Income distribution by age group and productive bubbles[*]

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Abstract

We study a three period OG model where productive investment done in the first period of life is a long term investment whose return occurs in the following two periods. A bubble is a short term speculative investment that facilitates intertemporal consumption smoothing. We show that the distribution of income by age group determines the existence of bubbles. Moreover, the effect of the emergence of bubbles on aggregate production also depends on the distribution of income by age group. We also show that fiscal policy, by changing the distribution of income, may cause or prevent the existence of bubbles and it may modify the effect that bubbles have on aggregate production.

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1. Introduction

A long debate in the literature that studies overlapping generation models (OG) is related to dynamic inefficiency, the existence of bubbles and their effect on production.[1] Dynamic inefficiency implies that increasing the steady state stock of capital reduces consumption in the long run. This form of inefficiency is explained by imperfections in the credit market that force individuals to use productive capital to postpone consumption. When the willingness of individuals to postpone consumption is large, they overaccumulate capital and, hence, the equilibrium is dynamically inefficient. Tirole (1985) show that in this situation individuals may use an asset with no fundamental value to postpone consumption. Therefore, Tirole (1985) shows that when the equilibrium without bubbles is dynamically inefficient, an equilibrium with bubbles may also exist.[2] These bubbles reduce the stock of productive capital and also GDP, as they are used to postpone consumption. However, more recently, Martin and Ventura (2012) provide convincing evidence showing that bubbles arise during economic booms. Obviously, this evidence suggests that gross domestic product (GDP) should be larger in the equilibrium with bubbles. To explain this evidence, they introduce the concept of productive bubbles. They are defined as bubbles that facilitate a larger accumulation of productive capital. Martin and Ventura (2012), Fahri and Tirole (2012) and Raurich and Seegmuller (2015) show that productive bubbles may emerge when there are heterogenous agents that are differentiated by their productivity of investment. Bubbles are productive when they transfer resources to the most productive agents.

The aforementioned literature considers OG models that mostly assume a simple life cycle structure, in which individuals live for two periods: in the first period they obtain labor income and in the second period they obtain capital income. Obviously, this is a simplifying assumption that disregards the fact that middle age individuals obtain both labor and capital income.[3] Table 1 provides empirical evidence for the US and several European economies on the distribution of income by age group.[4] The evidence provided in this table clearly shows that middle age individuals obtain both labor and capital income. Therefore, the introduction of this additional life period implies that total labor income is distributed between two generations, young and middle age, and total capital income is also distributed between two generations, middle age and old. The purpose of this paper is to contribute to the aforementioned debate by showing how the distribution of income by age group affects dynamic efficiency and the existence of

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[1] See Abel, et al. (1989) for an analysis of dynamic efficiency in OG models. The existence of bubbles has been studied in OG models by Samuelson (1958), Tirole (1985), and Weil (1987), and, more recently, by Bosi and Seegmuller (2010) and Caballero et al. (2006). Productive bubbles have been considered by Martin and Ventura (2012), and Fahri and Tirole (2012).

[2] There is a large literature that also studies the possibility of bubbles in the framework of the infinite horizon model. Some relevant examples of this literature are: Hirano and Yanagawa (2013), Kamihigashi (2008), Kocherlakota (1992 and 2009), Lansing (2012), Le Van and Pham (2015), Miao and Wang (2011).

[3] Fahri and Tirole (2012) or Raurich and Seegmuller (2015) among others assume that individuals live for three periods. However, either they assume that individuals work only in the first period or they do not consider how the distribution of labor income by age group affects the properties of the equilibrium.

[4] Appendix E contains a detailed explanation of the empirical strategy followed to elaborate these data.

productive bubbles.

We consider as a framework of analysis the version of the three period OG model studied in Raurich and Seegmuller (2015), where productive investment done in the first period is a long term investment whose return occurs in the following two periods of life. The bubble is a short term investment that facilitates intertemporal consumption smoothing. As in Raurich and Seegmuller (2015), we also assume that only the young individuals can invest in productive capital. This introduces an important distinction between young and middle age individuals. The former invest in productive capital, whereas the later only invest in financial assets to smooth consumption across generations.[5] We differentiate from the model in Raurich and Seegmuller (2015) by assuming that individuals also work when they are middle age. As a consequence, labor income is distributed between young and middle age individuals. We show that this introduces a crucial difference.

Our main contribution is to show that the existence of bubbles and their effect on production depends on the distribution of labor and capital income by age group. We first show that if the fraction of labor income obtained by the young individuals and the fraction of capital income obtained by the middle age individuals are sufficiently large, then the equilibrium of the model without bubbles is dynamically inefficient. Large values of these fractions imply that most of the labor income is obtained in the first period of life and most of the capital income is obtained in the second period of life. In this case, individuals are willing to transfer wealth to the last period of life. To this end, they overaccumulate capital, which makes the equilibrium dynamically inefficient. As in Tirole (1985), an equilibrium with bubbles may emerge in this case. These bubbles are unproductive as they are aimed to postpone consumption. Second, we show that for sufficiently small values of both the fraction of labor income obtained by the young individuals and the fraction of capital income obtained by the middle age individuals, an equilibrium with bubbles does not emerge. When these two fractions are small enough, neither the young nor the middle age individuals are interested in holding the speculative asset in order to postpone consumption. Finally, we show that when the value of only one of these two fractions is small, the equilibrium of the model without bubbles is dynamically efficient and bubbles may still emerge in this case. In this case, the GDP of the equilibrium with bubbles is larger than the GDP of the equilibrium without bubbles. This implies that bubbles will be productive in this case.

We show that bubbles can be productive in two different cases: when the income obtained by the middle age individuals is sufficiently large and when it is sufficiently small.[6] In the first case, bubbles are used to transfer consumption from the middle age period to the other two periods of life. The larger wealth obtained by the young individuals is invested in productive capital, which explains that the bubble is productive. In the second case, when the middle age individuals obtain a sufficiently small fraction of income, the bubble is used to transfer consumption from the young

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[5] Productive investment consists in investment in education and investment in new projects (start-up, new companies) that is mostly done when individuals are young.

[6] Note that middle age individuals obtain a large (small) fraction of total income when the fraction of labor income obtained by the young is small (large) and when the fraction of capital income obtained by the middle age is large (small). Thus, the two situations in which bubbles can be productive correspond to polar cases of the distribution of income by age group.

and the old periods of life to the middle age period. Young individuals increase savings to compensate the reduction that the bubble causes on the level of consumption in the last period of life. Part of the increase in this savings is invested in productive capital, which explains that bubbles are also productive in this case.

The distribution of income by age group is largely modified by fiscal policy. As this distribution determines the existence of productive bubbles, fiscal policies may cause or prevent the existence of bubbles and they may also modify the impact that bubbles have on aggregate production. This is analyzed in Section 5, where we introduce taxes that are differentiated by both source of income and by the age group of the tax payers, and we also introduce a transfer to the old in the form of a pension. Regarding the effect that taxes have on the existence of bubbles, we first show that the introduction of pensions hinders the existence of an equilibrium with bubbles, as individuals do not need to use bubbles to postpone consumption. Second, the introduction of labor income taxes paid by the young individuals reduce their income and, thus, savings are also reduced. Again, this limits the possibility to postpone consumption using bubbles, which also hinders the existence of bubbles. Third, the introduction of labor income taxes paid by the middle age individuals facilitates the emergence of an equilibrium with bubbles, as bubbles will be used to transfer wealth towards the middle age individuals. Finally, the introduction of a capital income tax reduces the income of both middle age and old individuals. As a consequence, the introduction of this tax facilitates the emergence of an equilibrium with bubbles that will be used to postpone consumption.

We also study the effect of fiscal policy on the stock of productive capital and we outline that it crucially depends on the existence of a speculative asset, the bubble. On the one hand, in the absence of bubbles, productive capital is used to smooth consumption. This explains the effect of labor income taxes and pensions on productive capital. First, an increase in the labor income tax paid by the young individuals reduces the income net of taxes, which causes a reduction in productive investment. Second, an increase in the labor income tax paid by the middle age individuals reduces the income of the middle age. Young individuals then increase investment in productive capital to smooth consumption. Third, an increase in the pensions reduces productive capital, as individuals willingness to postpone consumption declines. Finally, an increase in the taxes on capital income reduces the after tax return from investment in productive capital, which causes the reduction in productive capital. On the other hand, productive capital is not used to smooth consumption in the economy with bubbles. This explains that neither pensions, nor labor income taxes affect the stock of productive capital in this case. In contrast, capital income taxes reduce the stock of productive capital, as they reduce the net return from productive investment. To summarize, an increase in either the pension or the labor income taxes paid by the young individuals reduce the willingness to postpone consumption. Therefore, these fiscal policies make individuals use the bubbles to increase productive investment. Hence, these policies facilitate the emergence of productive bubbles. Obviously, the effect is the opposite when the fiscal policy consists of increasing the taxes paid by the middle age individuals. Finally, capital income taxes reduce the stock of productive capital both when the equilibrium exhibits bubbles and when it does not exhibit bubbles. This explains that an increase in the capital income taxes can make bubbles either productive or unproductive.

