On Stable Outcomes of the Multilateral Matching*

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Abstract

This paper considers the multilateral matching market, where two or more agents can make a contract on a joint venture multilaterally. The possible joint ventures are exogenously given, and the preference relation of each agent is represented by a quasilinear utility function consisting of the valuation on the joint venture and the monetary transfer. We investigate three stability concepts: the weak setwise stable outcome, the stable outcome, and the strongly group stable outcome. We show that if the structure of the possible joint ventures satisfies a condition called the acyclicity, then these three stability concepts are equivalent with each other, are efficient, and exist for any continuous valuation functions. We also show that the acyclicity is necessary to guarantee the equivalence and the efficiency of the stability concepts for any continuous and concave valuation functions. For the existence, the acyclicity is is a necessary condition for the stable and the strongly group stable outcomes. On the other hand, we need an additional condition to obtain a necessary condition for the existence of the weakly setwise stable outcome.

Keywords: multilateral matching, acyclic venture structure, stable outcome, weakly setwise stable outcome.
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1 Introduction

Multilateral agreements are made in many economic environments: Joint ventures are formed between two or more firms to produce new products or services by pooling resources; Clubs are associated with two or more agents; International agreements involves two or more countries. To analyze equilibrium agreements in such situations, Hatfield and Kominars (2015) introduced a model of multilateral matching. In their model, a joint project (or activity) of two or more agents is called a *venture*. The possible ventures are exogenously given. Agents are allowed to participate in multiple ventures with monetary transfers. Each agent has quasilinear utility function consisting of the valuation on the ventures that involve him/her and the monetary transfer. An agreement on a venture is represented by a multilateral contract that specifies the venture's participation (or output) level and monetary transfers. A feasible outcome can be represented as a set of multilateral contracts.

As an equilibrium notion, Hatfield and Kominars (2015) defined two stability concepts which are from matching theory originated with Gale and Shapley (1962).[1] The stronger stability concept is called *strongly group stability*.[2] An outcome is a strongly group stable if it is *individually rational*: no agent can be made strictly better off by dropping some of current contracts and it is not *strongly blocked*: no group of agents can be made strictly better off by signing new contracts possibly dropping some of current contracts. The weaker stability concept is simply called *stability*.[3] The only difference with the strong stability is that the blocking concept underlying the stability imposes an additional requirement into the strong blocking: The new contracts associated with the strong blocking should be contained in any optimal choice from the new and current contracts by the agents involved in the strong blocking. In general, every strongly group stable outcome is efficient while a stable outcome may be inefficient.[4]

[1]Their model does not subsume Gale and Shapley (1962)'s. However, their modeling enables us to use the matching theoretic stability concepts.

[2]The strongly group stability is originally introduced by Konishi and Ünver (2006) in the context of a many-to-many matching problem.

[3]This stability concept is used in Hatfield and Kominars (2012) and Hatfield et al. (2013) in the context of the model of trading networks with bilateral contracts.

[4]It is well recognized that the stability is incompatible with the efficiency in many-to-many matching models or more complex matching models. For example, Blair (1988) showed that a stable outcome may be inefficient in a many-to-many matching model. See also Sotomayor (1999), Echenique and Oviedo (2006), Konishi and Ünver (2006), Klaus and Walzl (2009), and Westkamp (2010) for related studies.

Hatfield and Kominars (2015) also defined the notion of competitive equilibrium. They showed that any competitive equilibrium outcome is strongly group stable and a competitive equilibrium outcome exists when agents have concave valuation functions. Therefore, a strongly group stable outcome and a stable outcome exist under the concavity assumption while they may not exist in general. Moreover, they claimed that any stable outcome is efficient when agents have concave valuation functions. Although the concavity assumption is crucial to guarantee the efficiency and existence of stable outcomes, it is not suitable in some situations. A typical example of violating the concavity assumption is increasing return to scale of the production function.

The purpose of this paper is to further investigate properties of stable outcomes under general valuation functions.[5] Specifically, we consider three stability concepts; the strong group stability, the stability and the weakly setwise stability. The last one is newly introduced in this paper. As Hatfield and Kominars (2015) pointed out, the blocking concept underlying the stability (or strongly group stability) allows somewhat unrealistic deviations: The deviating agents may disagree with the outcome after the deviation. The blocking concept underlying the weakly setwise stability excludes such a deviation. In general, the weakly setwise stability is a weaker notion than the stability.

We impose restrictions on the structure of possible ventures and allow general valuation functions, in contrast to Hatfield and Kominars (2015) who allowed general structure of possible ventures and restricted valuation functions to concave one. The key notion is an *acyclicity* of the venture structure. The acyclicity notion here is quite straightforward: For example, when agents i_1 and i_2 are involved in a certain venture and agents i_2 and i_3 are in another venture, there do not exist any ventures that involve i_1 and i_3. We show that the acyclicity is a necessary and sufficient condition for the equivalence of three stability concepts and efficiency of stable outcomes. More specifically, the venture structure is acyclic if and only if (1) the three stability concepts are mutually equivalent for all valuation functions, (2) any stable outcome is efficient for all concave valuation functions. Note that this result implies that without acyclicity, some stable outcomes may be inefficient even under the concavity assumption. Therefore, it shows that one of the statements of Hatfield and Kominers (2015) is incorrect.

[5]Hatfield and Kominars (2015) showed that without concavity, a competitive equilibrium may not exist even if there exists only one venture. This is the main reason to use the stability concepts as equilibrium notion instead of the competitive equilibrium.

We also show that the acyclicity is a necessary and sufficient condition for the existence of a strongly group stable outcome and stable outcome: The venture structure is acyclic if and only if a strongly group stable outcome (or stable outcome) exists for any valuation functions. It should be remarked that the existence of a strongly stable outcome (or stable outcome) under the acyclicity does not follow from the results of Hatfield and Kominers (2015) because we do not require the concavity assumption. The acyclicity is not necessary for the existence of a weakly setwise stable outcome. We provide some additional results for the existence (or nonexistence) of weak setwise stable outcomes.

Methodologically, there are some studies related to ours. Pápai (2004) analyzed a coalition formation model which can be considered as our model such that each agent can participate at most one venture without monetary transfers. In the setting of Pápai (2004), the stability always implies efficiency in contrast to our model. It was shown that an acyclicity of permissible coalitions is a necessary and sufficient condition for a stable coalition structure to uniquely exist. The stability and acyclicity notions in Pápai (2004) are essentially the same as in ours. However, the difference between the models prevents us to directly apply the technique used in Pápai (2004) for obtaining our results.

Our work is also related to Westkamp (2010). He showed that in a supply chain model introduced by Ostrovsky (2008), an acyclicity for a market structure is a necessary and sufficient condition for stable outcomes to be efficient.[6] His model assumes that agents preferences satisfy the condition called full substitutability. Moreover, the acyclicity notion used in Westkamp (2010) differs from ours. Therefore, a straightforward comparison of our results with his is not possible.

The rest of this paper is organized as follows: In Section 2, the model of multilateral matching is introduced and the stability concepts are also defined. In Section 3, our main results are presented. Section 4 concludes our study. Some proofs are postponed to the Appendices.

[6] More precisely, he considered the chain stability introduced by Ostrovsky (2008) instead of the stability. The result of Hatfield and Kominars (2012) implies that the chain stability are equivalent to the stability in the model of Westkamp (2010).

2 Model

2.1 Multilateral matching

We basically follow the definitions and notations of Hatfield and Kominars (2015) with some additional notations. Let I be the finite set of agents and Ω be the finite set of ventures. Each $\omega \in \Omega$ is associated with $a(\omega) \subseteq I$ where the cardinality of $a(\omega)$ is no less than 2 for all $\omega \in \Omega$. We call (I, Ω, a) the venture structure.

For each $\omega \in \Omega$, agents in $a(\omega)$ choose the participation level $r_\omega \in [0, r_\omega^{\max}]$ to ω. For each venture ω, the participation level is bounded from $r_\omega^{\max} > 0$. The tuple of the participation levels $r = (r_\omega)_{\omega \in \Omega}$ is called the allocation. For any allocation r and $\Omega' \subseteq \Omega$, denote $r_{\Omega'} = (r_\omega)_{\omega \in \Omega'}$. Each agent $i \in I$ is endowed with the valuation function $v^i(r)$ on $\times_{\omega \in \Omega}[0, r_\omega^{\max}]$. It is assumed that the valuation of an agent depends only on the participation levels to the ventures that are associated with that agent, *i.e.* for any $i \in I$, $\omega \in \Omega$, allocation r, and $r'_\omega \in [0, r_\omega^{\max}]$, $v^i(r) = v^i(r'_\omega, r_{\Omega \setminus \{\omega\}})$ if $i \notin a(\omega)$. It is also assume that the valuation function is continuous on $\times_{\omega \in \Omega}[0, r_\omega^{\max}]$ throughout the paper. An allocation $\hat{r}$ is efficient if $\hat{r} \in \arg\max_{r \in \times_{\omega \in \Omega}[0, r_\omega^{\max}]} \sum_{i \in I} v_i(r)$. We say $(I, \Omega, a, r^{\max}, v)$ a multilateral matching market, where $r^{\max} = (r_\omega^{\max})_{\omega \in \Omega}$ and $v = (v^i)_{i \in I}$.

We say that a venture structure (I, Ω, a) admits a cycle if there exist distinct $i_1, ..., i_k \in I$ and distinct $\omega_1, ..., \omega_k \in \Omega$ with $k \geq 2$ such that $\{i_h, i_{h+1}\} \subseteq a(\omega_h)$ for all $h = 1, ..., k$, where $k + 1 \equiv 1$. We sometimes refer such a cycle as a k-cycle. We say that a venture structure (I, Ω, a) is acyclic if it admits no cycle.

Hatfield and Kominars (2015) also gave a representation of a multilateral matching matket $(I, \Omega, a, r^{\max}, v)$ in terms of contracts. A contract $x = (\omega, r_\omega, s_\omega)$ is consisting of a venture ω, the participation level r_ω to ω, and a transfer vector $s_\omega = (s_\omega^i)_{i \in I}$ such that $\sum_{i \in I} s_\omega^i = 0$ and $s_\omega^i = 0$ for all $\omega \in \Omega$ and $i \notin a(\omega)$. Let

$$X = \left\{ (\omega, r_\omega, s_\omega) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^I \;\middle|\; r_\omega \leq r_\omega^{\max};\ \sum_{i \in a(\omega)} s_\omega^i = 0;\ s_\omega^i = 0, \forall i \notin a(\omega) \right\}$$

denote the set of all contracts. Given a contract $x = (\omega, r_\omega, s_\omega)$ denote $\tau(x) = \omega$ and $a(x) = a(\tau(x))$.

A subset $Y \subseteq X$ is said to be an outcome if for any $x, y \in Y$, $x \neq y$ implies $\tau(x) \neq \tau(y)$. For an outcome Y and $i \in I$, denote $Y_i = \{y \in Y | i \in a(y)\}$. For an outcome Y, denote $\tau(Y) = \bigcup_{x \in Y}\{\tau(x)\}$, $a(Y) = \bigcup_{\omega \in \tau(Y)} a(\omega)$, $\rho(Y) = (\rho_\omega(Y))_{\omega \in \Omega}$

where

$$\rho_\omega(Y) = \begin{cases} r_\omega & \text{if } (\omega, r_\omega, s_\omega) \in Y; \\ 0 & \text{otherwise,} \end{cases}$$

and $\sigma(Y) = (\sigma_\omega(Y))_{\omega \in \Omega}$ where

$$\sigma_\omega(Y) = \begin{cases} s_\omega & \text{if } (\omega, r_\omega, s_\omega) \in Y; \\ 0 & \text{otherwise.} \end{cases}$$

For each $i \in I$ and an outcome Y, the utility of i from Y is defined by

$$u^i(Y) = v^i(\rho(Y)) - \sum_{\omega \in \Omega} \sigma^i_\omega(Y) = v^i(\rho(Y)) - \sum_{\omega \in \tau(Y_i)} \sigma^i_\omega(Y).$$

An outcome A is said to be efficient if $\rho(A)$ is efficient.

The choice correspondence of $i \in I$ is defined by

$$C^i(Y) = \arg \max_{Z \subseteq Y_i, Z \text{ is an outcome}} \{u^i(Z)\},$$

where $Y \subseteq X$ may not be an outcome.

2.2 Stability concepts

We turn to defining the stability concepts in the multilateral matching models. We begin with the definition of the individual rationality, which is a basis for any stability concept. An outcome A is said to be individually rational for $i \in I$ if $A_i \in C^i(A)$. An outcome A is simply said to be individually rational if A is individually rational for all $i \in I$.

Hatfield and Kominars (2015) introduced the following two solution concepts..[7]

Definition 1

- *An outcome A is said to be blocked if there exists a nonempty $Z \subseteq X \setminus A$ such that for all $i \in a(Z)$ we have that $Z_i \subseteq Y^i$ for all $Y^i \in C^i(Z \cup A)$.*
- *An outcome A is said to be stable if it is individually rational and not blocked.*

We sometimes say that Z is a blocking set to A if it satisfies the first bullet of this definition.

Hatfield and Kominars (2015) also defined a stronger notion of stability

[7] Hatfield and Kominars (2015) further defined the core. However, we omit its definition because the present paper does not consider the core.

Definition 2 • *An outcome A is said to be strongly blocked if there exists a nonempty $Z \subseteq X \setminus A$ such that for all $i \in a(Z)$, there exists some $Y^i \subseteq Z \cup A$ such that $Z_i \subseteq Y^i$ and $u^i(Y^i) > u^i(A)$.*

- *An outcome A is said to be strongly group stable if it is individually rational and not strongly blocked.*

We sometimes say that Z is a strong blocking set to A if it satisfies the first bullet of this definition. Note that if Z is a blocking set to A, then it is a strong blocking set to A. To see this, suppose that Z is a blocking set to A. Then, we have that $u^i(Y^i) > u^i(A)$ for all $Y^i \in C^i(Z \cup A)$ and all $i \in a(Z)$. Otherwise, for some $i \in a(Z)$ and $Y^i \in C^i(Z \cup A)$, $u^i(Y^i) \leq u^i(A)$ which implies that $A_i \in C^i(Z \cup A)$, where $A_i \cap Z_i = \emptyset$, contradicting that Z is a blocking set. Therefore, if an outcome is blocked, then it is strongly blocked and hence any strongly group stable outcome is a stable outcome. Note that they may be different in a general case (See the example in pp. 187 in Hatfield and Kominars (2015)). We also note that any strongly group stable outcome is efficient by the definition. See also Theorem 6 of Hatfield and Kominars (2015).

As pointed out in Footnote 28 of Hatfield and Kominars (2015), the agents associated with (strong blocking) set Z are not required to agree on maintaining the contracts in the original outcome. The following example clearly describes this kind of disagreement.

Example 1 Let $I = \{i_1, i_2\}$, $\Omega = \{\omega_1, \omega_2\}$, $a(\omega_1) = a(\omega_2) = I$, $r^{\max}_{\omega_1} = r^{\max}_{\omega_2} = 1$, $v^{i_1}(r) = 2(r_{\omega_1} + r_{\omega_2})$, and $v^{i_2}(r) = -|1 - r_{\omega_1} - r_{\omega_2}|$. The valuation function of agent i_1 is linearly separable and monotone increasing in the participation levels, while the valuation function of agent i_2 is maximized at the intermediate total participation level. Let $a_1 = (\omega_1, 1/2, (1/2, -1/2))$, $a_2 = (\omega_2, 1, (0, 0))$, and $A = \{a_1, a_2\}$. It is easy to see that A is individually rational.

Let $\hat{z}_1 = (\omega_1, 1, (1/2 + \varepsilon, -1/2 - \varepsilon))$ and $\hat{Z} = \{\hat{z}_1\}$, where $\varepsilon > 0$ is a sufficiently small real number. From easy calculations, we have that $C^{i_1}(\hat{Z} \cup A) = \{\{\hat{z}_1, a_2\}\}$ and $C^{i_2}(\hat{Z} \cup A) = \{\hat{Z}\}$. Therefore, A is blocked via $\hat{Z}$ and is not stable, while players i_1 and i_2 cannot agree on whether they maintain contract a_2.

To avoid such a disagreement after the block, we introduce a weaker notion of the blocking concept that the agents associated with the blocking set are required to agree on some outcome and define the stability under this blocking concept.

Definition 3 • *An outcome A is said to be weakly setwise blocked if there exists a nonempty $Z \subseteq X \setminus A$ such that for all $i \in a(Z)$ we have that $Z_i \subseteq Y^i$ for all $Y^i \in C^i(Z \cup A)$ and there exists some outcome Y^* such that $Y_i^* \in C^i(Z \cup A)$ for all $i \in a(Z)$.*

- *An outcome A is weakly setwise stable if it is individually rational and not weakly setwise blocked.*

We sometimes say that Z is a weakly setwise blocking set to A if it satisfies the first bullet of this definition. Note that a blocking set, strong blocking set, or a weakly setwise blocking set Z is itself an outcome.

Remark 1 The notion of the weak setwise stability concept stems from the similar concept in many-to-many matching (with contracts) by Klaus and Walzl (2009). The strong group stability also has the similar source in the sense that it is a stronger concept than both the strong stability by Hatfield and Kominars (2016) and the group stability by Konishi and Ünver (2006) in many-to-many matching, where the latter is originated in Roth and Sotomayor (1990). See Hatfield et al. (2013) (Subsection IV.A.) or Hatfield and Kominars (2015) (Footnote 29) for more detailed arguments.

From the definition, if Z is a weakly setwise blocking set to A, then it is a blocking set to A. Therefore, any stable outcome is weakly setwise stable. The weakly setwise stability is essentially a weaker concept than the stability. Indeed, A appeared in Example 1 is weakly setwise stable although A was not stable.

Example 1. (Cont'd.) Recall the model and the outcome A in Example 1. Note that we have $u^{i_1}(A) = 5/2$ and $u^{i_2}(A) = 0$. Here, we give an intuitive arguments for the stability of A. A rigorous proof for a more general case will be given in Appendix A. Suppose that A is weakly setwise blocked via some Z. Let Y^* such that $Y_{i_1}^* \in C^{i_1}(Z \cup A)$ and $Y_{i_2}^* \in C^{i_2}(Z \cup A)$ be the resulting outcome at which agents i_1 and i_2 agree. Since Z is a weakly setwise blocking set to A, $u^{i_1}(Y^*) + u^{i_2}(Y^*) > 5/2 = u^{i_1}(A) + u^{i_2}(A)$. Then, it must be $\rho_{\omega_1}(Y^*) + \rho_{\omega_2}(Y^*) > 3/2$ because $u^{i_1}(Y^*) + u^{i_2}(Y^*) = v^{i_1}(\rho(Y^*)) + v^{i_2}(\rho(Y^*))$ and

$$
\begin{aligned}
v^{i_1}(r_{\omega_1}, r_{\omega_2}) + v^{i_2}(r_{\omega_1}, r_{\omega_2}) &= 2(r_{\omega_1} + r_{\omega_2}) - |1 - r_{\omega_1} - r_{\omega_2}| \\
&= \begin{cases} 3(r_{\omega_1} + r_{\omega_2}) - 1 & \text{if } r_{\omega_1} + r_{\omega_2} \leq 1 \\ r_{\omega_1} + r_{\omega_2} + 1 & \text{if } r_{\omega_1} + r_{\omega_2} > 1. \end{cases}
\end{aligned}
$$

Note that this also yields that $\tau(Y^*) = \Omega$ by $r^{\max}_{\omega_1} = r^{\max}_{\omega_2} = 1$. Let $Y^* = \{y_1^*, y_2^*\}$.

We claim that $\omega_2 \notin \tau(Z)$. Suppose that $\omega_2 \in \tau(Z)$. It follows that $y_2^* \neq a_2$ from $Z \subset Y^*$ and $Z \cap A = \emptyset$, in particular $\rho_{\omega_2}(Y^*) \leq 1$. Then, $v^{i_1}(\rho(\{y_1^*, a_2\})) - v^{i_1}(\rho(Y^*)) = 2(1 - \rho_{\omega_2}(Y^*))$ by the linearity of v^{i_1}. We also have

$$v^{i_2}(\rho(\{y_1^*, a_2\})) - v^{i_2}(\rho(Y^*)) = -\rho_{\omega_1}(Y^*) + |1 - \rho_{\omega_1}(Y^*) - \rho_{\omega_2}(Y^*)| \geq 1 - \rho_{\omega_2}(Y^*)$$

by the property of the absolute value. Summing up these (in)equalities, we have $(v^{i_1}(\rho(\{y_1^*, a_2\})) + v^{i_2}(\rho(\{y_1^*, a_2\}))) - (v^{i_1}(\rho(Y^*)) + v^{i_2}(\rho(Y^*))) \geq 1 - \rho_{\omega_2}(Y^*) \geq 0$. This yields either $u^{i_1}(\{y_1^*, a_2\}) \geq u^{i_1}(Y^*)$ or $u^{i_2}(\{y^*, a_2\}) \geq u^{i_2}(Y^*)$, violating $Z_i \subset Y^i$ for all $Y^i \in C^i(Z \cup A)$ for $i = i_1, i_2$. Hence $\tau(Z) = \{\omega_1\}$. Note that $\rho_{\omega_1}(Z) > 1/2$ since we need to have $\rho_{\omega_1}(Y^*) + \rho_{\omega_2}(Y^*) > 3/2$.

However, this kind of block cannot make agents i_1 and i_2 agree whether they maintain a_2 or not, like $\hat{Z}$ appeared in the former part of this example. Indeed, $a_2 \in Y^{i_1}$ for any $Y^{i_1} \in C^{i_1}(Z \cup A)$ by the linearity of v^{i_1} and $\tau(Z) = \{\omega_1\}$. In order to make $a_2 \in Y^{i_2}$ for any $Y^{i_2} \in C^{i_2}(Z \cup A)$ with $Z \subset Y^{i_2}$, we need

$$-\rho_{\omega_1}(Z) - \sigma^{i_1}_{\omega_1}(Z) = u^{i_2}(Z \cup \{a_2\}) > u^{i_2}(Z) = \rho_{\omega_1}(Z) - 1 - \sigma^{i_1}_{\omega_1}(Z).$$

It follows that $\rho_{\omega_1}(Z) < 1/2$, contradicting that $\rho_{\omega_1}(Z) > 1/2$. Hence, there is no weakly setwise blocking set to A.

3 Results

3.1 Relationship between stability concepts and efficiency

In the previous section, we observe that the three stability concepts are generally different with each other, and the efficiency of the stable and weakly setwise stable outcomes do not follow from the definitions. In this sebsection, we show that the acyclicity of the venture structure is not only the sufficient condition but also the necessary condition for guaranteeing the equivalence and efficiency of the stability concepts.

Theorem 1 *Let a venture structure (I, Ω, a) and a maximum participation vector $r^{\max}$ are given. The following statements are equivalent:*

(a) (I, Ω, a) *is acyclic.*

(b) *Any weakly setwise stable outcome is strongly stable for any tuple of valuation functions* v.

(c) *Any weakly setwise stable outcome is stable for any tuple of valuation functions* v.

(c') *Any weakly setwise stable outcome is stable for any tuple of concave valuation functions* v.

