

# THE SOCIAL COST PROBLEM, RIGHTS AND THE (NON)EMPTY CORE

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## Abstract

The Coase theorem holds only if two theses are verified. The efficiency thesis says that, in the absence of transaction costs and if property rights are well defined, individuals will always bargain to reach an optimal allocation of resources. The invariance (or neutrality) thesis says that the outcome of bargaining is independent of the distribution of rights. In this article we use a cooperative game model to show whether or not these two theses hold. We study a social cost problem between one polluter interacts and many potential victims. To analyze the solutions, we introduce three properties. First, *Core stability* indicates that the payoff vector of the solution belongs to the core of the associated cooperative game. Second *No veto power for a victim* says that no victim has the power to veto an agreement signed by the rest of the society. Third, *Full compensation* ensures that the victims are compensated for the damage caused by pollution. We then demonstrate two theorems. First, *Core stability* is satisfied if and only if the rights are assigned either to the polluter or to the entire set of victims. This means that the efficiency thesis can be satisfied but at the expense of the invariance thesis. Second, we show that no solution satisfied at the same time *Core stability*, *No veto power for a victim* and *Full compensation*. Hence, it is not possible to reach a core-stable agreement which fully compensates all the victims in a legal structure where no victim has a veto power.

*Keywords:* Coase theorem – Compensation – Core – Rights – (Im)possibility results – Veto power.

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## 1. Introduction

**The “Coase theorem”.** After George Stigler coined the term, and invented it (1966, p. 113), the “Coase theorem” has been re-formulated many times (see a list in Medema, 2011). All these versions more or less differ but nonetheless can be summarized by two theses that form what is now known as the “Coase theorem”. First, the *efficiency thesis* according to which in the absence of transaction costs and if property rights are well defined, individuals will always bargain to reach an optimal allocation of resources; second, the *invariance or neutrality thesis* according to which the outcome of bargaining is independent of the distribution of rights. Thus, *stricto sensu*, it can be said the “Coase theorem” holds if both theses are verified simultaneously.

The result has been discussed in a particularly important literature. Attempts were made to generalize the theorem – in particular because Ronald Coase, in “The Problem of Social Cost” (1960) reasoned with two agents only – or to demonstrate under which conditions it is verified – because of its assumption of zero transaction costs. The models have then been developed in terms of competitive equilibrium (e.g. Hurwicz, 1995, Chipman and Tian, 2012), of (perfect) Nash equilibrium of a strategic/bargaining game (e.g. Ellingsen, Pastreva, 2016, Anderlini, Felli, 2001, 2006, Lee, Sabourian, 2007), of the nonemptiness of the core of a cooperative game (e.g., Shapley and Shubik 1969, Aivazian and Callen, 1981, 1987, Coase, 1981, Ambec and Kervinio, 2016). In this paper we adopt a cooperative approach, that differs from the above-mentioned works on the social cost problem in that we provide an axiomatic study of it for the case where one polluter interacts with many potential victims.

**The model.** We consider a large class of social cost problems with zero transaction costs in which are involved a fixed and finite set of at least one victim and one polluter. The activity of the latter creates damage that affect the former. We assume that those damage can be described by nondecreasing functions on the set of activity levels of the polluter. In a similar way, the benefit function of the polluter is also assumed to be nondecreasing on his or her set of activity levels, so that victims and polluter have conflicting interests. In order to iron those conflicts out and to solve the problem of social cost, a negotiation will take place with the objective to sign a binding agreement about how much activity the polluter will be able to undertake and how much compensation victims should receive. Now, we interpret the possibility (permission) granted to each group of agents who want to form a coalition to sign such binding agreements about the level of activity of the polluter as a right; these rights form a *mapping* of rights. We then assume that a coalition that signs a binding agreement always achieves efficiency: the agreement allows the coalition to select the best contract for *itself*. This means, complementarily, that a right for a coalition is interpreted as the permission to sign efficient agreements for itself *without consideration for the interest of nonmembers*.

In addition, we impose some reasonable conditions on the mappings of rights. The first one is a condition of *sovereignty* saying that the grand coalition is entitled to sign

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binding agreements with the polluter. The second one is a condition of *monotonicity* saying that if a coalition of agents is entitled to negotiate binding agreements, then every larger coalition inherits this right. The third property is a condition of *effectivity of the law* saying that if a coalition is entitled to sign efficient binding agreements for itself, then the nonmembers cannot form a coalition to prevent this coalition from exercising its rights. The last property is a condition of *independence of law from the economic characteristics*, saying that the right to control an issue is independent on the benefit or damage functions of its members, though the set of feasible agreements depend on these functions.

For each social cost problem, a *solution*, on the class of social cost problems, is formed by a mapping of rights and a monetary payoff vector.

To determine the payoff vector, a cooperative game with transferable utility is constructed from the mapping of rights and the benefit and damage functions of the agents of the social cost problem.

**Three properties.** We introduce three properties for a solution on the class of social cost problems. The first property, *Core stability*, says that given the rights assignment of the solution, for any social cost problem, agents are able to sign a socially efficient agreement such that no coalition of agents can rationally block it. In the two-agent case, i.e. the case where one polluter interacts with one potential victim, *Core stability* expresses the “efficiency thesis” since, when it is satisfied, it means that both agents are able to reach a socially efficient agreement which is also individually rational.

Stability is only aspect of the problem. We take into account two other aspects that a solution may have to satisfy, that correspond to our second and third property.

Our second property relates to a minimum democratic requirement that should be guaranteed. We introduce it to avoid one of the consequences of the possibly large number of victims affected by pollution that exist in our model. Indeed, in this type of situation, any victim who is not in the grand coalition might have the possibility to unilaterally veto further negotiations between the polluter and the other victims. We claim that this would be anti-democratic. Hence, we introduce a condition, that corresponds to our second property, to avoid this problem. The property is defined as follows: a solution satisfies *No veto power for a victim* if the associated mapping of rights is such that any coalition formed by the polluter and the entire set of victims except one retains its right to negotiate binding agreements.

Our third property is a consequence of the introduction of a fairness principle. Victims of accidents, pollution or, more broadly, negative externalities suffer from losses while, in the meantime, the polluter earns benefits. This clearly, as it is well known, comes from the fact that the polluter does not bear the entire costs that result from his or her actions. We argue that, to guarantee fairness, victims should be fully compensated for their losses. Now, in terms of Hicks-Kaldor efficiency, one knows that the benefits made by one party should be enough to compensate the loss of the other(s) party(ies). We add a more specific property that aims at ensuring that the victims are fully compensated for the damage and that the polluter as an interest to produce. From this perspective, a solution satisfies the property of *Full compensation* if, for each social cost problem, the payoff component of the solution is a nonnegative vector.

**Main results** We obtain two main results. The first one, contained in Theorem 1, characterizes the subset of solutions that satisfy *Core stability*. It turns out that a solution satisfies *Core stability* if and only if either the polluter is not liable for the damage his or her activity generates or, on the contrary, if the polluter is liable for the damage he or she can generate and the entire set victims is the only coalition that has the right to sign binding agreements with the polluter. As it will become clear in section 4, Theorem 1 applied to the two-agent case, as in Coase’s original article, indicates that the core is nonempty whatever the rights assignment. Indeed, when one polluter interacts with only one potential victim, either the rights are assigned to the polluter or the rights are assigned to the victim. Therefore, our result can be viewed as a generalization of the “Coase theorem” from the two-agent case to the case where many potential victims interact with the polluter. To prove Theorem 1, we establish a useful intermediary result, contained in Proposition 3, saying that a solution satisfies the combination of *Core stability* and *No veto power for a victim* if and only if the polluter is not liable for the damage his or her activity generates. This means that the only way to guarantee a stable agreement in an environment where no victim has the power to veto an agreement signed by the entire population is to distribute the rights to the polluter and to each coalition containing it.