In this paper, fiscal policies have a large effect on production when they cause

the emergence of an equilibrium with bubbles. In order to illustrate numerically this conclusion, we compare the fiscal policies in the US and in several European economies. We show that capital income taxes are larger in the US, whereas labor income taxes are larger in European economies. As a consequence, the tax burden is more concentrated on the young individuals in European countries, which limits capital accumulation in these economies. We use the model to quantify the increase in the stock of productive capital that would occur in several European economies if these economies set tax rates at the level of the US economy. Our main finding is that this change in the fiscal policy would cause the emergence of productive bubbles in the European economies and, as a consequence, it would also cause a substantial increase in the stock of productive capital.

The paper is organized as follows. Section 2 presents the model. Section 3 studies the equilibrium without bubbles and characterizes dynamic efficiency. Section 4 studies the equilibrium with bubbles and obtains the distribution of income by age group for which bubbles are productive. Section 5 discusses the effect of fiscal policy on the existence of productive bubbles. Section 6 concludes the paper.

2. Model

Consider an economy that in period t is populated by N_t young individuals. Let $n = N_t/N_{t-1}$ be the constant ratio between the number of young and middle age individuals in period t. The utility of an individual born in period t is

$$u_t = \ln c_{1,t} + \beta \ln c_{2,t+1} + \beta^2 \ln c_{3,t+2}, \tag{2.1}$$

where $c_{1,t}$ is the consumption when young, $c_{2,t+1}$ is the consumption in the middle age, $c_{3,t+2}$ is the consumption when old and $\beta \in (0,1)$ is the subjective discount rate.

Young individuals work and obtain a labor income $\xi_1 w_t$ that use to consume $c_{1,t}$ and invest in both speculative assets, $b_{1,t}$, and non-speculative assets, a_{t+1}. The wage per efficiency unit is w_t and $\xi_1 > 0$ measures the efficiency units of a young worker. We assume that only the young individuals can invest in the non-speculative asset, which is a long term investment that provides returns in the following two periods of life. In the second period of life, agents also work and obtain a labor income $\xi_2 w_{t+1}$, where $\xi_2 > 0$ measures the efficiency units of a middle age worker. These workers also obtain capital income from the return on the non-speculative asset, $\phi_1 q_{t+1}$, and from selling the speculative asset, $R_{t+1} b_{1,t}$. This income is used to consume, $c_{2,t+1}$, and invest in speculative assets, $b_{2,t+1}$. The return of one unit of productive capital is q_{t+1} and ϕ_1 are the units of productive capital that middle age individuals obtain from one unit of investment. Finally, the return from selling the bubble, R_{t+1}, is the growth rate of the price of the bubble. In the last period of life, when old, individuals are retired and, hence, they only obtain capital income. They obtain capital income from selling the speculative asset, $R_{t+2} b_{2,t+1}$, and they obtain $\phi_2 q_{t+2}$ from the return on the non-speculative asset, where ϕ_2 are the units of productive capital that old individuals obtain from one unit of investment done in the first period of life. Old individuals consume $c_{3,t+2}$. It follows that the budget constraints of the young, middle age and old

individuals are, respectively,

$$c_{1,t} + a_{t+1} + b_{1,t} = \xi_1 w_t, \tag{2.2}$$

$$c_{2,t+1} + b_{2,t+1} = \xi_2 w_{t+1} + q_{t+1}\phi_1 a_{t+1} + R_{t+1}b_{1,t}, \tag{2.3}$$

$$c_{3,t+2} = R_{t+2}b_{2,t+1} + q_{t+2}\phi_2 a_{t+1}. \tag{2.4}$$

We introduce two important assumptions on the non-speculative investment. First, we assume that only the young individuals can invest in the non-speculative asset. This is a simplifying assumption aimed to introduce a relevant difference in the productivity of the investment decisions of the different age groups. In fact, it is a reasonable assumption once this productive investment is considered as investment in education or investment in new companies. These forms of productive investment clearly decline as individuals get older. Second, the return on investment depends on the period in which investment has been done. This is a consequence of assuming that the productivity of capital depends on the period in which investment has been done. We, therefore, introduce a simple form of vintage capital.

Technology is characterized by the following aggregate production function:

$$Y_t = AK_t^\alpha L_t^{1-\alpha},$$

where L_t is the total amount of efficiency units of labor and K_t is the stock of productive capital in the economy. Let $k_t = K_t/L_t$. Then, $Y_t/L_t = Ak_t^\alpha$ and competitive factor prices satisfy

$$w_t = (1 - \alpha) Ak_t^\alpha, \tag{2.5}$$

and

$$q_t = \alpha Ak_t^{\alpha-1}. \tag{2.6}$$

We complete the characterization of the model with the market clearing conditions for capital, labor and the speculative asset. The market clearing condition for capital is

$$K_t = N_{t-1}\phi_1 a_t + N_{t-2}\phi_2 a_{t-1},$$

where $\phi_1 a_t$ and $\phi_2 a_{t-1}$ measure, respectively, the units of productive capital owned by middle age and old individuals. The market clearing condition for efficiency units of labor is

$$L_t = N_t\xi_1 + N_{t-1}\xi_2,$$

where ξ_1 and ξ_2 measure, respectively, the efficiency units of labor provided by young and middle age workers. We use these two market clearing conditions to define the fraction of productive capital owned by the middle age individuals

$$\Omega_t = \frac{n\phi_1 a_t}{n\phi_1 a_t + \phi_2 a_{t-1}}, \tag{2.7}$$

and the fraction of efficiency units of employment provided by the young individuals

$$\Sigma = \frac{n\xi_1}{n\xi_1 + \xi_2}. \tag{2.8}$$

Note that these two fractions determine the distribution of labor and capital income by age group.

From the previous two market clearing conditions, we also obtain that capital per efficiency unit of labor is

$$k_t = \frac{N_{t-1}\phi_1 a_t + N_{t-2}\phi_2 a_{t-1}}{N_t\xi_1 + N_{t-1}\xi_2},$$

which can be rewritten as

$$k_t = \frac{\phi_1}{n\xi_1 + \xi_2}a_t + \frac{\phi_2}{n^2\xi_1 + n\xi_2}a_{t-1}. \tag{2.9}$$

We assume that the speculative asset is supplied in one unit at a price p_t in period t. New investments in this asset by young and middle age individuals are in quantities ϵ_t and $1 - \epsilon_t$, respectively. Therefore, the values of this asset bought or sold by these agents are $B_{1,t} = b_{1,t}N_t = p_t\epsilon_t$ and $B_{2,t} = b_{2,t}N_{t-1} = p_t(1 - \epsilon_t)$. Since this asset has no fundamental value, it is a bubble if $p_t = B_{1,t} + B_{2,t} > 0$, which happens when $nb_{1,t} + b_{2,t} > 0$. In contrast, there is no bubble if $p_t = B_{1,t} + B_{2,t} = 0$ and $b_{1,t} = b_{2,t} = 0$. Finally, the market clearing condition for the speculative asset at period $t + 1$ is

$$N_{t+1}b_{1,t+1} + N_t b_{2,t+1} = R_{t+1}\left(N_t b_{1,t} + N_{t-1}b_{2,t}\right).$$

The left hand side of the previous equation is the value of the speculative asset bought by young and middle age individuals, whereas the right hand side is the value of the speculative asset sold by middle age and old individuals. The speculative asset sold in period $t + 1$ is multiplied by the growth rate of the price, R_{t+1}, as it was purchased in period t. Finally, note that this equation can be rewritten as

$$nb_{1,t+1} + b_{2,t+1} = \frac{R_{t+1}}{n}\left(nb_{1,t} + b_{2,t}\right). \tag{2.10}$$

3. Equilibria without bubble

We start by analyzing the model when there is no bubble, i.e. $b_{1,t} = b_{2,t} = 0$. In this case, the household's budget constraint rewrites:

$$c_{1,t} = \xi_1 w_t - a_{t+1}, \tag{3.1}$$
$$c_{2,t+1} = \xi_2 w_{t+1} + q_{t+1}\phi_1 a_{t+1}, \tag{3.2}$$
$$c_{3,t+2} = q_{t+2}\phi_2 a_{t+1}. \tag{3.3}$$