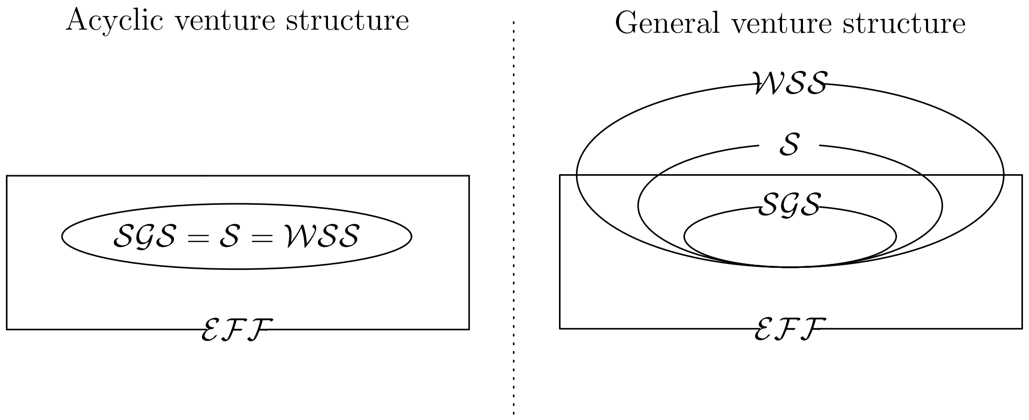
(d) *Any stable outcome is strongly group stable for any tuple of valuation functions* v.

(e) *Any stable outcome is efficient for any tuple of valuation functions* v.

(e') *Any stable outcome is efficient for any tuple of concave valuation functions* v.

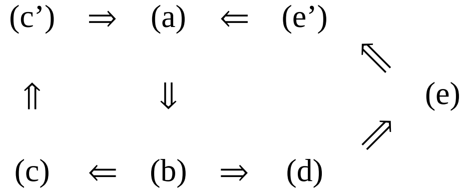
Theorem 1 is summarized in Figure 1, where $\mathcal{WSS}$, $\mathcal{S}$, $\mathcal{SGS}$, and $\mathcal{EFF}$ stand for the weakly setwise stable outcomes, the stable outcomes, the strongly group stable outcomes, and the efficient outcomes, respectively.

Figure 1: Relationship between the stability concepts.



The proof of Theorem 1 is done as Figure 2. Among those implications, (b)⇒(c), (b)⇒(d), and (d)⇒(e) follow from the definitions of the stability concepts; (c)⇒(c') and (e)⇒(e') are obvious. Among the remaining nontrivial implications, we prove implication (a)⇒(b) here, while the proofs of implications (c')⇒(a) and (e')⇒(a) are postponed to the Appendix. Indeed, the essence of implication (c')⇒(a) has been shown by Example 1 appeared in the previous section. We also show the essence of implication (e')⇒(a) Example 2 at the end of this subsection for the intuitive explanation.

Figure 2: An outline for the proof of Theorem 1.



Proof of Theorem 1(a)⇒(b). Let (I, Ω, a) be an acyclic venture structure and $r^{\max}$ be a maximum participation vector. Fix an arbitrary tuple of valuation functions v. Before proving this statement, we introduce a property related to the acyclicity. We say that (I, Ω, a) admits a weak cycle if there exist distinct $i_1, ..., i_k \in I$ with $k \geq 2$ and $\omega_1, ..., \omega_k \in \Omega$ such that (i) $\{i_h, i_{h+1}\} \subseteq a(\omega_h)$ for all $h = 1, ..., k$, where $k+1 \equiv 1$ and (ii) $\omega_1 \neq \omega_k$. We refer such a cycle as k-weak cycle.

Claim 1 (I, Ω, a) *admits a cycle if and only if it admits a weak cycle.*

Proof. The "only if" part is clear from the definition. To show the "if part", suppose that (I, Ω, a) admits a weak cycle. Consider a minimal weak cycle consisting of $i_1, ..., i_k \in I$ with $k \geq 2$ and $\omega_1, ..., \omega_k \in \Omega$, i.e., there exists no k'-weak cycle with $k' < k$. We show that $i_1, ..., i_k$ and $\omega_1, ..., \omega_k$ consistute a cycle. Suppose not. Then, either (a) $\omega_1 = \omega_h$ for some h with $1 < h < k$, (b) $\omega_h = \omega_k$ for some h with $1 < h < k$, or (c) $\omega_h = \omega_{h'}$ for some h, h' with $1 < h < h' < k$ holds. In the case (a), $i_1, i_{h+1} ..., i_k$ and $\omega_h, \omega_{h+1}, ..., \omega_k$ constitute a weak cycle. In the case (b), $i_1, ..., i_h$ and $\omega_1, ..., \omega_h$ constitute a weak cycle. In the case (c), $i_1, ., i_h, i_{h'+1}, ..., i_k$ and $\omega_1, ., \omega_h, \omega_{h'+1}, ..., \omega_k$ constitute a weak cycle. All cases contradict the minimality of $i_1, ..., i_k$ and $\omega_1, ..., \omega_k$. ■

Let A be an individually rational outcome. We will show that if A is strongly blocked, then it is weakly setwise blocked. Therefore, we assume that there exists a strong blocking set Z to A. Note that Z is an outcome. We also note that for each $i \in a(Z)$ and each $Y^i \in C^i(Z \cup A)$, $Y^i \setminus A_i \neq \emptyset$. To see this, suppose that for some $i \in a(Z)$ and some $Y^i \in C^i(Z \cup A)$, $Y^i \setminus A_i = \emptyset$ holds. This implies that $Y^i \subseteq A_i$. By the individual rationality of A, we have that $u^i(A) \geq u^i(Y^i)$. Because Z is a strong blocking set to A, there exists $\bar{Y}^i \subseteq Z_i \cup A_i$ with $u^i(\bar{Y}^i) > u^i(A)$ and hence $u^i(\bar{Y}^i) > u^i(Y^i)$. However, this contradicts $Y^i \in C^i(Z \cup A)$.

For each $i \in a(Z)$, let X^i be an element of $C^i(Z \cup A)$ such that there exists no $Y^i \in C^i(Z \cup A)$ with $Y^i \backslash A_i \subsetneq X_i \backslash A_i$. Then, for each $i \in a(Z)$, we have $\tilde{X}_i := X^i \backslash A_i \neq \emptyset$ and $\tilde{X}_i \subseteq Z_i$. Moreover, we have that for each $i \in a(Z)$,

$$\tilde{X}_i \subseteq Y^i \text{ for all } Y^i \in C_i(\tilde{X}_i \cup A_i). \tag{1}$$

To see this, suppose that $\tilde{X}_i \nsubseteq Y^i$ for some $Y^i \in C_i(\tilde{X}_i \cup A_i)$. By $X^i \subseteq \tilde{X}_i \cup A_i$, we have $u^i(Y^i) \geq u^i(X^i)$ from the definition of the choice. By $Y^i \subseteq \tilde{X}_i \cup A_i$, we have $Y^i \subseteq Z_i \cup X_i$. From the definition of the choice, we have $u^i(X^i) \geq u^i(Y^i)$ and hence $u^i(Y^i) = u^i(X^i)$ holds. This implies that $Y^i \in C^i(Z \cup A)$. By $Y^i \subseteq \tilde{X}_i \cup A_i$, we have $Y_i \setminus A_i \subseteq \tilde{X}_i$. By $\tilde{X}_i \nsubseteq Y^i$, there exists $\tilde{x} \in \tilde{X}_i$ such that $\tilde{x} \notin Y^i$, which implies that $Y_i \setminus A_i \subsetneq \tilde{X}_i$. However, this contradicts the definition of X^i. Therefore, we obtain (1).

For every $i, j \in a(Z)$ with $i \neq j$, we write $i \to j$ if and only if there exists $x \in \tilde{X}_i$ such that $j \in a(x)$. For each $i \in a(Z)$, let $\epsilon(i) := \{j \in a(Z) | i \to j\}$. We next state and prove the following claim.

Claim 2 *Let $S \subseteq a(Z)$ with $S \neq \emptyset$. If S satisfies the following two-conditions:*

(i) for all $i \in S$, $\epsilon(i) \subseteq S$, and

(ii) for all $i, j \in S$, $i \to j$ implies $j \to i$,

then $\bigcup_{i \in S} \tilde{X}_i$ is a blocking set to A.

Proof of Claim 2. Suppose that S is a nonempty subset of $a(Z)$ that satisfies conditions (i) and (ii). Let $\tilde{Z} := \bigcup_{i \in S} \tilde{X}_i$. Note that $\tilde{Z} \neq \emptyset$ because $S \neq \emptyset$ and $\tilde{X}_i \neq \emptyset$ for all $i \in S$.

We begin with proving $a(\tilde{Z}) = S$. We have $S \subseteq a(\tilde{Z})$ by the fact that $i \in a(\tilde{X}_i)$ for all $i \in S$. To show $a(\tilde{Z}) \subseteq S$, pick any $i \in a(\tilde{Z})$. This implies that for some $k \in S$ and some $\tilde{x} \in \tilde{X}_k$, $i \in a(\tilde{x})$. If $k = i$, we obtain $i \in S$. Suppose that $k \neq i$. This implies that $k \to i$ and hence $i \in \epsilon(k)$. By (i), $i \in \epsilon(k) \subseteq S$. Therefore, we obtain $a(\tilde{Z}) \subseteq S$.

Then, we turn to proving $\tilde{Z}_i = \tilde{X}_i$ for all $i \in S$. Let $i \in S$. Clearly, $\tilde{X}_i \subseteq \tilde{Z}_i$ holds. Suppose that $\tilde{X}_i \subsetneq \tilde{Z}_i$. Then, there exists $x \notin \tilde{X}_i$ but $x \in \tilde{Z}_i$. From the definition of $\tilde{Z}$, $x \in \tilde{X}_j$ for some $j \in S$ with $j \neq i$. This implies that $j \to i$. By (ii), we have $i \to j$ and hence there exists $\tilde{x} \in \tilde{X}_i$ such that $j \in a(\tilde{x})$. By $x \notin \tilde{X}_i$, we have $x \neq \tilde{x}$. Because Z is an outcome, we have that $\tau(x) \neq \tau(\tilde{x})$ by $x, \tilde{x} \in Z$. However, this means that there exists a 2-cycle, which contradicts the acyclicity.

To complete the proof of this claim, we show $\tilde{Z}$ is a blocking set to A. By $Z \cap A = \emptyset$ and $\tilde{Z} \subseteq Z$, we have $\tilde{Z} \cap A = \emptyset$. Moreover, by (1) together with the fact that $a(\tilde{Z}) = S$ and $\tilde{Z}_i = \tilde{X}_i$ for all $i \in S$, we have that for all $i \in a(\tilde{Z})$, $\tilde{Z}_i \subseteq Y^i$ for all $Y^i \in C^i(\tilde{Z} \cup A)$. □

We next construct nonempty $S \subseteq a(Z)$ that satisfies (i) and (ii) of Claim 2 as below. Define

$$\mathcal{S} := \{S \subseteq a(Z) | S \neq \emptyset \text{ and } \epsilon(i) \subseteq S \text{ for all } i \in S\}.$$

Note that $a(Z)$ itself is an element of $\mathcal{S}$ and hence $\mathcal{S}$ is nonempty. Let S^* be a minimal element of $\mathcal{S}$. Then, S^* is nonempty and satisfies that for all $i \in S^*$, $\epsilon(i) \subseteq S^*$. Take any $i \in S^*$. Define

$$S_i = \left\{ i' \in S^* \,\middle|\, \begin{array}{l} \exists \text{ distinct } i_1, ..., i_k (k \geq 2) \text{ such that } i = i_1, i' = i_k, \\ i_1 \to i_2 \to \cdots \to i_k. \end{array} \right\} \bigcup \{i\}.$$

By $i \in S_i$ and $i \in S^*$, we have that $S_i \neq \emptyset$ and $S_i \subseteq S^*$. We claim that $\epsilon(i') \subseteq S_i$ for all $i' \in S_i$. Pick any $i' \in S_i$ and any $j' \in \epsilon(i')$. When $i' = i$, we have $j' \in S_i$ by $i \to j'$. So, we assume that $i' \neq i$. Then, there exist distinct $i_1, ..., i_k (k \geq 2)$ such that $i = i_1, i' = i_k$, and $i_1 \to i_2 \to \cdots \to i_k$. Note that $j' \in S^*$ by $i' \in S^*$ and $j' \in \epsilon(i')$. When $j' \in \{i_1, ..., i_k\}$, $j' \in S_i$ clearly holds. When $j' \notin \{i_1, ..., i_k\}$, $i_1, ..., i_k, j'$ are distinct agents with $i_1 \to i_2 \to \cdots \to i_k \to j'$, which implies $j' \in S_i$. Therefore, we have $\epsilon(i') \subseteq S_i$ for all $i' \in S_i$ and hence $S_i \in \mathcal{S}$. By the minimality of S^*, we have that

$$S_i = S^* \text{ for all } i \in S^*. \tag{2}$$

We next show that for every $i, j \in S^*$, $i \to j$ implies $j \to i$. Let $i, j \in S^*$ with $i \to j$. Suppose that $j \to i$ does not hold. From (2), $i \in S_j$ holds. By $i \to j$, we have $i \neq j$. So, there exist distinct $i_1, ..., i_k$ with $k \geq 2$ such that $i_1 = j \to i_2 \to \cdots \to i_k = i$. This implies that for each $h = 1, 2, \cdots, k-1$, there exists $\tilde{x}_h \in \tilde{X}_{i_h}$ such that $\{i_h, i_{h+1}\} \subseteq a(\tilde{x}_h)$. By $i \to j$, there exists $\tilde{x}_k \in \tilde{X}_i$ such that $\{i, j\} \subseteq a(\tilde{x}_k)$. Because $j \to i$ does not hold, we have that $i \notin a(\tilde{x}_1)$ and hence $\tau(\tilde{x}_1) \neq \tau(\tilde{x}_k)$. Therefore, $i_1, ..., i_k$ and $\tau(\tilde{x}_1), ..., \tau(\tilde{x}_k)$ constitute a weak cycle. This contradicts the acyclicity. Therefore, S^* satisfies conditions (i) and (ii) of Claim 2 and hence $Z^* := \bigcup_{i \in S^*} \tilde{X}_i$ is a blocking set to A. Note that the proof of Claim 2 implies that $a(Z^*) = S^*$ and $Z^*_i = \tilde{X}_i$ for all $i \in S^*$.

We finally show that Z^* is a weakly setwise blocking set to A. For each $i \in S^*$, fix any $Y^i \in C^i(Z^* \cup A)$ and let $Y^* := \bigcup_{i \in S^*} Y^i$. Because Z^* is a blocking set to A, it is sufficient to show that Y^* is an outcome and $Y^*_i = Y^i$ for all $i \in S^*$. Suppose that Y^* is not an outcome. Then, there exist two different contracts $x, y \in Y^*$ such that $\tau(x) = \tau(y)$. Since both A and Z^* are outcomes, one of these two contracts is in Z^* and the other is in $Y^* \setminus Z^* \subseteq A$; say $x \in Z^*$ and $y \in Y^* \setminus Z^*$. By $y \in Y^*$, there exists $i \in a(Z^*)$ such that $y \in Y^i$. By $x \in Z^*$, we have $x \in Y^i$. However, this contradicts Y^i is an outcome. Hence, Y^* is an outcome.

To complete the proof, it remains to show that $Y^*_i = Y^i$ for all $i \in S^*$. Suppose that $Y^*_i \neq Y^i$ for some $i \in S^*$. Then, we have that $Y^i \subsetneq Y^*_i$ because $Y^i \subseteq Y^*_i$ holds from the definiton. Therefore, there exists $x \in Y^*_i$ such that $x \notin Y^i$. Note that $x \in Y^*_i$ and $x \notin Y^i$ imply $\tau(x) \notin \tau(Z^*)$. To see this, suppose that $\tau(x) \in \tau(Z^*)$. Then, there exists $z \in Z^*$ such that $\tau(z) = \tau(x)$. Because Z^* is a blocking set to A, we have $z \in Y^i$ and hence $z \in Y^*_i$. By $z \in Y^i$, we have $x \neq z$. However, this contradicts the fact that Y^* is an outcome. Therefore, $\tau(x) \notin \tau(Z^*)$. By $x \in Y^*$, there exists $j \in S^*$ with $j \neq i$ such that $x \in Y^j$. From (2), $i \in S_j$ holds. By $i \neq j$, there exist distinct $i_1, ..., i_k$ with $k \geq 2$ such that $i_1 = j \to i_2 \to \cdots \to i_k = i$. This implies that for each $h = 1, 2, \cdots, k-1$, there exists $\tilde{x}_h \in \tilde{X}_{i_h}$ such that $\{i_h, i_{h+1}\} \subseteq a(\tilde{x}_h)$. By $x \in Y^*_i$ and $x \in Y^j$, we have that $\{i, j\} \subseteq a(x)$. By $\tau(x) \notin \tau(Z^*)$, we have that $\tau(x) \neq \tau(x_1)$. Therefore, $i_1, ..., i_k$ and $\tau(\tilde{x}_1), ..., \tau(\tilde{x}_{k-1}), \tau(x)$ constitue a weak cycle. However, this contradicts the acyclicity, which completes the proof. ■

Here, we show a simple case of implication (c')⇒(a) of Theorem 1.

Example 2 Let $I = \{i_1, i_2\}$, $\Omega = \{\omega_1, \omega_2\}$, $a(\omega_1) = a(\omega_2) = I$, $r^{\max}_{\omega_1} = r^{\max}_{\omega_2} = 1$, $v^{i_1}(r) = 3\min\{r_{\omega_1}, r_{\omega_2}\}$, and $v^{i_2}(r) = -\max\{0, r_{\omega_1} + r_{\omega_2} - 1\}$. Let $a_1 = (\omega_1, 1/2, (0,0))$, $a_2 = (\omega_2, 1/2, (0,0))$, and $A = \{a_1, a_2\}$. It is easy to see that A is individually rational. Note that A is not efficient since $v^{i_1}(1/2, 1/2) + v^{i_2}(1/2, 1/2) = 3/2$, while $v^{i_1}(1,1) + v^{i_2}(1,1) = 2$. Indeed, $(1,1)$ is the efficient allocation since

$$v^{i_1}(r_{\omega_1}, r_{\omega_2}) + v^{i_2}(r_{\omega_1}, r_{\omega_2}) = \begin{cases} 3\min\{r_{\omega_1}, r_{\omega_2}\} \ (\leq 3/2) & \text{if } r_{\omega_1} + r_{\omega_2} \leq 1; \\ 3\min\{r_{\omega_1}, r_{\omega_2}\} - (r_{\omega_1} + r_{\omega_2} - 1) & \text{if } r_{\omega_1} + r_{\omega_2} > 1. \\ \qquad (\leq \min\{r_{\omega_1}, r_{\omega_2}\} + 1 \leq 2) & \end{cases}$$

We show that A is stable. Here, we only give an intuitive argument because the rigorous proof will be included in Appendix A. Suppose that $Z \subseteq X \setminus A$ is a blocking set to A. Then, there exists a pair of $Y^{i_1} \in C^{i_1}(Z \cup A)$ and $Y^{i_2} \in C^{i_2}(Z \cup A)$ such that $u^{i_1}(Y^{i_1}) + u^{i_2}(Y^{i_2}) > u^{i_1}(A) + u^{i_2}(A)$. To guarantee that this inequality holds, $u^{i_1}(Y^{i_1}) + u^{i_2}(Y^{i_2}) = v^{i_1}(\rho(Y^{i_1})) + v^{i_2}(\rho(Y^{i_2})) > 3/2$. Since $v^{i_2}(r) \leq 0$ for any r, $v^{i_1}(\rho(Y^{i_1}) > 3/2$. Then, we have $\rho_{\omega_1}(Y^{i_1}) > 1/2$ and $\rho_{\omega_2}(Y^{i_1}) > 1/2$. It is also necessary that $\tau(Z) = \Omega$ since $\rho_{\omega_1}(A) = \rho_{\omega_2}(A) = 1/2$. Since Z is a blocking set, $\{Z\} = C^{i_1}(Z \cup A)$ and $\{Z\} = C^{i_2}(Z \cup A)$. Note that $\rho_{\omega_1}(Z) > 1/2$ and $\rho_{\omega_2}(Z) > 1/2$ by the choice of Y^{i_1} and Y^{i_2} in the above arguments.

Therefore, $v^{i_1}(\rho_{\omega_1}(Z), \rho_{\omega_2}(Z)) > 3/2 = v^{i_1}(1/2, 1/2)$ and $v^{i_2}(\rho_{\omega_1}(Z), \rho_{\omega_2}(Z)) < 0 = v^{i_2}(1/2, 1/2)$. Then, agent i_1 must transfer a positive amount to agent i_2 in Z so that i_2 chooses Z from $Z \cup A$ and the utility of i_1 himself is increased from A. However, this is impossible. We confirm it for the case where $\rho_{\omega_1}(Z) = \rho_{\omega_2}(Z) = 1/2 + t$ $(0 < t \leq 1/2)$.[8]

By $v^{i_1}(1/2 + t, 1/2 + t) - v^{i_1}(1/2, 1/2) = 3t$, $\sigma^{i_1}_{\omega_1}(Z) + \sigma^{i_1}_{\omega_2}(Z) < 3t$. Without loss of generality, we may assume that $\sigma^{i_1}_{\omega_1}(Z) \leq \sigma^{i_1}_{\omega_2}(Z)$. Then, $\sigma^{i_1}_{\omega_1}(Z) \leq 3t/2$. Then,

$$\begin{aligned} u^{i_2}(Z) - u^{i_2}(\{(\omega_2, \rho_{\omega_2}(Z), s_{\omega_2}(Z))\}) &= (-2t - \sigma^{i_2}_{\omega_1}(Z) - \sigma^{i_2}_{\omega_2}(Z)) - (-\sigma^{i_2}_{\omega_2}(Z)) \\ &= -2t - \sigma^{i_2}_{\omega_1}(Z) \\ &\leq -t/2 \end{aligned}$$

by $-\sigma^{i_2}_{\omega_1}(Z) = \sigma^{i_1}_{\omega_1}(Z) \leq 3t/2$, contradicting that $Z \in C^{i_2}(Z \cup A)$.

In Appendix A, we will extend this kind of argument in a rigorous way for general cases where the venture structures have any k-cycle. Note that Theorem 1(e')⇒(a) corrects Theorem 8 of Hatfield and Kominars (2015), which stated that any stable outcome is efficient for any venture structure if the valuation functions are concave. Nevertheless, if the valuation functions are concave, then there exists at least one efficient stable outcome as the competitive equilibrium is shown to exist, be efficient, and be the stable outcome. (See Theorem 1,3, and 7 of Hatfield and Kominars (2015).)

3.2 Existence results

In this subsection, we consider the existence conditions for the strongly group stable, stable, and weakly setwise stable outcomes. Hatfield and Kominars (2015) proved the existence of the strongly group stable outcome under the concave valuation functions.

[8]In the range of the participation levels under consideration, the presence of the difference between the participation levels strictly decreases the payoff of i_2, while it does not increase the payoff of i_1. Therefore, taking the different participation levels in Z will make it more difficult for Z to be a blocking set to A.