The second main result, contained in Theorem 2, is an impossibility result: there is no solution on the set of social cost problems that satisfies *Core stability*, *No veto power for a victim* and *Full compensation*. To obtain this impossibility result, we first prove in Proposition 5 that a solution satisfies *Core stability* and *Full compensation* if and only if the rights are assigned to the set of victims. It is worth mentioning that Theorem 1 and Theorem 2 continue to hold on the subclass of social cost problems where benefit functions are concave and damage functions are convex.

To go back to the two theses of the “Coase theorem”, Theorem 1 says that a core-stable negotiated outcome to a problem of social cost exists; nevertheless, beyond the two-agent case, this outcome is not independent from the distribution of rights. This is equivalent to say that the *efficiency thesis* is not compatible with the *neutrality thesis* in situations where two or more potential victims interact with only one polluter. Theorem 2 indicates that it is not possible to reach a core-stable agreement which fully compensates all the victims in a legal structure where no victim has a veto power.

The article is organized as follows. Section 2 sets up our formal apparatus and introduce the three properties for a solution to the social cost problems. Section 3 contains our main results. Section 4 goes back to the two-agent case.

## 2. Preliminaries

### 2.1. Cooperative games

Let  $N$  be a fixed and finite set of  $n$  agents. Subsets of  $N$  are called *coalitions*, while  $N$  is called the *grand coalition*. A cooperative game with transferable utility or simply a *TU-game* is a pair  $(N, v)$ , where  $v$  is a function  $v : 2^N \rightarrow \mathbb{R}$  such that  $v(\emptyset) = 0$ . For each coalition  $S \subseteq N$ ,  $v(S)$  describes the *worth* of the coalition  $S$  when its members cooperate. The worth  $v(S)$  is thus what the members of  $S$  can accomplish together without the

assistance of members of  $N \setminus S$ . Let  $\mathbb{V}$  be the class of all TU-games. A *payoff vector*  $x \in \mathbb{R}^n$  of a TU-game  $(N, v) \in \mathbb{V}$  is an  $n$ -dimensional vector giving a payoff  $x_i \in \mathbb{R}$  to each agent  $i \in N$ . For each  $S \subseteq N$  and each  $x \in \mathbb{R}^n$ , denote by  $x(S)$  the sum of the payoffs  $x_i$ ,  $i \in S$ . A payoff vector is *efficient* with respect to  $(N, v)$  if  $x(N) = v(N)$ ; it is *coalitionally rational* if  $x(S) \geq v(S)$  for every possible coalition  $S$ .

The *core* of  $(N, v) \in \mathbb{V}$ , denoted by  $\mathbb{C}(N, v)$ , is the set, possibly empty, of efficient and coalitionally rational payoff vectors:

$$\mathbb{C}(N, v) = \left\{ x \in \mathbb{R}^n : \forall S \subseteq N, x(S) \geq v(S) \text{ and } x(N) = v(N) \right\}. \quad (1)$$

The interpretation of the core is that no group of agents has an incentive to split from the grand coalition  $N$  and form a smaller coalition  $S$  since they collectively receive at least as much as what they can obtain for themselves as coalition.

The so-called Bondareva-Shapley theorem (Bondareva, 1962; Shapley, 1967) provides a sufficient and necessary condition under which the core of a TU-game is nonempty. First, we introduce the concept of balanced map. A *balanced map*  $\lambda : 2^N \rightarrow [0, 1]$  is such that:

$$\lambda(\emptyset) = 0, \text{ and } \forall i \in N, \sum_{S \ni i} \lambda(S) = 1.$$

Denote by  $\mathbb{B}(N)$  the set of balanced maps over  $N$ .

**Proposition 1** (*Bondareva-Shapley Theorem*)

For each  $(N, v) \in \mathbb{V}$ ,  $\mathbb{C}(N, v) \neq \emptyset$  if and only if for each balanced map  $\lambda \in \mathbb{B}(N)$ , the following inequality holds:

$$\sum_{S \subseteq N} \lambda(S)v(S) \leq v(N). \quad (2)$$

## 2.2. Social cost problems

We analyze situations in which there are one polluter and more than one potential victim affected by the activity of the polluter. This means that we consider a finite and fixed set of agents  $M \cup \{p\}$  of size  $m + 1$ , where  $m \geq 2$ .<sup>1</sup>

Agent  $p$  undertakes an activity at level  $z \in Z$  that generates a *private benefit*  $B_p(z) \geq 0$ . The private benefit function  $B_p : Z \rightarrow \mathbb{R}$  is nonnegative and nondecreasing on a subset  $Z \subseteq \mathbb{R}$  containing 0. The production of  $z \in Z$  by  $p$  generates also a negative externality. This *external damage* potentially affects each agent  $i \in M$  through the function  $D_i(z)$ . Precisely, for each  $i \in M$ , the external damage function  $D_i : Z \rightarrow \mathbb{R}$  is nonnegative and nondecreasing on  $Z$ . We further assume that if  $p$  is not active, i.e.  $z = 0$ , then, for each  $i \in M$ ,  $D_i(0) = B_p(0) = 0$ .

A *social cost problem*  $P$  on  $M \cup \{p\}$  is described by a tuple  $(Z, B_p, (D_i)_{i \in M})$ . Let  $\mathbb{P}$  be the class of all social cost problems  $P$  on the finite set of agents  $M \cup \{p\}$ .

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<sup>1</sup>The case where  $m = 1$  is studied in section 4.

Let  $P = (Z, B_p, (D_i)_{i \in M}) \in \mathbb{P}$ . For each nonempty coalition  $S \subseteq N$ , denote by  $Z_{P,S}^*$ , the set of socially optimal levels of production for coalition  $S$ . We have:

$$Z_{P,S}^* = \arg \max_{z \in Z} \left\{ 1_p(S) B_p(z) - \sum_{i \in S \setminus \{p\}} D_i(z) \right\},$$

where  $1_p(S) = 1$  if  $S \ni p$ , and  $1_p(S) = 0$  otherwise. For each  $P$  and each  $S$ , we assume that  $Z_{P,S}^*$  is nonempty. This requirement is satisfied, for instance, if for each  $P$ ,  $Z$  is a finite set, or  $B_p$  and  $D_i$ ,  $i \in M$  are continuous on a compact subset  $Z$  of  $\mathbb{R}$ . In the sequel, for  $P \in \mathbb{P}$ , and each non empty coalition  $S$ , we will use the notation  $z_S^*$ , instead of  $z_{P,S}^*$ , to represent an element of  $Z_{P,S}^*$ . Note that  $0 \in Z_{P,S}^*$  whenever  $S \not\ni p$ .