Maximizing the utility under the budget constraints (3.1)-(3.3), we get:

$$\frac{1}{\xi_1 w_t - a_{t+1}} = \frac{\phi_1\beta}{\xi_2\frac{w_{t+1}}{q_{t+1}} + \phi_1 a_{t+1}} + \frac{\beta^2}{a_{t+1}},$$

which simplifies as follows:

$$w_{t+1} = \frac{q_{t+1}a_{t+1}\phi_1}{\xi_2}\left[\frac{\left(\beta + \beta^2\right)\xi_1 w_t - \left(1 + \beta + \beta^2\right)a_{t+1}}{\left(1 + \beta^2\right)a_{t+1} - \beta^2\xi_1 w_t}\right]. \tag{3.4}$$

From using (2.5) and (2.6), the previous equation can be rewritten as

$$k_{t+1} = \frac{\alpha a_{t+1}\phi_1}{(1-\alpha)\xi_2}\left[\frac{\left(\beta+\beta^2\right)\xi_1(1-\alpha)Ak_t^\alpha - \left(1+\beta+\beta^2\right)a_{t+1}}{\left(1+\beta^2\right)a_{t+1} - \beta^2\xi_1(1-\alpha)Ak_t^\alpha}\right], \qquad (3.5)$$

and we use (2.9) to obtain

$$k_{t+1} = \frac{\phi_1}{n\xi_1+\xi_2}a_{t+1} + \frac{\phi_2}{n^2\xi_1+n\xi_2}a_t. \qquad (3.6)$$

Definition 3.1. *Given k_0 and a_0, an equilibrium without bubbles is a path $\{k_t, a_t\}_{t=1}^{\infty}$ that solves the system of equations (3.5) and (3.6).*

3.1. Steady State

We use (3.5) and (3.6) to show that there is a unique steady state and the steady state values of productive investment, a^*, and capital, k^*, are

$$a^* = \left(\frac{n\xi_1+\xi_2}{n\phi_1+\phi_2}\right)nk^*, \qquad (3.7)$$

and

$$k^* = \left(\frac{(1-\alpha)\xi_2(n\phi_1+\phi_2)+\alpha(n\xi_1+\xi_2)\phi_1 n}{(1-\alpha)\xi_2\beta^2(n\phi_1+\phi_2)+\alpha(n\xi_1+\xi_2)\phi_1 n\left(\beta+\beta^2\right)}+1\right)^{\frac{1}{\alpha-1}}\left(\frac{n(n\xi_1+\xi_2)}{(n\phi_1+\phi_2)(1-\alpha)A\xi_1}\right)^{\frac{1}{\alpha-1}}. \qquad (3.8)$$

Using (2.7) and (2.8), we also obtain that the steady state value of the fraction of labor income obtained by the young is $\Sigma = n\xi_1/(n\xi_1+\xi_2)$ and the steady state value of the fraction of capital income obtained by the middle age is $\Omega = n\phi_1/(n\phi_1+\phi_2)$. Using these two fractions, (3.7) and (3.8) can be rewritten as

$$a^* = \frac{\Omega n\xi_1}{\phi_1\Sigma}k^*,$$

and

$$k^* = \left(\frac{(1-\alpha)(1-\Sigma)+\alpha\Omega}{(1-\alpha)\beta^2(1-\Sigma)+\alpha\Omega\left(\beta+\beta^2\right)}+1\right)^{\frac{1}{\alpha-1}}\left(\frac{\Omega n}{(1-\alpha)\phi_1 A\Sigma}\right)^{\frac{1}{\alpha-1}}.$$

Note that the capital stock at the steady state increases with the fraction of labor income obtained by the young individuals, Σ, and it also increases with the fraction of capital income obtained by the middle age individuals, Ω. On the one hand, an increase in Σ rises the income obtained by the young individuals, who then increase investment in productive capital. On the other hand, an increase in Ω reduces the income obtained by the old individuals. Young individuals then compensate this reduction by increasing the investment in the productive asset.

3.2. Dynamic efficiency

The steady state equilibrium is dynamically efficient when aggregate consumption increases with investment. As it is well known, this occurs when the return on investment is larger than population growth. In this model, this condition implies that $(\phi_1 + \phi_2/n)\, q > n$. Using (2.6) and the steady state value of capital, we obtain that the steady state is dynamically efficiency when the following condition holds:

$$\left(\frac{(1-\alpha)(1-\Sigma)+\alpha\Omega}{(1-\alpha)\beta^2(1-\Sigma)+\alpha\Omega(\beta+\beta^2)}+1\right)\left(\frac{\alpha}{1-\alpha}\right) > \Sigma. \tag{3.9}$$

The following result follows from using the previous condition.

Proposition 3.2. *The equilibrium is dynamically efficient if either (i) $\Sigma < \Sigma_1$ or (ii) $\Sigma \in (\Sigma_1, \Sigma_2)$ and $\Omega < \overline{\Omega}$, where $\Sigma_1 = \frac{\alpha}{1-\alpha}\frac{1+\beta+\beta^2}{\beta+\beta^2}$, $\Sigma_2 = \frac{\alpha}{1-\alpha}\frac{1+\beta^2}{\beta^2}$ and*

$$\overline{\Omega} = \left(\frac{\Sigma_2 - \Sigma}{\Sigma - \Sigma_1}\right)\left(\frac{\beta^2}{\beta + \beta^2}\right)\left(\frac{1-\Sigma}{\Sigma_2}\right).$$

Proof. *See Appendix A.*∎

The result in Proposition 3.2 implies that the equilibrium is dynamically inefficient when either Σ or Ω are sufficiently large. This result is obtained because there is a positive relationship between the savings rate and the values of both Σ and Ω. In order to illustrate this mechanism that relates dynamic efficiency with the distribution of income by age group and that it is based on savings, we next show the relation between the savings rate and condition (3.9). We first use (2.5) and (2.6) to obtain $w/q = (1-\alpha)\, k/\alpha$. We use this equation, (3.4) and (3.7) to obtain

$$\frac{1-\alpha}{\alpha} = \left(\frac{n\xi_1 + \xi_2}{\phi_1 + \frac{\phi_2}{n}}\right)\left(\frac{\phi_1}{\xi_2}\right)\left(\frac{(\beta+\beta^2)\xi_1 w - (1+\beta+\beta^2)\, a}{(1+\beta^2)\, a - \beta^2\xi_1 w}\right),$$

which simplifies as follows

$$\frac{\xi_1 w}{a} = \frac{(1-\alpha)(1-\Sigma)+\alpha\Omega}{(1-\alpha)(1-\Sigma)\beta^2 + \alpha\Omega(\beta+\beta^2)} + 1, \tag{3.10}$$

where $a/\xi_1 w_t$ is the savings rate defined as the ratio between savings and the labor income of the young. Note that condition (3.9) can then be written as

$$\frac{\alpha}{1-\alpha} > \frac{a\Sigma}{\xi_1 w} = \frac{na}{(n\xi_1 + \xi_2)\, w}.$$

The right hand side of the previous expression is the savings rate defined as the ratio between aggregate savings and total labor income. Therefore, condition (3.9) holds and the steady state equilibrium is dynamically efficient when the savings rate is smaller than $\alpha/(1-\alpha)$. This is exactly the same condition that the literature has obtained for dynamic efficiency. In fact, if $\Sigma = 1$ then condition (3.9) simplifies to

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$\alpha / (1 - \alpha) > (\beta + \beta^2) / (1 + \beta + \beta^2)$, which is the condition obtained in Raurich and Seegmuller (2015). However, in this case, the savings rate is independent from the distribution of income by age group. In contrast, as follows from (3.10), the savings rate depends on the distribution of both labor and capital income when $\Sigma < 1$. This is a crucial difference that explains that dynamic efficiency depends on the income distribution by age group and it will also explain some of the main results in the following section.

