The degree measure as a utility function for positions in weighted networks*

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Abstract. One of the most famous network centrality measures is the degree measure which assigns to every position in a weighted network the sum of the weights of all links with its neighbours. We show that this degree measure can be seen as a von Neumann-Morgenstern utility function. We do this in three steps. First, we characterize the degree measure as a centrality measure for weighted networks using four natural axioms (anonymity, the isolated node property, scale invariance and additivity). Second, we relate these network centrality axioms to properties of preference relations over positions in networks. In particular, we consider the property of neutrality to ordinary risk. Third, we prove that the utility function is equal to a multiple of the degree measure if and only if it represents a preference relation that is neutral to ordinary risk. In this way we build a bridge between the social network literature on network centrality, and the economic literature on preferences and utility.

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1 Introduction

The study of network centrality originates from the social network literature where different types of network centrality are distinguished, such as degree, closeness, betweenness, etc.; see e.g. Bavelas (1948, 1950); Beauchamp (1965); Sabidussi (1966); Freeman (1977, 1979). Various centrality measures are developed measuring these types of centrality; for surveys see e.g. Borgatti (2005); Goyal (2007); Jackson (2008); Newman (2010). More recently, these centrality measures are used to measure centrality in economic networks. However, there is no utility foundation of network centrality. Since economic decision making is based on preferences of economic decision makers, a utility foundation is fundamental for the application of centrality measures in economic models. Our aim is to develop such a utility foundation for network centrality by considering network centrality measures as von Neumann-Morgenstern utility functions (von Neumann and Morgenstern

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(1944)) reflecting preferences over positions in networks. In this way, we can evaluate different positions in different networks. Questions that can be addressed in this context are, for example, does an agent prefer to be the top of a small organization or a middle manager in a large organization?

The present work is inspired by Roth (1977a) who motivates the Shapley value (Shapley (1953)) as a von Neumann-Morgenstern utility function over being particular players in different games. He distinguishes between two kinds of risk: ordinary risk and strategic risk. While ordinary risk involves the uncertainty that arises from lotteries, strategic risk involves the uncertainty that arises from the strategic interaction of the players in a game. Roth (1977a) shows that the Shapley value of a game is equal to the utility of playing the game if and only if the underlying preferences are neutral to both ordinary and strategic risk. We find that adding neutrality to ordinary risk to standard axioms of preferences, essentially characterizes the *degree measure* as a von Neumann-Morgenstern utility function. This is one of the most natural centrality concepts which can be seen as an index of the node's communication ability and is based on the *degree*, i.e., the number of links formed by a node. Inspired by some experimental studies, Shaw (1954) presents the degree centrality as a measure to be used for predicting the behavior of individuals in small groups, and Nieminen (1974) and van den Brink and Gilles (2000) analyze it from an axiomatic point of view. Weighted networks where each link has a nonnegative weight can appear more suitable if one wishes to analyze different strengths of interaction or relation between two individuals. The degree of a node is then extended in a natural way to weighted networks, where it is defined as the sum of the weights of all links formed by the given node.

In the present paper, we show that the degree measures are utility functions for positions in weighted networks. More precisely, first, we characterize the degree measure as a centrality measure for weighted networks using the following four axioms. *Anonymity* imposes that the labeling of the nodes has no effect on the measure. The *isolated node property* states that the value assigned to an isolated node (i.e., a node with degree zero) is equal to zero. *Scale invariance* says that if a weighted network is rescaled, i.e., the weights of all its links are multiplied by a common factor, then the measure is also multiplied by the same factor. *Additivity* states that if we add two weighted networks, then the measure of the obtained 'sum' network is equal to the sum of the measures of the two weighted networks. Next, we provide the interpretation of the degree measure as a utility function for positions in weighted networks. We show that the utility function is equal to a multiple of the degree measure if and only if it represents a regular preference relation that is neutral to ordinary risk, the last property meaning that an agent is indifferent between taking a position in a convex combination of two networks and playing a lottery over the two networks with the corresponding probabilities.