**Example 1** *In many applications the benefit function is concave and the external damage functions are convex. A specific instance of such social problems is the social cost problem  $\tilde{P}$  where  $Z = [0, 1]$ ,  $B_p(z) = \sqrt{z}$ , and for  $i \in M$ ,  $D_i(z) = c_i z$  for some  $c_i > 0$ .  $\square$*

### 2.3. Coalitions and rights assignment

We aim to model the impact of rights on the activity of the polluter in such a way that the right to control an issue for a coalition is not the consequence of the internal structure of this coalition but is the consequence of a certain distribution of rights given *ex ante* to the coalition. Therefore, the set of agreements that a coalition can sign is determined by the distribution of property rights. The assumption that property rights determine economic possibilities is consistent with the law and economics literature and the idea that one must take the existing legal structure – the existing distribution of rights – as a starting point of the analysis. This is not only one of – if not the – most fundamental and well known insights of law and economics analyses, brought up by Coase (1959, 1960) and Buchanan (1972) among others.

Formally, this means that the set of feasible levels of production that a coalition can choose is summarized by a *mapping of rights*  $\phi$  which assigns a subset  $\phi_P(S) \in \{\emptyset, Z_{P,S}^*\}$  to each  $P = (Z, B_p, (D_i)_{i \in M}) \in \mathbb{P}$  and each  $S \subseteq M \cup \{p\}$ . The situation  $\phi_P(S) = Z_{P,S}^*$  means that  $S$  is entitled to control the activity of  $p$  either to negotiate a socially optimal level of production if  $S \ni p$  or to prevent  $p$  from being active if  $S \not\ni p$ . In case  $\phi_P(S) = \emptyset$ , coalition  $S$  is neither entitled to produce if  $S \ni p$  nor entitled to prevent the activity of  $p$  if  $S \not\ni p$ . Thus,  $\phi$  assigns rights to some coalitions, which allow them to implement an optimal contract without consideration of the nonmembers of this coalition. This means that, in our framework, a right is viewed as the permission to choose some actions that reduce the set of possible outcomes in a specific way.

We assume that  $\phi$  satisfies five independent and natural conditions: for each  $P = (Z, B_p, (D_i)_{i \in M}) \in \mathbb{P}$ , we have:

$$(C.1) \quad \phi_P(\emptyset) = \emptyset;$$

$$(C.2) \quad \phi_P(M \cup \{p\}) = Z_{P, M \cup \{p\}}^*;$$

**(C.3)** If  $\phi_P(S) = Z_{P,S}^*$ , then, for each  $T \supseteq S$ ,  $\phi_P(T) = Z_{P,T}^*$ ;

**(C.4)** If  $\phi_P(S) = Z_{P,S}^*$ , then  $\phi_P(N \setminus S) = \emptyset$ .

**(C.5)** For each  $P, P' \in \mathbb{P}$  and each  $S \subseteq M \cup \{p\}$ , if  $\phi_P(S) \neq \emptyset$ , then  $\phi_{P'}(S) \neq \emptyset$ .

Condition **(C.1)** is clear: there is no assignment of rights associated with the empty coalition.

Condition **(C.2)** is a condition of sovereignty. It indicates that the grand coalition  $M \cup \{p\}$  has always the right to implement a socially optimal level of production for the whole society. The consequences of this condition differ, depending on the initial distribution of rights. In the case where  $p$  has the right to pollute – is not liable for the damage its activity can create –, then **(C.2)** means that all the victims can form a coalition *with*  $p$  and bargain to reach an agreement; conversely, if the victims have the right not to be polluted – that is if  $p$  is liable for the damage its activity can generate – then **(C.2)** means that  $p$  can try to form a coalition with all the victims and try to bargain with them. Condition **(C.3)** is a monotonicity condition. It is straightforward in the case of two agents, a tortfeasor and a victim. In such a situation **(C.3)** means that if the polluter has the right to pollute, then the coalition made of the polluter and the victim – with the objective to bargain – has also the right to control pollution; conversely, if the victim have the right to prevent pollution – by controlling the activity of  $p$  –, then the polluter must form a coalition with the victim to continue its activity. Here, we generalize it. Thus, if a coalition  $S$  has the right to control the activity of the polluter  $p$ , then every coalition  $T$  containing  $S$  is also entitled to control the activity of  $p$ . For instance, assume that  $p$  is liable for the damage generated by his or her activity. If  $S$  does not contain  $p$  and  $S$  is entitled to control the activity of  $p$ , then the members of  $S$  have the right to prevent  $p$  to pollute since  $p$  does not belong to  $S$  and is liable for the damage he or she can induce. If coalition  $T \supseteq S$  contains  $p$ , then, by virtue of monotonicity,  $p$  is now a member of a coalition that has the right to control the level of pollution. It follows that the victims in  $T$  and  $p$  will bargain to choose an optimal contract for  $T$ . Reciprocally, if  $p$  has the right to pollute, any coalition containing  $p$  has the right to bargain with  $p$  in order to control the level of pollution.

Condition **(C.4)** is a condition of effectivity of the law, meaning that if  $S$  has the right to implement an optimal contract without consideration of the members of  $(M \cup \{p\}) \setminus S$ , then coalition  $(M \cup \{p\}) \setminus S$  cannot prevent  $S$  to exercise its right. Once again, this is obvious when, as it is the case in “The Problem of Social Cost”, there are only two agents: the tortfeasor and the victim cannot have the same rights simultaneously. Our condition **(C.4)** generalizes the case to many agents.

Then, combining **(C.3)** and **(C.4)**, we obtain that if  $\phi_P(S) = Z_{P,S}^*$ , then  $\phi_P(T) = \emptyset$  for each  $T \subseteq (M \cup \{p\}) \setminus S$ . Indeed, if victims have the right not to be polluted, and if a coalition  $S$  of victims is formed to which  $p$  does not belong, i.e.  $p \in (M \cup \{p\}) \setminus S$ , then  $p$  cannot have the right to pollute and the victims have the right to prevent  $p$  to pollute. In that case, for  $p$  to go on with its activity is to form the coalition  $S \cup \{p\}$  with the victims. The consequence is then that now  $p$  will no longer be able to be liable for the damage its

activity generates. Monotonicity allows to transfer the right of the victims to the polluter. The other victims in  $M \setminus S$  have no right to oppose to  $p$ . They are also obliged to negotiate with the members of  $S \cup \{p\}$ . Reciprocally, if  $p$  has the right to pollute, then no group of victims has the right not to be polluted; the only way for the victims to control pollution is to form a coalition with  $p$ .<sup>2</sup>

Finally, **(C.5)** is a condition of independence of law from the economic characteristics. It indicates that the shape of  $B_p$  and  $D_i$ ,  $i \in M$ , do not influence the right to control the level of pollution, even if the shape of these functions affects the set of optimal contracts. Indeed, if  $\phi_P(S) = Z_{P,S}^*$ , then **(C.5)**,  $\phi_{P'}(S) = Z_{P',S}^*$ , but  $Z_{P,S}^*$  is not necessarily equal to  $Z_{P',S}^*$ .

Let  $\Phi_P$  be the set of all correspondences of rights satisfying **(C.1)**, **(C.2)**, **(C.3)**, **(C.4)** and **(C.5)** that we can construct from  $P \in \mathbb{P}$ .