We examine the existence conditions on the venture structures rather than the conditions on the valuation functions. Therefore, we do not require the concavity of the valuation functions at all.

First, we state that the acyclicity of the venture structure is also a necessary and sufficient condition for the stable outcome to exist.

Theorem 2 *Let a venture structure* (I, Ω, a) *and a maximum participation vector* $r^{\max}$ *are given. The following statements are equivalent:*

(a) (I, Ω, a) *is acyclic.*

(b) *A strong group stable outcome exists for any tuple of valuation functions* v*.*

(c) *A stable outcome exists for any tuple of valuation functions* v*.*

Theorem 2 is proved by three implications (a)⇒(b), (b)⇒(c), and (c)⇒(a). Implication (b)⇒(c) follows from the definitions of strongly group stable and stable outcomes. To show implications (a)⇒(b) and (c)⇒(a), we state two lemmas, the proofs of which are postponed to Appendix B.

Lemma 1 *Let* $(I, \Omega, a, r^{\max}, v)$ *be a multilateral matching market. If* (I, Ω, a) *is acyclic, then the strongly group stable outcome exists.*

Lemma 2 *Let* $(I, \Omega, a, r^{\max}, v)$ *be a multilateral matching market. If* (I, Ω, a) *admits a* 2*-cycle, then a stable outcome does not exist for some tuple of valuation functions* v*.*

Implication (a)⇒(b) follows from Lemma 1, while implication (c)⇒(a) is shown by Lemma 2 and Proposition 2, which will be stated at the end of this subsection. Lemma 1 will be proved by means of the mathematical induction. Here, we show the induction step for explaining the intuition of the proof.

The existnece of the strongly group stable outcome in the multilateral matching market $M = (I, \Omega, a, r^{\max}, v)$ with k ventures will be proved by assuming that the multilateral matching market with $k-1$ ventures possesses the strongly group stable outcome for any tuple of valuation functions. To this end, we construct a multilateral matching market with $k-1$ ventures from that with k ventures by excluding one extreme venture, say ω_0, and agents who are not associating any venture in $\Omega \setminus \{\omega_0\}$. We often write $v^i(r_{\Omega\setminus\{\omega_0\}}; r_{\omega_0})$ instead of $v^i(r)$ for each $i \in I$ and each $r \in \times_{\omega\in\Omega}[0, r^{\max}_\omega]$.

Since ω_0 is extreme, there exists at most one agent, say i_0, who is associating a venture $\omega \neq \omega_0$.[9] Let M_{k-1} be a multilateral matching market $(\bar{I}, \bar{\Omega}, a, (r_\omega^{\max})_{\omega\in\bar{\Omega}\}}, (\bar{v}^i)_{i\in\bar{I}})$ constructed from M_k by excluding ω_0 and the agents in $a(\omega_0)\backslash\{i_0\}$ where $\bar{I} = (I\backslash a(\omega_0))\cup\{i_0\}$ and $\bar{\Omega} = \Omega \setminus \{\omega_0\}$. The components except for $\bar{v} = (\bar{v})_{i\in\bar{I}}$ are constructed in a quite natural way. We need an additional explanation for the construction of $\bar{v}$.

Because each agent i in $\bar{I}$ except for i_0 is irrelevant to ω_0, i's function $\bar{v}_i$ can be defined as the essentially same as in the original problem. Formally, for any $r \in \times_{\omega\in\bar{\Omega}}[0, r_\omega^{\max}]$,

$$\bar{v}^i(r) = v^i(r; 0) \text{ for all } i \in \bar{I} \setminus \{i_0\}.$$

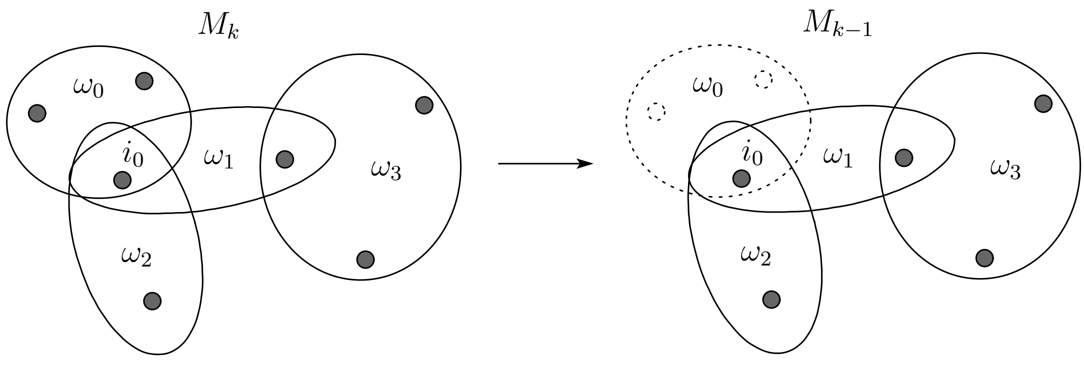
On the other hand, the valuation function of i_0 is defined as follows. For any $r \in \times_{\omega\in\bar{\Omega}}[0, r_\omega^{\max}]$,

$$\bar{v}^{i_0}(r) = \max_{r'\in[0,r_{\omega_0}^{\max}]} \left(v^{i_0}(r; r') - \sum_{j\in a(\omega_0)\backslash\{i_0\}} (v^j(0_{\bar{\Omega}}; r') - v^j(0; 0)) \right). \tag{3}$$

In the function $\bar{v}^{i_0}$, the information of the excluded agents' valuation functions are integrated in an appropriate way. More specifically, i_0 evaluates $r_{\bar{\Omega}}$ as the maximal value of the original function v^{i_0} so that the excluded agents in $a(\omega_0) \setminus \{i_0\}$ can satisfy the individually rational constraints in M_k by taking r_{ω_0} appropriately.

Let A be a strongly group stable outcome in M_{k-1}, which is assumed to exist by the induction hypothesis. Let A^* be an outcome obtained by adding a contract $(\omega_0, r_{\omega_0}^*, s_{\omega_0}^*)$

Figure 3: Construction of M_{k-1} from M_k: an example when $k = 4$.



[9] If there exists no agent like i_0, we can divide the market into one consisting of agents in $a(\omega_0)$ and venture $\{\omega_0\}$ and one consisting of agents in $I \setminus a(\omega_0)$ and ventures $\Omega \setminus \{\omega_0\}$, and show the existence of the strong group stable outcome independently.

to A, where $r^*_{\omega_0}$ maximizes (3) when $r = \rho_{\bar{\Omega}}(A)$, $s^{*i}_{\omega_0} = v^i(0; r^*_{\omega_0}) - v^i(0; 0)$ for all $i \in a(\omega_0)$, and $s^{*i_0}_{\omega_0} = -\sum_{i \in a(\omega_0) \setminus \{i_0\}} (v^i(0; r^*_{\omega_0}) - v^i(0; 0))$. Note that in this contract, the payoffs obtained by the agents in $a(\omega_0) \setminus \{i_0\}$ are integrated to i_0 through the transfer so that the agents can satisfy the individually rational constraint, just like (3).

The strong group stability of A^* will be formally proved in Appendix B (Claim 9) as we mentioned above. Intuitively, suppose that there exists a strongly blocking set Z to A^* in M_k. Here, we only consider the case where Z includes a contract for venture ω_0, say z^0. Consider the set of contracts $\bar{Z}$ that is obtained by excluding the contract for venture ω_0 from Z. Then, any agent in $a(\bar{Z}) \setminus \{i_0\}$ can take Y^i again so that $\bar{Z}_i \subseteq Y^i \subseteq \bar{Z}_i \cup A_i$ and be made better off in M_{k-1} since her valuation function is essentially same as that in M_k. For i^0, he also can be made better of by taking $Y^i \setminus \{z^0\}$ since valuation $\bar{v}^{i_0}(\rho(Y^i \setminus \{z^0\}))$ is no less than $v^{i_0}(Y^i) - \sum_{i \in a(\omega_0) \setminus \{i_0\}} \sigma^i_{\omega_0}(Y^i)$ by the construction of $\bar{v}^{i_0}$. Therefore, $\bar{Z}$ becomes a strongly blocking set to A in M_{k-1}, which yields a contradiction.

The following corollary is immediate from Theorem 2, which states only the sufficient condition for the existence of the weakly setwise stable outcome.

Corollary 1 *Let $(I, \Omega, a, r^{\max}, v)$ be a multilateral matching market. A weakly setwise stable outcome exists for any tuple of valuation functions v if (I, Ω, a) is acyclic.*

Indeed, the acyclicity of the venture structure is no longer a necessary condition for the existence of the weakly setwise stable outcome. To see this, we first show a special case where the weakly setwise stable outcome exists for any tuple of valuation functions though the venture structure admits a cycle.

Proposition 1 *Let $(I, \Omega, a, r^{\max}, v)$ be a multilateral matching market. If $a(\omega) = I$ for all $\omega \in \Omega$, then a weakly setwise stable outcome exists.*

The proof of Proposition 1 is postponed to Appendix B. By this proposition, we can easily see that there is a venture structure with a cycle where the weakly setwise stable outcome exists for any tuple of valuation functions. For example, let $I = \{i_1, i_2, i_3\}$, $\Omega = \{\omega_1, \omega_2, \omega_3\}$, and $a(\omega_1) = a(\omega_2) = a(\omega_3) = I$. In this example, the condition in Proposition 1 is satisfied and thus the weakly setwise stable outcome exists. However, it admits a cycle consisting of (i_1, i_2, i_3) and $(\omega_1, \omega_2, \omega_3)$. Note that the condition in

Proposition 1 neither guarantee the efficiency of the weakly setwise stable outcome nor existence of the strongly group and stable outcome since Example 1 and 2 satisfies the condition.

We need an additional assumption to specify the venture structure where the weakly setwise stable outcome fails to exist for some tuple of valuation functions. Below, we show a contraposition of the necessary condition for the existence of the weakly setwise stable outcome.

Proposition 2 *Let $(I, \Omega, a, r^{\max}, v)$ be a multilateral matching market. If (I, Ω, a) admits a ℓ-cycle with $\ell \geq 3$ and admits no 2-cycle, then no weakly setwise stable outcome exists for some tuple of valuation functions.*

The proof is postponed to Appendix B. Note that the above example admits both 2-cycle and 3-cycle. Therefore, the example does not satisfy the condition in Proposition 2.

4 Concluding remarks

This paper considered three stability concepts, the weakly setwise stable outcome, the stable outcome, and the strongly group stable outcome, in the multilateral matching market. The first one is introduced in this paper and the latter two were introduced by Hatfield and Kominars (2015). The acyclicity of the venture structure played the central role in our analysis. We showed that the acyclicity of the venture structure is sufficient for the equivalence, the efficiency, and the existence of the three stability concepts for any tuple of valuation functions. We also showed that the acyclicity of the venture structure is also the necessary condition for those results except for the existence of the weakly setwise stable outcome. Moreover, the acyclicity of the venture structure is still a necessary condition for the equivalence and the efficiency though we restrict to the concave valuation functions.

The analyses in the present paper and Hatfield and Kominars (2015) are two extremes: we allowed general valuation functions by restricting the venture structures to acyclic, while Hatfield and Kominars (2015) allowed general venture structure by restricting the valuation functions concave. The concavity of the valuation function may seem rather general, but it excludes some valuation function where the choice among the multilateral contracts is substitutable, which is a quite popular assumption in the standard matching theory. Investigating the multilateral matching market under a more general venture structures by excluding less plausible valuation functions may make the analysis more applicable to at least some specific market. We remain such investigations for future research.

Appendix A. Remaining proofs of Theorem 1.

We prove implications (c')⇒(a) and (e')⇒(a) of Theorem 1 stated in Subsection 3.1. We begin with proving some fundamental facts and claims that are commonly used in the both proofs. Since (I, Ω, a) is not acyclic, it admits a cycle consisting of distinct $i_1, ..., i_k \in I$ with $k \geq 2$ and distinct $\omega_1, ..., \omega_k \in \Omega$. Without loss of generality, we may assume that this cycle is minimal, *i.e.* there is no cycle $i_1, ..., i_{k'} \in I$ and $\omega_1, ..., \omega_{k'} \in \Omega$ with $k' < k$. Throughout this appendix, this cycle plays an important role. As a convention, we will frequently use the following notations: $\omega_{k+1} \equiv \omega_1$, $\omega_0 \equiv \omega_k$, $i_{k+1} \equiv i_1$, and $i_0 \equiv i_k$.

Claim 3 *For any $h = 1, ..., k$, $a(\omega_h) \cap \{i_1, ..., i_k\} = \{i_h, i_{h+1}\}$.*

Proof of Claim 3. Suppose that there exists some $h = 1, ..., k$ and $\ell \neq h, h+1$ such that $i_\ell \in a(\omega_h)$. Assume that $\ell < h$. Consider the sequence of players $i_\ell, ..., i_h$ and the sequence of ventures $\omega_\ell, ..., \omega_h$. By the choice of these sequences, the players are distinct with each other, and the ventures are distinct with each other. Further, $\{i_{\ell'}, i_{\ell'+1}\} \subset a(\omega'_\ell)$ for each $\ell' = \ell, ..., h-1$, and $\{i_h, i_\ell\} \subset a(\omega_h)$. Thus, $\{i_\ell, ..., i_h\}$ and $\{\omega_\ell, ..., \omega_h\}$ consist a $(h-\ell+1)$-cycle. If $h = k$, then $\ell \neq 1$, and we have $h - \ell + 1 \leq k - 1$. If $h \neq k$, then $h - \ell + 1 < k - \ell + 1 \leq k$. Either case contradicts the minimality of $\{i_1, ..., i_k\}$ and $\{\omega_1, ..., \omega_k\}$. The case where $\ell > h+1$ can be proved in a similar manner by considering the sequence of agents $\{i_{h+1}, ..., i_\ell\}$ and the sequence of ventures $\{\omega_h, ..., \omega_{\ell-1}\}$, which is omitted. Hence, $a(\omega_h) \cap \{i_1, ..., i_k\} = \{i_h, i_{h+1}\}$ for any $h = 1, ..., k$. □

By Claim 3, $\{i_h, i_{h+1}\} \subseteq a(\omega_h)$ and $i_{h'} \notin a(\omega_h)$ for all $h = 1, ..., k$ and $h' \neq h, h+1$. For each $h = 1, ..., k$, let $v^{i_h}(r)$ be any valuation function that depends only on $r_{\omega_{h-1}}$ and r_{ω_h}, that is, for any r, r' with $r_{\omega_{h-1}} = r'_{\omega_{h-1}}$ and $r_{\omega_h} = r'_{\omega_h}$, we have $v^{i_h}(r) = v^{i_h}(r')$. For each $i \notin \{i_1, ..., i_k\}$, let v^i is a constant function such that $v^i(r) = 0$ for any r.

Let A be an individually rational outcome such that $\tau(A) = \{\omega_1, ..., \omega_k\}$. Denote $a_h \in A$ a contract such that $\tau(a_h) = \omega_h$. Suppose that A is blocked. Then, there exists a nonempty $Z \subseteq X \setminus A$ such that for all $i \in a(Z)$ we have that $Z_i \subseteq Y^i$ for all $Y^i \in C^i(Z \cup A)$. Note that $u^i(Y^i) > u^i(A)$ for all $Y^i \in C^i(Z \cup A)$ and all $i \in a(Z)$; otherwise, $A_i \in C^i(Z \cup A)$, where $A_i \setminus Z_i \neq \emptyset$, contradicting that Z is a blocking set. We begin with proving two claims.

Claim 4 *For any $h = 1, ..., k$, if $\omega_h \in \tau(Z)$, then $\sigma^{i_h}_{\omega_h}(Z) + \sigma^{i_{h+1}}_{\omega_h}(Z) \geq 0$.*

Proof of Claim 4. Fix an arbitrary $h = 1, ..., k$. Assume that $\omega_h \in \tau(Z)$. If $a(\omega_h) = \{i_h, i_{h+1}\}$, then $\sigma^{i_h}_{\omega_h}(Z) + \sigma^{i_{h+1}}_{\omega_h}(Z) = 0$ by the definition. Therefore, assume that $a(\omega_h) \setminus \{i_h, i_{h+1}\} \neq \emptyset$. Fix an arbitrary $i \in a(\omega_h) \setminus \{i_h, i_{h+1}\}$. Note that $i \notin \{i_1, ..., i_k\}$ by Claim 3, and thus, $v^i(r) = 0$ for any allocation r. Fix an arbitrary $Y^i \in C^i(Z \cup A)$.

Suppose that $\sigma^i_{\omega_h}(Z) \geq 0$. Denote $y_h \in Y^i$ such that $y_h = (\omega_h, \rho_{\omega_h}(Y^i), \sigma_{\omega_h}(Y^i))$. Since v^i is constant to 0 and $y_h \in Z \subseteq Y^i$,

$$u^i(Y^i) = -\sum_{\omega \in \tau(Y^i)} \sigma^i_\omega(Y^i) \leq -\sum_{\omega \in \tau(Y^i) \setminus \{\omega_h\}} \sigma^i_\omega(Y^i) = u^i(Y^i \setminus \{y_h\}).$$

Then, $Y^i \setminus \{y_h\} \in C^i(Z \cup A)$, contradicting that $Z_i \subseteq Y^i$ for all $Y^i \in C^i(Z \cup A)$. Thus, $\sigma^i_{\omega_h}(Z) > 0$. By $\sum_{j \in a(\omega_h)} \sigma^j_{\omega_h}(Z) = 0$, we have $\sigma^{i_h}_{\omega_h}(Z) + \sigma^{i_{h+1}}_{\omega_h}(Z) > 0$. □

Claim 5 $\tau(Z) \subseteq \{\omega_1, ..., \omega_k\}$.

Proof of Claim 5. Fix an arbitrary $\omega' \in \Omega \setminus \{\omega_1, ..., \omega_k\}$. Suppose that $\omega' \in \tau(Z)$. By $\omega' \notin \{\omega_1, ..., \omega_k\}$, v^i is independent to $r_{\omega'}$. Then, $\sigma^i_{\omega'}(Z) < 0$ for all $i \in a(\omega')$ in order to guarantee that $Z_i \subseteq Y^i$ for all $Y^i \in C^i(Z \cup A)$ for all $i \in a(\omega')$. This contradicts the definition of the transfer vector. Hence, $\omega' \notin \tau(Z)$. □

Throughout this appendix, denote $z_h = (\omega_h, \rho_{\omega_h}(Z), \sigma_{\omega_h}(Z))$ for each $h = 1, ..., k$ whenever $\omega_h \in \tau(Z)$.

Proof of Theorem 1(c')⇒(a). We prove by constructing a tuple of concave valuation functions at which there exists a weakly setwise stable outcome that is not stable. Let

$$v^{i_1}(r) = 2\left(\frac{r_{\omega_1}}{r^{\max}_{\omega_1}} + \frac{r_{\omega_k}}{r^{\max}_{\omega_k}}\right);$$

$$v^{i_h}(r) = -2\left|\frac{r_{\omega_{h-1}}}{r^{\max}_{\omega_{h-1}}} - \frac{r_{\omega_h}}{r^{\max}_{\omega_h}}\right| \text{ for any } h = 2, ..., k-1 \text{ if exist;}$$

$$v^{i_k}(r) = -\left|1 - \frac{r_{\omega_{k-1}}}{r^{\max}_{\omega_{k-1}}} - \frac{r_{\omega_k}}{r^{\max}_{\omega_k}}\right|;$$

$$v^i(r) = 0 \text{ for any } i \in I \setminus \{i_1, ... i_k\} \text{ if exist.}$$

We can easily confirm that all of these functions are concave. Note that each valuation function depends only on the ratio of the venture participation levels to the maximum participation levels. Hereafter, we normalize to $r^{\max}_{\omega_h} = 1$ for all $h = 1, ..., k$ for the simplicity, which does not change any feature of the model.

Consider an outcome $A = \{(\omega_h, 1/2, s_{\omega_h}) | h = 1, ..., k-1\} \cup \{(\omega_k, 1, (0, ..., 0))\}$, where for each $h = 1, ..., k-1$,

$$s^i_{\omega_h} = \begin{cases} \frac{1}{2} & \text{if } i = i_h; \\ -\frac{1}{2} & \text{if } i = i_{h+1}; \\ 0 & \text{otherwise.} \end{cases}$$

Note that $u^{i_1}(A) = 5/2$ and $u^i(A) = 0$ for all $i \in I \setminus \{i_1\}$. It is easy to see that A is inefficient since $\sum_{i\in I} v^i(\rho(A)) = 5/2 < 3 = \sum_{i\in I} v^i(1, ..., 1)$. Outcome A is individually rational since

$$\begin{array}{llll} u^{i_1}(\{a_1\}) = 1/2, & u^{i_1}(\{a_k\}) = 2, & u^{i_1}(\emptyset) = 0; & \\ u^{i_h}(\{a_h\}) = -3/2, & u^{i_h}(\{a_{h-1}\}) = -1/2, & u^{i_h}(\emptyset) = 0 & \text{for all } h = 2, ..., k-1; \\ u^{i_k}(\{a_k\}) = 0, & u^{i_k}(\{a_{k-1}\}) = 0, & u^{i_k}(\emptyset) = -1. & \end{array}$$

Claim 6 *A is not stable.*

Proof of Claim 6. Let $\varepsilon_1 > 0$ be a sufficiently small positive real number so that $0 < \varepsilon_1 < 1$. For each $h = 1, ..., k-1$, define $\varepsilon_{h+1} = \varepsilon_h / |a(\omega_h)|$. Thus, $\varepsilon_{h+1} \leq \varepsilon_h/2$ for all $h = 1, ..., k-1$.

Let $Z^* = \{(\omega_h, 1, s_{\omega_h}) | h = 1, ..., k-1\}$, where

$$\sigma^i_{\omega_h}(Z^*) = \begin{cases} \frac{1}{2} + \varepsilon_h & \text{if } i = i_h; \\ -\frac{1}{2} - 2\varepsilon_{h+1} & \text{if } i = i_{h+1}; \\ -\varepsilon_{h+1} & \text{if } i \in a(\omega_h) \setminus \{i_h, i_{h+1}\} \end{cases}$$

for each $h = 1, ..., k-1$. Note that $\sum_{i\in a(\omega_h)} \sigma^i_{\omega_h}(Z^*) = 0$ by the definition. For each $h = 1, ..., k-1$, let $z^*_h = (\omega_h, \rho_{\omega_h}(Z^*), \sigma_{\omega_h}(Z^*))$. For any $i \in a(Z^*) \setminus \{i_1, ..., i_k\}$ (if exists), u^i depends only on the transfers. Then, it is easy to see that $\{Z^*_i\} = C^i(Z^* \cup A)$ for any $i \in a(Z^*) \setminus \{i_1, ..., i_k\}$ since $\sigma^i_\omega(Z^*) > 0$ for any $\omega \in \tau(Z^*)$ with $i \in a(\omega)$ and $\sigma^i_\omega(A) = 0$ for any $\omega \in \tau(A)$ with $i \in a(\omega)$.