Related literature As already mentioned, we use a similar approach to that of Roth (1977a), building on Herstein and Milnor (1953). Roth (1977a) develops a preference relation which permits to compare positions in a game and in different games, by extending the utility function for lotteries used to define the games, and shows that the Shapley value is a utility function reflecting preferences neutral to both *ordinary* and *strategic* risk; see also Roth (1977b,c, 1988) for related studies on this issue. Other attitudes towards risk lead to different utility functions. This concerns, for instance, the Banzhaf index (Banzhaf (1965)) and the Shapley-Shubik index (Shapley and Shubik (1954)) proposed in the context of simple games; see also Coleman (1971); Dubey (1975); Dubey and Shapley (1979); Owen (1975). Roth (1977c, 1988) considers how the Shapley-Shubik index can be uniquely characterized as a risk-neutral utility function defined on the class of simple games. The Banzhaf index is an extended utility function reflecting preferences averse to strategic risk and neutral to ordinary risk (Roth (1977d)). For an overview of Roth's approach to the Shapley value, see also Pintér (2014).

The paper is also related to the literature on social networks and centrality, in particular, to works using the *axiomatic approach to centrality measures*. This stream of literature focusses mainly on specific centrality measures. For instance, Garg (2009) characterizes axiomatically the degree, decay and closeness centralities. Some prestige and eigenvector-related centrality measures are characterized in Palacios-Huerta and Volij (2004); Slutzki and Volij (2006); Dequiedt and Zenou (2014); Kitti (2016). Bloch et al. (2016) characterize the standard centrality measures within a unified framework and show that they all are characterized by a common set of axioms. Although the present paper also uses the axiomatic approach to centrality measures and characterizes the degree measure, our main aim is to show that this measure can be interpreted as a utility function for positions in weighted networks.

An issue closely related to centrality is the ranking of nodes which is treated by using a *ranking method*. Formally, a ranking method assigns to every (weighted) network a (complete) preorder on the set of nodes. This (pre)order is a ranking of the nodes in order of 'importance' or 'centrality' in the network. Various ranking methods are characterized in the literature, in particular, methods based on directed networks and weighted directed networks, see e.g. Rubinstein (1980) for the ranking by outdegree on the class of tournaments, also Henriet (1985) and Bouyssou (1992) for the ranking by Copeland score (Copeland (1951)), Bouyssou and Perny (1992), van den Brink and Gilles (2003) for the ranking by outdegree for arbitrary directed networks, and van den Brink and Gilles (2009, 2000) for the outflow ranking method for weighted directed networks. There exist recent studies that characterize ranking methods based on evaluations or citations which consider one-sided settings (e.g. Demange (2014)) and ranking methods in two-sided settings (e.g. Demange (2016)). An important difference between such ranking methods and the topic of this paper, is that ranking methods only compare the positions in one and the same network. This is useful if one wants to rank, for example, teams in a sports competition, alternatives in a preference relation, web pages on the internet, etc. Besides such comparisons within one network, we also want to compare positions in different networks. For example, we want to know if an agent prefers a 'central' position in a small network to a position in the fringe of a large network. In order to answer these questions we need to be able to compare positions in different networks.

This paper is organized as follows. In Section 2 we discuss preliminaries on weighted networks and Herstein and Milnor (1953)'s expected utility theory over mixture sets. In Section 3 we present our main result, characterizing the degree measure as a von Neumann-Morgenstern utility function, using as intermediary results an axiomatization of the degree measure as a centrality measure, and relating properties of network centrality measures to properties of preference relations over network positions. Section 4 contains concluding remarks.

2 Preliminaries

In this section we present basic concepts and notation that will be used in the paper.

Weighted undirected networks A *weighted undirected network* is a pair (N, ω) consisting of a finite set of *nodes* $N \subset \mathbb{N}$ that can represent individuals or agents, and a *weight function* $\omega\colon L^c \to \mathbb{R}_+$, where $L^c = \{\{i,j\} \mid i,j \in N,\ i \neq j\}$ denotes the *complete undirected network* on N. An element $\{i,j\} \in L^c$ is a subset of N of size 2 representing a certain relationship between i and j, and is called a *link*. A weight function gives a nonnegative weight $\omega(\{i,j\})$ to every link that can be interpreted as the 'importance' of that relationship. By $\mathcal{WG}^N$ we denote the collection of all weight functions on N. A weighted undirected network with $\omega(\{i,j\}) \in \{0,1\}$ for all $\{i,j\} \in L^c$, is usually called a simple undirected network. Since N is assumed to be fixed, we represent a weighted undirected network (N, ω) by the weight function ω. When there is no confusion, in this paper we refer to a weighted undirected network simply as a network.