#### 2.4. Social cost TU-games

The results a coalition can achieve also depend on the behavior of the nonmembers. More precisely, there are situations in which the behavior of the nonmembers of  $S$  affects the gains of the members of  $S$ , and situations in which the behavior of the nonmembers of  $S$  does not affect the gains of the members of  $S$ . Then, the worth of a coalition  $S$  will depend on whether or not they believe the nonmembers will behave in a way that will affect them. This represents a large number of cases. We propose to restrict our attention to the so-called pessimistic (expectation) rule. This rule, inspired by Aumann (1967) and discussed by Hart and Kurz (1983), assumes that the members of any coalition  $S$  believe or expect that nonmembers  $S$  will form the partition which minimizes the worth of  $S$ . As noted by Bloch and van Nouweland (2014), the pessimistic rule is most in line with the idea that a coalition should consider the worth that it can guarantee itself independent of the behavior of nonmembers of  $S$  - an idea that underlines the very definition of a TU-game. From this assumption, the TU-game  $(M \cup \{p\}, v_{\phi_P})$  is defined as:

$$v_{\phi_P}(S) = \begin{cases} B_p(z_S^*) - \sum_{i \in S \setminus \{p\}} D_i(z_S^*) & \text{if } S \ni p \text{ and } \phi_P(S) \neq \emptyset, \\ -\max_{T \subseteq (M \cup \{p\}) \setminus S} \sum_{i \in S \setminus \{p\}} D_i(z_T^*) & \text{if } S \not\ni p \text{ and } \phi_P((M \cup \{p\}) \setminus S) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

Several comments are in order.

If  $S \ni p$  and  $\phi_P(S) \neq \emptyset$ , then the members of  $S$  has the right to negotiate an optimal production level in  $Z_{S,P}^*$ . By **(C.3)** and **(C.4)**, for each  $T \subseteq M \setminus S$ , it holds that  $\phi_P(T) = \emptyset$ . So, whatever the partition of  $M \setminus S$ , coalition  $S$  can guarantee

$$v_{\phi_P}(S) = B_p(z_S^*) - \sum_{i \in S \setminus \{p\}} D_i(z_S^*).$$

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<sup>2</sup>As the reader may remark conditions **(C.1)** – **(C.4)** generate a structure of coalitions similar to the structure of winning coalitions of a simple game which is proper.



If  $S \not\ni p$  and  $\phi_P((M \cup \{p\}) \setminus S) \neq \emptyset$ , then the polluter  $p$  does not cooperate with the members of coalition  $S$ , and there is at least one coalition in  $(M \cup \{p\}) \setminus S$  including  $p$  which can cooperate with the polluter to negotiate an optimal level of production for itself. Under the pessimistic rule, agents in  $S$  expect that nonmembers of  $S$  will form the partition which minimizes the worth of  $S$ . Because every coalition of the partition not containing  $p$  has no impact on the worth  $S$ , the maximal impact of pollution caused by the nonmembers of  $S$  is determined by the coalition  $T \ni p$ ,  $T \subseteq (M \cup \{p\}) \setminus S$ , such that  $\phi_P(T) \neq \emptyset$  and which maximizes the total damage of members of  $S$ , i.e.

$$v_{\phi_P}(S) = - \max_{T \subseteq (M \cup \{p\}) \setminus S} \sum_{i \in S \setminus \{p\}} D_i(z_T^*).$$

Note that this coalition  $T$  is necessarily minimal (with respect to set inclusion) among the coalitions in  $(M \cup \{p\}) \setminus S$  which have the right to negotiate an optimal level of production.

In all other cases, either  $p$  does not belong to  $S$  and  $\phi_P((M \cup \{p\}) \setminus S) = \emptyset$ , or  $p$  belongs to  $S$  and  $\phi_P(S) = \emptyset$ . In both situations,  $p$  has not the right to pollute or to contract an optimal level of production with some victims, and so no activity is undertaken. Therefore, the worth of  $S$  is equal to zero.

### 2.5. Solutions and properties for a solution

A solution to a class of problems of social cost is a pair formed by a mapping of rights and payoff vectors.

Formally, a (*singled-valued*) solution on  $\mathbb{P}$  is a function  $F$  defined on  $\mathbb{P}$  which assigns a mapping of rights and a payoff vector  $F(P) = (\phi_P, x_P) \in \Phi_P \times \mathbb{R}^{M \cup \{p\}}$  to each social cost problem  $P \in \mathbb{P}$ . The rights assignment part of the solution has been discussed above. Regarding the “monetary” part of the solution, i.e. the payoff vector  $x_P$ , the interpretation is as follows. First,  $x_{P,i}$ ,  $i \in M$ , represents the quantity of money received by the victim  $i$  after having suffered a damage represented by  $D_i$  and having received a transfer from  $p$ . Second,  $x_{P,p}$  represents what  $p$  obtains after having produced a certain quantity of goods, received a certain benefit  $B_p$ , and transferred a certain amount to the victim. If  $x_{P,i} \geq 0$  for  $i \in M$ , then this means that  $i$  has been integrally compensated for the activity of  $p$ , and  $x_{P,p} \geq 0$  means that after having produced at a certain level for a benefit represented by  $B_p$ , and proceeded to the transfers,  $p$  obtains a surplus (otherwise,  $p$  will not produce).

We now introduce three properties for a solution  $F$  on  $\mathbb{P}$ . The first one is a propriety of efficiency and stability captured by the core. It indicates that, for each social cost problem  $P$ , acceptable payoff vectors are those who belong to the core of the TU-game  $(M \cup \{p\}, v_{\phi_P})$ . Thus, this property implies that, for each social cost problem, the core of the corresponding TU-game must be nonempty.

*Core stability* A solution  $F$  satisfies Core stability on  $\mathbb{P}$  if, for each  $P \in \mathbb{P}$ ,  $F(P) = (\phi_P, x_P)$  is such that  $x_P \in C(M \cup \{p\}, v_{\phi_P})$ .

The second property contains a principle of democracy, which is captured by the idea that no victim can veto the negotiations between the rest of the victims and the polluter.

It indicates that if a victim leaves the grand coalition, which is sovereign by hypothesis, the rest of the victims and the polluter retain their right to negotiate binding agreements. This property echoes the property of No veto power introduced by Maskin (1999) in the context of implementation of social choice functions. Note that No veto power for a victim is satisfied by any solution assigning rights to coalitions formed by a majority of the population, i.e.  $F$  is such that, for each  $P$ ,  $\phi_P(S) \neq \emptyset$  if  $|S| > (m + 1)/2$ .

*No veto power for a victim* A solution  $F$  satisfies No veto power for a victim on  $\mathbb{P}$ , if, for each  $P \in \mathbb{P}$ ,  $F(P) = (\phi_P, x_P)$  is such that, for each  $i \in M$ ,  $\phi_P((M \cup \{p\}) \setminus \{i\}) = Z_{P, (M \cup \{p\}) \setminus \{i\}}^*$ .

Let us note that, by condition **(C.4)**, No veto power for a victim implies that each victim, individually, has no right to oppose to other coalitions: for  $i \in M$ ,  $\phi_P(\{i\}) = \emptyset$ .

The last property reflects a principle of fairness that bears on the amount of compensation that victims receive. Here, fairness is captured by the idea that each victim should be fully compensated for the damage the polluter generates by his or her activities, and that the latter earns, having received a certain benefit and transferred a certain amount of money to each victim, a nonnegative payoff.

*Full compensation* A solution  $F$  satisfies Full compensation on  $\mathbb{P}$  if, for each  $P \in \mathbb{P}$ ,  $F(P) = (\phi_P, x_P)$  is such that  $x_P \geq (0, \dots, 0)$ .

### 3. Results

In order to state our main results, we need to define two (sub)sets of solutions.