4. Equilibria with a bubble

We introduce in this section the portfolio decision of the consumer between a short term speculative asset, $b_{1,t}$ and $b_{2,t+1}$, and a long term productive asset, a_{t+1}. Hence, the consumer decides a_{t+1}, $b_{1,t}$ and $b_{2,t+1}$ to maximize (2.1) subject to (2.2), (2.3) and (2.4). The solution to this maximization problem is characterized by the first order conditions with respect to $b_{1,t}$, $b_{2,t+1}$, and a_{t+1}, which are, respectively,

$$\frac{1}{c_{1,t}} = \beta \frac{R_{t+1}}{c_{2,t+1}}, \tag{4.1}$$

$$\frac{\beta}{c_{3,t+2}} = \frac{1}{R_{t+2}c_{2,t+1}}, \tag{4.2}$$

$$\frac{1}{c_{1,t}} = \beta \phi_1 \frac{q_{t+1}}{c_{2,t+1}} + \phi_2 \beta^2 \frac{q_{t+2}}{c_{3,t+2}}. \tag{4.3}$$

From combining (4.1)-(4.3) and using (2.6), we obtain the following non-arbitrage condition between the returns from investing one unit in the speculative asset and the returns from investing the same unit in productive capital:

$$R_{t+1} = \phi_1 \alpha A k_{t+1}^{\alpha-1} + \frac{\phi_2 \alpha A k_{t+2}^{\alpha-1}}{R_{t+2}}. \tag{4.4}$$

In Appendix B, we combine (2.2)-(2.6), (4.1), (4.2) and (4.4) to obtain the following two equations that determine the value of the speculative asset:

$$b_{1,t} = \frac{(\beta + \beta^2) \xi_1 (1 - \alpha) A k_t^{\alpha} - \frac{\xi_2(1-\alpha)Ak_{t+1}^{\alpha}}{R_{t+1}}}{1 + \beta + \beta^2} - a_{t+1}, \tag{4.5}$$

$$b_{2,t+1} = \frac{\beta^2 \xi_2(1-\alpha)Ak_{t+1}^{\alpha}+\beta^2\xi_1(1-\alpha)Ak_t^{\alpha}R_{t+1}}{1+\beta+\beta^2} + a_{t+1}\left(\phi_1 \alpha A k_{t+1}^{\alpha-1} - R_{t+1}\right). \tag{4.6}$$

Definition 4.1. *Given k_{-1}, k_0, an equilibrium is a path of $\{a_t, k_t, b_{1,t}, b_{2,t} R_t\}_{t=1}^{\infty}$ that solves the system of difference equations (4.4), (4.5) and (4.6) and the market clearing conditions (2.9) and (2.10).*

We proceed to obtain the steady state and then we study when an equilibrium with bubbles can emerge and when these bubbles are productive.

4.1. Steady state

We first use (2.10) to obtain $R = n$. Next, from (4.4), we obtain that the steady state value of capital in the equilibrium with bubbles, k, is

$$k = \left(\frac{\phi_1 \alpha A}{\Omega n} \right)^{\frac{1}{1-\alpha}}.$$

We use (2.9) to obtain the steady state value of productive investment, a, which is

$$a = \frac{n \xi_1 \Omega}{\Sigma \phi_1} k.$$

From (4.5), we obtain the steady state value of the bubbles owned by the young individuals

$$b_1 = \frac{(n\xi_1 + \xi_2)(1-\alpha) A k^\alpha}{n} \left[\Sigma - \frac{1}{1+\beta+\beta^2} - \frac{\alpha}{1-\alpha} \right],$$

and, from (4.6), we obtain the steady state value of the bubbles owned by the middle age individuals

$$b_2 = (n\xi_1 + \xi_2)(1-\alpha) A k^\alpha \left[\frac{\beta^2}{1+\beta+\beta^2} - \frac{\alpha}{1-\alpha} (1-\Omega) \right].$$

Finally, the price of the bubble is $N_{t-1} (nb_1 + b_2)$, where

$$nb_1 + b_2 = (n\xi_1 + \xi_2)(1-\alpha) A k^\alpha \left[\Sigma + \frac{\beta^2 - 1}{1+\beta+\beta^2} - \frac{\alpha}{1-\alpha} (2-\Omega) \right].$$

Note that b_1 is used to smooth consumption between young and middle age individuals, whereas b_2 is used to smooth consumption between middle age and old individuals. This explains that the sign of b_1 depends on Σ, whereas the sign of b_2 depends on Ω. A large value of Σ implies that most labor income is obtained by the young individuals. The bubble is then used to transfer wealth to the second period of life, i.e. $b_1 > 0$. In contrast, a small value of Σ implies that labor income is obtained by middle age individuals. The bubble is then used to transfer wealth to the first period of life, $b_1 < 0$. Similarly, a large value of Ω implies that most capital income is obtained by the middle age individuals. These individuals use the bubble to transfer wealth to the last period of life, i.e. $b_2 > 0$. Obviously, the opposite may occur when Ω is small.

We next obtain conditions for which an equilibrium with bubbles may exist.

Proposition 4.2. *A steady state with a bubble may exist if*

$$\Sigma + \frac{\alpha}{1-\alpha} \Omega > \frac{1-\beta^2}{1+\beta+\beta^2} + \frac{2\alpha}{1-\alpha}. \tag{4.7}$$

Proof. A bubble can exist when its price is positive, which occurs when $nb_1 + b_2 > 0$. This inequality implies condition (4.7). ∎

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Proposition 4.2 clearly shows that the existence of a bubble depends on the distribution of income by age group. As follows from condition (4.7), a bubble does not exist when either Σ or Ω are sufficiently small. A bubble may only exist if either the young individuals buy the speculative asset $(b_1 > 0)$, or the middle age individuals buy this asset $(b_2 > 0)$. As already explained, the young individuals buy the speculative asset if they obtain a sufficiently large income, which requires large Σ. Similarly, middle age individuals buy this asset when they obtain a sufficiently large amount of income, which requires a sufficiently large value of Ω.

Note also that condition (4.7) implies that population growth affects the existence of a bubble only through its impact on income distribution. In contrast, population growth affects the steady state price of the bubble even if Ω and Σ are kept constant. In particular, if we do not consider the effects on distribution, an increase in population growth reduces the price of the bubble. To see this, note that an increase in population growth causes a reduction in wages and, hence, it reduces savings. Obviously, this reduces the demand of bubbles, which explains the reduction in the price of the bubble. In our model, population growth coincides with the ratio between the size of the young population and the middle age population. Geanakoplos, et al. (2004) provide evidence showing that this ratio and the price of financial assets are negatively related. They rationalize this finding by arguing that the demand of assets decreases with this ratio. In this paper, we show that this rationalization also applies when we consider speculative assets.

4.2. Productive bubbles

Bubbles are a financial instrument that facilitates consumption smoothing and, hence, individuals do not need to use productive capital to smooth consumption. As a consequence, the introduction of bubbles modifies the stock of productive capital, which may either increase or decrease. We claim that bubbles are productive when the stock of productive capital is larger in the bubbly steady state than in the bubbleless one, i.e. $k > k^*$. In the following proposition, it is shown that this property is related to dynamic efficiency of the steady state without bubbles.

Proposition 4.3. *The bubble is productive if and only if the equilibrium without bubbles is dynamically efficient.*

Proof. Follows from the comparison between Condition (3.9) and the steady state values of capital, k and k^*.■

We have shown that a bubble may exist when the young generation obtains a large fraction of the labor income and when the middle age generation obtains a large fraction of the capital income. We have also shown that if these two fractions are not too large then the steady state without bubbles is dynamically efficient and, hence, the bubble is productive. The following proposition summarizes these findings and provides a complete characterization of the conditions implying the existence of productive bubbles.

Proposition 4.4. *Let us define*

$$\widetilde{\Omega} = \left(\frac{1-\alpha}{\alpha}\right)(\Sigma_3 - \Sigma),$$

where $\Sigma_3 = \frac{1-\beta^2}{1+\beta+\beta^2} + \frac{2\alpha}{1-\alpha}$. We obtain:

1. *If $\Sigma < \Sigma_1$, then (i) the bubble is productive when $\Omega > \widetilde{\Omega}$ and (ii) the bubble does not exist when $\Omega < \widetilde{\Omega}$.*

2. *If $\Sigma > \Sigma_1$, then (i) the bubble is not productive when $\Omega > \max\left\{\widetilde{\Omega}, \overline{\Omega}\right\}$, (ii) the bubble is productive when $\Omega \in \left(\widetilde{\Omega}, \overline{\Omega}\right)$ and (iii) the bubble does not exist when $\Omega < \widetilde{\Omega}$.*

Proof. Using (4.7), it is immediate to show that the bubble exists if $\Omega > \widetilde{\Omega}$. From Propositions 3.2 and 4.3, it is easy to show that the bubble is productive if either $\Sigma < \Sigma_1$ or $\Sigma > \Sigma_1$ and $\Omega < \overline{\Omega}$, where the expressions of $\overline{\Omega}$ and Σ_1 are defined in Proposition 3.2.∎

This proposition shows that the distribution of income by age group crucially determines the existence of productive bubbles. Raurich and Seegmuller (2015) already show that bubbles can increase the stock of productive capital when productive investment is a long term investment. In that paper, it is shown that the bubble is productive when $b_1 < 0$ and, hence, the bubble is used by the young agents to finance productive investment. In contrast, in this model where labor income is distributed between two generations, bubbles can be productive even if $b_1 > 0$. This is shown in the following proposition that characterizes the parametric regions for which productive bubbles satisfy $b_1 < 0$ and the regions for which productive bubbles satisfy $b_1 > 0$.