The *degree* of node $i \in N$ in weighted network ω is defined by

$$d_i(\omega) = \sum_{j \in N \setminus \{i\}} \omega(\{i,j\}) \tag{1}$$

For a network $\omega \in \mathcal{WG}^N$ and a permutation $\pi \in \Pi(N)$, the permuted network $\pi\omega \in \mathcal{WG}^N$ is given by $\pi\omega(\{\pi(i), \pi(j)\}) = \omega(\{i,j\})$ for every $\{i,j\} \in L^c$.

We denote the set of networks where $i \in N$ is an isolated node by

$$\mathcal{WG}_i^N = \{\omega \in \mathcal{WG}^N \mid \omega(\{i,j\}) = 0 \text{ for all } j \in N \setminus \{i\}\}$$

and $\omega^0 \in \mathcal{WG}^N$ denotes the 'empty' network given by $\omega^0(\{i,j\}) = 0$ for all $i,j \in N$.

By $\omega^i \in \mathcal{WG}^N$ we denote the standard star network with i as center given by $\omega^i(\{i,j\}) = 1$ for all $j \in N \setminus \{i\}$ and $\omega^i(\{h,j\}) = 0$ otherwise.

Expected utility We recapitulate the utility theory on mixture sets of Herstein and Milnor (1953) (for some related works and literature on the linear utility representation theorems, see e.g., Trockel (1989, 1992); Neuefeind and Trockel (1995)). A set M is a *mixture set* if for any $a, b \in M$ and any $p \in [0,1]$, we can associate another element of M, called a *lottery* between a and b, and denoted by $[pa; (1-p)b]$. It is assumed that for all $a, b \in M$, $p, q \in [0,1]$ the following holds:

$$[1a; 0b] = a, \quad [pa; (1-p)b] = [(1-p)b; pa], \quad [q[pa; (1-p)b]; (1-q)b] = [pqa; (1-pq)b] \tag{2}$$

A *preference relation* on M is a binary relation $\succeq$ with the interpretation that $a \succeq b$ meaning that "a is at least as good as b".

A function $u\colon M \to \mathbb{R}$ is a *utility function* representing the preference relation $\succeq$ if for all $a, b \in M$ and $p \in [0,1]$, it holds that

$$\begin{aligned} &(i)\ \ u(a) > u(b) \text{ if and only if } a \succ b, \text{ and} \\ &(ii)\ u([pa; (1-p)b]) = pu(a) + (1-p)u(b). \end{aligned} \tag{3}$$

We write $[a \succ b]$ if and only if $[a \succeq b$ and $b \not\succeq a]$, and $[a \sim b]$ if and only if $[a \succeq b$ and $b \succeq a]$.

The following axioms secure the existence of a utility function representing $\succeq$.

Axiom 1 *The preference relation* $\succeq$ *on* M *is complete and transitive.*[1]

Axiom 2 *For any* $a, b, c \in M$*, the sets* $\{p \mid [pa; (1-p)b] \succeq c\}$ *and* $\{p \mid c \succeq [pa; (1-p)b]\}$ *are closed.*

Axiom 3 *If* $a, a' \in M$ *and* $a \sim a'$ *then for any* $b \in M$*,* $[\frac{1}{2}a; \frac{1}{2}b] \sim [\frac{1}{2}a'; \frac{1}{2}b]$.

The utility function that represents $\succeq$ is unique up to an affine transformation. For any $x \in M$, we can write

$$u(x) = \frac{p_{ab}(x) - p_{ab}(r_0)}{p_{ab}(r_1) - p_{ab}(r_0)}$$

where $a, b, r_0, r_1 \in M$ with $a \succeq x \succeq b$ and $a \succeq r_1 \succeq r_0 \succeq b$, and for any $y \in M$ such that $a \succeq y \succeq b$, $p_{ab}(y)$ is defined by

$$y \sim [p_{ab}(y)a; (1 - p_{ab}(y))b]. \tag{4}$$

Note that the p_{ab} are well defined and u is independent of the choice of a and b. Moreover, r_1 and r_0 determine the origin and scale of the utility function: $u(r_1) = 1$ and $u(r_0) = 0$.