First, for each social cost problem  $P = (Z, B_p, (D_i)_{i \in M}) \in \mathbb{P}$ , consider the mapping of rights  $\phi_P^d \in \Phi_P$  defined as follows:

$$\phi_P^d(\{p\}) = Z_{P, \{p\}}^* \text{ and } \phi_P^d(S) = \emptyset \text{ if } S \not\ni p.$$

Each  $\phi_P^d$  indicates that the polluter  $p$  is not liable for the pollution damage and so has the right to select an optimal level of production in  $Z_{P, \{p\}}^*$ . By the monotonicity condition **(C.3)**, every coalition of victims  $S$  containing the polluter  $p$  has also the right to select an optimal level of production in  $Z_{P, S}^*$ , i.e.  $\phi_P(S) = Z_{P, S}^*$  for each  $S \ni p$ . By condition **(C.4)**, if  $S \not\ni p$ , then  $S$  is not entitled to prevent the polluter  $p$  to produce, which means that  $p$  can exercise his or her right to pollute. All in all,  $\phi_P^d$  says that  $p$  can veto any coalition of victims in the sense that if  $p$  leaves such a coalition, the latter will suffer the maximal damage. From (3), for each social cost problem  $P = (Z, B_p, (D_i)_{i \in M}) \in \mathbb{P}$ , the TU-game  $(M \cup \{p\}, v_{\phi_P^d})$  rewrites as:

$$v_{\phi_P^d}(S) = \begin{cases} B_p(z_S^*) - \sum_{i \in S \setminus \{p\}} D_i(z_S^*) & \text{if } S \ni p. \\ - \sum_{i \in S \setminus \{p\}} D_i(z_p^*) & \text{if } S \not\ni p. \end{cases} \quad (4)$$

From these considerations, define the set of solutions  $\mathbb{F}^d$  as follows:

$$F \in \mathbb{F}^d \quad \text{if} \quad \forall P \in \mathbb{P}, \quad F(P) = (\phi_P^d, x_P) \text{ for some } x_P \in \mathcal{C}(M \cup \{p\}, v_{\phi_P^d}).$$

Second, for each  $P = (Z, B_p, (D_i)_{i \in M}) \in \mathbb{P}$ , define the (two) mappings of rights  $\phi_P^u \in \Phi_P$  as follows:

$$\phi_P^u(M \cup \{p\}) = Z_{P, M \cup \{p\}}^*, \quad \phi_P^u(M) \in \{\emptyset, Z_{P, M}^*\}, \text{ and } \phi_P^u(S) = \emptyset \text{ otherwise.}$$

The mappings of rights  $\phi_P^u$  indicate that the polluter  $p$  needs the agreement of the set of victims  $M$  to produce at some activity level, meaning that an agreement requires unanimity. Denote by  $\Phi_P^u$  these two mappings of rights. For each  $P \in \mathbb{P}$ , and whatever the value of  $\phi_P^u(M)$ ,  $(M \cup \{p\}, v_{\phi_P^u})$  is such that:

$$v_{\phi_P^u}(M \cup \{p\}) = B_p(z_{M \cup \{p\}}^*) - \sum_{i \in M} D_i(z_{M \cup \{p\}}^*) \geq 0, \quad \text{and} \quad v_{\phi_P^u}(S) = 0 \quad \text{otherwise.}$$

Next, let us introduce the set of solutions  $\mathbb{F}^u$  as follows:

$$F \in \mathbb{F}^u \quad \text{if} \quad \forall P \in \mathbb{P}, \quad F(P) = (\phi_P^u, x_P) \text{ for some } x_P \in \mathcal{C}(M \cup \{p\}, v_{\phi_P^u}).$$

Our first main result characterizes the set of solutions that satisfy Core stability. Theorem 1 below states that a solution satisfies Core stability if and only if it belongs to  $\mathbb{F}^u \cup \mathbb{F}^d$ .

**Theorem 1** *The solutions in  $\mathbb{F}^u \cup \mathbb{F}^d$  are the only solutions on  $\mathbb{P}$  satisfying Core stability.*

To establish Theorem 1, we proceed in three steps. In a first step, we show in Proposition 2 below that when the polluter is not liable for the externalities he or she generates, the core of the TU-game constructed from the associated correspondence of rights and defined as in (3) is nonempty whatever the social cost problem. From this result, we conclude that the subset of solutions  $\mathbb{F}^d$  is well-defined. In a second step, we establish a stronger result by showing that the solutions in  $\mathbb{F}^d$  are the only solutions that satisfy Core stability and No veto power for a victim. This result is contained in Proposition 3. In a third step, we prove Theorem 1.

**Proposition 2** *For each  $P = (Z, B_p, (D_i)_{i \in M}) \in \mathbb{P}$ , it holds that  $\mathcal{C}(M \cup \{p\}, v_{\phi_P^d}) \neq \emptyset$ .*

To prove Proposition 2, we need an intermediary result.

**Lemma 1** *Let  $\lambda \in \mathbb{B}(N)$  be a balanced map on the agent set  $N$  containing at least two elements. Consider any two distinct agents  $i$  and  $j$  in  $N$ . It holds that:*

$$\sum_{\substack{S \ni i \\ S \not\ni j}} \lambda(S) = \sum_{\substack{S \not\ni i \\ S \ni j}} \lambda(S).$$

*Proof.* Take  $\lambda$ ,  $i$  and  $j$  as hypothesized. We have:

$$\begin{aligned}
1 &= \sum_{S \ni i} \lambda(S) \\
&= \sum_{\substack{S \ni i \\ S \ni j}} \lambda(S) + \sum_{\substack{S \ni i \\ S \not\ni j}} \lambda(S) \\
&= \sum_{S \ni j} \lambda(S) - \sum_{\substack{S \ni j \\ S \not\ni i}} \lambda(S) + \sum_{\substack{S \ni i \\ S \not\ni j}} \lambda(S) \\
&= 1 - \sum_{\substack{S \ni j \\ S \not\ni i}} \lambda(S) + \sum_{\substack{S \ni i \\ S \not\ni j}} \lambda(S),
\end{aligned}$$

and so

$$\sum_{\substack{S \ni i \\ S \not\ni j}} \lambda(S) = \sum_{\substack{S \not\ni i \\ S \ni j}} \lambda(S),$$

as desired. ■

We have the material to prove Proposition 2.

*Proof.* (Proposition 2) Pick any  $P \in \mathbb{P}$ . By the Bondareva-Shapley theorem (see Proposition 1), it suffices to prove that, for each balanced map  $\lambda \in \mathbb{B}(M \cup \{p\})$ , it holds that:

$$\sum_{S \subseteq M \cup \{p\}} \lambda(S) v_{\phi_P^d}(S) \leq v_{\phi_P^d}(M \cup \{p\}),$$

where  $(M \cup \{p\}, v_{\phi_P^d})$  is given by (4). So, pick any balanced map  $\lambda \in \mathbb{B}(M \cup \{p\})$ . By definition of  $(M \cup \{p\}, v_{\phi_P^d})$ , we have:

$$\begin{aligned}
\sum_{S \subseteq M \cup \{p\}} \lambda(S) v_{\phi_P^d}(S) &= \sum_{S \ni p} \lambda(S) v_{\phi_P^d}(S) + \sum_{S \not\ni p} \lambda(S) v_{\phi_P^d}(S) \\
&= \sum_{S \ni p} \lambda(S) B_p(z_S^*) - \sum_{S \ni p} \lambda(S) \sum_{\substack{i \in S \\ i \neq p}} D_i(z_S^*) - \sum_{S \not\ni p} \lambda(S) \sum_{i \in S} D_i(z_p^*) \\
&= \sum_{S \ni p} \lambda(S) B_p(z_S^*) - \sum_{S \ni p} \lambda(S) \sum_{\substack{i \in S \\ i \neq p}} D_i(z_S^*) - \sum_{i \in M} D_i(z_p^*) \left( \sum_{\substack{S \not\ni p \\ S \ni i}} \lambda(S) \right).
\end{aligned} \tag{5}$$