By the choice of ε_1,

$$2\rho_{\omega_1}(Z^*) - \sigma^{i_1}_{\omega_1}(Z^*) = \frac{3}{2} - \varepsilon_1 > \frac{1}{2} = 2\rho_{\omega_1}(A) - \sigma^{i_1}_{\omega_1}(A).$$

Therefore, $\{\{z^*_1, a_k\}\} = C^{i_1}(Z^* \cup A)$ by the linearity of v^{i_1}.

We next show that $\{Z^*_{i_h}\} = C^{i_h}(Z^* \cup A)$ for each $h = 2, ..., k-1$. Fix an arbitrary $h = 2, ..., k-1$. It is easy to see that

$$u^{i_h}(Z^*) = -2|1-1| - \left(-\frac{1}{2} - 2\varepsilon_h\right) - \left(\frac{1}{2} + \varepsilon_h\right) = \varepsilon_h.$$

Thus, $u^{i_h}(Z^*_{i_h}) > u^{i_h}(A)$. Further, since we have $\varepsilon_h < 1/2$ by the choice of ε_h and $h \geq 2$,

$$u^{i_h}(\{z^*_{h-1}\}) = -2 - \left(-\frac{1}{2} - 2\varepsilon_h\right) = 2\varepsilon_h - \frac{3}{2} < 0;$$

$$u^{i_h}(\{z^*_h\}) = -2 - \left(\frac{1}{2} + \varepsilon_h\right) = -\frac{5}{2} - \varepsilon_h < 0;$$

$$u^{i_h}(\{z^*_{h-1}, a_h\}) = -2\left|1 - \frac{1}{2}\right| - \left(-\frac{1}{2} - 2\varepsilon_h\right) - \frac{1}{2} = -1 + 2\varepsilon_h < 0$$

$$u^{i_h}(\{a_{h-1}, z^*_h\}) = -2\left|\frac{1}{2} - 1\right| + \frac{1}{2} - \left(\frac{1}{2} + \varepsilon_h\right) = -1 - \varepsilon_h < 0.$$

Together with $A_{i_h} \in C^{i_h}(A)$, $u^{i_h}(Z^*) > u^{i_h}(Z')$ for any $Z' \subsetneq Z^* \cup A$. Hence, $\{Z^*_{i_h}\} = C^{i_h}(Z^* \cup A)$.

We finally show that $\{Z^*_k\} = \{\{z^*_{k-1}\}\} = C^{i_k}(Z^* \cup A)$. We have

$$u^{i_k}(\{z^*_{k-1}\}) = -|1 - \rho_{\omega_{k-1}}(Z^*)| - \sigma^{i_k}_{\omega_{k-1}}(Z^*) = -|1-1| - \left(-\frac{1}{2} - \varepsilon_k\right) = \frac{1}{2} + \varepsilon_k > 0.$$

Thus, $u^{i_k}(\{z^*_{k-1}\}) > u^{i_k}(A)$. Further,

$$u^{i_k}(\{z^*_{k-1}, a_k\}) = -|1-1-1| - \left(-\frac{1}{2} - \varepsilon_k\right) = -\frac{1}{2} + \varepsilon_k < \frac{1}{2} + \varepsilon_k.$$

Thus, $u^{i_k}(\{z^*_{k-1}\}) > u^{i_k}(\{z^*_{k-1}, a_k\})$. Together with $A_{i_k} \in C^{i_k}(A)$, $\{Z^*_k\} = \{\{z^*_{k-1}\}\} = C^{i_k}(Z^* \cup A)$.

Since $\{\{z^*_1, a_k\}\} = C^{i_1}(Z^* \cup A)$ and $\{Z^*_{i_h}\} = C^{i_h}(Z^* \cup A)$ for all $h = 2, ..., k$, Z^* is a blocking set on A. Note that players i_1 and i_k do not agree whether they maintain a_k or not since i_1 chooses $\{z^*_1, a_k\}$ while i_k chooses $\{z^*_{k-1}\}$ and drops a_k. □

Now, we turn to proving that A is weakly setwise stable. Suppose that A is weakly setwise blocked via $Z \subseteq X \setminus A$. Thus, for any $i \in a(Z)$ we have that $Z_i \subseteq Y^i$ for all $Y^i \in C^i(Z \cup A)$ and there exists some outcome Y^* such that $Y^*_i \in C^i(Z \cup A)$ for all $i \in a(Z)$. Note that $u^i(Y^i) > u^i(A)$ for all $Y^i \in C^i(Z \cup A)$ and all $i \in a(Z)$. Note that Z is itself an outcome.

Since any weakly setwise blocking set is also a blocking set. Thus, we have $\tau(Z) \subseteq \{\omega_1, ..., \omega_k\}$ by Claim 5. We claim that $\omega_k \notin \tau(Z)$. Suppose that there exists some

$z_k \in Z$. Then, $2\rho_{\omega_k}(Z) - \sigma^{i_1}_{\omega_k}(Z) > 2$ is necessary in order to guarantee that $z_k \in Y^{i_1}$ for all $Y^{i_1} \in C^{i_1}(Z \cup A)$ by the linearity of v^{i_1}. By Claim 4, $\sigma^{i_k}_{\omega_k}(Z) > 2 - 2\rho_{\omega_k}(Z)$. First, consider the case where $\omega_{k-1} \notin \tau(Z)$. Then,

$$\begin{aligned} u^{i_k}(\{a_{k-1}, z_k\}) &= -\left|\frac{1}{2} - \rho_{\omega_k}(Z)\right| + \frac{1}{2} - \sigma^{i_k}_{\omega_k}(Z) \\ &< -\left|\frac{1}{2} - \rho_{\omega_k}(Z)\right| + \frac{1}{2} - (2 - 2\rho_{\omega_k}(Z)) \\ &= \begin{cases} 3\rho_{\omega_k}(Z) - 2 & \text{if } \rho_{\omega_k}(Z) \leq \frac{1}{2}; \\ \rho_{\omega_k}(Z) - 1 & \text{if } \rho_{\omega_k}(Z) > \frac{1}{2}. \end{cases} \end{aligned}$$

Therefore, $u^{i_k}(\{a_{k-1}, z_k\}) < 0 = u^{i_k}(\{a_{k-1}\})$ with irrespective to the choice of $\rho_{\omega_k}(Z)$. Thus, $\{a_{k-1}, z_k\} \notin C^{i_k}(Z \cup A)$. Also, by

$$\begin{aligned} u^{i_k}(\{z_k\}) &= -|1 - \rho_{\omega_k}(Z)| - \sigma^{i_k}_{\omega_k}(Z) \\ &< -(1 - \rho_{\omega_k}(Z)) - (2 - 2\rho_{\omega_k}(Z)) \\ &= 3\rho_{\omega_k}(Z) - 3 \\ &\leq 0 \\ &= u^{i_k}(\{a_{k-1}\}), \end{aligned}$$

$\{z_k\} \notin C^{i_k}(Z \cup A)$. Thus, z_k is never chosen by i_k, contradicting the choice of Z.

Next, consider the case where $\omega_{k-1} \in \tau(Z)$. Since Z is a weakly setwise blocking set and $\omega_k \in \tau(Z)$ is also assumed, $\{Z_{i_k}\} = C^{i_k}(Z \cup A)$. Then,

$$\begin{aligned} u^{i_k}(\{z_{k-1}, z_k\}) &= -|1 - \rho_{\omega_{k-1}}(Z) - \rho_{\omega_k}(Z)| - \sigma^{i_k}_{\omega_{k-1}}(Z) - \sigma^{i_k}_{\omega_k}(Z) \\ &< -|1 - \rho_{\omega_{k-1}}(Z) - \rho_{\omega_k}(Z)| - \sigma^{i_k}_{\omega_{k-1}}(Z) - (2 - 2\rho_{\omega_k}(Z)). \end{aligned} \tag{4}$$

Consider the case where $1 \geq \rho_{\omega_{k-1}}(Z) + \rho_{\omega_k}(Z)$. Then, by (4)

$$\begin{aligned} u^{i_k}(\{z_{k-1}, z_k\}) &< -(1 - \rho_{\omega_k}(Z) - \rho_{\omega_{k-1}}(Z)) - \sigma^{i_k}_{\omega_{k-1}}(Z) - (2 - 2\rho_{\omega_k}(Z)) \\ &= \rho_{\omega_{k-1}}(Z) + 3\rho_{\omega_k}(Z) - \sigma^{i_k}_{\omega_{k-1}}(Z) - 3 \\ &\leq 2\rho_{\omega_k}(Z) - \sigma^{i_k}_{\omega_{k-1}}(Z) - 2. \end{aligned}$$

By this inequality and $u^{i_k}(\{z_{k-1}, a_k\}) = -\rho_{\omega_{k-1}}(Z) - \sigma^{i_k}_{\omega_{k-1}}(Z)$,

$$\begin{aligned} &u^{i_k}(\{z_{k-1}, a_k\}) - u^{i_k}(\{z_{k-1}, z_k\}) \\ &\quad > -\rho_{\omega_{k-1}}(Z) - \sigma^{i_k}_{\omega_{k-1}}(Z) - (2\rho_{\omega_k}(Z) - \sigma^{i_k}_{\omega_{k-1}}(Z) - 2) \\ &\quad = -\rho_{\omega_{k-1}}(Z) - 2\rho_{\omega_k}(Z) + 2 \\ &\quad \geq -\rho_{\omega_k}(Z) + 1 \\ &\quad \geq 0, \end{aligned}$$

contradicting that $Z_{i_k} \in C^{i_k}(Z \cup A)$.

Consider the case where $1 < \rho_{\omega_k}(Z) + \rho_{\omega_{k-1}}(Z)$. Then by (4)

$$\begin{aligned} u^{i_k}(\{z_{k-1}, z_k\}) &< -(\rho_{\omega_{k-1}}(Z) + \rho_{\omega_k}(Z) - 1) - \sigma^{i_k}_{\omega_{k-1}}(Z) - (2 - 2\rho_{\omega_k}(Z)) \\ &= -\rho_{\omega_{k-1}}(Z) + \rho_{\omega_k}(Z) - \sigma^{i_k}_{\omega_{k-1}}(Z) - 1. \end{aligned}$$

By this inequality and $u^{i_k}(\{z_{k-1}, a_k\}) = -\rho_{\omega_{k-1}}(Z) - \sigma^{i_k}_{\omega_{k-1}}(Z)$,

$$\begin{aligned} &u^{i_k}(\{z_{k-1}, a_k\}) - u^{i_k}(\{z_{k-1}, z_k\}) \\ &\quad > -\rho_{\omega_{k-1}}(Z) - \sigma^{i_k}_{\omega_{k-1}}(Z) - (-\rho_{\omega_{k-1}}(Z) + \rho_{\omega_k}(Z) - \sigma^{i_k}_{\omega_{k-1}}(Z) - 1) \\ &\quad = -\rho_{\omega_k}(Z) + 1 \\ &\quad \geq 0, \end{aligned}$$

contradicting that $Z_{i_k} \in C^{i_k}(Z \cup A)$. Hence, $\omega_k \notin \tau(Z)$.

Let $\hat{h} = 1, ..., k-1$ be the minimum integer such that $\omega_{\hat{h}} \in \tau(Z)$, and $\hat{\ell} = \hat{h}, ..., k-1$ be the minimum integer no less than $\hat{h}$ such that $\omega_{\hat{\ell}+1} \notin \tau(Z)$. Therefore, note that $\{\omega_{\hat{h}}, ..., \omega_{\hat{\ell}}\} \subseteq \tau(Z)$ and $\omega_{\hat{h}-1}, \omega_{\ell+1} \notin \tau(Z)$. Note also that $C^{i_{\hat{h}}}(Z \cup A) \subseteq \{\{z_{\hat{h}}\}, \{a_{\hat{h}-1}, z_{\hat{h}}\}\}$, $C^{i_{\hat{\ell}+1}}(Z \cup A) \subseteq \{\{z_{\hat{\ell}}\}, \{z_{\hat{\ell}}, a_{\hat{\ell}+1}\}\}$, and $C^{i_h}(Z \cup A) = \{Z_{i_h}\}$ for all $h = \hat{h}+1, ..., \ell$.

Case 1. $\hat{h} \neq 1$ and $\hat{\ell} \neq k-1$.

Since Z is a weakly setwise blocking set and $u^{i_h}(A) = 0$ for all $h \neq 1$, for any $\bar{Y}^{i_{\hat{h}}} \in C^{i_{\hat{h}}}(Z \cup A)$ and $\bar{Y}^{i_{\hat{\ell}+1}} \in C^{i_{\hat{\ell}+1}}(Z \cup A)$,

$$\begin{aligned} 0 &< \sum_{h=\hat{h}+1}^{\hat{\ell}} u^{i_h}(Z) + u^{i_{\hat{h}}}(\bar{Y}^{i_{\hat{h}}}) + u^{i_{\hat{\ell}+1}}(\bar{Y}^{i_{\hat{\ell}+1}}) \\ &= \sum_{h=\hat{h}+1}^{\hat{\ell}} (v^{i_h}(\rho(Z)) - \sigma^{i_h}_{\omega_{h-1}}(Z) - \sigma^{i_h}_{\omega_h}(Z)) + (v^{i_{\hat{h}}}(\rho(\bar{Y}^{i_{\hat{h}}})) - \sigma^{i_{\hat{h}}}_{\omega_{\hat{h}-1}}(\bar{Y}^{i_{\hat{h}}}) - \sigma^{i_{\hat{h}}}_{\omega_{\hat{h}}}(Z)) \\ &\qquad + (v^{i_{\hat{\ell}+1}}(\rho(\bar{Y}^{i_{\hat{\ell}+1}})) - \sigma^{i_{\hat{\ell}+1}}_{\omega_{\hat{\ell}}}(Z) - \sigma^{i_{\hat{\ell}+1}}_{\omega_{\hat{\ell}+1}}(\bar{Y}^{i_{\hat{\ell}+1}})) \\ &\leq \sum_{h=\hat{h}+1}^{\hat{\ell}} v^{i_h}(\rho(Z)) + v^{i_{\hat{h}}}(\rho(\bar{Y}^{i_{\hat{h}}})) - \sigma^{i_{\hat{h}}}_{\omega_{\hat{h}-1}}(\bar{Y}^{i_{\hat{h}}}) + v^{i_{\hat{\ell}+1}}(\rho(\bar{Y}^{i_{\hat{\ell}+1}})) - \sigma^{i_{\hat{\ell}+1}}_{\omega_{\hat{\ell}+1}}(\bar{Y}^{i_{\hat{\ell}+1}}). \end{aligned} \tag{5}$$

Note that the first term is 0 when $\hat{h} = \hat{\ell}$. Note also that the last inequality follows from Claim 4. There are four subcases.

Subcase 1(a). $u^{i_{\hat{h}}}(\{a_{\hat{h}-1}, z_{\hat{h}}\}) \geq u^{i_{\hat{h}}}(\{z_{\hat{h}}\})$ and $u^{i_{\hat{\ell}+1}}(\{z_{\hat{\ell}}, a_{\hat{\ell}+1}\}) \geq u^{i_{\hat{\ell}+1}}(\{z_{\hat{\ell}}\})$.

In this subcase, $\{a_{\hat{h}-1}, z_{\hat{h}}\} \in C^{i_{\hat{h}}}(Z \cup A)$ and $\{z_{\hat{\ell}}, a_{\hat{\ell}+1}\} \in C^{i_{\hat{\ell}+1}}(Z \cup A)$. By (5),

$$
\begin{aligned}
0 &< \sum_{h=\hat{h}+1}^{\hat{\ell}} (-2|\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z)|) - 2\left|\frac{1}{2} - \rho_{\omega_{\hat{h}}}(Z)\right| - 2\left|\rho_{\omega_{\hat{\ell}}}(Z) - \frac{1}{2}\right| \\
&\leq \sum_{h=\hat{h}+1}^{\hat{\ell}} (-2(\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z))) - 2\left(\frac{1}{2} - \rho_{\omega_{\hat{h}}}(Z)\right) - 2\left(\rho_{\omega_{\hat{\ell}}}(Z) - \frac{1}{2}\right) \\
&= 0,
\end{aligned}
$$

a contradiction, where the second inequality follows from the property of the absolute value.

Subcase 1(b). $u^{i_{\hat{h}}}(\{a_{\hat{h}-1}, z_{\hat{h}}\}) \geq u^{i_{\hat{h}}}(\{z_{\hat{h}}\})$ and $u^{i_{\hat{\ell}+1}}(\{z_{\hat{\ell}}, a_{\hat{\ell}+1}\}) < u^{i_{\hat{\ell}+1}}(\{z_{\hat{\ell}}\})$.

In this subcase, $\{a_{\hat{h}-1}, z_{\hat{h}}\} \in C^{i_{\hat{h}}}(Z \cup A)$ and $\{z_{\hat{\ell}}\} \in C^{i_{\hat{\ell}+1}}(Z \cup A)$. By (5),

$$
\begin{aligned}
0 &< \sum_{h=\hat{h}+1}^{\hat{\ell}} (-2|\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z)|) - 2\left|\frac{1}{2} - \rho_{\omega_{\hat{h}}}(Z)\right| + \frac{1}{2} - 2\rho_{\omega_{\hat{\ell}}}(Z) \\
&\leq \sum_{h=\hat{h}+1}^{\hat{\ell}} (-2(\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z))) - 2\left(\frac{1}{2} - \rho_{\omega_{\hat{h}}}(Z)\right) + \frac{1}{2} - 2\rho_{\omega_{\hat{\ell}}}(Z) \\
&= -\frac{1}{2},
\end{aligned}
$$

a contradiction, where the second inequality follows from the property of the absolute value.

Subcase 1(c). $u^{i_{\hat{h}}}(\{a_{\hat{h}-1}, z_{\hat{h}}\}) < u^{i_{\hat{h}}}(\{z_{\hat{h}}\})$ and $u^{i_{\hat{\ell}+1}}(\{z_{\hat{\ell}}, a_{\hat{\ell}+1}\}) \geq u^{i_{\hat{\ell}+1}}(\{z_{\hat{\ell}}\})$.

In this subcase, $\{z_{\hat{h}}\} \in C^{i_{\hat{h}}}(Z \cup A)$ and $\{z_{\hat{\ell}}, a_{\hat{\ell}+1}\} \in C^{i_{\hat{\ell}+1}}(Z \cup A)$. By (5),

$$
\begin{aligned}
0 &< \sum_{h=\hat{h}+1}^{\hat{\ell}} (-2|\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z)|) - 2\rho_{\omega_{\hat{h}}}(Z) - 2\left|\rho_{\omega_{\hat{\ell}}}(Z) - \frac{1}{2}\right| - \frac{1}{2} \\
&\leq \sum_{h=\hat{h}+1}^{\hat{\ell}} (-2(\rho_{\omega_h}(Z) - \rho_{\omega_{h-1}}(Z))) - 2\rho_{\omega_{\hat{h}}}(Z) - 2\left(\frac{1}{2} - \rho_{\omega_{\hat{\ell}}}(Z)\right) - \frac{1}{2} \\
&= -\frac{3}{2},
\end{aligned}
$$

a contradiction, where the second inequality follows from the property of the absolute value.

Subcase 1(d). $u^{i_{\hat{h}}}(\{a_{\hat{h}-1}, z_{\hat{h}}\}) < u^{i_{\hat{h}}}(\{z_{\hat{h}}\})$ and $u^{i_{\hat{\ell}+1}}(\{z_{\hat{\ell}}, a_{\hat{\ell}+1}\}) < u^{i_{\hat{\ell}+1}}(\{z_{\hat{\ell}}\})$.

In this subcase, $\{z_{\hat{h}}\} \in C^{i_{\hat{h}}}(Z \cup A)$ and $\{z_{\hat{\ell}}\} \in C^{i_{\hat{\ell}+1}}(Z \cup A)$. By (5),

$$\begin{aligned} 0 &< \sum_{h=\hat{h}+1}^{\hat{\ell}} (-2|\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z)|) - 2\rho_{\omega_{\hat{h}}}(Z) - 2\rho_{\omega_{\hat{\ell}}}(Z) \\ &\leq \sum_{h=\hat{h}+1}^{\hat{\ell}} (-2(\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z))) - 2\rho_{\omega_{\hat{h}}}(Z) - 2\rho_{\omega_{\hat{\ell}}}(Z) \\ &= -4\rho_{\omega_{\hat{h}}}(Z) \\ &\leq 0, \end{aligned}$$

a contradiction, where the second inequality follows from the property of the absolute value.

Case 2. $\hat{h} \neq 1$ and $\hat{\ell} = k - 1$.

In this case, (5) is again necessary in order to guarantee that Z is a weakly setwise blocking set. Further,

$$\begin{aligned} u^{i_k}(\{z_{k-1}, a_k\}) - u^{i_k}(\{z_{k-1}\}) &= -\rho_{\omega_{k-1}}(Z) + (1 - \rho_{\omega_{k-1}}(Z)) \\ &= 1 - 2\rho_{\omega_{k-1}}(Z) \\ &\begin{cases} \geq 0 & \text{if } \rho_{\omega_{k-1}}(Z) \leq \frac{1}{2}; \\ < 0 & \text{if } \rho_{\omega_{k-1}}(Z) > \frac{1}{2}. \end{cases} \end{aligned} \tag{6}$$

There are four subcases.

Subcase 2(a). $u^{i_{\hat{h}}}(\{a_{\hat{h}-1}, z_{\hat{h}}\}) \geq u^{i_{\hat{h}}}(\{z_{\hat{h}}\})$ and $u^{i_k}(\{z_{k-1}, a_k\}) \geq u^{i_k}(\{z_{k-1}\})$. In this subcase, $\{a_{\hat{h}-1}, z_{\hat{h}}\} \in C^{i_{\hat{h}}}(Z \cup A)$ and $\{z_{k-1}, a_k\} \in C^{i_k}(Z \cup A)$. By (5),

$$\begin{aligned} 0 &< \sum_{h=\hat{h}+1}^{k-1} (-2|\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z)|) - 2\left|\frac{1}{2} - \rho_{\omega_{\hat{h}}}(Z)\right| + \frac{1}{2} - \rho_{\omega_{k-1}}(Z) \\ &\leq \sum_{h=\hat{h}+1}^{k-1} (-2(\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z))) - 2\left(\frac{1}{2} - \rho_{\omega_{\hat{h}}}(Z)\right) + \frac{1}{2} - \rho_{\omega_{k-1}}(Z) \\ &= \rho_{\omega_{k-1}}(Z) - \frac{1}{2} \\ &\leq 0, \end{aligned}$$

a contradiction, where the second inequality follows from the property of the absolute value, and the last line follows from (6).

Subcase 2(b). $u^{i_{\hat{h}}}(\{a_{\hat{h}-1}, z_{\hat{h}}\}) \geq u^{i_{\hat{h}}}(\{z_{\hat{h}}\})$ and $u^{i_k}(\{z_{k-1}, a_k\}) < u^{i_k}(\{z_{k-1}\})$.