Proposition 4.5. *We distinguish among the following cases:*

1. *If $\frac{\alpha}{1-\alpha} \notin \left(\frac{\beta^2}{1+\beta+\beta^2}, \frac{\beta/2+\beta^2}{1+\beta+\beta^2}\right)$ then productive bubbles satisfy $b_1 < 0$ and $b_2 > 0$.*

2. *If $\frac{\alpha}{1-\alpha} \in \left(\max\left\{\frac{(1-\beta^2)(\beta+2\beta^2)}{(1+\beta+\beta^2)(2+\beta)}, \frac{\beta^2}{1+\beta+\beta^2}\right\}, \frac{\beta/2+\beta^2}{1+\beta+\beta^2}\right)$ then productive bubbles satisfy $b_1 < 0$ and $b_2 > 0$ when $\Sigma < \frac{1}{1+\beta+\beta^2} + \frac{\alpha}{1-\alpha}$ and $b_1 > 0$ and $b_2 < 0$ otherwise.*

3. *If $\frac{\alpha}{1-\alpha} \in \left(\frac{\beta^2}{1+\beta+\beta^2}, \frac{(1-\beta^2)(\beta+2\beta^2)}{(1+\beta+\beta^2)(2+\beta)}\right)$ then productive bubbles satisfy $b_1 > 0$ and $b_2 < 0$.*

Proof.   See Appendix C. ∎

The previous proposition shows that depending on the values of α and β we can distinguish among three possible cases. In Case 1, bubbles are productive only when

$b_1 < 0$ and $b_2 > 0$.[7] Figures 1, 2 and 3 show this case by displaying the relationship between Ω and Σ implied by the functions $\widetilde{\Omega}$ and $\overline{\Omega}$ in three different situations in which $b_1 < 0$ and $b_2 > 0$ is the only possible productive bubble. In Appendix C, we explain in detail these different situations and how the parameter conditions are determined. As it is obvious from these Figures, productive bubbles arise when $b_1 < 0$ and $b_2 > 0$ if the middle age individuals obtain a sufficiently large fraction of total income.[8] In this case, individuals use the bubble to transfer wealth from the middle age to the other two periods of life. On the one hand, middle age individuals postpone consumption, which implies that $b_2 > 0$. On the other hand, middle age individuals transfer wealth to the young individuals, which implies that $b_1 < 0$.

In Case 2 of the previous proposition, bubbles can be productive when either $b_1 < 0$ and $b_2 > 0$ or when $b_1 > 0$ and $b_2 < 0$. This case is displayed in Figure 4. This figure shows that, as in the previous case, bubbles are productive when $b_1 < 0$ and $b_2 > 0$ if the middle age obtains a sufficiently large fraction of income. The figure also shows that bubbles are productive when $b_1 > 0$ and $b_2 < 0$ if the middle age individuals obtain a small fraction of income (Σ large and Ω small). In this case, consumption smoothing implies that wealth is transferred from the young and old individuals to the middle age individuals. Finally, in Case 3 of the previous proposition, bubbles can be productive only when $b_1 < 0$ and $b_2 > 0$. This case is displayed in Figure 5. As in Case 2, this productive bubble arises when the middle age individuals obtain a small fraction of total income.

Proposition 4.5 shows that bubbles can be productive in two very different situations: (i) when $b_1 < 0$ and $b_2 > 0$, and (ii) when $b_1 > 0$ and $b_2 < 0$. In the first situation, the bubble is productive as it is used to transfer wealth to the young, who then increase productive investment. In the second situation, the bubble reduces the consumption of the old individuals and, as a consequence, young individuals increase savings in order to keep their consumption when old. If the positive effect on the savings of the young is large enough, productive investment increases even though part of the savings are used to transfer wealth to the middle age individuals $(b_1 > 0)$. In order to show more explicitly this argument, we compare the savings rate in the economy with bubbles and the savings rate in the economy without bubbles. This savings rate is defined as the ratio between assets accumulated when young and the income of the young individuals. We first use (2.5) and (4.5) to obtain the savings rate in the economy with bubbles,

$$\frac{a + b_1}{\xi_1 w} = \frac{\beta + \beta^2}{1 + \beta + \beta^2} - \frac{1 - \Sigma}{\Sigma}.$$

Note that in the economy with bubbles young individuals accumulate both productive assets and speculative assets. Using (3.10), we obtain the savings rate in the economy without bubbles, where the young individuals only accumulate productive assets. This

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[7]The productive bubbles obtained in Raurich and Seegmuller (2015) are an example of the bubbles in Case 1. To see this, note that in that paper bubbles arise when $\alpha / (1 - \alpha) > (\beta + \beta^2) / (1 + \beta + \beta^2)$. This parameter condition is included in Case 1 of Proposition 4.5.

[8]Observe from Figures 1, 2 and 3 that productive bubbles no not emerge when Σ is large and Ω is small.

savings rate is

$$\frac{a}{\xi_1 w} = \frac{(1-\alpha)(1-\Sigma)\beta^2 + \alpha\Omega\left(\beta+\beta^2\right)}{(1-\alpha)(1-\Sigma)\left(1+\beta^2\right) + \alpha\Omega\left(1+\beta+\beta^2\right)}.$$

Note that both expressions of the savings rate are different when $\Sigma < 1$, whereas they coincide when $\Sigma = 1$. As a consequence, when $\Sigma = 1$, productive capital is larger with bubbles if and only if $b_1 < 0$. On the contrary, when $\Sigma < 1$, capital can be larger with bubbles even if $b_1 > 0$, as the savings rates can be larger in the economy with bubbles. From the comparison between the two savings rates, it follows that the savings rate of the economy with bubbles is larger when the following condition on the distribution of income by age group holds:

$$\Omega < \left(\frac{\beta\Sigma}{1+\beta+\beta^2} - (1-\Sigma)\left(1+\beta^2\right)\right)\left(\frac{1-\alpha}{\alpha\left(1+\beta+\beta^2\right)}\right).$$

This condition implies that the savings rate is larger in the economy with bubbles when either Σ is sufficiently large or when Ω is sufficiently small. Therefore, these two conditions show that the savings rate is larger when the middle age individuals are sufficiently poor, which is precisely the condition that makes bubbles be productive when $b_1 > 0$ and $b_2 < 0$.

The parametric regions in the three cases shown in Proposition 4.5 only depend on two parameters: β and $\alpha/(1-\alpha)$. The first one is the subjective discount rate, whereas the second one determines the distribution of national income between capital and labor. A large value of β implies that individuals prefer to postpone consumption. This increases the value of both b_1 and b_2 that then can be positive. A large value of α implies that a larger fraction of national income is devoted to compensate capital and, hence, a larger fraction of income is obtained by the middle age and old individuals. Obviously, this reduces the value of both b_1 and b_2, which can then be negative.

Figure 6 displays the three parametric regions of Proposition 4.5 in the space β, $\alpha/(1-\alpha)$. As follows from this figure, both types of productive bubbles may emerge for any possible value of β. The figure also shows that $b_1 > 0$ and $b_2 < 0$ is only possible when $\alpha/(1-\alpha)$ is sufficiently small. Plausible values of the ratio $\alpha/(1-\alpha)$ belong to the interval $(0.4, 0.66)$. Clearly, these excludes Case 3 in Proposition 4.5. However, Cases 1 and 2 are both possible and they depend on the value of β. In order to see this, we consider two different parameter scenarios: (i) $\alpha = 0.3$ and $\beta = 0.6$ and (ii) $\alpha = 0.3$ and $\beta = 0.9$. The first scenario corresponds to Case 1 and, hence, productive bubbles satisfy $b_1 < 0$ and $b_2 > 0$. Productive bubbles emerge when $\Omega > 0.4$ and $\Sigma \in (0.6, 0.85)$. These values of the distribution of income by age group are similar to the ones shown in Table 1. In the second scenario, bubbles can be productive either when $b_1 < 0$ and $b_2 > 0$ or when $b_1 > 0$ and $b_2 < 0$. Productive bubbles with $b_1 < 0$ and $b_2 > 0$ emerge when the distribution of income satisfies $\Omega > 0.3$ and $\Sigma \in (0.45, 0.8)$. These values of the distribution of income are also similar to the values of the distribution of income shown in Table 1. In contrast, bubbles can be productive when $b_1 > 0$ and $b_2 < 0$ only when the distribution of income satisfies that $\Omega < 0.015$ and $\Sigma \in (0.92, 0.957)$. These values of the distribution of income are clearly different from the distributions shown in Table 1.