3 Centrality and utility in networks

We assume that a preference relation $\succeq$ is defined on the set $\mathcal{WG}^N \times N$ of strategic positions in weighted networks. We interpret $(\omega, i) \succeq (\omega', j)$ as "it is at least as good to be in position i in network ω as to be in position j in network ω'". We extend this preference relation to the mixture set M that also contains all lotteries $[p(\omega, i); (1-p)(\omega', j)]$ with (ω, i), $(\omega', j) \in \mathcal{WG}^N \times N$ and $p \in [0, 1]$. The lottery $[p(\omega, i); (1-p)(\omega', j)]$ considers a type of risk with respect to taking a position in a network. It means that with probability p the agent takes position i in network ω, and with probability $(1-p)$ he takes position j in network ω'.

Besides the standard axioms on mixture sets stated in the preliminaries (Axioms 1 - 3), we consider the following axioms on mixtures of network positions.

The first three axioms are similar to those of Roth (1977a) (specifically Conditions 6, 7 and 8), but in terms of network positions instead of being a player in a game. The first requires that relabeling the nodes in a network yields a corresponding reordering in the preference relation.

Axiom 4 *For all* $\omega \in \mathcal{WG}^N$*,* $i \in N$ *and* $\pi \in \Pi(N)$*, it holds that* $(\omega, i) \sim (\pi\omega, \pi(i))$.

The second axiom compares different network positions, expressing preference with respect to connectedness. More specifically, an agent (i) is indifferent between being isolated in any network, and being in the empty network, (ii) weakly prefers any position in any network above being in the empty network, and (iii) strictly prefers to to be the center of the star than being in the empty network. [2]

[1] A preference relation $\succeq$ on M is *complete* if for any $a, b \in M$ either $a \succeq b$ or $b \succeq a$. A preference relation $\succeq$ on M is *transitive* if for any $a, b, c \in M$ such that $a \succeq b$ and $b \succeq c$ it holds that $a \succeq c$.

[2] This axiom is inspired by Condition 7 of Roth (1977a) but (i) the role of null player replaced by being isolated, (ii) the role of the null game replaced by the empty network, and (iii) the role of a dictator replaced by being the centre of the star.

Axiom 5 *For all $i \in N$, $\omega \in \mathcal{WG}^N$ and $\omega' \in \mathcal{WG}_i^N$, it holds that (i) $(\omega', i) \sim (\omega^0, i)$, (ii) $(\omega, i) \succeq (\omega^0, i)$, and (iii) $(\omega^i, i) \succ (\omega^0, i)$, where ω^i is the standard star network with i as center.*

Conditions (i) and (ii) express the importance of being connected in the sense that the worst that can happen is to be isolated. Note that condition (iii) does not imply that it is best to be the centre of a star, but only that it is better to be the centre of a start than to be in the empty network.[3]

The third axiom requires that an agent is indifferent between a position in a network and a lotery between a multiple of that network and the empty network, where the probabilities are determined by the multiplication factor.

Axiom 6 *For all $\omega \in \mathcal{WG}^N$, $i \in N$ and $c > 1$, it holds that $(\omega, i) \sim [(\frac{1}{c}(c\omega, i); (1 - \frac{1}{c})(\omega^0, i)]$.*

From now on, we refer to preference relations that satisfy Axioms 1 - 6 as *regular* preference relations.

To axiomatize the Shapley value for TU-games as a von Neumann-Morgenstern utility function, Roth (1977a) introduces two types of risk neutrality: neutrality to ordinary risk and neutrality to strategic risk. In this paper we need to consider only the first type of risk. Consider position i in a convex combination $p\omega + (1 - p)\omega'$ of two networks. Neutrality to ordinary risk requires that an agent is indifferent between taking a position in network $p\omega + (1 - p)\omega'$ and playing a lottery over the networks ω and ω' with the corresponding probabilities.