In this subcase, $\{a_{\hat{h}-1}, z_{\hat{h}}\} \in C^{i_{\hat{h}}}(Z \cup A)$ and $\{z_{k-1}\} \in C^{i_k}(Z \cup A)$. By (5),

$$\begin{aligned}
0 &< \sum_{h=\hat{h}+1}^{k-1} (-2|\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z)|) - 2\left|\frac{1}{2} - \rho_{\omega_{\hat{h}}}(Z)\right| + \frac{1}{2} - |1 - \rho_{\omega_{k-1}}(Z)| \\
&\leq \sum_{h=\hat{h}+1}^{k-1} (-2(\rho_{\omega_h}(Z) - \rho_{\omega_{h-1}}(Z))) - 2\left(\rho_{\omega_{\hat{h}}}(Z) - \frac{1}{2}\right) + \frac{1}{2} - (1 - \rho_{\omega_{k-1}}(Z)) \\
&= -\rho_{\omega_{k-1}}(Z) + \frac{1}{2} \\
&< 0,
\end{aligned}$$

a contradiction, where the second inequality follows from the property of the absolute value, and the last line follows from (6).

Subcase 2(c). $u^{i_{\hat{h}}}(\{a_{\hat{h}-1}, z_{\hat{h}}\}) < u^{i_{\hat{h}}}(\{z_{\hat{h}}\})$ and $u^{i_k}(\{z_{k-1}, a_k\}) \geq u^{i_k}(\{z_{k-1}\})$.

In this subcase, $\{z_{\hat{h}}\} \in C^{i_{\hat{h}}}(Z \cup A)$ and $\{z_{k-1}, a_k\} \in C^{i_k}(Z \cup A)$. By (5),

$$\begin{aligned}
0 &< \sum_{h=\hat{h}+1}^{k-1} (-2|\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z)|) - 2\rho_{\omega_{\hat{h}}}(Z) - \rho_{\omega_{k-1}}(Z) \\
&\leq \sum_{h=\hat{h}+1}^{k-1} (-2(\rho_{\omega_h}(Z) - \rho_{\omega_{h-1}}(Z))) - 2\rho_{\omega_{\hat{h}}}(Z) - \rho_{\omega_{k-1}}(Z) \\
&= -3\rho_{\omega_{k-1}}(Z) \\
&\leq 0,
\end{aligned}$$

a contradiction, where the second inequality follows from the property of the absolute value, and the last line follows from the nonnegativity of the participation level.

Subcase 2(d). $u^{i_{\hat{h}}}(\{a_{\hat{h}-1}, z_{\hat{h}}\}) < u^{i_{\hat{h}}}(\{z_{\hat{h}}\})$ and $u^{i_k}(\{z_{k-1}, a_k\}) < u^{i_k}(\{z_{k-1}\})$.

In this subcase, $\{z_{\hat{h}}\} \in C^{i_{\hat{h}}}(Z \cup A)$ and $\{z_{k-1}\} \in C^{i_k}(Z \cup A)$. By (5),

$$\begin{aligned}
0 &< \sum_{h=\hat{h}+1}^{k-1} (-2|\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z)|) - 2\rho_{\omega_{\hat{h}}}(Z) - |1 - \rho_{\omega_{k-1}}(Z)| \\
&\leq \sum_{h=\hat{h}+1}^{k-1} (-2(\rho_{\omega_h}(Z) - \rho_{\omega_{h-1}}(Z))) - 2\rho_{\omega_{\hat{h}}}(Z) - (1 - \rho_{\omega_{k-1}}(Z)) \\
&= -\rho_{\omega_{k-1}}(Z) - 1 \\
&< 0,
\end{aligned}$$

a contradiction, where the second inequality follows from the property of the absolute value.

Case 3. $\hat{h} = 1$.

In this case, $\{\{z_1, a_k\}\} = C^{i_1}(Z \cup A)$ since Z is a weakly setwise blocking set and the linearity of v^{i_1}. Since Z is a weakly setwise blocking set, for any $\bar{Y}^{i_{\hat{\ell}+1}} \in C^{i_{\hat{\ell}+1}}(Z \cup A)$,

$$\begin{aligned}
\frac{5}{2} &< \sum_{h=2}^{\hat{\ell}} u^{i_h}(Z) + u^{i_1}(\{z_1, a_k\}) + u^{i_{\hat{\ell}+1}}(\bar{Y}^{i_{\hat{\ell}+1}}) \\
&= \sum_{h=2}^{\hat{\ell}} (v^{i_h}(\rho(Z)) - \sigma^{i_h}_{\omega_{h-1}}(Z) - \sigma^{i_h}_{\omega_h}(Z)) + (2 + 2\rho_{\omega_1}(Z) - \sigma^{i_1}_{\omega_1}(Z)) \\
&\qquad + (v^{i_{\hat{\ell}+1}}(\rho(\bar{Y}^{\hat{\ell}+1})) - \sigma^{i_{\hat{\ell}+1}}_{\omega_{\hat{\ell}}}(Z) - \sigma^{i_{\hat{\ell}+1}}_{\omega_{\hat{\ell}+1}}(\bar{Y}^{\hat{\ell}+1})) \\
&\leq \sum_{h=2}^{\hat{\ell}} v^{i_h}(\rho(Z)) + 2 + 2\rho_{\omega_1}(Z) + v^{i_{\hat{\ell}+1}}(\rho(\bar{Y}^{\hat{\ell}+1})) - \sigma^{i_{\hat{\ell}+1}}_{\omega_{\hat{\ell}+1}}(\bar{Y}^{\hat{\ell}+1}).
\end{aligned} \tag{7}$$

Note that the first term is 0 when $\ell = 1$. Note also that the last inequality follows from Claim 4. We distinguish three subcases.

Subcase 3(a). $\hat{\ell} \neq k - 1$ and $u^{i_{\hat{\ell}+1}}(\{z_{\hat{\ell}}, a_{\hat{\ell}+1}\}) \geq u^{i_{\hat{\ell}+1}}(\{z_{\hat{\ell}}\})$.

In this subcase, $\{z_{\hat{\ell}}, a_{\hat{\ell}+1}\} \in C^{\hat{\ell}+1}(Z \cup A)$. By (7),

$$\begin{aligned}
\frac{5}{2} &< \sum_{h=2}^{\hat{\ell}} (-2|\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z)|) + (2 + 2\rho_{\omega_1}(Z)) - 2\left|\rho_{\omega_{\hat{\ell}}}(Z) - \frac{1}{2}\right| - \frac{1}{2} \\
&\leq \sum_{h=2}^{\hat{\ell}} (-2(\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z))) + (2 + 2\rho_{\omega_1}(Z)) - 2\left(\rho_{\omega_{\hat{\ell}}}(Z) - \frac{1}{2}\right) - \frac{1}{2} \\
&= \frac{5}{2},
\end{aligned}$$

a contradiction, where the second inequality follows from the property of the absolute value.

Subcase 3(b). $\hat{\ell} \neq k - 1$ and $u^{i_{\hat{\ell}+1}}(\{z_{\hat{\ell}}, a_{\hat{\ell}+1}\}) < u^{i_{\hat{\ell}+1}}(\{z_{\hat{\ell}}\})$.

In this subcase, $\{z_{\hat{\ell}}\} \in C^{i_{\hat{\ell}}}(Z \cup A)$. By (7),

$$\begin{aligned}
\frac{5}{2} &< \sum_{h=2}^{\hat{\ell}} (-2|\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z)|) + (2 + 2\rho_{\omega_1}(Z)) - 2\rho_{\omega_{\hat{\ell}}}(Z) \\
&\leq \sum_{h=2}^{\hat{\ell}} (-2(\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z))) + (2 + 2\rho_{\omega_1}(Z)) - 2\rho_{\omega_{\hat{\ell}}}(Z) \\
&= 2,
\end{aligned}$$

a contradiction, where the second inequality follows from the property of the absolute value.

Subcase 3(c). $\hat{\ell} = k - 1$.

In this subcase, $\{z_{k-1}, a_k\} \in C^{i_k}(Z \cup A)$ since Z is a weakly setwise blocking set and $\{\{z_1, a_k\}\} = C^{i_1}(Z \cup A)$. By (6), $\rho_{\omega_{k-1}}(Z) \leq 1/2$. Then, by (7),

$$\begin{aligned}
\frac{5}{2} &< \sum_{h=2}^{k-1}(-2|\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z)|) + (2 + 2\rho_{\omega_1}(Z)) - \rho_{\omega_{k-1}}(Z) \\
&\leq \sum_{h=2}^{k-1}(-2(\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z))) + (2 + 2\rho_{\omega_1}(Z)) - \rho_{\omega_{k-1}}(Z) \\
&= 2 + \rho_{\omega_{k-1}}(Z) \\
&\leq \frac{5}{2},
\end{aligned}$$

a contradiction, where the second inequality follows from the property of the absolute value.

Any of the three cases leads a contradiction. Therefore, A is stable. □

Proof of Theorem 1(e')⇒(a). We prove by constructing a tuple of valuation functions at which there exists an inefficient stable outcome. Let

$$\begin{aligned}
v^{i_1}(r) &= 3\min\left\{\frac{r_{\omega_1}}{r^{\max}_{\omega_1}}, \frac{r_{\omega_k}}{r^{\max}_{\omega_k}}\right\}; \\
v^{i_h}(r) &= -3\left|\frac{r_{\omega_{h-1}}}{r^{\max}_{\omega_{h-1}}} - \frac{r_{\omega_h}}{r^{\max}_{\omega_h}}\right| \text{ for all } h = 2, ..., k-1 \text{ if exist;} \\
v^{i_k}(r) &= -\max\left\{0, \frac{r_{\omega_{k-1}}}{r^{\max}_{\omega_{k-1}}} + \frac{r_{\omega_k}}{r^{\max}_{\omega_k}} - 1\right\}; \\
v^i(r) &= 0 \text{ for any } i \in I \setminus \{i_1, ...i_k\} \text{ if exist.}
\end{aligned}$$

We can easily confirm that all of these functions are concave. Note that each valuation function depends only on the ratio of the venture participation levels relative to the maximum participation levels. Hereafter, we normalize to $r^{\max}_{\omega_h} = 1$ for all $h = 1, ..., k$ for the simplicity, which does not change any feature of the model.

Consider an outcome $A = \{(\omega_i, 1/2, (0, ..., 0)) | i = 1, ..., k\}$. Note that $u^{i_1}(A) = 3/2$ and $u^i(A) = 0$ for all $i \in I \setminus \{i_1\}$. It is easy to see that A is inefficient since

$\sum_{i\in I} v^i(\rho(A)) = 3/2 < 2 = \sum_{i\in I} v^i(1,...,1)$. For any $A' \subsetneq A$,

$$u^{i_1}(A) = 3/2 > 0 = u^{i_1}(A');$$
$$u^{i_h}(A) = 0 \geq u^{i_h}(A') \text{ for each } h = 2,...,k-1;$$
$$u^{i_k}(A) = 0 = u^{i_k}(A');$$
$$u^{i}(A) = 0 = u^{i}(A') \text{ for all } i \in I \setminus \{i_1,...,i_k\}.$$

Thus, $A_i \in C^i(A)$ for all $i \in I$. Hence, A is individually rational.

Suppose that A is blocked via $Z \subseteq X \setminus A$. Then, by Claim 5, $\tau(Z) \subseteq \{\omega_1,...,\omega_k\}$.

We claim that $\{\omega_1,\omega_k\} \subseteq \tau(Z)$ and both $\rho_{\omega_1}(Z) > 1/2$ and $\rho_{\omega_k}(Z) > 1/2$. Suppose not. Then, $v^{i_1}(\rho(Y)) \leq v^{i_1}(\rho(A)) = 3/2$ for all $Y \subseteq Z \cup A$ by the definition of v^{i_1}. Also by the definition of v^i, $v^i(r) \leq 0 = v^i(\rho(A))$ for all $i \in I \setminus \{i_1\}$ and any allocation r. Thus, $\sum_{i\in a(Z)} v^i(\rho(Y^i)) \leq \sum_{i\in a(Z)} v^i(\rho(A))$ for any $Y^i \in C^i(Z\cup A)$ for each $i \in a(Z)$. Since $\sigma_\omega(A) = 0$ for any $\omega \in \tau(A)$, $\sum_{i\in a(Z)} u^i(A) = \sum_{i\in a(Z)} v^i(\rho(A))$. Similarly, for any $Y^i \in C^i(Z\cup A)$ for each $i \in a(Z)$, by $Z_i \subset Y^i$ for any $i \in a(Z)$,

$$\begin{aligned}\sum_{i\in a(Z)} u^i(Y^i) &= \sum_{i\in a(Z)}\left(v^i(\rho(Y^i)) - \sum_{\omega\in\tau(Z)} \sigma^i_\omega(Z) - \sum_{\omega\in\tau(Y^i)\setminus\tau(Z)} \sigma^i_\omega(A)\right)\\ &= \sum_{i\in a(Z)} v^i(\rho(Y^i)) - \sum_{\omega\in\tau(Z)}\sum_{i\in a(Z)} \sigma^i_\omega(Z)\\ &= \sum_{i\in a(Z)} v^i(\rho(Y^i)).\end{aligned}$$

Therefore, $\sum_{i\in a(Z)} u^i(Y^i) \leq \sum_{i\in a(Z)} u^i(A)$ for any $Y^i \in C^i(Z\cup A)$ for each $i \in a(Z)$. It follows that $A_j \in C^j(Z\cup A)$ for some $j \in a(Z)$, contradicting that Z is a blocking set. Hence, $\{\omega_1,\omega_k\} \subseteq \tau(Z)$ and both $\rho_{\omega_1}(Z) > 1/2$ and $\rho_{\omega_k}(Z) > 1/2$. Since $Z_{i_1} \subseteq Y^{i_1}$ for all $Y^{i_1} \in C^{i_1}(Z\cup A)$ and $\{\omega_1,\omega_k\} \subseteq \tau(Z) \subseteq \{\omega_1,...,\omega_k\} = \tau(A)$, $C^{i_1}(Z\cup A) = \{Z_{i_1}\}$.

We claim that $\sigma^{i_k}_{\omega_k}(Z) < 0$. Suppose that $\sigma^{i_k}_{\omega_k}(Z) \geq 0$. Then, since

$$u^{i_k}(\{z_k\}) = 0 - \sigma^{i_k}_{\omega_k}(Z) \leq 0 = u^i(\emptyset);$$
$$u^{i_k}(\{a_{k-1},z_k\}) \leq 0 - \sigma^{i_k}_{\omega_{k-1}}(A) - \sigma^{i_k}_{\omega_k}(Z) \leq 0 - \sigma^{i_k}_{\omega_{k-1}}(A) = u^i(\{a_{k-1}\});$$
$$u^{i_k}(\{z_{k-1},z_k\}) \leq 0 - \sigma^{i_k}_{\omega_{k-1}}(Z) - \sigma^{i_k}_{\omega_k}(Z) \leq 0 - \sigma^{i_k}_{\omega_{k-1}}(Z) = u^i(\{z_{k-1}\}),$$

where the last inequality makes sense whenever $\omega_{k-1} \in \tau(Z)$, there exists some $Y^{i_k} \in C^{i_k}(Z\cup A)$ such that $z_k \notin Y^{i_k}$, contradicting the choice of Z. Thus, $\sigma^{i_k}_{\omega_k}(Z) < 0$. By Claim 4, $\sigma^{i_1}_{\omega_k}(Z) > 0$.

We claim that $\omega_h \in \tau(Z)$ for all $h = 2, ..., k-1$. Suppose not. Let $\hat{h} = 2, ..., k-1$ be the minimum integer such that $\omega_{\hat{h}} \notin \tau(Z)$. For each $h = 2, ..., \hat{h}-1$, $Z_{i_h} \subset Y^{i_h}$ for all $Y^{i_h} \in C^{i_h}(Z \cup A)$ since Z is a blocking set. By $\{\omega_k, \omega_1, ..., \omega_{\hat{h}-1}\}\} \subseteq \tau(Z) \subseteq \{\omega_1, ..., \omega_k\} = \tau(A) = \tau(Z \cup A)$ and Claim 3, $\{Z_{i_h}\} = \{\{z_{h-1}, z_h\}\} = C^{i_h}(Z \cup A)$ for any $h = 2, ..., \hat{h}-1$ and $C^{i_{\hat{h}}}(Z \cup A) \subseteq \{\{z_{\hat{h}-1}\}, \{z_{\hat{h}-1}, a_{\hat{h}}\}\}$.

In order to guarantee that Z is a blocking set, we need $\sum_{h=1}^{\hat{h}-1} u^{i_h}(Z_{i_h}) + u^{\hat{h}}(Y^{i_{\hat{h}}}) > 3/2 = \sum_{h=1}^{\hat{h}} u^{i_h}(A)$ for each $Y^{i_{\hat{h}}} \in \{\{z_{\hat{h}-1}\}, \{z_{\hat{h}-1}, a_{\hat{h}}\}\}$. However, by $\sigma^{i_1}_{\omega_k}(Z) > 0$ and Claim 4, if $Y^{i_{\hat{h}}} = \{z_{\hat{h}-1}\}$, then we have that

$$\begin{aligned}
&\sum_{h=1}^{\hat{h}-1} u^{i_h}(Z_{i_h}) + u^{\hat{h}}(\{z_{\hat{h}-1}\}) \\
&\qquad = \sum_{h=1}^{\hat{h}-1} v^{i_h}(\rho(Z)) + v^{i_{\hat{h}}}(\rho(\{z_{\hat{h}-1}\})) - \sigma^{i_1}_{\omega_k}(Z) - \sum_{h=1}^{\hat{h}-1}(\sigma^{i_h}_{\omega_h}(Z) + \sigma^{i_{h+1}}_{\omega_h}(Z)) \\
&\qquad < \sum_{h=1}^{\hat{h}-1} v^{i_h}(\rho(Z)) + v^{i_{\hat{h}}}(\rho(\{z_{\hat{h}-1}\})) \\
&\qquad \leq 3\rho_{\omega_1}(Z) - 3\sum_{h=2}^{\hat{h}-1} |\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z)| - 3\rho_{\omega_{\hat{h}-1}}(Z) \\
&\qquad \leq 3\rho_{\omega_1}(Z) - 3\sum_{h=2}^{\hat{h}-1} (\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z)) - 3\rho_{\omega_{\hat{h}-1}}(Z) \\
&\qquad = 0,
\end{aligned}$$

while if $Y^{i_{\hat{h}}} = \{z_{\hat{h}-1}, a_{\hat{h}}\}$, then we have that

$$\begin{aligned}
&\sum_{h=1}^{\hat{h}-1} u^{i_h}(Z_{i_h}) + u^{\hat{h}}(\{z_{\hat{h}-1}, a_{\hat{h}}\}) \\
&\quad = \sum_{h=1}^{\hat{h}-1} v^{i_h}(\rho(Z)) + v^{i_{\hat{h}}}(\rho(\{z_{\hat{h}-1}, a_{\hat{h}}\})) - \sigma^{i_1}_{\omega_k}(Z) - \sum_{h=1}^{\hat{h}-1}(\sigma^{i_h}_{\omega_h}(Z) + \sigma^{i_{h+1}}_{\omega_h}(Z)) - \sigma^{i_{\hat{h}}}_{\omega_{\hat{h}}}(A) \\
&\quad < \sum_{h=1}^{\hat{h}-1} v^{i_h}(\rho(Z)) + v^{i_{\hat{h}}}(\rho(\{z_{\hat{h}-1}, a_{\hat{h}}\})) \\
&\quad \leq 3\rho_{\omega_1}(Z) - 3\sum_{h=2}^{\hat{h}-1} |\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z)| - 3\left|\rho_{\omega_{\hat{h}-1}}(Z) - \frac{1}{2}\right| \\
&\quad \leq 3\rho_{\omega_1}(Z) - 3\sum_{h=2}^{\hat{h}-1}(\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z)) - 3\left(\rho_{\omega_{\hat{h}-1}}(Z) - \frac{1}{2}\right) \\
&\quad = \frac{3}{2}.
\end{aligned}$$

Note that terms $3\sum_{h=2}^{\hat{h}-1} |\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z)|$ as well as $3\sum_{h=2}^{\hat{h}-1}(\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z))$ in both inequalities become 0 when $\hat{h} = 2$. Anyway, both inequalities contradict that $\sum_{h=1}^{\hat{h}-1} u^{i_h}(Z_{i_h}) + u^{\hat{h}}(Y^{i_{\hat{h}}}) > 3/2$ for each $Y^{i_{\hat{h}}} \in \{\{z_{\hat{h}-1}\}, \{z_{\hat{h}-1}, a_{\hat{h}}\}\}$. Hence, $\omega_{i_h} \in \tau(Z)$ for all $i = 2, ..., k-1$. Therefore, $\tau(Z) = \{\omega_1, ..., \omega_k\}$. Note that $C^{i_h}(Z \cup A) = \{Z_{i_h}\}$ for all $h = 1, ..., k$ by $\tau(Z) = \tau(A) = \{\omega_1, ..., \omega_k\}$ and $Z_{i_h} \subseteq Y^{i_h}$ for all $Y^{i_h} \in C^{i_h}(Z \cup A)$. Further, recall that we have confirmed that both $\rho_{\omega_1}(Z) > 1/2$ and $\rho_{\omega_k}(Z) > 1/2$.

Denote $\rho^* = \min\{\rho_{\omega_1}(Z), \rho_{\omega_k}(Z)\}$. By $u^{i_1}(A) = 3/2$,

$$\sigma^{i_1}_{\omega_1}(Z) + \sigma^{i_1}_{\omega_k}(Z) < 3(\rho^* - 1/2) \tag{8}$$

in order to guarantee that $u^{i_1}(Z) > u^{i_1}(A)$. For all $h = 2, ..., k-1$, by $u^{i_h}(A) = 0$, it is necessary that

$$\sigma^{i_h}_{\omega_{h-1}}(Z) + \sigma^{i_h}_{\omega_h}(Z) < -3|\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z)| \leq 0 \tag{9}$$

in order to guarantee that $u^{i_h}(Z) > u^{i_h}(A)$. Then,

$$\begin{aligned}
3\left(\rho^* - \frac{1}{2}\right) &- 3\sum_{h=2}^{k-1} |\rho_{\omega_{h-1}}(Z_{i_h}) - \rho_{\omega_h}(Z_{i_h})| \\
&> (\sigma^{i_1}_{\omega_k}(Z) + \sigma^{i_1}_{\omega_1}(Z)) + \sum_{h=2}^{k-1}(\sigma^{i_h}_{\omega_{h-1}}(Z) + \sigma^{i_h}_{\omega_h}(Z)) \qquad (10) \\
&= \sigma^{i_1}_{\omega_k}(Z) + \sigma^{i_{k-1}}_{\omega_{k-1}}(Z) + \sum_{h=1}^{k-2}(\sigma^{i_h}_{\omega_h}(Z) + \sigma^{i_{h+1}}_{\omega_h}(Z)).
\end{aligned}$$

Note that each of terms $3\sum_{h=2}^{k-1}|\rho_{\omega_{h-1}}(Z_{i_h})-\rho_{\omega_h}(Z_{i_h})|$, $\sum_{h=2}^{k-1}(\sigma^{i_h}_{\omega_{h-1}}(Z)+\sigma^{i_h}_{\omega_h}(Z))$, and $\sum_{h=1}^{k-2}(\sigma^{i_h}_{\omega_h}(Z)+\sigma^{i_{h+1}}_{\omega_h}(Z))$ becomes 0 when $k=2$.