Consider the polluter  $p$  and any victim  $i \in M$ . By Lemma 1, we have:

$$\sum_{\substack{S \not\ni p \\ S \ni i}} \lambda(S) = \sum_{\substack{S \ni p \\ S \not\ni i}} \lambda(S).$$

Therefore, equality (5) can be rewritten as:

$$\sum_{S \subseteq M \cup \{p\}} \lambda(S) v_{\phi_P^d}(S) = \sum_{S \ni p} \lambda(S) B_p(z_S^*) - \sum_{S \ni p} \lambda(S) \sum_{\substack{i \in S \\ i \neq p}} D_i(z_S^*) - \sum_{S \ni p} \lambda(S) \left( \sum_{i \in (M \cup \{p\}) \setminus S} D_i(z_p^*) \right). \quad (6)$$

By construction, for each  $z_S^* \in Z_{P, S^*}$  and each  $z_p^* \in Z_{P, \{p\}}$ ,  $z_p^* \geq z_S^*$ . Because, for each  $i \in M$ ,  $D_i$  are nondecreasing functions,  $z_p^* \geq z_S^*$  implies  $D_i(z_p^*) \geq D_i(z_S^*)$ . Therefore, from equality (6), we obtain:

$$\begin{aligned} \sum_{S \subseteq M \cup \{p\}} \lambda(S) v_{\phi_P^d}(S) &= \sum_{S \ni p} \lambda(S) B_p(z_S^*) - \sum_{S \ni p} \lambda(S) \sum_{\substack{i \in S \\ i \neq p}} D_i(z_S^*) - \sum_{S \ni p} \lambda(S) \left( \sum_{i \in (M \cup \{p\}) \setminus S} D_i(z_p^*) \right) \\ &\leq \sum_{S \ni p} \lambda(S) B_p(z_S^*) - \sum_{S \ni p} \lambda(S) \sum_{\substack{i \in S \\ i \neq p}} D_i(z_S^*) - \sum_{S \ni p} \lambda(S) \left( \sum_{i \in (M \cup \{p\}) \setminus S} D_i(z_S^*) \right) \\ &= \sum_{S \ni p} \lambda(S) \left( B_p(z_S^*) - \sum_{i \in M} D_i(z_S^*) \right) \\ &\leq \sum_{S \ni p} \lambda(S) \max_{z \in Z} \left( B_p(z) - \sum_{i \in M} D_i(z) \right) \\ &= \max_{z \in Z} \left( B_p(z) - \sum_{i \in M} D_i(z) \right) \\ &= v_{\phi_P^d}(M \cup \{p\}), \end{aligned}$$

where the third equality comes from the fact that, by definition of a balanced map,  $\sum_{S \ni p} \lambda(S) = 1$ . This completes the proof of Proposition 2.  $\blacksquare$

Proposition 2 helps to show that the only way to obtain a solution satisfying Core stability and No veto power for a victim is to pick a solution in  $\mathbb{F}^d$ .

**Proposition 3** *The solutions in  $\mathbb{F}^d$  are the only solutions on  $\mathbb{P}$  satisfying Core stability and No veto power for a victim.*

*Proof.* By Proposition 2,  $\mathbb{F}^d$  are well-defined. By definition of  $\mathbb{F}^d$ , any  $F \in \mathbb{F}^d$  satisfies Core stability. By definition of  $\phi^d$ , for each  $P = (Z, B_p, (D_i)_{i \in M}) \in \mathbb{P}$ ,  $\phi_P^d((M \cup \{p\}) \setminus \{i\}) \neq \emptyset$ . Therefore, any  $F \in \mathbb{F}^d$  satisfies No veto power for a victim.

It remains to show that  $F \notin \mathbb{F}^d$  violates Core stability or No veto power for a victim. It suffices to show that if  $F \notin \mathbb{F}^d$  satisfies No veto power for a victim, then it violates Core stability. To this end, pick any  $P = (M \cup \{p\}, Z, B_p, (D_i)_{i \in M}) \in \mathbb{P}$ , and any  $\phi_P \in \Phi_P \setminus \{\phi_P^d\}$ . Among the elements of  $\Phi_P \setminus \{\phi^d\}$  only two types of mappings of rights can be used to construct a solution  $F$  satisfying No veto power for a victim. To show this point, consider the following two remaining and exclusive cases.

- (a) Assume that  $\phi_P(\{i\}) = Z_{P,\{i\}}^*$  for some victim  $i \in M$ . By **(C.4)**,  $\phi_P((M \cup \{p\}) \setminus \{i\}) = \emptyset$ . Therefore, any solution  $F$  constructed from such a correspondence of rights will violate No veto power for a victim.
- (b) Assume  $\phi_P(\{i\}) = \emptyset$  for each victim  $i \in M$ . Note that  $\phi_P(\{p\}) = Z_{P,\{p\}}^*$  is not possible since otherwise  $\phi_P = \phi_P^d$ . So,  $\phi_P(\{p\}) = \emptyset$  as well. If, for some  $i \in M$ ,  $\phi_P(M \cup \{p\} \setminus \{i\}) = \emptyset$ , then the corresponding solution  $F$  will violate No veto power for a victim. Thus, we must have  $\phi_P((M \cup \{p\}) \setminus \{i\}) = Z_{P,(M \cup \{p\}) \setminus \{i\}}^*$  for each  $i \in M$ . At this step, we conclude that either  $\phi_P(M) = Z_{P,M}^*$  or  $\phi_P(M) = \emptyset$  are possible.

From (a) and (b), conclude that  $\phi$  is either of the form  $\phi_P(\{i\}) = \emptyset$  for each  $i \in M \cup \{p\}$  and  $\phi_P(M) = \emptyset$  or of the form  $\phi_P(\{i\}) = \emptyset$  for each  $i \in M \cup \{p\}$ , and  $\phi_P(M) = Z_{P,M}^*$ . Next, consider the social cost problem  $\tilde{P}$  given in Example 1, and any such above mentioned mappings of rights  $\phi_{\tilde{P}}$ . The worth of the grand coalition and the worth of each coalition of size  $m$  in  $(M \cup \{p\}, v_{\phi_{\tilde{P}}})$  are given by:

- $v_{\phi_{\tilde{P}}}(M \cup \{p\}) = \left(4 \sum_{i \in M} c_i\right)^{-1}$  ;
- For each  $i \in M$ ,  $v_{\phi_{\tilde{P}}}((M \cup \{p\}) \setminus \{i\}) = \left(4 \sum_{j \in M \setminus \{i\}} c_j\right)^{-1}$  ;
- $v_{\phi_{\tilde{P}}}(M) = 0$  whatever the value of  $\phi_{\tilde{P}}(M) \in \{\emptyset, Z_{P,M}^*\}$ .