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5. Fiscal Policy

Most fiscal policies cause large changes in the distribution of income by age group. The analysis in the previous sections shows that these changes may affect both the existence of bubbles and their effect on GDP. Therefore, the model analyzed introduces a new framework to study the effects of fiscal policy on GDP. In this section, we analyze the effects of a fiscal policy that consists of taxes on labor income paid by the young individuals, τ_w^1, taxes on labor income paid by the middle age individuals, τ_w^2, taxes on capital income paid by the middle age individuals, τ_k^1, taxes on capital income paid by the old individuals, τ_k^2, and a pension, p_{t+2}, that is paid to the old retired individuals.[9] We assume that tax rates depend on the age group of the tax payers to introduce progressive taxes. We also assume that the pension is proportional to the wages of the middle age individuals and, hence, $p_{t+2} = \xi_2 \sigma w_{t+2}$, where $\sigma \in (0,1)$ is the replacement ratio. Finally, we assume that government revenues are used to pay the pension and a useless government spending, G_t. Thus, an increase in the tax rates will cause an increase in this useless government spending that will not affect individuals decisions, as government spending is assumed to be useless. The government budget constraint is

$$\tau_w^1 \xi_1 w_t N_t + \tau_w^2 \xi_2 w_t N_{t-1} + \tau_k^1 q_t \phi_1 a_t N_{t-1} + \tau_k^2 q_t \phi_2 a_t N_{t-2} = p_{t+2} N_{t-2} + G_t.$$

The budget constraints of the individuals, equations (2.2)-(2.4), are modified as follows:

$$c_{1,t} + a_{t+1} + b_{1,t} = \left(1 - \tau_w^1\right) \xi_1 w_t, \tag{5.1}$$

$$c_{2,t+1} + b_{2,t+1} = \left(1 - \tau_w^2\right) \xi_2 w_{t+1} + \left(1 - \tau_k^1\right) q_{t+1} \phi_1 a_{t+1} + R_{t+1} b_{1,t}, \tag{5.2}$$

$$c_{3,t+2} = R_{t+2} b_{2,t+1} + \left(1 - \tau_k^2\right) q_{t+2} \phi_2 a_{t+1} + p_{t+2}. \tag{5.3}$$

5.1. Steady state without bubbles

We start by analyzing the equilibrium of the model when there are no bubbles. In this case, the household maximizes the utility under the budget constraints (5.1)-(5.3) when $b_{1,t} = b_{2,t} = 0$. We get

$$\frac{1}{\left(1 - \tau_w^1\right) \xi_1 w_t - a_{t+1}} = \frac{\left(1 - \tau_k^1\right) \phi_1 \beta}{\left(1 - \tau_w^2\right) \xi_2 \frac{w_{t+1}}{q_{t+1}} + \left(1 - \tau_k^1\right) \phi_1 a_{t+1}} + \frac{\beta^2 \left(1 - \tau_k^2\right) \phi_2}{\left(1 - \tau_k^2\right) \phi_2 a_{t+1} + \frac{p_{t+2}}{q_{t+2}}}.$$

In order to obtain the steady state without bubbles, we substitute in the previous equation (2.5), (2.6), (3.7) and the definition of the pensions. We obtain

$$\frac{1}{\left(1 - \tau_w^1\right) \xi_1 \left(1 - \alpha\right) A k^{\alpha-1} - n \left(\frac{n\xi_1 + \xi_2}{n\phi_1 + \phi_2}\right)} = \frac{\phi_1 \beta}{\xi_2 \lambda + \phi_1 n \left(\frac{n\xi_1 + \xi_2}{n\phi_1 + \phi_2}\right)} + \frac{\beta^2 \phi_2}{\phi_2 n \left(\frac{n\xi_1 + \xi_2}{n\phi_1 + \phi_2}\right) + \frac{\sigma \xi_2 (1-\alpha)}{\left(1 - \tau_k^2\right) \alpha}},$$

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[9] Taxes on bubble returns could have been introduced. If they were introduced, the after tax return from the bubbles would be $\widetilde{R} = 1 + (R-1)(1 - \tau_b)$, where τ_b is the tax rate on bubble returns. As $R = n$ then $\widetilde{R} = 1 + (n-1)(1 - \tau_b)$. However, these taxes will reduce the cost of the bubble and, hence, they would be a subsidy when bubbles are negative. To avoid this problem, we do not introduce these taxes.

where $\lambda = \left(1 - \tau_w^2\right)(1 - \alpha)/[\left(1 - \tau_k^1\right)\alpha]$. Using the expressions of Σ and Ω, this equation simplifies as follows

$$\frac{\Omega n}{\left(1 - \tau_w^1\right)\Sigma\left(1 - \alpha\right)Ak^{\alpha-1}\phi_1 - n\Omega} = \frac{\beta\Omega}{\left(1 - \Sigma\right)\lambda + \Omega} + \frac{\beta^2\left(1 - \Omega\right)}{\left(1 - \Omega\right) + \left(1 - \Sigma\right)v},$$

and it can be rewritten as

$$k^* = \left(\frac{n\Omega\left(1 + \frac{[(1-\Sigma)\lambda+\Omega][(1-\Omega)+(1-\Sigma)v]}{\beta\Omega[(1-\Omega)+(1-\Sigma)v]+[(1-\Sigma)\lambda+\Omega]\beta^2(1-\Omega)}\right)}{\left(1 - \tau_w^1\right)\Sigma\left(1 - \alpha\right)A\phi_1}\right)^{\frac{1}{\alpha-1}},$$

where $v = \sigma\left(1 - \alpha\right)/[\left(1 - \tau_k^2\right)n\alpha]$. Note that Ω and Σ measure the distribution by age group of before taxes labor and capital income.

5.2. Steady state with bubbles

We assume that the consumer can smooth consumption using bubbles. Hence, the consumer maximizes the utility function subject to (5.1)-(5.3). The first order conditions of the household problem are:

$$c_{2,t+1} = \beta R_{t+1}c_{1,t} \tag{5.4}$$

$$R_{t+2}\beta R_{t+1}c_{1,t}\beta = c_{3,t+2} \tag{5.5}$$

$$\frac{1}{c_{1,t}} = \beta\phi_1\frac{\left(1 - \tau_k^1\right)q_{t+1}}{c_{2,t+1}} + \phi_2\beta^2\frac{\left(1 - \tau_k^2\right)q_{t+2}}{c_{3,t+2}}. \tag{5.6}$$

From combining (5.4)-(5.6) and using (2.6), we obtain the following non-arbitrage condition between the returns from the speculative asset and the returns from productive capital:

$$R_{t+1} = \left(1 - \tau_k^1\right)\phi_1\alpha Ak_{t+1}^{\alpha-1} + \frac{\left(1 - \tau_k^2\right)\phi_2\alpha Ak_{t+2}^{\alpha-1}}{R_{t+2}}. \tag{5.7}$$

In Appendix D, we use (3.7), (5.1)-(5.3), (5.4), (5.5) and (5.7) to obtain the following steady state values:

$$k = \left[\left(n\left(1 - \tau_k^1\right)\phi_1 + \left(1 - \tau_k^2\right)\phi_2\right)\frac{A\alpha}{n^2}\right]^{\frac{1}{1-\alpha}},$$

$$b_1 = \frac{\alpha\left(n\xi_1 + \xi_2\right)\left(1 - \tau_k^1\right)Ak^{\alpha}}{n}\left[\frac{\beta(1+\beta)\pi\Sigma-\lambda(1-\Sigma)-(1-\Sigma)v\eta}{1+\beta+\beta^2} - \Omega - \eta\left(1 - \Omega\right)\right], \tag{5.8}$$

and

$$b_2 = \alpha\left(n\xi_1 + \xi_2\right)\left(1 - \tau_k^1\right)Ak^{\alpha}\left[\frac{\beta^2\lambda(1-\Sigma)+\beta^2\pi\Sigma-(1+\beta)\eta(1-\Sigma)v}{1+\beta+\beta^2} - \eta\left(1 - \Omega\right)\right], \tag{5.9}$$

where $\pi = \left(1 - \tau_w^1\right)\left(1 - \alpha\right)/[\left(1 - \tau_k^1\right)\alpha]$ and $\eta = \left(1 - \tau_k^2\right)/\left(1 - \tau_k^1\right)$.

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The steady state price of the bubble is positive if $nb_1 + b_2 > 0$, where

$$nb_1 + b_2 = Ak^\alpha \alpha \left(n\xi_1 + \xi_2\right) \left(1 - \tau_k^1\right) F,$$

and

$$F = \frac{\left(\beta^2 - 1\right)\lambda(1-\Sigma)}{1+\beta+\beta^2} - \frac{(2+\beta)(1-\Sigma)\eta v}{1+\beta+\beta^2} + \frac{\beta(1+2\beta)\pi\Sigma}{1+\beta+\beta^2} - \Omega - 2\eta\left(1-\Omega\right).$$

Note that a steady state with a bubble may exist if $F > 0$. Therefore, the effects of fiscal policy on the existence of bubbles are obtained from a simple comparative static analysis on the function F. The results obtained from this analysis are summarized in the following proposition:

Proposition 5.1. *An equilibrium with bubbles may emerge as a consequence of the following policies: (i) a reduction in the pensions paid to the old individuals; (ii) a reduction in the labor income taxes paid by the young individuals; (iii) an increase in the labor income taxes paid by the middle age individuals; (iv) an increase in the capital income taxes paid by either middle age or old individuals.*

Proof. The results follow directly from using the function F. ∎

The introduction of pensions increase the income of the old generation. Hence, individuals do not need to use bubbles to postpone consumption, which hinders the emergence of an equilibrium with bubbles. Labor income taxes paid by the young individuals reduce their income and, thus, savings are reduced. Again, this limits the possibility to postpone consumption using bubbles. Labor income taxes paid by the middle age individuals reduce their income. As a consequence, individuals may use bubbles to postpone consumption towards the middle age individuals. Thus, the introduction of these taxes facilitates the existence of an equilibrium with bubbles. Finally, capital income taxes reduce the income of either middle age or old individuals. Therefore, the introduction of these taxes facilitates the emergence of bubbles that will be used to postpone consumption.