Axiom 7 (Neutrality to ordinary risk) *For all $\omega, \omega' \in \mathcal{WG}^N$ and $i \in N$, it holds that $((p\omega + (1 - p)\omega'), i) \sim [p(\omega, i); (1 - p)(\omega', i)]$.*

Next, we state the main result of this section characterizing the class of utility functions that represent a regular preference relation that is neutral to ordinary risk as those that correspond to a multiple of the degree measure. Let $\overline{\mathcal{WG}^N}$ be the mixture set of all network positions $(\omega, i) \in \mathcal{WG}^N \times N$. Then a utility function for network positions is a function $\phi\colon \overline{\mathcal{WG}^N} \to \mathbb{R}$ assigning a utility value to every mixture of network positions.

Theorem 1 *The utility function ϕ represents a regular preference relation that is neutral to ordinary risk if and only if there exists an $\alpha > 0$ such that $\phi(\omega, i) = \alpha d_i(\omega)$ for all $(\omega, i) \in \mathcal{WG}^N \times N$.*

This theorem gives the degree measure, which is a well-known centrality measure in social network theory, an interpretation as a von Neumann-Morgenstern utility function. We prove this theorem by (i) characterizing the (class of multiples of the) degree measure as those centrality measures $f\colon \mathcal{WG}^N \to \mathbb{R}$ that satisfy the following four properties of centrality measures for networks, and (ii) relating those four network centrality properties to properties of preference relations.

First, anonymity says that the labeling of the nodes in a network has no effect on the centrality of positions in a network.

[3] Our results are also valid if this axiom is strengthened by requiring that being the centre of the connected star is strictly better than any position in any network on a fixed set of positions. This is often required when measuring centrality in the social network literature, see e.g. Gómez et al. (2003).

Property 1 (Anonymity) *For every* $\omega \in \mathcal{WG}^N$ *and permutation* $\pi \in \Pi(N)$*, it holds that* $f_i(\omega) = f_{\pi(i)}(\pi(\omega))$*.*

Second, the isolated node property states that the centrality of an isolated node (i.e., a node with degree zero) is zero. Almost all centrality measures from the literature satisfy this property.

Property 2 (Isolated node property) *For every* $\omega \in \mathcal{WG}_i^N$*, it holds that* $f_i(\omega) = 0$*.*

Third, scale invariance states that if the weights of all links in a network are multiplied by a common factor, then the centralities of the positions in that network are multiplied by the same factor.

Property 3 (Scale invariance) *Let* $\omega, \omega' \in \mathcal{WG}^N$ *be such that there exists an* $\alpha \in \mathbb{R}$ *with* $\omega'(\{i,j\}) = \alpha\ \omega(\{i,j\})$ *for all* $\{i,j\} \in L^c$*. Then* $f(\omega') = \alpha f(\omega)$*.*

Finally, additivity means that the centralities in the network obtained by adding two networks is equal to the sum of the centralities of these two networks.

Property 4 (Additivity) *For* $\omega, \omega' \in \mathcal{WG}^N$ *it holds that* $f(\omega + \omega') = f(\omega) + f(\omega')$*, where* $(\omega + \omega')(\{i,j\}) = \omega(\{i,j\}) + \omega'(\{i,j\})$ *for all* $\{i,j\} \in L^c$*.*

It turns out that the class of centrality measures that is characterized by these four properties is exactly the class of multiples of the degree measure.

Proposition 1 *A centrality measure* f *satisfies anonymity, scale invariance, additivity and the isolated node property if and only if there exists an* $\alpha \in \mathbb{R}$ *such that*

$$f_i(\omega) = \alpha d_i(\omega) \text{ for all } (\omega, i) \in \mathcal{WG}^N \times N. \quad (5)$$