By $\rho_{\omega_1}(Z)\geq\rho^*$ and $|\rho_{\omega_{h-1}}(Z)-\rho_{\omega_h}(Z)|\geq\rho_{\omega_{h-1}}(Z)-\rho_{\omega_h}(Z)$ for each $h=2,...,k-1$,

$$\begin{aligned}\text{The most LHS of (10)} &\leq 3\left(\rho_{\omega_1}(Z)-\frac{1}{2}\right)-3\sum_{h=2}^{k-1}(\rho_{\omega_{h-1}}(Z)-\rho_{\omega_h}(Z))\\ &=3\rho_{\omega_{k-1}}(Z)-\frac{3}{2}.\end{aligned}\tag{11}$$

For any $h=1,...,k-1$, we have $\sigma^{i_h}_{\omega_h}(Z)+\sigma^{i_{h+1}}_{\omega_h}(Z)\geq 0$ by Claim 4. Then,

$$\text{The most RHS of (10)}\geq\sigma^{i_1}_{\omega_k}(Z)+\sigma^{i_{k-1}}_{\omega_{k-1}}(Z).\tag{12}$$

By combining (10)-(12), we obtain $3\rho_{\omega_{k-1}}(Z)-(3/2)-\sigma^{i_1}_{\omega_k}(Z)>\sigma^{i_{k-1}}_{\omega_{k-1}}(Z)$. Then, by Claim 4,

$$\sigma^{i_k}_{\omega_{k-1}}(Z)>\sigma^{i_1}_{\omega_k}(Z)+\frac{3}{2}-3\rho_{\omega_{k-1}}(Z).\tag{13}$$

We distinguish two cases.

Case 1. $\rho_{\omega_{k-1}}(Z)+\rho_{\omega_k}(Z)\leq 1$.

In this case,

$$\begin{aligned}u^{i_k}(Z_{i_k}) &= -\sigma^{i_k}_{\omega_{k-1}}(Z)-\sigma^{i_k}_{\omega_k}(Z)\\ &< -\left(\sigma^{i_1}_{\omega_k}(Z)+\frac{3}{2}-3\rho_{\omega_{k-1}}(Z)\right)-\sigma^{i_k}_{\omega_k}(Z)\\ &\leq 3\rho_{\omega_{k-1}}(Z)-\frac{3}{2}\\ &\leq 3(1-\rho_{\omega_k}(Z))-\frac{3}{2}\\ &< \frac{3}{2}-\frac{3}{2}\\ &= 0,\end{aligned}$$

where the first line follows from $\tau(Z)=\{\omega_1,...,\omega_k\}$, the second line follows from (13), the third line follows from Claim 4, the fourth line follows from the assumption of Case 1, and the last line follows from $\rho_{\omega_k}(Z)>1/2$. This contradicts that $u^{i_k}(Z_{i_k})>u^{i_k}(A)=0$.

Case 2. $\rho_{\omega_{k-1}}(Z)+\rho_{\omega_k}(Z)>1$.

In this case, we need

$$1-\rho_{\omega_{k-1}}(Z)-\rho_{\omega_k}(Z)-\sum_{\omega\in\tau(Z)}\sigma^{i_k}_{\omega}(Z)=u^{i_k}(Z)>u^{i_k}(Z\setminus\{z_h\})=-\sum_{\omega\in\tau(Z\setminus\{z_h\})}\sigma^{i_k}_{\omega}(Z)$$

for $h=k-1,k$ in order to guarantee that $\{Z_{i_k}\}=C^{i_k}(Z\cup A)$. Therefore,

$$1-\rho_{\omega_{k-1}}(Z)-\rho_{\omega_k}(Z)>\sigma^{i_k}_{\omega_h}(Z) \text{ for } h=k-1,k. \tag{14}$$

Then,

$$\begin{aligned}
0 &\leq \sum_{h=1}^{k}(\sigma^{i_h}_{\omega_h}(Z)+\sigma^{i_{h+1}}_{\omega_h}(Z))\\
&= \sum_{h=1}^{k}(\sigma^{i_h}_{\omega_{h-1}}(Z)+\sigma^{i_h}_{\omega_h}(Z))\\
&= \sum_{h=2}^{k-1}(\sigma^{i_h}_{\omega_{h-1}}(Z)+\sigma^{i_h}_{\omega_h}(Z))+\sum_{h=1,k}(\sigma^{i_h}_{\omega_{h-1}}(Z)+\sigma^{i_h}_{\omega_h}(Z))\\
&< -3\sum_{h=2}^{k-1}|\rho_{\omega_{h-1}}(Z)-\rho_{\omega_h}(Z)|+\left(3\rho^*-\frac{3}{2}\right)+2(1-\rho_{\omega_{k-1}}(Z)-\rho_{\omega_k}(Z))
\end{aligned} \tag{15}$$

where the first line follows from Claim 4, and last line follows from (8), (9), and (14).

We further distinguish two subcases.

Subcase 2(a). Either $\rho_{\omega_{k-1}}(Z)\leq\rho_{\omega_1}(Z)$ or $\rho_{\omega_{k-1}}(Z)\leq\rho_{\omega_k}(Z)$.

In this subcase, by $|\rho_{\omega_{h-1}}(Z)-\rho_{\omega_h}(Z)|\geq\rho_{\omega_{h-1}}(Z)-\rho_{\omega_h}(Z)$ for all $h=2,...,k-1$,

$$\begin{aligned}
&\text{The most RHS of (15)}\\
&\leq -3\sum_{h=2}^{k-1}(\rho_{\omega_{h-1}}(Z)-\rho_{\omega_h}(Z))+\left(3\rho^*-\frac{3}{2}\right)+2(1-\rho_{\omega_{k-1}}(Z)-\rho_{\omega_k}(Z))\\
&= -3(\rho_{\omega_1}(Z)-\rho_{\omega_{k-1}}(Z))+3\rho^*+\frac{1}{2}-2\rho_{\omega_{k-1}}(Z)-2\rho_{\omega_k}(Z)\\
&= -3\rho_{\omega_1}(Z)+\rho_{\omega_{k-1}}(Z)+3\rho^*-2\rho_{\omega_k}(Z)+\frac{1}{2}.
\end{aligned} \tag{16}$$

If $\rho^*=\rho_{\omega_1}(Z)$, then either $\rho_{\omega_{k-1}}(Z)\leq\rho_{\omega_1}(Z)\leq\rho_{\omega_k}(Z)$ or $\rho_{\omega_{k-1}}(Z)\leq\rho_{\omega_k}(Z)$. Anyway, $\rho_{\omega_{k-1}}(Z)\leq\rho_{\omega_k}(Z)$. Then, the most RHS of (16) turns out to be

$$\rho_{\omega_{k-1}}(Z)-2\rho_{\omega_k}(Z)+\frac{1}{2}<0$$

by $\rho_{\omega_k}(Z)>1/2$. This contradicts that the most RHS of (15) is greater than 0.

If $\rho^*=\rho_{\omega_k}(Z)$, then either $\rho_{\omega_{k-1}}(Z)\leq\rho_{\omega_1}(Z)$ or $\rho_{\omega_{k-1}}(Z)\leq\rho_{\omega_k}(Z)\leq\rho_{\omega_1}(Z)$. Anyway, $\rho_{\omega_{k-1}}(Z)\leq\rho_{\omega_1}(Z)$. Then, the most RHS of (16) turns out to be

$$-3\rho_{\omega_1}(Z)+\rho_{\omega_{k-1}}(Z)+\rho_{\omega_k}(Z)+\frac{1}{2}<0$$

by $\rho_{\omega_k}(Z) \leq \rho_{\omega_1}(Z)$, $\rho_{\omega_{k-1}}(Z) \leq \rho_{\omega_1}(Z)$, and $\rho_{\omega_1}(Z) > 1/2$. This contradicts that the most RHS of (15) is greater than 0.

Subcase 2(b). Both $\rho_{\omega_{k-1}}(Z) > \rho_{\omega_1}(Z)$ and $\rho_{\omega_{k-1}}(Z) > \rho_{\omega_k}(Z)$.

In this subcase, by $|\rho_{\omega_{h-1}}(Z) - \rho_{\omega_h}(Z)| \geq \rho_{\omega_h}(Z) - \rho_{\omega_{h-1}}(Z)$ for all $h = 2, ..., k-1$,

$$\begin{aligned}
&\text{The most RHS of (15)} \\
&\leq -3\sum_{h=2}^{k-1}(\rho_{\omega_h}(Z) - \rho_{\omega_{h-1}}(Z)) + \left(3\rho^* - \frac{3}{2}\right) + 2(1 - \rho_{\omega_{k-1}}(Z) - \rho_{\omega_k}(Z)) \\
&= -3(\rho_{\omega_{k-1}}(Z) - \rho_{\omega_1}(Z)) + 3\rho^* + \frac{1}{2} - 2\rho_{\omega_{k-1}}(Z) - 2\rho_{\omega_k}(Z) \\
&= 3\rho_{\omega_1}(Z) - 5\rho_{\omega_{k-1}}(Z) + 3\rho^* - 2\rho_{\omega_k}(Z) + \frac{1}{2}.
\end{aligned} \tag{17}$$

If $\rho^* = \rho_{\omega_1}(Z)$, then the most RHS of (17) turns out to be

$$6\rho_{\omega_1}(Z) - 5\rho_{\omega_{k-1}}(Z) - 2\rho_{\omega_k}(Z) + \frac{1}{2} < 0$$

by $\rho_{\omega_{k-1}}(Z) > \rho_{\omega_1}(Z)$ and $1/2 < \rho_{\omega_1}(Z) \leq \rho_{\omega_k}(Z)$. This contradicts that the most RHS of (15) is greater than 0.

If $\rho^* = \rho_{\omega_k}(Z)$, then the most RHS of (17) turns out to be

$$3\rho_{\omega_1}(Z) - 5\rho_{\omega_{k-1}}(Z) + \rho_{\omega_k}(Z) + \frac{1}{2} < 0$$

by $1/2 < \rho_{\omega_1}(Z) < \rho_{\omega_{k-1}}(Z)$ and $\rho_{\omega_k}(Z) < \rho_{\omega_{k-1}}(Z)$. This contradicts that the most RHS of (15) is greater than 0.

Each of Case 1 and 2 yields a contradiction. Hence, A is a stable outcome.

■

Appendix B. Proofs in Subsection 3.2

We give the proofs for the results in Subsection 3.2. The following table shows an outline of the proofs, where "→" indicates "is used in", and "↠" indicates "immediately implies".

Proposition 1 → Lemma 1 ↠ [Theorem 2(a) ⇒ (b)]

Lemma 2, Proposition 2 } ↠ [Theorem 2(c) ⇒ (a)]

Therefore, we begin with the proof of Proposition 1.

Proof of Proposition 1.

We denote $I = \{i_1, ..., i_k\}$ and $\Omega = \{\omega_1, ..., \omega_m\}$. Assume that $I = a(\omega)$ for all $\omega \in \Omega$.

Define

$$X^* = \left\{ ((\rho_{\omega_1}(Y), \sigma_{\omega_1}(Y)), ..., (\rho_{\omega_m}(Y), \sigma_{\omega_m}(Y))) \in \mathbb{R}^{(k+1)m} \;\middle|\; \begin{array}{l} Y \text{ is an individually} \\ \text{rational outcome} \end{array} \right\}.$$

The nonemptiness of X^* is straightforward since $((0, (0, ..., 0)), ..., (0, (0, ..., 0))) \in X^*$. The boundedness follows from the boundedness of $r_\omega^{\max} \geq 0$ for each $\omega \in \Omega$ and the individual rationality of Y. To see that X^* is closed, let $((\rho_{\omega_1}(Y_t), \sigma_{\omega_1}(Y_t)), ..., (\rho_{\omega_m}(Y_t), \sigma_{\omega_m}(Y_t)))_{t=1}^{\infty}$ be a convergent sequence in X^* that converges to $((r_{\omega_1}, s_{\omega_1}), ..., (r_{\omega_m}, s_{\omega_m}))$ where Y_t is an individually rational oucome for all t. Note that for all $h = 1, ..., m$, $r_{\omega_h} \in [0, r_{\omega_h}^{\max}]$ and $\sum_{\ell=1}^{k} s_{\omega_h}^{i_\ell} = 0$ since $\rho_{\omega_h}(Y_t) \in [0, r_{\omega_h}^{\max}]$ and $\sum_{\ell=1}^{k} \sigma_{\omega_h}^{i_\ell}(Y_t) = 0$ for any t. Thus, there exists an outcome Y such that $\rho_{\omega_h}(Y) = r_{\omega_h}$ and $\sigma_{\omega_h}(Y) = s_{\omega_h}$ for all $h = 1, ..., m$.

We show that Y is individually rational, which implies that $((r_{\omega_1}, s_{\omega_1}), ..., (r_{\omega_m}, s_{\omega_m})) \in X^*$ and hence X^* is closed. Pick any $i \in I$ and any $\Omega' \subseteq \{\omega \in \Omega | i \in a(\omega)\}$. For each t, let $r^t = (\rho_{\omega_1}(Y_t), ..., \rho_{\omega_m}(Y_t)) \in \mathbb{R}^m$. We also denote $(r_{\omega_1}, ..., r_m)$ by $r \in \mathbb{R}^m$. By the individulal rationality of $(Y_t)_{t=1}^{\infty}$, we have that

$$v_i(r^t) - \sum_{\omega \in \Omega} \sigma_\omega^i(Y_t) \geq v_i(0_{\Omega'}, r^t_{\Omega \setminus \Omega'}) - \sum_{\omega \in \Omega \setminus \Omega'} \sigma_\omega^i(Y_t) \text{ for all } t.$$

Since v_i is continous, by $r^t \to r$ and $(\sigma_{\omega_1}(Y_t), ..., \sigma_{\omega_m}(Y_t)) \to (s_{\omega_1}, ..., s_{\omega_m})$ $(t \to \infty)$, we have that

$$v_i(r) - \sum_{\omega \in \Omega} s_\omega^i \geq v_i(0_{\Omega'}, r_{\Omega \setminus \Omega'}) - \sum_{\omega \in \Omega \setminus \Omega'} s_\omega^i.$$

Therefore, Y is individually rational. Hence X^* is nonempty and compact.

Then, there exists $((r^*_{\omega_1}, s^*_{\omega_1}), ..., (r^*_{\omega_m}, s^*_{\omega_m})) \in X^*$ such that $\sum_{\ell=1}^{k} v^{i_\ell}(r^*) \geq \sum_{\ell=1}^{k} v^{i_\ell}(r)$ for any $((r_{\omega_1}, s_{\omega_1}), ..., (r_{\omega_m}, s_{\omega_m})) \in X^*$ by the continuity of the valuation functions. Let A^* be the individually rational outcome such that $\rho_{\omega_h}(A^*) = r^*_{\omega_h}$ and $\sigma_{\omega_h}(A^*) = s^*_{\omega_h}$ for all $h = 1, ..., m$.

Suppose that A^* is not weakly setwise stable. Then, there exists a nonempty $Z \subseteq X \setminus A^*$ such that for all $i \in I$ we have that $Z_i \subseteq Y^i$ for all $Y^i \in C^i(Z \cup A^*)$ and there exists some outcome Y^* such that $Y_i^* \in C^i(Z \cup A^*)$ for all $i \in I$. Note that $Y_i^* \in C^i(Y^*)$ for all $i \in I$ from the definition of the choice and hence $(\rho(Y^*), \sigma(Y^*)) \in X^*$. We also have that $u^i(Y^*) > u^i(A^*)$ for all $i \in I$, which implies that $\sum_{i \in I} v_i(\rho(Y^*)) > \sum_{i \in I} v_i(A^*))$. This contradicts the choice of A^*. Hence, A^* is stable.

Proof of Lemma 1. Let $(I, \Omega, a, r^{\max}, v)$ be a multilateral matching market with acyclic venture structure. The proof is done by a mathematical induction of the number of ventures. We begin with the proof for the induction base.

Claim 7 *Consider any venture structure (I, Ω, a) with $|\Omega| = 1$. For any maximum participation vector r^{max} and any tuple of valuation functions $v = (v^i)_{i \in I}$, a strongly group stable outcome exists in $(I, \Omega, a, r^{max}, v)$.*

Proof of Claim 7. Any venture structure with a single venture is obviously acyclic and $I = a(\omega)$ for the unique $\omega \in \Omega$. Then, the existence of the strongly group stable outcome immediately follows from Theorem 1(a)⇒(b) and Proposition 1. □

We assume that for any acyclic venture structure (I, Ω, a) with $|\Omega| = k - 1$ $(k \geq 2)$, any maximum participation vector $r^{\max}$ and any tuple of valuation functions $v = (v^i)_{i \in I}$, a strongly group stable outcome exists in $(I, \Omega, a, r^{\max}, v)$ (Induction hypothesis). We will show that this statement holds when the number of ventures is k. Consider any acyclic venture structure (I, Ω, a) with $|\Omega| = k$, any maximum participation vector $r^{\max}$ and any tuple of valuation functions $v = (v^i)_{i \in I}$. We denote by M_k the multilateral matching market $(I, \Omega, a, r^{\max}, v)$.

For each $i \in I$, we denote $\{\omega \in \Omega | i \in a(\omega)\}$ by Ω_i. We say that $\omega \in \Omega$ is an extreme venture if $|\{i \in a(\omega) | |\Omega_i| \geq 2\}| \leq 1$. By the acyclicity, there exists at least one extreme venture. Fix an extreme venture ω_0 arbitrary. We often write $v^i(r)$ by $v^i(r_{\Omega \setminus \{\omega_0\}}; r_{\omega_0})$ for each $i \in I$ and each $r \in \times_{\omega \in \Omega}[0, r_\omega^{\max}]$.

We construct a multilateral matching market $M_{k-1} = (\bar{I}, \Omega \setminus \{\omega_0\}, \bar{a}, \bar{r}^{\max}, \bar{v})$, where $\bar{I} = \bigcup_{\omega \in \Omega \setminus \{\omega_0\}} a(\omega)$, $\bar{a}$ is a restriction of a to $\Omega \setminus \{\omega_0\}$, and $\bar{r}^{\max} = (r_\omega^{\max})_{\omega \in \Omega \setminus \{\omega_0\}}$. The tuple of valuation function $\bar{v}$ will be specified later. The acyclicity of (I, Ω, a) implies that $(\bar{I}, \Omega \setminus \{\omega_0\}, \bar{a})$ is also acyclic. Suppose that $|\Omega_i| = 1$ for all $i \in a(\omega_0)$ which implies that $a(\omega_0) \cap a(\Omega \setminus \{\omega_0\}) = \emptyset$. Then, the induction hypothesis and the existence result of a strongly group stable outcome for a market with one venture (Claim 7) implies the existence of a strongly group stable outcome in M_k. Therefore, we assume that there exists $i_0 \in a(\omega_0)$ such that $|\Omega_{i_0}| \geq 2$. Note that such an agent uniquely exists since ω_0 is extreme, and therefore, $\bar{I} = (I \setminus a(\omega_0)) \cup \{i_0\}$ holds. Note also that for each $j \in a(\omega_0) \setminus \{i_0\}$, $\Omega_j = \{\omega_0\}$ holds.

We next define a tuple of valuation functions $\bar{v} = (\bar{v}^i)_{i \in \bar{I}}$ which are defined on $\times_{\omega \in \Omega \setminus \{\omega_0\}} [0, r_\omega^{\max}]$ as follows. For each $j \in a(\omega_0) \setminus \{i_0\}$ and each $x \in [0, r_{\omega_0}^{\max}]$, let $s^j(x) := v^j(0; x) - v^j(0; 0)$ and $s^{i_0}(x) := -\sum_{j \in a(\omega_0) \setminus \{i_0\}} s^j(x)$. Then, $\bar{v}_{i_0}$ is defined as follows: for each $\tilde{r} \in \times_{\omega \in \Omega \setminus \{\omega_0\}} [0, r_\omega^{\max}]$,

$$\bar{v}^{i_0}(\tilde{r}) := \max_{r \in [0, r_{\omega_0}^{\max}]} v^{i_0}(\tilde{r}; r) - s^{i_0}(r).$$

For each $i \in \bar{I} \setminus \{i_0\}$ and each $\tilde{r} \in \times_{\omega \in \Omega \setminus \{\omega_0\}} [0, r_\omega^{\max}]$,

$$\bar{v}^i(\tilde{r}) := v^i(\tilde{r}; 0).$$

Then, for all $i \in \bar{I}$, $\bar{v}_i$ is continuous on $\times_{\omega \in \Omega \setminus \{\omega_0\}} [0, r_\omega^{\max}]$ from the continuity of v_i. For each $i \in \bar{I}$, we denote by $\bar{u}^i(A)$ the utility from an outcome A in M_{k-1}. It should be remarked that (i) for any outcome A in M_{k-1}, A is an outcome in M_k and $\bar{u}^i(A) = u^i(A)$ for all $i \in \bar{I} \setminus \{i_0\}$, and (ii) for any $i \in I \setminus a(\omega_0)$ and any outcome A in M_k, A_i is an outcome in M_{k-1} and $\bar{u}^i(A_i) = u^i(A)$ because $i \in I \setminus a(\omega_0)$ implies $\omega_0 \notin \Omega_i$.

By the induction hypothesis, there exists a strongly group stable outcome A in M_{k-1}. We denote $(\rho_\omega(A))_{\omega \in \Omega \setminus \{\omega_0\}}$ by r^A, and $\sigma_\omega(A)$ by s_ω for each $\omega \in \tau(A)$. Fix an arbitrary

$$r_{\omega_0}(r^A) \in \arg \max_{x \in [0, r_{\omega_0}^{\max}]} v^{i_0}(r^A; x) - s^{i_0}(x).$$

Define a transfer vector s_{ω_0} by $s_{\omega_0}^i := s^i(r_{\omega_0}(r^A))$ if $i \in a(\omega_0)$, and $s_{\omega_0}^i := 0$ if $i \notin a(\omega_0)$. Note that $\sum_{i \in a(\omega_0)} s_{\omega_0}^i = 0$ holds from the definition. Let $A^* := A \cup \{(\omega_0, r_{\omega_0}(r^A), s_{\omega_0})\}$.

We will show that A^* is a strongly group stable outcome in M_k. Note that

$$\begin{aligned}
&u^j(A^*) = v^j(0; r_{\omega_0}(r^A)) - s^j(r_\omega(r^A)) = v^j(0; 0) \text{ for all } j \in a(\omega_0) \setminus \{i_0\}, \\
&u^i(A^*) = \bar{u}^i(A) \text{ for all } i \in I \setminus a(\omega_0), \\
&u^{i_0}(A^*) = \bar{u}^{i_0}(A),
\end{aligned}$$

where the first equation follows from $\Omega_j = \{\omega_0\}$ for all $j \in a(\omega_0)$ and the last equation follows from

$$
\begin{aligned}
u^{i_0}(A^*) &= v^{i_0}(r^A; r_{\omega_0}(r^A)) - \sum_{\omega \in \tau(A^*_{i_0})} s^i_\omega \\
&= v^{i_0}(r^A; r_{\omega_0}(r^A)) - s^{i_0}(r_{\omega_0}(r^A)) - \sum_{\omega \in \tau(A_{i_0})} s^{i_0}_\omega \\
&= \bar{v}^{i_0}(r^A) - \sum_{\omega \in \tau(A_{i_0})} s^{i_0}_\omega = \bar{u}^{i_0}(A_{i_0}).
\end{aligned}
$$

Claim 8 *A^* is individually rational.*

Proof of Claim 8. Clearly, A^* is individually rational for all $j \in a(\omega_0) \setminus \{i_0\}$. The individual rationality of A in M_{k-1} directly implies that A^* is individual rational for all $i \in I \setminus a(\omega_0)$. Therefore, it remains to show that letting $r^* := (r^A; r_{\omega_0}(r^A))$, for each $\Omega' \subseteq \tau(A^*_i)$,

$$
u^{i_0}(A^*) = v^{i_0}(r^*) - \sum_{\omega \in \tau(A^*_{i_0})} s^{i_0}_\omega \geq v^{i_0}(0_{\Omega'}, r^*_{-\Omega'}) - \sum_{\omega \in \tau(A^*_{i_0}) \setminus \Omega'} s^{i_0}_\omega.
$$

Pick any $\Omega' \subseteq \tau(A^*_i)$. We have two cases to consider.