5.3. Analysis of the effects of fiscal policy

We proceed to study the effect of fiscal policy on production both in the economy without bubbles and in the economy with bubbles. At this point, it is important to clarify that, in this simple model, the effect of fiscal policy on production follows directly from the effect that fiscal policy has on the stock of productive capital.

Proposition 5.2. *The steady state stock of productive capital of the economy without bubbles decreases when (i) the tax rate on the labor income of the young individuals increases; (ii) the tax rate on the labor income of the middle age individuals decreases; (iii) the pension obtained by the old individuals increases; and (iv) the tax rates on capital income paid by either the middle age or the old individuals increase.*

Proof. The results follow from using the expression of k^*. ∎

In the absence of bubbles, productive capital is used to smooth consumption. This explains the effects of labor income taxes and pensions on productive capital. First, an

increase in the labor income tax paid by the young individuals reduces their income net of taxes, which causes a reduction in savings and, hence, on capital. Second, an increase in the labor income tax paid by the middle age individuals reduces their after tax income. Young individuals then increase investment in productive capital to smooth consumption. Third, an increase in the pensions reduces the willingness to postpone consumption and, hence, productive capital decreases. Finally, taxes on capital income reduce the return from productive capital, which causes the reduction in productive investment.

Proposition 5.3. *The steady state stock of productive capital of the economy with bubbles decreases when the tax rates on capital income paid by either the middle age or the old individuals increase. This stock of productive capital does not depend on the tax rates on labor income, nor on the pension.*

Proof. The results follow from using the expression of k.∎

In the economy with bubbles, productive capital is not used to smooth consumption. As a consequence, pensions and labor income taxes do not affect productive capital. In contrast, productive capital decreases with the taxes on capital income that reduce the return on productive investment.

The previous two results show that the effect on the stock of capital of taxes on labor income and pensions depend on the existence of bubbles. As a consequence, fiscal policy may make bubbles productive or unproductive. To study the effect of fiscal policy, we compare the stocks of capital k and k^* and we show that $k^* < k$ when $\Psi > 0$, where

$$\Psi = 1 + \frac{[(1-\Sigma)\lambda + \Omega][(1-\Omega) + (1-\Sigma)v]}{\beta\Omega[(1-\Omega) + (1-\Sigma)v] + [(1-\Sigma)\lambda + \Omega]\beta^2(1-\Omega)} - \frac{\pi\Sigma}{\Omega + \eta(1-\Omega)}.$$

Straightforward comparative statics on the function Ψ show the effects that fiscal policy has on the existence of productive bubbles. The results from this analysis are summarized in the following proposition.

Proposition 5.4. *Bubbles may become productive as a consequence of the following fiscal policies: (i) an increase in the pensions paid to the old; (ii) an increase in the labor income taxes paid by the young individuals; (iii) a reduction in the labor income taxes paid by the middle age individuals. Finally, the effect of capital income taxes on the existence of productive bubbles is ambiguous.*

Proof. The results follow from using the function Ψ.∎

As explained before, an increase in either the pension or the labor income taxes paid by the young individuals make individuals use the bubbles to transfer wealth to the first period of life, which makes bubbles either vanish or become productive. Obviously, the effect is the opposite when the fiscal policy consists of increasing the taxes paid by the middle age individuals. Finally, the ambiguous effects of the taxes on capital income is explained by the fact that these taxes reduce the stock of productive capital both when the equilibrium exhibits bubbles and when it does not exhibit bubbles.

The results displayed in the previous proposition are summarized in Figure 7 that shows how both stocks of productive capital depend on the different tax rates. Panel a shows that if the tax rate on the labor income of the young is sufficiently small, then the bubble will be used to postpone consumption and, hence, it will be unproductive. To see this, note that $k > k^*$ for low values of this tax rate. As the tax rate increases, the income of the young decreases and the bubble becomes productive. Finally, the bubble vanishes for a sufficiently large tax rate. Panel b shows the effects of the tax rate on the income of the middle age individuals. These effects are the opposite from the ones displayed in Panel a. When this tax rate is sufficiently small, the bubble does not exist. When the tax rate increases, a productive bubble arises. Finally, for sufficiently large values of the tax rate, $k^* > k$ and, hence, the bubble becomes unproductive. The last panel, Panel c, shows the effect of the pensions by displaying how the stock of productive capital depends on the replacement rate. When this rate is sufficiently small, bubbles are unproductive as they are used to postpone consumption. As the replacement rate increases, bubbles become productive as they are used to increase productive investment. Finally, for a sufficiently large replacement rate, no one holds the bubble and, hence, the equilibrium with bubbles vanishes.

Figure 7 introduces an important implication for fiscal policy. It shows that an increase in the labor income tax rates or in the replacement rates that does not affect the existence of bubbles has no effect on the stock of productive capital in the economy with bubbles. However, a sufficiently large increase in the tax rate on the labor income of the young or in the replacement rate that eliminates bubbles will cause a dramatic reduction in the stock of productive capital. A large decline in the stock of productive capital would also occur if we instead consider a large reduction in the tax rate on the labor income of the middle age individuals that also eliminates bubbles. These results point out an important discontinuity in the effects that fiscal policy has on production.

The effect on productive capital of taxes on capital income depends on the value of the parameters. Hence, we cannot obtain general conclusions regarding the effects of these taxes. At this point, it is important to clarify that the patterns illustrated in Figure 7 also depend on the value of the parameters. In particular, this figure has been constructed assuming that a region of productive bubbles exists. However, this depends on the value of the other parameters and, mainly, on the distribution of income by age group. In fact, if we had assumed that a region of productive bubbles does not exist, then the effects of fiscal policy would be reversed. As an example, an increase in the tax rate paid by the young individuals that eliminates unproductive bubbles would cause an increase instead of a decrease of productive capital. Therefore, the conclusions on the effects of fiscal policy crucially depend on the value of the parameters. It follows that in order to obtain clear results on the effects of fiscal policy we need to perform a numerical analysis based on a plausible parametrization. This is the purpose of the following section.

5.4. Numerical Analysis

We fix the value of the parameters as follows. First, A and ϕ_1 are normalized to one, without loss of generality. Second, $\alpha = 0.3$, which implies a labor income share equal to

70%.[10] Third, $\beta = 0.9$, which is consistent with a high value of the subjective discount rate during a period of 20 years.[11] The values of Σ and Ω are taken from Table 1 and computed as it is explained in Appendix E. Table 2 shows the tax rates and the population growth rate. All the data in this table is obtained from the OECD data set. The population growth rate n is obtained from the OECD data set as the ratio between the size of the young population and the size of the middle age population. Finally, we assume that there are no pensions. As explained in Appendix E, pensions are not included in the gross distribution of income shown in Table 1.

The results from this calibration are displayed in Table 3. This table shows the value of the capital stock in both economies (bubble and no bubble) and the value of F. The sign of F determines the existence of the bubble, with a negative sign implying that a bubble cannot exist. As it is clear from this table, only the US economy may exhibit a bubble. This bubble is productive, as follows from the comparison between the capital stocks. In contrast, none of the European economies may exhibit a bubble according to this analysis. From the comparison between the fundamentals of the European economies and those of the US economy, displayed in Tables 1 and 2, it follows that the main difference is fiscal policy. In fact, there are no relevant differences between US and European economies in the population growth rate, nor in the distribution of income by age. The only clear difference with respect to European economies are the larger taxes on capital income and the smaller taxes on labor income, mainly among the young individuals. Therefore, the tax burden in European economies is more concentrated on the young individuals, which limits investment in productive capital and prevents the existence of an equilibrium with bubbles, as follows from Proposition 5.1.

The previous arguments suggest that if European economies change their fiscal policy then they may exhibit productive bubbles and, hence, productive capital may increase. This is studied in Table 4. To this end, we set the value of the tax rates on capital income and of the tax rate on the labor income of the middle age individuals as in the US. The tax rate on the labor income of the young is set to the maximum value compatible with the possibility of bubbles. We observe that all countries except one, Luxemburg, may exhibit bubbles when the tax rate on the labor earnings of the young is sufficiently small. We also observe that in 6 countries this tax rate is above 20% and very close to the value of the US. This clearly shows that moving from the tax policy that most Europeans economies currently follow to a fiscal policy similar to the US economy may drive the emergence of bubbles. Moreover, these bubbles will be productive. According to the model, changing the fiscal policy will move the Europeans economies towards a new a steady state with a larger capital stock. The results in Table 4 show that the average increase in the stock of capital of these European economies would be 50% if the economy remains in an equilibrium without bubbles. However, if a bubble equilibrium emerges, the increase in the capital stock will be substantially large and on average it will be 112%.

