNE* of the whole game (the solution of program $(\mathcal{P}_2)$).

With this vector of parameters, we can see that the price p^1 that maximizes $\Pi(p, n)$ does not correspond to a *SPNE*.

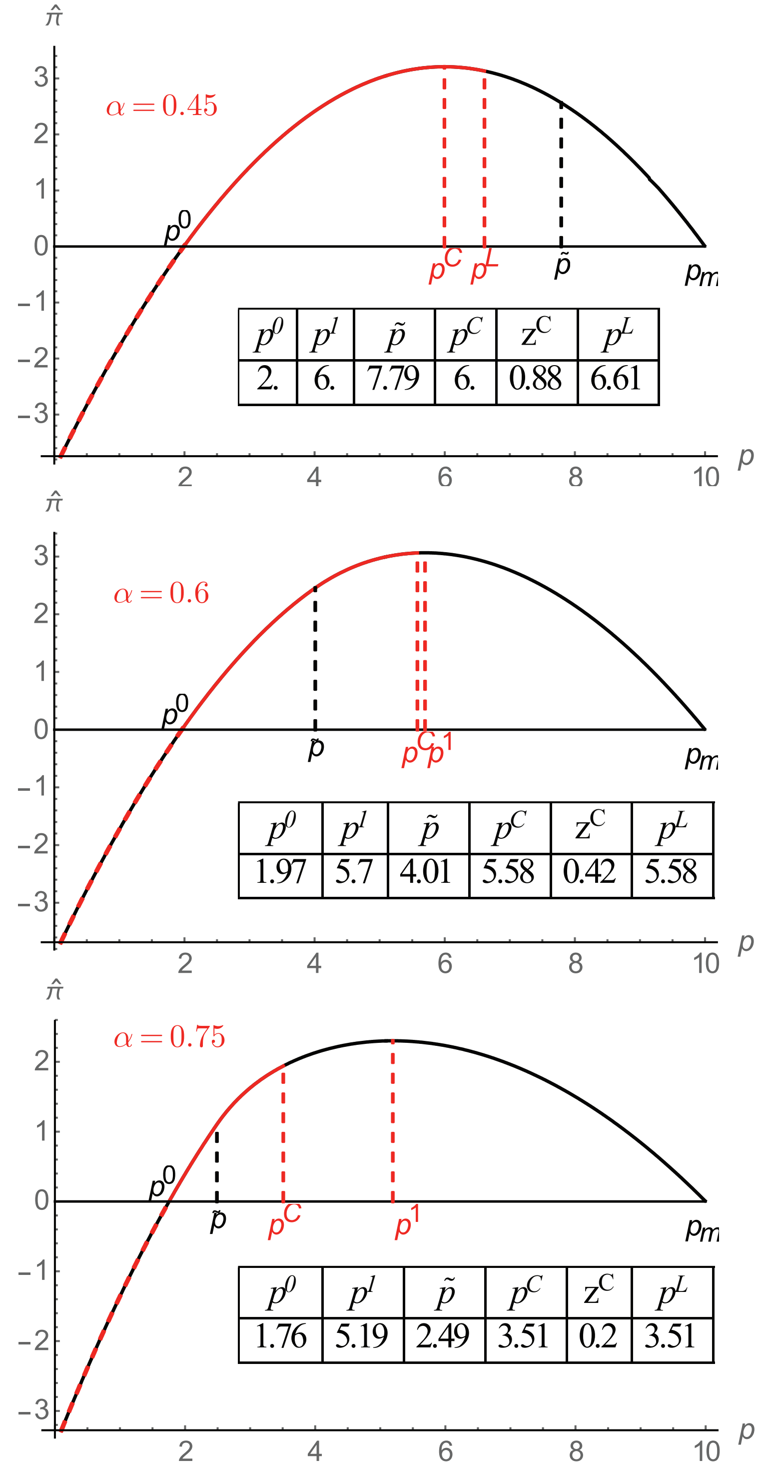
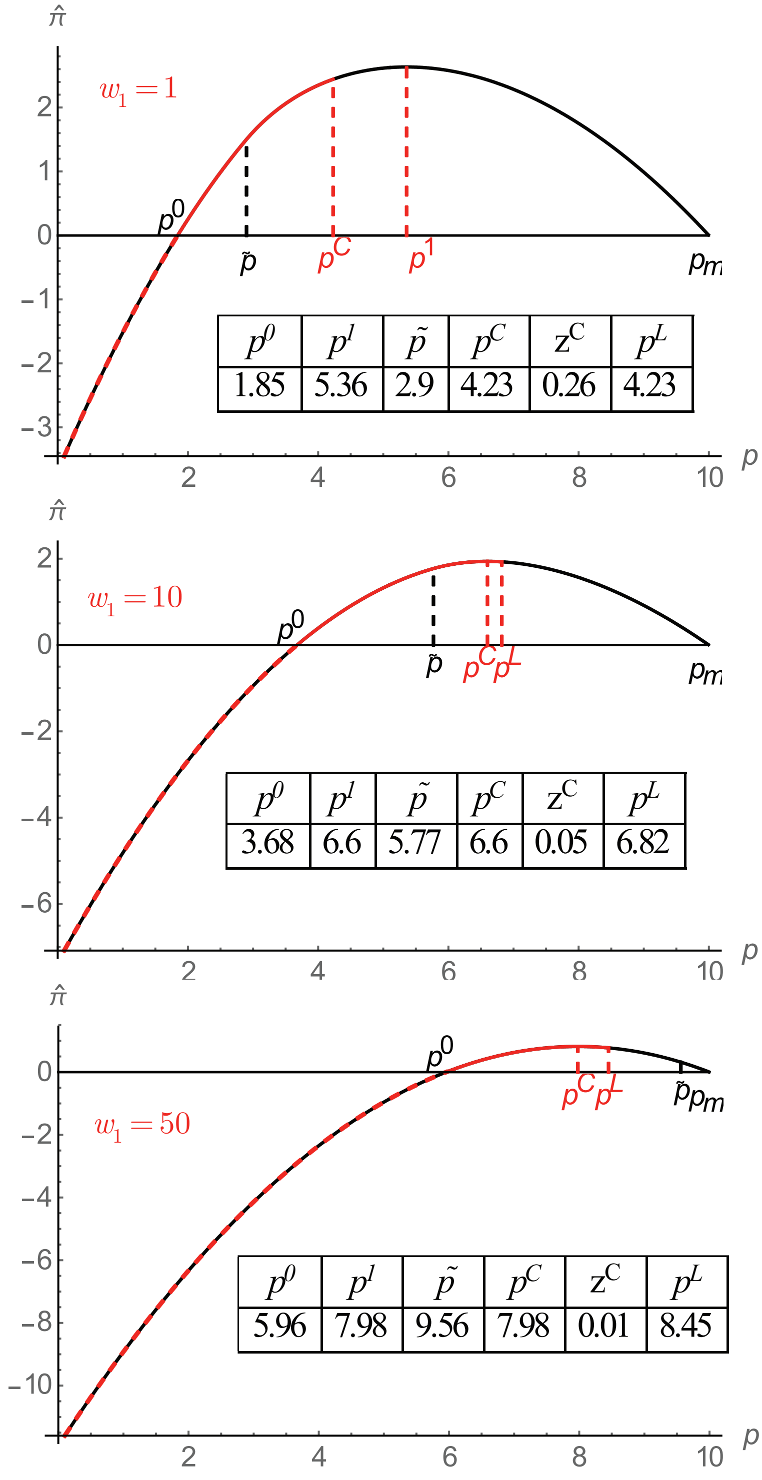