Proof
It is straightforward to verify that centrality measures as given by (5) satisfy the four properties. To show uniqueness, suppose that centrality measure f satisfies the four axioms, and consider $\omega \in \mathcal{WG}^N$.
First, consider the empty network ω^0. Additivity implies that $f_i(\omega^0+\omega^0) = f_i(\omega^0)+f_i(\omega^0)$. Since $\omega^0 + \omega^0 = \omega^0$, this implies that $f_i(\omega^0) = f_i(\omega^0) + f_i(\omega^0)$, and thus $f_i(\omega^0) = 0$. By the isolated node property, $f_i(\omega) = 0$ for all $\omega \in \mathcal{WG}_i^N$.
Next, take a pair $i, j \in N$, $i \neq j$, and define $\mathcal{WG}_{ij}^N = \{\omega \in \mathcal{WG}^N \mid \omega(\{i,j\}) \neq 0$ and $\omega(\{h,g\}) = 0$ for all $\{h,g\} \neq \{i,j\}\}$, being the class of networks where only link $\{i,j\}$ has a nonzero weight.
Consider any $\overline{\omega} \in \mathcal{WG}_{ij}^N$. Anonymity implies that $f_i(\overline{\omega}) = f_j(\overline{\omega})$.
By scale invariance, there exists an $\alpha \in \mathbb{R}$ such that $f_i(\omega) = f_j(\omega) = \alpha\omega(\{i,j\})$ for any $\omega \in \mathcal{WG}_{ij}^N$.
Now take any $\{h,g\} \subset N, h \neq g,\ \{h,g\} \neq \{i,j\}$, and $\omega' \in \mathcal{WG}_{hg}^N$. By anonymity and the class $\mathcal{WG}_{ij}^N$ discussed above, we have $f_h(\omega') = f_g(\omega') = \alpha\omega'(\{h,g\})$.
Finally, consider any $\omega \in \mathcal{WG}^N$. For every $i, j \in N$, $i \neq j$, define $\omega^{ij}(\{i,j\}) = \omega(\{i,j\})$ and $\omega^{ij}(\{h,g\}) = 0$ for all $\{h,g\} \neq \{i,j\}$. Then additivity implies that for all $i \in N$ $f_i(\omega) = \sum_{\substack{h,g \in N \\ h \neq g}} f_i(\omega^{hg}) = \sum_{j \in N\setminus\{i\}} \alpha\omega(\{i,j\}) = \alpha \sum_{j \in N\setminus\{i\}} \omega(\{i,j\}) = \alpha d_i(\omega)$.
□

This proposition characterizes a social network centrality measure. Our main result (Theorem 1) is an economic result interpreting such centrality measures as vNM utility functions. To prove Theorem 1 from Proposition 1, we need a result that 'bridges' social network theory with economic utility theory. This is done by the following lemma which shows how the four properties for network centrality measures are implied by the regularity Axioms 4 - 6 and neutrality to ordinary risk (Axiom 7) on preferences introduced before.

Lemma 1 *Consider a utility function $\phi\colon \mathcal{WG}^N \times N \to \mathbb{R}$ for positions in a network that is determined by a centrality measure f as follows: $\phi(\omega, i) = f_i(\omega)$.*

(i) *If utility function ϕ represents a preference relation $\succeq$ satisfying Axiom 4 then centrality measure f satisfies anonymity.*
(ii) *If utility function ϕ represents a preference relation $\succsim$ satisfying Axiom 5 then centrality measure f satisfies the isolated node property.*
(iii) *If utility function ϕ represents a preference relation $\succeq$ satisfying Axioms 4-6 then centrality measure f satisfies scale invariance.*
(iv) *If utility function ϕ represents a preference relation $\succeq$ satisfying Axioms 4-6 and is neutral to ordinary risk (Axiom 7) then centrality measure f satisfies additivity.*

Proof
The proof of parts (i)-(iii) of this lemma is similar to the proofs of corresponding lemma's in Roth (1977a)).

(i) This follows immediately from Axiom 4.
(ii) If utility function ϕ represents a preference relation $\succsim$ satisfying Axiom 5 then $(\omega, i) \sim (\omega', i)$, and thus $\phi_i(\omega) = \phi_i(\omega')$ for all $\omega, \omega' \in \mathcal{WG}_i^N$.
(iii) We can derive from Herstein and Milnor (1953) (see preliminaries), that a utility function ϕ over the positions in a weighted network ω can be written as

$$\phi(\omega, i) = \frac{p_{ab}((\omega, i)) - p_{ab}(r_0)}{p_{ab}(r_1) - p_{ab}(r_0)} \tag{6}$$