As a final step, choose the balanced map  $\lambda \in \mathbb{B}(M \cup \{p\})$  given by:

$$\lambda(S) = m^{-1} \text{ if } |S| = m \text{ and } \lambda(S) = 0 \text{ otherwise.}$$

Straightforward computations give:

$$\sum_{S \subseteq M \cup \{p\}} \lambda(S) v_{\phi_{\tilde{P}}}(S) = \sum_{i \in M} \left(4m \sum_{j \in M \setminus \{i\}} c_j\right)^{-1} > \left(4 \sum_{i \in M} c_i\right)^{-1} = v_{\phi_{\tilde{P}}}(M \cup \{p\}).$$

By the Bondareva-Shapley theorem (see Proposition 1),  $(M \cup \{p\}, v_{\phi_{\tilde{P}}})$  has an empty core. Therefore, we conclude from (a) and (b) that if  $F \notin \mathbb{F}^d$  satisfies No veto power for a victim, then it violates Core stability. This completes the proof of Proposition 3.  $\blacksquare$

Proposition 3 states that the *efficiency thesis* is valid even if one imposes the requirement that no victim has the right to veto the rest of the society. But, it also indicates that under this democratic requirement, there is only one possible rights assignment compatible with the *efficiency thesis*. Under this assignment, the polluter is not liable for the damage his or her activity generates. It should be noted that Proposition 3 continues to be true if one restricts the domain of social cost problems to the subset of instances where the benefit functions are concave and the damage functions are convex. To see this, it suffices

to note that Proposition 2 remains true on this subdomain and the Example 1 used in the Proposition 3 to prove the emptiness of the core belongs to this subdomain. So, we have the following corollary.

**Corollary 1** *Assume that the domain of social cost problems under consideration is restricted to instances where the benefit functions are concave and the damage functions are convex. Then, the solutions in  $\mathbb{F}^d$  are the only solutions on this subdomain satisfying Core stability and No veto power for a victim.*

We are now in position to prove Theorem 1.

*Proof.* (of Theorem 1) First note that each social cost problem  $P \in \mathbb{P}$ ,  $C(M \cup \{p\}, v_{\phi_P^u}) \neq \emptyset$ , so that  $\mathbb{F}^u$  is well-defined. Combining this fact with Proposition 2, we obtain that each solution in  $\mathbb{F}^u \cup \mathbb{F}^d$  satisfies Core stability. To prove that  $\mathbb{F}^u \cup \mathbb{F}^d$  is the largest set of solutions satisfying Core stability, take any  $F$  on  $\mathbb{P}$  which satisfies Core stability but violates No veto power for a victim. To show:  $F \in \mathbb{F}^u$ . By Proposition 3,  $F \notin \mathbb{F}^d$ . So, there is  $P = (M \cup \{p\}, Z, B_p, (D_i)_{i \in M}) \in \mathbb{P}$  for which  $F(P) = (\phi_P, x_P)$  is such that  $\phi_P((M \cup \{p\}) \setminus \{i\}) = \emptyset$  for at least one  $i \in M$ . There are two cases to consider.

- (a)  $\phi_P((M \cup \{p\}) \setminus \{i\}) = \emptyset$  for each  $i \in M$ . By condition (C.3),  $\phi_P(S) = \emptyset$  for each  $S \ni p$ ,  $S \neq M \cup \{p\}$ . Next, assume that there is  $S \subsetneq M$  such that  $\phi_P(S) = Z_{P,S}^*$ . This means that there is  $i \in M$  such that  $S \subseteq M \setminus \{i\} \subseteq (M \cup \{p\}) \setminus \{i\}$ . Applying (C.3),  $\phi_P(S) = Z_{P,S}^*$  implies  $\phi_P((M \cup \{p\}) \setminus \{i\}) = Z_{P,(M \cup \{p\}) \setminus \{i\}}^*$ , a contradiction. Therefore, we necessarily have, for each  $S \subsetneq M$ ,  $\phi_P(S) = \emptyset$ . It follows that  $\phi_P \in \Phi_P^u$ . By definition of a solution  $F$ , for each  $P' \in \mathbb{P} \setminus \{P\}$ , we have  $\phi_{P'}(S) = \emptyset$  if and only if  $\phi_P(S) = \emptyset$ . Conclude that  $F \in \mathbb{F}^u$ .
- (b)  $\phi_P((M \cup \{p\}) \setminus \{j\}) = Z_{P,(M \cup \{p\}) \setminus \{j\}}^*$  for some  $j \in M$ . Define by  $I$  and  $J$  the following nonempty subsets of agents:

$$I = \left\{ i \in M : \phi_P((M \cup \{p\}) \setminus \{i\}) = \emptyset \right\},$$

and

$$J = \left\{ j \in M : \phi_P((M \cup \{p\}) \setminus \{j\}) \neq \emptyset \right\}.$$

By condition (C.5), the sets  $I$  and  $J$  do not depend on  $P$ . For the rest of the proof assume, without loss of generality, that  $2 \in I$  and  $1 \in J$ . Consider the social problem  $P = (M \cup \{p\}, Z, B_p, (D_i)_{i \in M}) \in \mathbb{P}$  defined as:

$$\forall z \in Z = [0, 1], B_p(z) = z^{3/4}, D_1(z) = 4mz, D_2(z) = z, \text{ and } D_i(z) = 0, \forall i \in M \setminus \{1, 2\}.$$

For the grand coalition, coalition  $M$ , and coalitions of size  $m$  containing  $p$ , we have:

$$- v_{\phi_P}(M \cup \{p\}) = \frac{27}{256(4m+1)^3};$$

- $v_{\phi_P}(M) = 0$ ;
- $v_{\phi_P}((M \cup \{p\}) \setminus \{1\}) = \frac{27}{256}$ ;
- For each  $i \in I$ ,  $v_{\phi_P}(M \cup \{p\} \setminus \{i\}) = 0$ ;
- For each  $j \in J \setminus \{1\}$ ,  $v_{\phi_P}(M \cup \{p\} \setminus \{j\}) = v_{\phi_P}(M \cup \{p\})$ .

Take the balanced map  $\lambda \in \mathbb{B}(M \cup \{p\})$  given by:

$$\lambda(S) = m^{-1} \text{ if } |S| = m \text{ and } \lambda(S) = 0 \text{ otherwise.}$$

We have:

$$\begin{aligned} \sum_{S \subseteq M \cup \{p\}} \lambda(S) v_{\phi_P}(S) &= \sum_{\ell \in M \cup \{p\}} \frac{v_{\phi_P}((M \cup \{p\}) \setminus \{\ell\})}{m} \\ &= \sum_{j \in J \setminus \{1\}} \frac{v_{\phi_P}((M \cup \{p\}) \setminus \{j\})}{m} + \frac{v_{\phi_P}((M \cup \{p\}) \setminus \{1\})}{m} \\ &= \frac{|J| - 1}{m} v_{\phi_P}(M \cup \{p\}) + \frac{v_{\phi_P}((M \cup \{p\}) \setminus \{1\})}{m}. \end{aligned}$$

By the Bondareva-Shapley theorem (see Proposition 1), if

$$\frac{|J| - 1}{m} v_{\phi_P}(M \cup \{p\}) + \frac{v_{\phi_P}((M \cup \{p\}) \setminus \{1\})}{m} > v_{\phi_P}(M \cup \{p\}),$$

that is, if

$$v_{\phi_P}((M \cup \{p\}) \setminus \{1\}) > v_{\phi_P}(M \cup \{p\})(m - |J| + 1),$$

then the social cost TU-game  $(M \cup \{p\}, v_{\phi_P})$  has an empty core. Because  $|J| \in \{1, \dots, m - 1\}$ ,

$$m v_{\phi_P}(M \cup \{p\}) \geq v_{\phi_P}(M \cup \{p\})(m - |J| + 1) \geq 2 v_{\phi_P}(M \cup \{p\}).$$

Therefore, to show that the core of the above social cost TU-game is empty whatever the value taken by  $|J|$ , that is whatever the number of victims without veto power, it is enough to verify that:

$$v_{\phi_P}(M \cup \{p\} \setminus \{1\}) > m v_{\phi_P}(M \cup \{p\}),$$

which is a routine exercise left to the reader.