Figure 2: Effect of α ($\rho = 1$, $n = 5$, w_1=1, w_2=1).

In our model, the convexity of the cost function in the second stage is crucial to obtain our results. When the production function is Cobb-Douglas, in the second stage (with z fixed), this convexity is determined by the product $\alpha\rho$. When $\alpha\rho$ tends to one, the variable cost function tends to be linear. For a given level of the scale elasticity, ρ, a lower level of α coresponds to "more convex" production function. Figure 2 shows the effect of different levels of convexity on the equilibrium prediction of the whole game. In the upper graphic, $\alpha = .45$ corresponding to a significant level of convexity of the variable cost function. In this case, $\tilde{p}$ tends to be high: The more convex variable cost function implies that price deviation in the second stage is more costly (here, with five firms, the deviating firm will have to produce approximatively five times more.) Thus, p^C and p^L will not be $\bar{p}$, and, the corresponding z^* will be higher. That is the reason why $p^L > p^1 = p^C$. The maximum of $\Pi(p, n)$ corresponds to the solution of Program ($\mathcal{P}2$). In the intermediary graphic, $\alpha = .6$. We can see that, $\tilde{p}$ is now lower than p^1 and $p^L = p^C$. Thus p^1 and $p^L = p^C$ are $\bar{p}$ (the constraint of non-profitable deviation in second stage is binding). z^C is much lower. In p^1, limit pricing strategies are profitable, p^1 is not a *SPNE*. In the lower graph, $\alpha = .75$, the properties are essentially the same, with a p^C much more lower than p^1. The **non-existence of limit pricing strategy** constraint excludes more than half of the price in the interval $[p^0, p^1]$ from being a *SPNE*. Finally, in this example, when $\alpha \to 0.45 \to 0.6 \to 0.75$, $p^C \to 6 \to 5.58 \to 3.51$ and $z^C \to 0.88 \to 0.42 \to 0.20$. The lower is the convexity, the lower are the equilibrium price and level of fixed factor.