for some $a, b, r_0, r_1 \in \mathcal{WG}^N \times N$ with $a \succeq (\omega, i) \succeq b$ and $a \succeq r_1 \succeq r_0 \succeq b$ with probabilities $p_{ab}((\omega, i))$ defined such that for all $(\omega, i) \in \mathcal{WG}^N \times N$ with $a \succeq y \succeq b$, we have $y \sim [p_{ab}(y)a; (1 - p_{ab}(y)b]$. By Axiom 5, we can take $b = r_0$ such that $p_{ab}(r_0) = 0$ for all $a \in \mathcal{WG}^N \times N$. We distinguish the following two cases.
Case 1: Suppose that $(c\omega, i) \succeq r_1$.[4]
Take $a = (c\omega, i)$ and $b = r_0 = (\omega^0, i)$. Then by (6), $\phi_i(c\omega) = \frac{p_{ab}((c\omega,i))}{p_{ab}(r_1)} = \frac{1}{p_{ab}(r_1)}$.
By Axiom 6, we have $(\omega, i) \sim [\frac{1}{c}(c\omega, i); (1 - \frac{1}{c})(\omega^0, i)]$, so $p_{ab}((\omega, i)) = \frac{1}{c}$. But then $\phi_i(\omega) = \frac{p_{ab}((\omega,i))}{p_{ab}(r_1)} = \frac{1}{c}\frac{1}{p_{ab}(r_1)} = \frac{1}{c}\phi_i(c\omega)$.

Case 2: Suppose that $r_1 \succeq (c\omega, i)$.

[4] Although Roth takes specifically that r_1 is the unanimity game with player i as only nonnull player, we do not specify r_1. It is sufficient that there exists a weighted network that is strictly preferred to the empty network, as is guaranteed by Axiom 5 with the star network (ω^i, i).

Take $a = r_1$ and $b = r_0 = (\omega^0, i)$. Then $p_{ab}(r_1) = 1$, and so $\phi_i(c\omega) = p_{ab}((c\omega, i))$. But by Axiom 6, we have $(\omega, i) \sim [\frac{1}{c}(c\omega, i); (1 - \frac{1}{c})(\omega^0, i)] \sim [\frac{1}{c}[p_{ab}((c\omega, i))a; 1 - p_{ab}((c\omega, i))b]; (1 - \frac{1}{c})(\omega^0, i)]$. So $\phi_i(\omega) = p_{ab}((\omega, i)) = \frac{1}{c}p_{ab}((c\omega, i)) = \frac{1}{c}\phi_i(c\omega)$.

(iv) For every $i \in N$, it holds $\phi_i(\omega + \omega') = \phi_i(2(\frac{1}{2}\omega + \frac{1}{2}\omega')) = 2\phi_i(\frac{1}{2}\omega + \frac{1}{2}\omega')$, where the second equality follows from part (iii). But then $\phi_i(\frac{1}{2}\omega + \frac{1}{2}\omega') = \frac{1}{2}\phi_i(\omega + \omega') = \frac{1}{2}\phi_i(\omega) + \frac{1}{2}\phi_i(\omega')$, where the second equality follows from neutrality to ordinary risk. Hence, we have $\phi_i(\omega + \omega') = \phi_i(\omega) + \phi_i(\omega')$.

□

Now we can give the proof of the main theorem of this section.

Proof of Theorem 1

To prove the 'only if' part, note that it follows from Lemma 1 and Proposition 1 that, if utility function ϕ represents a regular preference relation that is neutral to ordinary risk then $\phi(\omega, i) = \alpha d_i(\omega)$ for some $\alpha \in \mathbb{R}$. By Axiom 5 it must hold that $\phi(\omega^i, i) = \alpha(|N| - 1) > 0 = \phi(\omega^0, i)$, and thus $\alpha > 0$.