From (a) and (b), we conclude that if  $F$  on  $\mathbb{P}$  satisfies Core stability but violates No veto power for a victim, then  $F \in \mathbb{F}^u$ . Therefore,  $\mathbb{F}^d \cup \mathbb{F}^u$  is the largest subclass of solutions satisfying Core stability, as desired. ■

As for Proposition 3, it is easy to notice that Theorem 1 continues to hold if one restricts the domain to instances where the benefit functions are concave and the damage functions are convex.



**Corollary 2** *Assume that the domain of social cost problems under consideration is restricted to instances where the benefit functions are concave and the damage functions are convex. Then, the solutions in  $\mathbb{F}^d \cup \mathbb{F}^u$  are the only solutions on this subdomain satisfying Core stability.*

Our second main result is an impossibility result saying that there is no solution satisfying Core stability, No veto power for a victim and Full compensation. To show this, we first prove that there is no solution in  $F \in \mathbb{F}^d$  that satisfies Full compensation.

**Proposition 4** *Each solution  $F \in \mathbb{F}^d$  violates Full compensation.*

*Proof.* Pick any  $F \in \mathbb{F}^d$  and any  $P \in \mathbb{P}$ . By (4) the corresponding TU-game  $(M \cup \{p\}, v_{\phi^d})$  is such that:

$$v_{\phi^d}(\{p\}) = B_p(z_p^*) \quad \text{and} \quad v_{\phi^d}(M \cup \{p\}) = B_p(z_{M \cup \{p\}}^*) - \sum_{i \in M} D_i(z_{M \cup \{p\}}^*).$$

Of course,  $v_{\phi_P^d}(\{p\}) \geq v_{\phi_P^d}(M \cup \{p\}) \geq 0$ . Combining the core constraints on coalitions  $\{p\}$  and  $M \cup \{p\}$  with the latter inequality, yields:

$$x_{P,p} \geq v_{\phi_P^d}(\{p\}) \geq v_{\phi_P^d}(M \cup \{p\}) = x_{P,p} + \sum_{i \in M} x_{P,i} \geq 0.$$

In  $\mathbb{P}$  there exist instances  $P$  such that  $v_{\phi_P^d}(\{p\}) > v_{\phi_P^d}(M \cup \{p\})$ . For such a  $P$ , we have:

$$x_{P,p} > x_{P,p} + \sum_{i \in M} x_{P,i} \geq 0 \implies \sum_{i \in M} x_{P,i} < 0,$$

which is not compatible with Full compensation. ■

Combining Proposition 4 with Theorem 1, we obtain the following characterization.

**Proposition 5** *The solutions in  $\mathbb{F}^u$  are the only solutions on  $\mathbb{P}$  satisfying Core stability and Full compensation.*

*Proof.* By Theorem 1,  $\mathbb{F}^d \cup \mathbb{F}^u$  is the largest subset of solutions satisfying Core stability. On the one hand, by Proposition 4, solutions in  $\mathbb{F}^d$  violate Full compensation. On the other hand, it is easy to see that each solution in  $\mathbb{F}^u$  satisfies Full compensation. This completes the proof. ■

Proposition 3 and Proposition 5 lead to the following impossibility result.

**Theorem 2** *There is no solution on  $\mathbb{P}$  satisfying Core stability, No veto power for a victim and Full compensation.*

Theorem 2 means that the *efficiency thesis* is no longer true in presence of Full compensation and No veto power for a victim. From Proposition 3, we know that the *neutrality thesis* does not hold.

#### 4. The two-agent case

Theorem 1 states that the set of agents can reach an efficient and core-stable agreement if and only if the rights are assigned either to the set of victims or to the polluter. In the two-agent case – a polluter and a potential victim – as in Coase’s original article, our result reduces to the statement that *the core is nonempty whatever the rights assignment*. Indeed, when one polluter interacts with one potential victim three cases arise:

- the rights are assigned to the polluter according to  $\phi^d$ ;
- the rights are assigned to the victim according to one of two mappings  $\phi^u$ ;
- the rights are assigned to the grand coalition formed by the polluter and the victim according to the second mapping  $\phi^u$ .

In the two-agent case, there is no other mappings of rights satisfying (C.1) – (C.5). This difference with Coase’s original article has no consequence because the induced TU-games in the last two cases coincide. Regarding the coalition functions  $v_{\phi^d}$  and  $v_{\phi^u}$  (whatever  $\phi^u$ ) when  $m = 1$ , we have the following worths:

$S$	$\{i\}$	$\{p\}$	$\{i, p\}$
$v_{\phi^d}(S)$	$-D_i(z_{\{i\}}^*)$	$B_p(z_{\{i\}}^*)$	$B_p(z_{\{i,p\}}^*) - D_i(z_{\{i,p\}}^*)$
$v_{\phi^u}(S)$	0	0	$B_p(z_{\{i,p\}}^*) - D_i(z_{\{i,p\}}^*)$

On the one hand,  $B_p(z_{\{i,p\}}^*) - D_i(z_{\{i,p\}}^*) \geq 0$ , which implies that  $C(\{i, p\}, v_{\phi^u}) \neq \emptyset$ . On the other hand, by definition of the maximum:

$$v_{\phi^d}(\{i, p\}) = B_p(z_{\{i,p\}}^*) - D_i(z_{\{i,p\}}^*) = \max_{z \in Z} \left\{ B_p(z) - D_i(z) \right\} \geq B_p(z_{\{i\}}^*) - D_i(z_{\{i\}}^*),$$

which also implies that  $C(\{i, p\}, v_{\phi^d}) \neq \emptyset$ .

Hence, Theorem 1 can be viewed as a generalization of the “Coase theorem” from the two-agent case to the case where many potential victims interact with the polluter.

#### 5. Conclusion

This article is a contribution to the already large literature that has been devoted to the “Coase theorem”. The originality of our approach first lies in the method we use – axiomatic and cooperative game theory – and second in the result we demonstrate. Certainly, our results are partial – limited to the case of one polluter and many victims and we do not take transaction costs into account. These are clearly the next steps that should be taken – taking into account transaction costs, having more than one polluters, for instance. But we nonetheless put forward important results.

We show that the two theses, efficiency and neutrality, upon which the theorem rests cannot be satisfied at the same time: the *efficiency thesis* can be satisfied but under only two specific assignment of rights. More precisely, we show that the payoff vector that solve a problem of social cost belongs to the core of the associated cooperative game – what we call *Core stability* – if and only if the rights are assigned either to the polluter or to the entire set of victims.

However, and this is the second result we reach, this efficiency thesis is incompatible with democracy and justice. Indeed, we demonstrate that no solution satisfies at the same time *Core stability*, *No veto power for a victim* and *Full compensation*. This is what we refer to as an impossibility result: it is impossible to reconcile efficiency with a right to veto negotiations and with the full compensation of victims.

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