Figure 3 shows the effect of the price of the fixed factor on the price prediction of the model. In the upper graphic, both factors have the same price ($w_1 = w_2 = 1$). In the intermediary graphic $w_1 = 10$ and in the lower $w_1 = 50$! As we can see, $p^C \to 4.23 \to 6.6 \to 7.98$ and $z^C \to 0.26 \to 0.05 \to 0.01$. The higher the cost of fixed factor, the less you use it, and the higher is the equilibrium price.



p^0	p^1	$\tilde{p}$	p^C	z^C	p^L
1.85	5.36	2.9	4.23	0.26	4.23

p^0	p^1	$\tilde{p}$	p^C	z^C	p^L
3.68	6.6	5.77	6.6	0.05	6.82

p^0	p^1	$\tilde{p}$	p^C	z^C	p^L
5.96	7.98	9.56	7.98	0.01	8.45

Figure 3: Effect of w_1 ($\rho = 1$, $\alpha = 0.7$, $n = 5$, w_2=1).

4.3 Effect of the number of firms and returns to scale

Now, we are going to study how the price varies with the number of firms. We will show that the nature of the returns to scale have a qualitative impact on this relation.

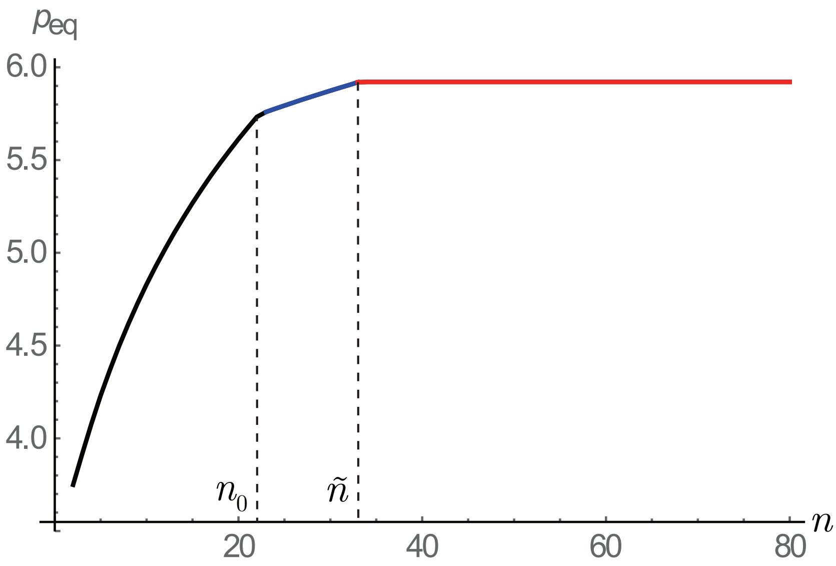


Figure 4: Effect of the number of firms on price($\rho = 1$, $\alpha = 0.7$,w_1=1 , w_2=1).

Let us start with the case of constant returns to scale. Figure 4 puts in relation the equilibrium prediction p^C and the number of firms when ($\rho = 1$, $\alpha = 0.7$,w_1=1, w_2=1). Two important thresholds appear. When the number of firm is low (between 2 and n_0), the **non-existence of limit pricing strategy** constraint is binding. The less is n, the more effective is this constraint. The price increases with n. When n reaches n_0 and beyond, the **non-existence of limit pricing strategy** constraint is no more binding. Between n_0 and $\tilde{n}$, the constraint of non-profitable deviation in the second stage is binding. The corresponding prices are $\bar{p}$. The higher n, the less effective is this constraint. The price continues to increase with n. Beyond $\tilde{n}$, no more constraints are binding at the equilibrium. p^C corresponds to a purely collusive outcome. Because of the constant returns to scale, the market price p^C is independent from the size of the firm and thus, from the number of firms sharing the market.

The result of our model is very unusual! Price increases with the number of firms. The insights behind this property is still the same: the convexity of cost. In our model, a deviating firm (either in the second stage by lowering its price or, in the first stage, by playing limit pricing strategy in z) will catch all the market (i.e. operate the market alone). The increase of the production will be proportional to the number of firms. Because of the convexity of variable cost, the higher is the increase, the lower is the incentive to deviate. However, we have to be careful in the explanations, because of the endogeneity of z. For example, as we have shown in Lemma 5, $\hat{p}$, $\bar{p}$ and p^* decrease with z for a given n and in n for a given z. But, in the equilibrium, for a given p, z is a z^* and decreases with n.