Case 1: $\omega_0 \in \Omega'$.

Let $\Omega'' := \Omega' \setminus \{\omega_0\} (\subseteq \tau(A_i))$. Then, we have that

$$
\begin{aligned}
u^{i_0}(A^*) = \bar{u}^{i_0}(A) = \bar{v}^{i_0}(r^A) &- \sum_{\omega \in \tau(A_{i_0})} s^{i_0}_\omega \\
&\geq \bar{v}^{i_0}(0_{\Omega''}, r^A_{-\Omega''}) - \sum_{\omega \in \tau(A_{i_0}) \setminus \Omega''} s^{i_0}_\omega \\
&= \bar{v}^{i_0}(0_{\Omega''}, r^A_{-\Omega''}) - \sum_{\omega \in \tau(A^*_{i_0}) \setminus \Omega'} s^{i_0}_\omega \\
&\geq v^{i_0}(0_{\Omega''}, r^A_{-\Omega''}; 0) - s^{i_0}(0) - \sum_{\omega \in \tau(A^*_{i_0}) \setminus \Omega'} s^{i_0}_\omega \\
&= v^{i_0}(0_{\Omega'}, r^*_{-\Omega'}) - \sum_{\omega \in \tau(A^*_{i_0}) \setminus \Omega'} s^{i_0}_\omega,
\end{aligned}
$$

where the first line follows from the individual rationality of A in M_{k-1}, the third line follows from $\omega_0 \in \Omega'$, the fourth line follows from the definition of $\bar{v}^{i_0}$, and the last line follows from $s^{i_0}(0) = 0$. Therefore, we obtain $u^{i_0}(A^*) \geq v^{i_0}(0_{\Omega'}, r^*_{-\Omega'}) - \sum_{\omega \in \tau(A^*_{i_0}) \setminus \Omega'} s^{i_0}_\omega$.

Case 2: $\omega_0 \notin \Omega'$.

In this case, $\Omega' \subseteq \tau(A_{i_0})$ holds. Then, we have that

$$\begin{aligned}
u^{i_0}(A^*) = \bar{u}^{i_0}(A) &= \bar{v}^{i_0}(r^A) - \sum_{\omega \in \tau(A_{i_0})} s^{i_0}_{\omega} \\
&\geq \bar{v}^{i_0}(0_{\Omega'}, r^A_{-\Omega'}) - \sum_{\omega \in \tau(A_{i_0}) \setminus \Omega'} s^{i_0}_{\omega} \\
&\geq v^{i_0}(0_{\Omega'}, r^A_{-\Omega'}; r_{\omega_0}(r^A)) - s^{i_0}(r_{\omega_0}(r^A)) - \sum_{\omega \in \tau(A_{i_0}) \setminus \Omega'} s^{i_0}_{\omega} \\
&= v^{i_0}(0_{\Omega'}, r^*_{-\Omega'}) - s^{i_0}_{\omega_0} - \sum_{\omega \in \tau(A_{i_0}) \setminus \Omega'} s^{i_0}_{\omega} \\
&= v^{i_0}(0_{\Omega'}, r^*_{-\Omega'}) - \sum_{\omega \in \tau(A^*_{i_0}) \setminus \Omega'} s^{i_0}_{\omega},
\end{aligned}$$

where the second line follows from the individual rationality of A in M_{k-1}, the third line follows from the definition of $\bar{v}_{i_0}$, and the last line follows from $\omega_0 \notin \Omega'$. Therefore, we obtain $u^{i_0}(A^*) \geq v^{i_0}(0_{\Omega'}, r^*_{-\Omega'}) - \sum_{\omega \in \tau(A^*_{i_0}) \setminus \Omega'} s^{i_0}_{\omega}$. Hence, A^* is individually rational. □

Claim 9 *A^* is not strongly blocked in M_k.*

Proof of Claim 9. Suppose that there exists a strong blocking set Z to A^*. By the strong group stability of A in M_{k-1}, $a(Z)$ must contain i_0. For all $i \in a(Z)$, fix any $Y^i \subseteq A^*_i \cup Z_i$ such that $u^i(Y^i) > u^i(A^*)$ and $Z_i \subseteq Y^i$. Then, we have two cases to consider.

Case 1: $\omega_0 \notin \tau(Z)$.

This assumption guarantees that Z is an outcome in M_{k-1}. We will show that Z is a strong blocking set to A in M_{k-1}, which contradicts the strong group stability of A in M_{k-1}. Because $A \cap Z = \emptyset$ holds by $A^* \cap Z = \emptyset$, it is sufficient to show that for all $i \in a(Z)$, there exists $\bar{Y}^i \subseteq A_i \cup Z_i$ such that $\bar{u}^i(\bar{Y}^i) > \bar{u}^i(A^*)$ and $Z_i \subseteq \bar{Y}^i$. Consider any $i \in a(Z)$ with $i \neq i_0$. By $\omega_0 \notin \tau(Z)$ and $i \neq i_0$, we have that $i \notin a(\omega_0)$ and hence Y^i satisfies that $Y^i \subseteq A_i \cup Z_i$, $\bar{u}^i(Y^i) > \bar{u}^i(A)$ and $Z_i \subseteq Y^i$. Therefore, it remains to show that there exists $\bar{Y}^{i_0} \subseteq A_{i_0} \cup Z_{i_0}$ such that $\bar{u}^{i_0}(\bar{Y}^{i_0}) > \bar{u}^{i_0}(A)$ and $Z_{i_0} \subseteq \bar{Y}^{i_0}$. We further distinguish two cases.

Subcase 1(a): $\omega_0 \in \tau(Y^{i_0})$

In this case, there exists $y^0 \in Y^{i_0}$ with $\tau(y^0) = \omega_0$. Let $\bar{Y}^{i_0} := Y^{i_0} \setminus \{y_0\}$. Then, we have that $\bar{Y}^{i_0} \subseteq A_{i_0} \cup Z_{i_0}$ and $Z_{i_0} \subseteq \bar{Y}^{i_0}$ by $Y^{i_0} \subseteq A^*_{i_0} \cup Z_{i_0}$ and $\omega_0 \notin \tau(Z)$. Note that

$\omega_0 \notin \tau(Z)$ implies $y_0 \in A^*_{i_0}$ and hence $y_0 = (\omega_0, \rho_{\omega_0}(Y^{i_0}), \sigma_{\omega_0}(Y^{i_0})) = (\omega_0, r_{\omega_0}(r^A), s_{\omega_0})$ holds. Then, we have that

$$\begin{aligned}
\bar{u}^{i_0}(\bar{Y}^{i_0}) &= \bar{v}^{i_0}((\rho_\omega(Y^{i_0}))_{\omega\in\Omega\setminus\{\omega_0\}}) - \sum_{\omega\in\tau(\bar{Y}^{i_0})} \sigma^{i_0}_\omega(\bar{Y}^{i_0}) \\
&\geq v^{i_0}((\rho_\omega(Y^{i_0}))_{\omega\in\Omega\setminus\{\omega_0\}}; \rho_{\omega_0}(Y^{i_0})) - s^{i_0}(\rho_{\omega_0}(Y^{i_0})) - \sum_{\omega\in\tau(\bar{Y}^{i_0})} \sigma^{i_0}_\omega(\bar{Y}^{i_0}) \\
&= v^{i_0}((\rho_\omega(Y^{i_0}))_{\omega\in\Omega\setminus\{\omega_0\}}; \rho_{\omega_0}(Y^{i_0})) - s^{i_0}(r_{\omega_0}(r^A)) - \sum_{\omega\in\tau(\bar{Y}^{i_0})} \sigma^{i_0}_\omega(\bar{Y}^{i_0}) \\
&= v^{i_0}((\rho_\omega(Y^{i_0}))_{\omega\in\Omega\setminus\{\omega_0\}}; \rho_{\omega_0}(Y^{i_0})) - \sigma^{i_0}_{\omega_0}(Y^{i_0}) - \sum_{\omega\in\tau(\bar{Y}^{i_0})} \sigma^{i_0}_\omega(\bar{Y}^{i_0}) \\
&= v^{i_0}((\rho(Y^{i_0})) - \sum_{\omega\in\tau(Y^{i_0})} \sigma^{i_0}_\omega(Y^{i_0}) \\
&= u^{i_0}(Y^{i_0}),
\end{aligned}$$

where the second line follows from the definition of $\bar{v}_{i_0}$. By the choice of Y^{i_0}, $\bar{u}^{i_0}(\bar{Y}^{i_0}) \geq u^{i_0}(Y^{i_0}) > u^{i_0}(A^*)$ which implies that $\bar{u}^{i_0}(\bar{Y}^{i_0}) > \bar{u}^{i_0}(A)$ by $u^{i_0}(A^*) = \bar{u}^{i_0}(A)$, contradicting that A is strongly group stable in M_{k-1}.

Subcase 1(b): $\omega_0 \notin \tau(Y^{i_0})$

In this case, $Y^{i_0} \subseteq A_{i_0} \cup Z_{i_0}$ and $Z_{i_0} \subseteq Y^{i_0}$ hold. We also have that

$$\begin{aligned}
\bar{u}^{i_0}(Y^{i_0}) &= \bar{v}^{i_0}(\rho_\omega(Y^{i_0})_{\omega\in\Omega\setminus\{\omega_0\}}) - \sum_{\omega\in\tau(Y^{i_0})} \sigma^{i_0}_\omega(Y^{i_0}) \\
&\geq v^{i_0}(\rho_\omega(Y^{i_0})_{\omega\in\Omega\setminus\{\omega_0\}}; 0) - s^{i_0}(0) - \sum_{\omega\in\tau(Y^{i_0})} \sigma^{i_0}_\omega(Y^{i_0}) \\
&= v^{i_0}(\rho(Y^{i_0})) - \sum_{\omega\in\tau(Y^{i_0})} \sigma^{i_0}_\omega(Y^{i_0}) \\
&= u^{i_0}(Y^{i_0}),
\end{aligned}$$

where the second line follows from the definition of $\bar{v}_{i_0}$ and the third line follows from $\omega_0 \notin \tau(Y^{i_0})$ and $s^{i_0}(0) = 0$. By the choice of Y^{i_0}, $\bar{u}^{i_0}(\bar{Y}^{i_0}) \geq u^{i_0}(Y^{i_0}) > u^{i_0}(A^*)$ which implies that $\bar{u}^{i_0}(Y^{i_0}) > \bar{u}^{i_0}(A)$ by $u^{i_0}(A^*) = \bar{u}^{i_0}(A)$, contradicting that A is strongly group stable in M_{k-1}.

Case 2: $\omega_0 \in \tau(Z)$.

In this case, there exists $z_0 \in Z$ such that $\tau(z_0) = \omega_0$ which we denote by $z_0 = (\omega_0, \gamma_{\omega_0}, \hat{s}_{\omega_0})$. Note that for any $j \in a(\omega_0) \setminus \{i_0\}$, $Y^j = \{z_0\}$ and hence

$$v^j(0; \gamma_{\omega_0}) - \hat{s}^j_{\omega_0} = u^j(Y^j) > u^j(A^*) = v^j(0; 0) = v^j(0; \gamma_{\omega_0}) - s^j(\gamma_{\omega_0}).$$

Therefore, $s^j(\gamma_{\omega_0}) > \hat{s}^j_{\omega_0}$ for all $j \in a(\omega_0) \setminus \{i_0\}$. By $s^{i_0}(\gamma_{\omega_0}) = -\sum_{j\in a(\omega_0)\setminus\{i_0\}} s^j(\gamma_{\omega_0})$ and $\hat{s}^{i_0}_{\omega_0} = -\sum_{j\in a(\omega_0)\setminus\{i_0\}} \hat{s}^j_{\omega_0}$, we have $\hat{s}^{i_0}_{\omega_0} > s^{i_0}(\gamma_{\omega_0})$ which implies that

$$\begin{aligned} v^{i_0}(\rho(Y^{i_0})) - s^{i_0}(\gamma_{\omega_0}) - \sum_{\omega\in\tau(Y^{i_0})\setminus\{\omega_0\}} \sigma^{i_0}_\omega(Y^{i_0}) &> v^{i_0}(\rho(Y^{i_0})) - \hat{s}^{i_0}_{\omega_0} - \sum_{\omega\in\tau(Y^{i_0})\setminus\{\omega_0\}} \sigma^{i_0}_\omega(Y^{i_0}) \\ &= u^{i_0}(Y^{i_0}), \end{aligned}$$

where the equality follows from $z_0 \in Y^{i_0}$. Therefore, by $u^{i_0}(Y^{i_0}) > u^{i_0}(A^*)$,

$$v^{i_0}(\rho(Y^{i_0})) - s^{i_0}(\gamma_{\omega_0}) - \sum_{\omega\in\tau(Y^{i_0})\setminus\{\omega_0\}} \sigma^{i_0}_\omega(Y^{i_0}) > u^{i_0}(A^*). \tag{18}$$

We next show that $Z \setminus \{z_0\} \neq \emptyset$. Suppose that $Z = \{z_0\}$. This implies that $Y^{i_0} \setminus \{z\} \subseteq A_{i_0}$ and $\sigma^{i_0}_\omega(Y^{i_0}) = s^{i_0}_\omega$ for all $\omega \in \tau(Y^{i_0}) \setminus \{\omega_0\}$. Therefore, by (18),

$$\begin{aligned} v^{i_0}(\rho(Y^{i_0})) - s^{i_0}(\gamma_{\omega_0}) - \sum_{\omega\in\tau(Y^{i_0})\setminus\{\omega_0\}} & s^{i_0}_\omega \\ &> u^{i_0}(A^*) \\ &= v^{i_0}(r^A; r_{\omega_0}(r^A)) - s^{i_0}(r_{\omega_0}(r^A)) - \sum_{\omega\in\tau(A^*_{i_0})\setminus\{\omega_0\}} s^{i_0}_\omega. \end{aligned} \tag{19}$$

Suppose that $Y^{i_0} \setminus \{z\} = A_{i_0}$. Then, we have that $v^{i_0}(\rho(Y^{i_0})) = v^{i_0}(r^A; \gamma_{\omega_0})$ and $\tau(Y^{i_0}) \setminus \{\omega_0\} = \tau(A^*_{i_0}) \setminus \{\omega_0\}$. Therefore, from the above inequality, we have that

$$v^{i_0}(r^A; \gamma_{\omega_0}) - s^{i_0}(\gamma_{\omega_0}) > v^{i_0}(r^A; r_{\omega_0}(r^A)) - s^{i_0}(r_{\omega_0}(r^A)),$$

which contradicts the definition of $r_{\omega_0}(r^A)$. Therefore, $Y^{i_0} \setminus \{z\} \subsetneq A_{i_0}$ holds. This implies that $\Omega' := \{\omega \in \tau(A_{i_0}) | \omega \notin \tau(Y^{i_0})\} \neq \emptyset$. Then, $v_{i_0}(\rho(Y^{i_0})) = v_{i_0}(0_{\Omega'}, r^A_{-\Omega'}; \gamma_{\omega_0})$ and $\tau(Y^{i_0}) \setminus \{\omega_0\} = \tau(A_{i_0}) \setminus \Omega'$ hold. By (19), we have that

$$v^{i_0}(0_{\Omega'}, r^A_{-\Omega'}; \gamma_{\omega_0}) - s^{i_0}(\gamma_{\omega_0}) - \sum_{\omega\in\tau(A_{i_0})\setminus\Omega'} s^{i_0}_\omega > u^{i_0}(A^*).$$

From the definition of $\bar{v}_{i_0}$ and $u^{i_0}(A^*) = \bar{u}^{i_0}(A)$,

$$\bar{v}^{i_0}(0_{\Omega'}, r^A_{-\Omega'}) - \sum_{\omega\in\tau(A_{i_0})\setminus\Omega'} s^{i_0}_\omega > \bar{u}^{i_0}(A).$$

However, this contradicts the individual rationality of A in M_{k-1}. Therefore, $Z \setminus \{z_0\} \neq \emptyset$.

Note that $\bar{Z} := Z \setminus \{z_0\}$ is an outcome in M_{k-1} and $\bar{Z} \cap A = \emptyset$ holds by $Z \cap A^* = \emptyset$. We will show that $\bar{Z}$ is a strong blocking set to A in M_{k-1} i.e, for any $i \in a(\bar{Z})$, there exists $\bar{Y}^i \subseteq A_i \cup \bar{Z}_i$ such that $\bar{u}^i(\bar{Y}^i) > \bar{u}^i(A)$, which contradicts the strong group stability of A in M_{k-1}. Consider any $i \in a(\bar{Z})$ with $i \neq i_0$. Then, we have $i \notin a(\omega_0)$. Therefore, Y^i satisfies that $Y^i \subseteq A_i \cup \bar{Z}_i$, $\bar{u}^i(Y^i) > \bar{u}^i(A)$. For i_0, consider $\bar{Y}^{i_0} := Y^{i_0} \setminus \{z_0\}$. Then, $\bar{Y}^{i_0} \subseteq A_i \cup \bar{Z}_i$ holds. Moreover, we have

$$\begin{aligned}
\bar{u}^{i_0}(\bar{Y}^i) &= \bar{v}^{i_0}((\rho_\omega(Y^{i_0}))_{\omega \in \Omega \setminus \{\omega_0\}}) - \sum_{\omega \in \tau(\bar{Y}_{i_0})} \sigma^{i_0}_\omega(\bar{Y}^{i_0}) \\
&\geq v^{i_0}((\rho_\omega(Y^{i_0}))_{\omega \in \Omega \setminus \{\omega_0\}}; \gamma_{\omega_0}) - s^{i_0}(\gamma_{\omega_0}) - \sum_{\omega \in \tau(\bar{Y}_{i_0})} \sigma^{i_0}_\omega(\bar{Y}^{i_0}) \\
&= v^{i_0}(\rho(Y^{i_0})) - s^{i_0}(\gamma_{\omega_0}) - \sum_{\omega \in \tau(\bar{Y}_{i_0})} \sigma^{i_0}_\omega(\bar{Y}^{i_0}) \\
&= v^{i_0}(\rho(Y^{i_0})) - s^{i_0}(\gamma_{\omega_0}) - \sum_{\omega \in \tau(Y_{i_0}) \setminus \{\omega_0\}} \sigma^{i_0}_\omega(Y^{i_0}) \\
&> u^{i_0}(A^*),
\end{aligned}$$

where the second line follows from the definition of $\bar{v}^{i_0}$, the third line follows from $\gamma_{\omega_0} = \rho_{\omega_0}(Y^{i_0})$ (by definition), and the strict inequality follows from (18). Therefore, by $u^{i_0}(A^*) = \bar{u}^{i_0}(A)$, we have that $\bar{u}^{i_0}(\bar{Y}^{i_0}) > \bar{u}^{i_0}(A)$, contradicting that A is strongly group stable in M_{k-1}. □

Claim 8 and 9 complete the proof of Lemma 1. ■

Proof of Lemma 2. Let $(I, \Omega, a, r^{\max}, v)$ be a multilateral matching market such that (I, Ω, a) admits a 2-cycle. Without loss of generality, we may assume that $\{i_1, i_2\}$ and $\{\omega_1, \omega_2\}$ consists the cycle. For any $r \in \times_{\omega \in \Omega}[0, r^{\max}_\omega]$, let

$$\begin{aligned}
v^{i_1}(r) &= \max\left\{\frac{r_{\omega_1}}{r^{\max}_{\omega_1}}, \frac{r_{\omega_2}}{r^{\max}_{\omega_2}}\right\} - 2\min\left\{\frac{r_{\omega_1}}{r^{\max}_{\omega_1}}, \frac{r_{\omega_2}}{r^{\max}_{\omega_2}}\right\}; \\
v^{i_2}(r) &= \min\left\{\frac{r_{\omega_1}}{r^{\max}_{\omega_1}}, \frac{r_{\omega_2}}{r^{\max}_{\omega_2}}\right\}; \\
v^i(r) &= 0 \text{ for any } i \in I \setminus \{i_1, i_2\} \text{ if exist.}
\end{aligned}$$

We normalize $r^{\max}_{\omega_h} = 1$ for $h = 1, 2$ as is the case for the proofs in Appendix A.

Let Y be an individually rational outcome. We assume that $\rho_{\omega_1}(Y) \leq \rho_{\omega_2}(Y)$. The case where $\rho_{\omega_1}(Y) > \rho_{\omega_2}(Y)$ can be proved in a similar way as the following proof.

Suppose that $\rho_{\omega_1}(Y) > 0$. By the individual rationality of Y and $\rho_{\omega_1}(Y) \leq \rho_{\omega_2}(Y)$, $\sigma^{i_1}_{\omega_1}(Y) \leq -2\rho_{\omega_1}(Y)$. By the individual rationality of Y, $\sigma^i_{\omega_1}(Y) \leq 0$ for all $i \in$

$a(\omega_1) \setminus \{i_1, i_2\}$. Thus, $\sigma^{i_2}_{\omega_1}(Y) \geq 2\rho_{\omega_1}(Y)$. Then, $u^{i_2}(Y) \leq -\rho_{\omega_1}(Y) - \sigma^{i_2}_{\omega_2}(Y) < -\sigma^{i_2}_{\omega_2}(Y) = u^{i_2}(\{(\omega_2, \rho_{\omega_2}(Y), \sigma_{\omega_2}(Y)\})$, contradicting the individual rationality of Y. Hence, $\rho_{\omega_1}(Y) = 0$.

By the individual rationality of Y, $\sigma^{i}_{\omega_1}(Y) = 0$ for all $i \in a(\omega_1)$. Hereafter, we consider the case where $\tau(Y) = \{\omega_2\}$. Thus, $u^{i_1}(Y) = \rho_{\omega_2}(Y) - \sigma^{i_1}_{\omega_2}(Y)$ and $u^{i_2}(Y) = -\sigma^{i_2}_{\omega_2}(Y)$. By the individual rationality of Y, $\sigma^{i}_{\omega_2}(Y) \leq 0$ for all $i \in a(\omega_2) \setminus \{i_1\}$. Hence, $\sigma^{i_1}_{\omega_2}(Y) \geq 0$.