To prove the 'if' part, let $\succeq$ be the preference relation based on $\phi(\omega, i) = \alpha d_i(\omega)$: $(\omega, i) \succeq (\omega', j)$ if and only if $\alpha d_i(\omega) \geq \alpha d_j(\omega')$. It is straightforward to check that $\succeq$ satisfies Axiom 4. Axiom 5 follows since $d_i(\omega') = 0$ for all $\omega' \in \mathcal{WG}_i^N$, $d_i(\omega^i) > 0$ and $d_i(\omega) \geq 0$ for all $\omega \in \mathcal{WG}^N$. Axiom 6 follows since, by (3), $\phi([(\frac{1}{c}(c\omega, i); (1 - \frac{1}{c})(\omega^0, i)]) = \frac{1}{c}\phi(c\omega, i) + (1 - \frac{1}{c})\phi(\omega^0, i) = \frac{1}{c}\alpha d_i(c\omega) + 0 = \alpha d_i(\omega) = \phi(\omega, i)$. Finally, to prove neutrality to ordinary risk, consider $\omega, \omega' \in \mathcal{WG}^N$ and $i \in N$. Then, for $p \in [0, 1]$ we have $\phi(p\omega + (1-p)\omega', i) = d_i(p\omega + (1-p)\omega') = pd_i(\omega) + (1-p)d_i(\omega') = p\phi(\omega, i) + (1-p)\phi(\omega', i) = \phi([p(\omega, i); (1-p)(\omega', i)], i)$, where the last equality follows from (3). □

Note that compared to Roth's Theorem on the Shapley value for TU-games (Roth (1977a)), which equals its utility if and only if the underlying preferences are neutral to both ordinary and strategic risk, we do not use neutrality to strategic risk for the degree measure as a utility function for networks.

4 Concluding remarks

In this paper we showed that (multiples of) the degree measure in weighted (undirected) networks are von Neumann-Morgenstern utility functions. The approach is of crucial importance, since the interpretation of the degree measures as utility functions for positions permits to compare different positions in networks. Despite its simplicity, the degree measure is sufficient for measuring involvement or communication ability of an agent in the network. Moreover, the simplicity of the degree measure is an advantage, since only the local structure around a node must be known for calculations, for instance, when using social survey data.

Summarizing, we showed that

(i) (Theorem 1) a utility function ϕ represents a regular preference relation that is neutral to ordinary risk if and only if it is a multiple of the degree measure

(ii) (Proposition 1) a centrality measure f satisfies anonymity, scale invariance, additivity and the isolated node property if and only if it is a multiple of the degree measure

(iii) (Lemma 1) relating properties of preference relations to properties of centrality measures.

These three steps make clear the connection that we make between economics and social networks. Our main result is Theorem 1 which is an *economic* result which characterizes the degree measures as von Neuman-Morgenstern utility functions. Proposition 1 is a *social network* result which gives an axiomatic characterization of the degree measures. Finally, Lemma 1 *bridges economics with social networks* by relating properties of economic preference relations to properties of social networks.

We plan a number of follow-up research projects. While in the paper the set of nodes is assumed to be fixed, we could consider utility functions over nodes in networks of different size. Another interesting extension would be to analyze processes on a network and to combine utility of positions in a network with utility generation from processes on a network. Different centrality measures usually capture complementary aspects of an agent's position and ability, and therefore can give different agents as the most central ones in the network. We intend to relax the assumption of risk neutrality to find utility foundations of other centrality measures.

Another theoretical question that we want to address is how to incorporate externalities in measuring network centrality. For example, does your utility depend on the way how agents in other components of the network (i.e. agents with whom you are not connected, not directly nor indirectly) are linked? In almost all centrality measures the centrality of a position does not depend on the structure of other components. For example, isolated nodes usually have zero centrality irrespective of the rest of the network. But when, for example, you are isolated (i.e. have no neighbors), it might still make a difference whether other agents are linked to each other or not. We plan to incorporate this type of externality in measuring network centrality, in particular when interpreting these measures as utility functions. Besides this theoretical study, we plan to do an experimental study, both testing measures of centrality without externalities, as well as taking account of externalities.

Finally, we mention an extension to directed networks. This brings around new questions. For example, under the regularity axioms of this paper, the utility function assigns zero to every isolated position in any network. For undirected networks being isolated is also the worst position (as reflected by Axiom 5). However, for directed networks it is not obvious whether it is worse to be isolated or be connected but connected only by having 'predecessors' and no 'successors', i.e. having a positive indegree and zero outdegree. Which position is more preferred depends on the application of the network. In some cases it might be better to be connected, even with only ingoing arcs. But in other cases it might be better to be isolated and independent than to be connected with only ingoing arcs.

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