With the same values of parameters, excepted that the returns to scale are decreasing ($\rho = 0.9$), Figure 5 shows the same qualitative properties. The only difference is that for the right part of the curve (in red), the price decreases with the number of firms. That is the direct effect of decreasing returns to scale. Smaller firms will be more efficient and will have interest, when the outcome is purely collusive, to sustain slightly lower prices.

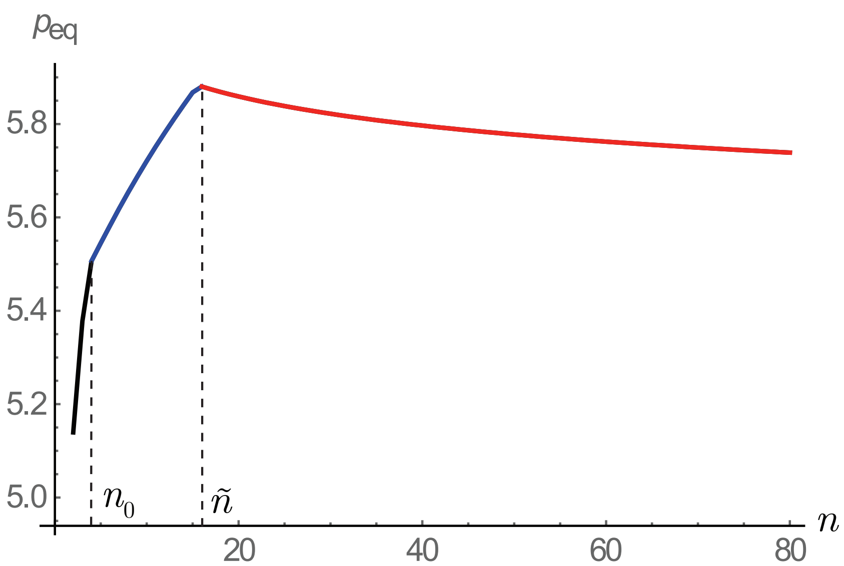


Figure 5: Effect of the number of firms on price($\rho = 0.9$, $\alpha = 0.7$,w_1=1 , w_2=1).

Figure 6 illustrates the case of increasing returns to scale ($\rho = 1.02$). The left part of the curve (take the lower envelop of the three curves) until the thresholds n_1 shows the same qualitative results as before, with the purely collusive part (in red, between $\tilde{n}$ and n_1 being slightly increasing because of the increasing returns to scale (in increasing number of smaller firms sharing the market is less efficient). The novelty is that, beyond the threshold n_1 the **non-existence of limit pricing strategy** constraint is binding again. There is an efficiency gain of the limit pricing strategy due to increasing returns to scale. Beyond n_1, this gain is sufficient to cancel the effect of the convexity of variable cost described previously.

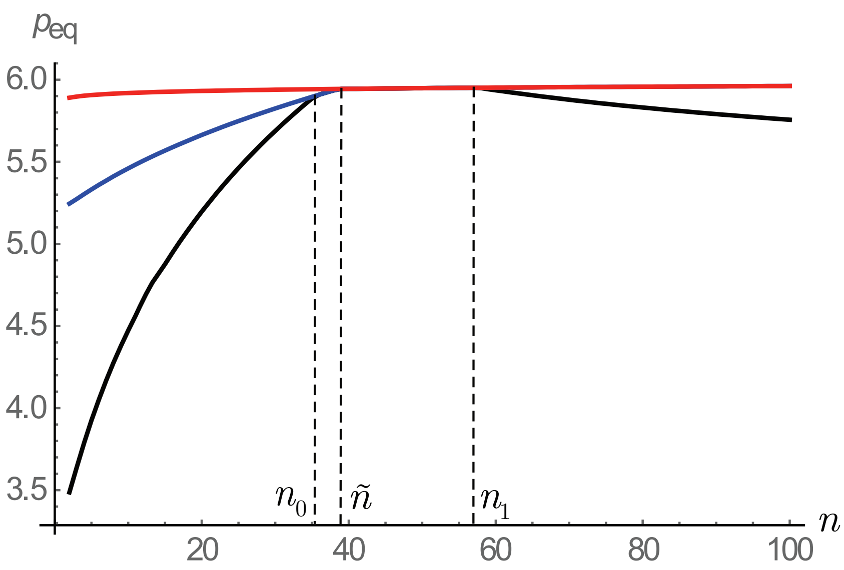


Figure 6: Effect of the number of firms on price($\rho = 1.02$, $\alpha = 0.7$,w_1=1 , w_2=1).

5 Conclusion

The general model of price competition with soft capacity constraint is a simple and realistic extension of standard literature in price competition that bridges three lines of literature: capacity constraints, cost convexity and limit pricing strategy. We illustrate that an equilibrium may exist whatever the number of firms and the nature of the returns to scale. Moreover, price may increase with the number of firms.

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