Suppose that $\rho_{\omega_2}(Y) = 0$. Then, $\sigma^{i}_{\omega_2}(Y) = 0$ for all $i \in a(\omega_2)$ by the individual rationality of Y. Thus, $u^{i_1}(Y) = u^{i_2}(Y) = 0$. Let $Z = \{(\omega_1, 1, s_{\omega_1})\}$, where

$$s^{i}_{\omega_1} = \begin{cases} 1/2 & \text{if } i = i_1; \\ -\frac{1}{2(|a(\omega_1)|-1)} & \text{if } i \in a(\omega_1) \setminus \{i_1\}. \end{cases}$$

Then, $\{Z\} = C^{i}(Z \cup Y)$ for all $i \in a(\omega_1)$, and thus, Y is blocked via Z.

Assume, therefore, that $\rho_{\omega_2}(Y) > 0$. Note that $\sigma^{i}_{\omega_2}(Y) \leq 0$ for all $i \in a(\omega_2) \setminus \{i_1, i_2\}$ by the individual rationality of Y. Also, $\sigma^{i_2}_{\omega_2}(Y) \leq 0$ by $\rho_{\omega_1}(Y) = 0$, and hence $\sigma^{i_1}_{\omega_2} \geq 0$.

Define $\hat{Z} = \{(\omega_1, 1, \hat{s}_{\omega_1})\}$ such that

$$\hat{s}^{i}_{\omega_1} = \begin{cases} \sigma^{i}_{\omega_2}(Y) - \frac{\varepsilon}{|a(\omega_1)|-1} & \text{if } i = i_1; \\ \sum_{\ell \in a(\omega_2) \setminus \{i_1\}} \sigma^{\ell}_{\omega_2}(Y) + \varepsilon & \text{if } i = i_2; \\ -\frac{\varepsilon}{|a(\omega_1)|-1} & \text{if } i \in a(\omega_1) \setminus \{i_1, i_2\}, \end{cases}$$

where $\varepsilon > 0$ is sufficiently small so that $\varepsilon < \rho_{\omega_2}(Y)$.

We show that Y is blocked via $\hat{Z}$. For any player $i \in a(\omega_1) \setminus \{i_1, i_2\}$, $\hat{Z} \subseteq Y^{i}$ for all $Y^{i} \in C^{i}(\hat{Z} \cup Y)$ since the payoff of i increases by choosing $(\omega_1, 1, \hat{s}_{\omega_1})$ regardless of other chosen contracts. Also, we have $\{\hat{Z}\} = C^{i_1}(\hat{Z} \cup Y)$ since

$$u^{i_1}(\hat{Z}) = 1 - \sigma^{i_1}_{\omega_2}(Y) + \frac{\varepsilon}{|a(\omega_1)| - 1} > \rho_{\omega_2}(Y) - \sigma^{i_1}_{\omega_2}(Y) = u^{i_1}(Y) \geq u^{i_1}(\emptyset);$$
$$u^{i_1}(\hat{Z}) = 1 - \sigma^{i_1}_{\omega_2}(Y) + \frac{\varepsilon}{|a(\omega_1)| - 1} > 1 - 2\rho_{\omega_2}(Y) - 2\sigma^{i_1}_{\omega_2}(Y) + \frac{\varepsilon}{|a(\omega_1)| - 1} = u^{i_1}(\hat{Z} \cup Y)$$

by $0 < \rho_{\omega_2}(Y) \leq 1$, the individual rationality of Y, and $\sigma^{i_1}_{\omega_2}(Y) \geq 0$. Further, we have $\{\hat{Z} \cup Y\} = C^{i_2}(\hat{Z} \cup Y)$ since

$$u^{i_2}(\hat{Z} \cup Y) = \rho_{\omega_2}(Y) - \sum_{\ell \in a(\omega_2) \setminus \{i_1\}} \sigma^{\ell}_{\omega_2}(Y) - \varepsilon - \sigma^{i_2}_{\omega_2}(Y) > -\sigma^{i_2}_{\omega_2}(Y) = u^{i_2}(Y) \geq u^{i_2}(\emptyset);$$
$$u^{i_2}(\hat{Z} \cup Y) = \rho_{\omega_2}(Y) - \sum_{\ell \in a(\omega_2) \setminus \{i_1\}} \sigma^{\ell}_{\omega_2}(Y) - \varepsilon - \sigma^{i_2}_{\omega_2}(Y) > - \sum_{\ell \in a(\omega_2) \setminus \{i_1\}} \sigma^{\ell}_{\omega_2}(Y) - \varepsilon = u^{i_2}(\hat{Z})$$

by $\sigma^i_{\omega_2}(Y) \leq 0$ for all $i \in a(\omega_2) \setminus \{i_1\}$, individual rationality of Y, and $\rho_{\omega_2}(Y) > \varepsilon > 0$. Hence, Y is blocked via $\hat{Z}$. $\square$

Proof of Proposition 2. Let $(I, \Omega, a, r^{\max}, v)$ be a multilateral matching market, where (I, Ω, a) admits a cycle consisting of $\{i_1, ..., i_k\} \subset I$ and $\{\omega_1, ..., \omega_k\} \subset \Omega$ with $k \geq 3$. Without loss of generality, we may assume that this cycle is irreducible since we are assuming that (I, Ω, a) admits no 2-cycle. For any $r \in \times_{\omega\in\Omega}[0, r^{\max}_\omega]$, let

$$
\begin{aligned}
v^{i_1}(r) &= \max\left\{\frac{r_{\omega_1}}{r^{\max}_{\omega_1}}, \frac{r_{\omega_k}}{r^{\max}_{\omega_k}}\right\} - 2\min\left\{\frac{r_{\omega_1}}{r^{\max}_{\omega_1}}, \frac{r_{\omega_k}}{r^{\max}_{\omega_k}}\right\};\\
v^{i_\ell}(r) &= -\left|1 - \frac{r_{\omega_{\ell-1}}}{r^{\max}_{\omega_{\ell-1}}} - \frac{r_{\omega_\ell}}{r^{\max}_{\omega_\ell}}\right| \text{ for } \ell = 2, k;\\
v^{i_h}(r) &= -2\left|\frac{r_{\omega_{h-1}}}{r^{\max}_{\omega_{h-1}}} - \frac{r_{\omega_h}}{r^{\max}_{\omega_h}}\right| \text{ for all } h = 3, ..., k-1 \text{ whenever } k > 3;\\
v^i(r) &= 0 \text{ for any } i \in I \setminus \{i_1, ...i_k\} \text{ if exist.}
\end{aligned}
$$

We normalize $r^{\max}_\omega = 1$ for all $\omega \in \Omega$ as is the case for the proofs in Appendix A.

Let Y be an individually rational outcome.

Claim 10 $\sum_{i\in I} u^i(Y) \leq 0$. *Moreover, the equality holds only if (i) either* $\rho_{\omega_1}(Y) = 0$ *or* $\rho_{\omega_k}(Y) = 0$*; and (ii)* $\rho_{\omega_{h-1}}(Y) = \rho_{\omega_h}(Y)$ *for all* $h = 3, ..., k-1$.

Proof of Claim 10. It suffices to show that $\sum_{\ell=1}^k v^{i_\ell}(r) \leq 0$ for any r. Fix an arbitrary r. Then,

$$
\begin{aligned}
\sum_{\ell=2}^{k} v^{i_\ell}(r) &= -|1 - r_{\omega_1} - r_{\omega_2}| - 2\sum_{h=3}^{k-1}|r_{\omega_{h-1}} - r_{\omega_h}| - |1 - r_{k-1} - r_k|\\
&\leq -|1 - r_{\omega_1} - r_{\omega_2}| - \sum_{h=3}^{k-1}|r_{\omega_{h-1}} - r_{\omega_h}| - |1 - r_{k-1} - r_k|\\
&\leq -(1 - r_{\omega_1} - r_{\omega_2}) - \sum_{h=3}^{k-1}(r_{\omega_{h-1}} - r_{\omega_h}) - (r_{\omega_{k-1}} + r_{\omega_k} - 1)\\
&= r_{\omega_1} - r_{\omega_k}
\end{aligned}
\tag{20}
$$

by the property of the absolute value. Similarly,

$$\begin{aligned}\sum_{\ell=2}^{k} v^{i_\ell}(r) &= -|1-r_{\omega_1}-r_{\omega_2}| - 2\sum_{h=3}^{k-1}|r_{\omega_{h-1}}-r_{\omega_h}| - |1-r_{k-1}-r_k| \\ &\le -|1-r_{\omega_1}-r_{\omega_2}| - \sum_{h=3}^{k-1}|r_{\omega_{h-1}}-r_{\omega_h}| - |1-r_{k-1}-r_k| \\ &\le -(r_{\omega_1}+r_{\omega_2}-1) - \sum_{h=3}^{k-1}(r_{\omega_h}-r_{\omega_{h-1}}) - (1-r_{\omega_{k-1}}-r_{\omega_k}) \\ &= r_{\omega_k}-r_{\omega_1}.\end{aligned} \tag{21}$$

By (20) and (21),

$$\sum_{\ell=1}^{k} v^{i_\ell}(r) \le \begin{cases}(r_{\omega_1}-2r_{\omega_k})+(r_{\omega_k}-r_{\omega_1}) = -r_{\omega_k} & \text{if } r_{\omega_1}\ge r_{\omega_k};\\ (r_{\omega_k}-2r_{\omega_1})+(r_{\omega_1}-r_{\omega_k}) = -r_{\omega_1} & \text{if } r_{\omega_1}< r_{\omega_k}.\end{cases} \tag{22}$$

Thus, $\sum_{\ell=1}^{k} v^{i_\ell}(r) \le 0$.

For the latter part, (i) follows from (22), and (ii) follows from the first inequality in each of (20) and (21). □

By the individual rationality of Y, $u^{i_1}(Y) \ge 0$, $u^{i_h}(Y) \ge -1$ for $h = 2, k$, and $u^{i_h}(Y) \ge 0$ for all $h = 3, ..., k-1$. First, suppose that $\sum_{i\in a(\omega_1)} u^i(Y) < 1$. Let $Z = \{(\omega_1, 1, s_{\omega_1})\}$ such that $s^{i_1}_{\omega_1} = 1 - u^{i_1}(Y) - \varepsilon/|a(\omega_1)|$, $s^{i_2}_{\omega_1} = -u^{i_2}(Y) - \varepsilon/|a(\omega_1)|$, and $s^{i}_{\omega_1} = -u^{i}(Y) - \varepsilon/|a(\omega_1)|$ for all $i \in a(\omega_1) \setminus \{i_1, i_2\}$, where $\varepsilon = 1 - \sum_{i\in a(\omega_1)} u^i(Y) > 0$. Note that $\sum_{i\in a(\omega_1)} s^i_{\omega_1} = 0$. Note also that $u^i(Z) - u^i(Y) = \varepsilon/|a(\omega_1)| > 0$ for all $i \in a(\omega_1)$. Then, since Z is a singleton, $Z \subseteq Y^i$ for any $Y^i \in C^i(Z \cup Y)$ for all $i \in a(\omega_1)$. Let $\hat{Z} = \bigcup_{i\in a(\omega_1)} \hat{Y}^i$, where $\hat{Y}^i \in C^i(Z \cup Y)$ is arbitrary chosen for all $i \in a(\omega_1)$.

Since (I, Ω, a) admits no 2-cycle, there is no $\omega' \in \Omega\setminus\{\omega_1\}$ such that $|a(\omega_1)\cap a(\omega')| > 1$. Then, $\tau(\hat{Y}^i) \cap \tau(\hat{Y}^j) = \{\omega_1\}$ for any distinguished $i, j \in a(\omega_1)$. Thus, $\hat{Z}$ is an outcome and $\hat{Z}_i = \hat{Y}^i$ for all $i \in a(\omega_1)$. Therefore, Y is weakly setwise blocked via Z. Similarly, we can prove that Y is weakly setwise blocked if $\sum_{i\in a(\omega_k)} u^i(Y) < 1$. Therefore, assume that both $\sum_{i\in a(\omega_1)} u^i(Y) \ge 1$ and $\sum_{i\in a(\omega_k)} u^i(Y) \ge 1$.

Second, suppose that $u^{i_2}(Y) > -1$. Then, by $\sum_{i\in a(\omega_k)} u^i(Y) \ge 1$ and $u^i(Y) \ge 0$ for all $i \in a(\omega_1) \setminus (\{i_2\} \cup a(\omega_k))$, $\sum_{i\in a(\omega_1)\cup a(\omega_k)} u^i(Y) > 0$. Since $u^i(Y) \ge 0$ for all $i \in I \setminus (a(\omega_1) \cup a(\omega_k))$ by the individual rationality of Y, $\sum_{i\in I} u^i(Y) > 0$, contradicting Claim 10. Thus, $u^{i_2}(Y) = -1$ by the individual rationality of Y. Similarly, we can prove that $u^{i_k}(Y) = -1$. Thus, $\sum_{i\in a(\omega_1)\setminus\{i_2\}} u^i(Y) \ge 2$ and $\sum_{i\in a(\omega_k)\setminus\{i_k\}} u^i(Y) \ge 2$. If

$\sum_{i\in a(\omega_1)\setminus\{i_2\}} u^i(Y) > 2$, then $\sum_{i\in(I\setminus a(\omega_1))\cup\{i_2\}} u^i(Y) < -2$ by Claim 10. By $u^{i_2}(Y) = u^{i_k}(Y) = -1$, $\sum_{i\in I\setminus(a(\omega_1)\cup\{i_k\})} < 0$. It follows that some $i \in I \setminus (a(\omega_1) \cup \{i_k\})$ exists and $u^i(Y) < 0$, contradicting the individual rationality of Y. Thus, $\sum_{i\in a(\omega_1)\setminus\{i_2\}} u^i(Y) = 2$. We can prove show $\sum_{i\in a(\omega_k)\setminus\{i_k\}} u^i(Y) = 2$ in a similar way. Therefore,

$$\sum_{i\in a(\omega_1)\setminus\{i_2\}} u^i(Y) = \sum_{i\in a(\omega_k)\setminus\{i_k\}} u^i(Y) = 2. \tag{23}$$

Third, suppose that $u^{i_1}(Y) \neq 2$. Note that $\{i_1, i_2\} \subsetneq a(\omega_1)$ and $\{i_1, i_k\} \subsetneq a(\omega_k)$ by (23). Then, $u^{i_1}(Y) < 2$ by the individual rationality of Y and (23). By (23), $\sum_{i\in a(\omega_1)\setminus\{i_1,i_2\}} u^i(Y) = \sum_{i\in a(\omega_k)\setminus\{i_1,i_k\}} u^i(Y) = 2 - u^{i_1}(Y)$. By $u^{i_1}(Y) < 2$,

$$u^{i_1}(Y) + \sum_{i\in a(\omega_1)\setminus\{i_1,i_2\}} u^i(Y) + \sum_{i\in a(\omega_k)\setminus\{i_1,i_k\}} u^i(Y) = 4 - u^{i_1}(Y) > 2.$$

Since $u^{i_2}(Y) = u^{i_k}(Y) = -1$ and $u^i(Y) \geq 0$ for all $i \in I\setminus(a(\omega_1)\cup a(\omega_k))$, $\sum_{i\in I} u^i(Y) > 0$, contradicting Claim 10. Hence $u^{i_1}(Y) = 2$. Then, by Claim 10 and the individual rationality of Y, $u^i(Y) = 0$ for all $i \in I \setminus \{i_1, i_2, i_k\}$.

To summarize,

$$(u^{i_1}(Y), u^{i_2}(Y), u^{i_3}(Y)..., u^{i_{k-1}}(Y), u^{i_k}(Y)) = (2, -1, 0, ..., 0, -1)$$

and $u^i(Y) = 0$ for all $i \in I \setminus \{i_1, ..., i_k\}$. Note that $\sigma^i_\omega(Y) = 0$ for all $i \in I \setminus \{i_1, ..., i_k\}$ and all $\omega \in \Omega$ together with the individual rationality of Y.

By (ii) of the latter part of Claim 10, $\sum_{\ell=1}^k u^{i_\ell}(Y) = 0$ implies $\rho_{\omega_2}(Y) = \cdots = \rho_{\omega_{k-1}}(Y)$. Since $u^{i_\ell}(Y) = 0$ for all $\ell = 3, ..., k-1$, $\sigma^{i_\ell}_{\omega_{\ell-1}}(Y) + \sigma^{i_\ell}_{\omega_\ell} = 0$. Since $\sigma^i_\omega(Y) = 0$ for all $i \in I \setminus \{i_1, ..., i_k\}$ and all $\omega \in \Omega$, $\sigma^{i_\ell}_{\omega_\ell}(Y) + \sigma^{i_{\ell+1}}_{\omega_\ell}(Y) = 0$ for all $\ell = 2, ..., k-1$. These imply

$$\sigma^{i_{\ell-1}}_{\omega_{\ell-1}}(Y) = -\sigma^{i_\ell}_{\omega_{\ell-1}}(Y) = \sigma^{i_\ell}_{\omega_\ell}(Y) = -\sigma^{i_{\ell+1}}_{\omega_\ell}(Y) \text{ for all } \ell = 3, ..., k-1; \tag{24}$$

By (i) of the latter part of Claim 10, $\sum_{\ell=1}^k u^{i_\ell}(Y) = 0$ implies either $\rho_{\omega_1}(Y) = 0$ or $\rho_{\omega_k}(Y) = 0$. Hereafter, assume that $\rho_{\omega_k}(Y) = 0$. The case where $\rho_{\omega_1}(Y) = 0$ can be proved in a similar way as the following proof. By the individual rationality of Y, $\sigma^i_{\omega_k}(Y) = 0$ for all $i \in a(\omega_k)$. By $u^{i_k}(Y) = -1$, $\rho_{\omega_{k-1}}(Y) = \sigma^{i_k}_{\omega_{k-1}}(Y)$. Then,

$$\rho_{\omega_2}(Y) = \rho_{\omega_{k-1}}(Y) = \sigma^{i_k}_{\omega_{k-1}}(Y) = \sigma^{i_3}_{\omega_2}(Y) = -\sigma^{i_2}_{\omega_2}(Y)$$

by (24). Then, $u^{i_2}(\{\omega_2, \rho_{\omega_2}(Y), \sigma_{\omega_2}(Y)\}) = -|1-\rho_{\omega_2}(Y)| + \rho_{\omega_2}(Y) = -1+2\rho_{\omega_2}(Y)$. By the individual rationality of Y and $u^{i_2}(Y) = -1$, $\rho_{\omega_2}(Y) = 0$. Therefore, $\rho_{\omega_h}(Y) = 0$ for all $h = 2, ..., k-1$. Then, $\sum_{h=3}^{k} v^{i_h}(\rho(Y)) = -1$. By $\sum_{h=1}^{k} v^{i_h}(\rho(Y)) = 0$, we have $v^{i_1}(\rho(Y)) + v^{i_2}(\rho(Y)) = 1$. It follows that $\rho_{\omega_1}(Y) - (1-\rho_{\omega_1}(Y)) = 1$ from $\rho_{\omega_2}(Y) = \rho_{\omega_k}(Y) = 0$. Thus, $\rho_{\omega_1}(Y) = 1$. By $u^{i_1}(Y) = 2$ and $u^{i_2}(Y) = -1$, $\sigma^{i_1}_{\omega_1}(Y) = -1$ and $\sigma^{i_2}_{\omega_2}(Y) = 1$.

To summarize,

$$
\begin{aligned}
(\rho_{\omega_1}(Y), \rho_{\omega_2}(Y), ..., \rho_{\omega_k}(Y)) &= (1, 0, ..., 0);\\
(\sigma^{i_1}_{\omega_1}(Y), \sigma^{i_1}_{\omega_k}(Y)) &= (-1, 0);\\
(\sigma^{i_2}_{\omega_1}(Y), \sigma^{i_2}_{\omega_2}(Y)) &= (1, 0);\\
(\sigma^{i_h}_{\omega_{h-1}}(Y), \sigma^{i_h}_{\omega_h}(Y))) &= (0, 0) \text{ for all } h = 3, ..., k.
\end{aligned}
$$

Then, we may consider that $Y = \{y_1\} = \{(\omega_1, 1, \sigma_{\omega_1}(Y))\}$, where $\sigma_{\omega_1}(Y)$ is described above.

Now, we construct a weakly setwise blocking set Z. Let Z be an outcome such that $\tau(Z) = \{\omega_2, ..., \omega_{k-1}\}$. Let $\rho_{\omega_h}(Z) = 1$ for all $h = 2, ..., k-1$. Let $\varepsilon_2 > 0$ be a sufficiently small real number. For each $h = 3, ..., k-1$, define $\varepsilon_h = \varepsilon_{h-1}/|a(\omega_h)|$. For each $h = 2, ..., k-1$, define

$$
\sigma^i_{\omega_h}(Z) = \begin{cases} \varepsilon_h & \text{if } i = i_h;\\ -2\varepsilon_{h+1} & \text{if } i = i_{h+1};\\ -\varepsilon_{h+1} & \text{if } i \in a(\omega_h) \setminus \{i_h, i_{h+1}\}. \end{cases}
$$

The proof completes if we prove $Z_i \in C^i(Z \cup Y)$ and $Z_i \subseteq Y^i$ for all $Y^i \in C^i(Z \cup Y)$ for all $i \in a(Z)$ since Z is an outcome. It is straightforward that $\{Z_i \cup Y'_i |\ Y'_i \subseteq Y_i\} = C^i(Z \cup Y)$ for all $i \in a(Z) \setminus \{i_2, ..., i_k\}$ since $\sigma^i_\omega(Z) < 0$ for all $\omega \in \tau(Z)$ and $\sigma^i_\omega(Y) = 0$ for all $\omega \in \Omega$.

We turn to the proof for players $i_2, ..., i_k$. For all $h = 2, ..., k-1$, let $z_h \in Z$ be a contract such that $\tau(z_h) = \omega_h$ for the notational simplicity. We have

$$
\begin{aligned}
u^{i_2}(\{z_2\}) &= -\varepsilon_2;\\
u^{i_2}(\{y_1, z_2\}) &= -2-\varepsilon_2;\\
u^{i_2}(Y') &\le -1 \text{ for any } Y' \subseteq Y.
\end{aligned}
$$

Hence, $\{Z_{i_2}\} = \{\{z_2\}\} = C^{i_2}(Z \cup Y)$. For each $h = 3, ..., k-1$, $Y_{i_h} = \emptyset$. Then, for each $h = 3, ..., k-1$, we have $\{Z_{i_h}\} = \{\{z_{h-1}, z_h\}\} = C^{i_h}(Z) = C^{i_h}(Z \cup Y)$ by

$$u^{i_h}(\{z_{h-1}, z_h\}) = \varepsilon_h;$$
$$u^{i_h}(\{z_h\}) = -2 - \varepsilon_h;$$
$$u^{i_h}(\{z_{h-1}\}) = -2 + 2\varepsilon_h;$$
$$u^{i_h}(\emptyset) = 0.$$

Finally, we have $\{Z_{i_k}\} = \{\{z_{k-1}\}\} = C^{i_k}(Z) = C^{i_k}(Z \cup Y)$ by

$$u^{i_k}(\{z_{k-1}\}) = u^{i_k}(\{z_{k-1}, y_k\}) = 2\varepsilon_{k-1};$$
$$u^{i_k}(\emptyset) = -1.$$

Hence, Y is weakly setwise blocked via Z. □

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