At this point, it is important to introduce some words of caution on the large

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[10] There is not a consensus in the literature on the value of the labor income share. In a recent paper, Koh, Santaeulalia-Llopis and Zheng (2016) show that in the US the labor income share is stable and close to 70% if intellectual property capital is not considered as a form of capital income.

[11] The savings rate implies by this value of β is 26% in the US when the equilibrium does not exhibit bubbles.

effects of fiscal policy obtained in the previous analysis. First, the changes in the stock of capital are obtained by comparing two different steady states. Thus, these effects of fiscal policy will occur in the very long run. Second, as explained in the previous sections, the effects of fiscal policy crucially depend on the value of Σ and Ω. To obtain these values, we have introduced assumptions on the distribution of labor and capital income by age group that, as explained in Appendix E, may introduce small biases on the values of both Ω and Σ. Therefore, the results in Table 4 should only be considered as illustrative of the large effects that fiscal policies may have.

6. Concluding remarks

We have studied a three period OG model, where productive investment done in the first period is a long term investment whose return occurs in the following two period of life. In this model, the bubble is a short term speculative investment that facilitates intertemporal consumption smoothing. Our main result shows that the distribution of labor and capital income by age group is a crucial determinant of both the existence of bubbles and of their effect on production. We show that when the values of the fraction of labor income obtained by the young individuals and the fraction of capital income obtained by the middle age individuals are sufficiently large, bubbles exist and they are used to transfer wealth to the last period of life. Obviously, in this case, bubbles are unproductive and, thus, the stock of capital is smaller in the equilibrium with bubbles. We also show that if the fraction of labor income obtained by the young individuals is too small and the fraction of capital income obtained by the middle age individuals is also too small, then bubbles do not emerge. Finally, we show that productive bubbles may arise in two different situations: when the middle age individuals obtain a large fraction of total income and when these individuals obtain a small fraction of total income. In the first case, bubbles are productive because they are used to transfer wealth to the young individuals, who then increase investment in the productive asset. In the second case, bubbles are productive because young individuals increase the savings rate in the equilibrium with bubbles.

Fiscal policies cause large changes in the distribution of income by age group. According to our findings, these changes may modify the effect that bubbles have on production and may even prevent or cause the emergence of productive bubbles. In particular, we show that large capital income taxes facilitate the emergence of an equilibrium with bubbles. We also show that an increase in the labor income tax paid by the young may make bubbles productive. However, if this increase is sufficiently large then the equilibrium with bubbles may vanish. Finally, we show that an increase in the labor income taxes paid by middle age individuals has the opposite effects.

The distribution of income by age group depends on other economic fundamentals. A clear example is the size of each age group. The current population aging will increase the size of the oldest age group. As a consequence, it will dramatically reduce the value of Ω in the following years, which will reduce the stock of productive capital. Our result suggest that population aging can be particularly harmful in those economies where productive bubbles finance a large stock of productive capital, as these bubbles, due to the reduction in Ω, may not be sustainable.

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A. Proof of Proposition 3.2

We rewrite condition (3.9) as

$$(1 - \alpha)(1 - \Sigma)\beta^2 (\Sigma_2 - \Sigma) > \alpha\Omega (\beta + \beta^2)(\Sigma - \Sigma_1),$$

where $\Sigma_2 = \frac{\alpha}{1-\alpha}\frac{1+\beta^2}{\beta^2}$ and $\Sigma_1 = \frac{\alpha}{1-\alpha}\frac{1+\beta+\beta^2}{\beta+\beta^2}$.

As $(1 + \beta^2)/\beta^2 > (1 + \beta + \beta^2)/(\beta + \beta^2)$, there are only three possibilities: (i) $\Sigma > \Sigma_2$ and condition (3.9) is not satisfied; (ii). $\Sigma_1 < \Sigma < \Sigma_2$ and condition (3.9) is satisfied when $\Omega < \overline{\Omega}$, where $\overline{\Omega}$ is obtained from (3.9); and (iii).$\Sigma < \Sigma_1$ and condition (3.9) is always satisfied.

B. Equilibrium with bubbles

We first use (2.2), (2.3) and (2.4), to rewrite equations (4.1) and (4.2) as

$$b_{1,t} = \frac{(\beta + \beta^2)(\xi_1 w_t - a_{t+1}) - \frac{\xi_2 w_{t+1}}{R_{t+1}} - \left[q_{t+1}\phi_1 + \frac{q_{t+2}}{R_{t+2}}\phi_2\right]\frac{a_{t+1}}{R_{t+1}}}{1 + \beta + \beta^2}, \qquad \text{(B.1)}$$

$$b_{2,t+1} = \frac{\beta^2\xi_2 w_{t+1} + \beta^2 q_{t+1}\phi_1 a_{t+1} + \beta^2(\xi_1 w_t - a_{t+1})R_{t+1} - (1+\beta)\frac{q_{t+2}}{R_{t+2}}\phi_2 a_{t+1}}{1+\beta+\beta^2}. \qquad \text{(B.2)}$$

From using (2.6) and (2.5), equations (B.1) and (B.2) can be rewritten as

$$b_{1,t} = \frac{(\beta+\beta^2)(\xi_1(1-\alpha)Ak_t^\alpha - a_{t+1}) - \frac{\xi_2(1-\alpha)Ak_{t+1}^\alpha}{R_{t+1}} - \left[\alpha Ak_{t+1}^{\alpha-1}\phi_1 + \frac{\alpha Ak_{t+2}^{\alpha-1}}{R_{t+2}}\phi_2\right]\frac{a_{t+1}}{R_{t+1}}}{1+\beta+\beta^2}, \qquad \text{(B.3)}$$

$$b_{2,t+1} = \frac{\beta^2\xi_2(1-\alpha)Ak_{t+1}^\alpha + \beta^2\alpha Ak_{t+1}^{\alpha-1}\phi_1 a_{t+1} + \beta^2[\xi_1(1-\alpha)Ak_t^\alpha - a_{t+1}]R_{t+1} - \frac{(1+\beta)\alpha Ak_{t+2}^{\alpha-1}\phi_2 a_{t+1}}{R_{t+2}}}{1+\beta+\beta^2} \qquad \text{(B.4)}$$

We use (4.4) to rewrite (B.3) and (B.4) as (4.5) and (4.6) in the main text.

C. Proof of Proposition 4.5

Note first that $\overline{\Omega}(\Sigma_2) = 0$, $\overline{\Omega}(1) = 0$, $\overline{\Omega}(\Sigma_1) = \infty$, $\widetilde{\Omega}(\Sigma_3) = 0$, $\widetilde{\Omega}'(\Sigma) < 0$, and if $\Sigma \in (\Sigma_1, \Sigma_2)$ then $\overline{\Omega}'(\Sigma) < 0$. Next, from using the expressions of Σ_1 and Σ_2, it can be shown that $\Sigma_1 < \Sigma_2$. We define $\Sigma' = \frac{1-\beta^2}{1+\beta+\beta^2} + \frac{\alpha}{1-\alpha}$, which is such that $\widetilde{\Omega}(\Sigma') = 1$.

We use the steady state expression of b_1 to show that $b_1 < 0$ if and only if $\Sigma < \Sigma_{b_1}$, where

$$\Sigma_{b_1} = \frac{1}{1 + \beta + \beta^2} + \frac{\alpha}{1 - \alpha}.$$

Σ_{b_1} satisfies the following two properties. First, if $\Sigma_1 < 1$ then $\Sigma_{b_1} \in (\Sigma_1, 1)$. Second, bubbles are productive when $\Sigma \geq \Sigma_{b_1}$ if and only if $\Omega < \overline{\Omega}(\Sigma_{b_1})$. This condition simplifies as

$$\frac{\beta^2}{1 + \beta + \beta^2} - \frac{\alpha}{1 - \alpha}(1 - \Omega) < 0,$$

which implies that $b_2 < 0$. This proves that if bubbles are productive and $b_1 \geq 0$ then $b_2 < 0$. It also proves that if bubbles are productive and $b_2 > 0$ then $\Omega > \overline{\Omega}(\Sigma_{b_1})$,

which implies that $\Sigma < \Sigma_{b_1}$ and, hence, $b_1 < 0$. This analysis shows that if bubbles are productive, then it cannot be that both $b_1 > 0$ and $b_2 > 0$.

Finally, we obtain the following relationships which will be necessary to characterize the different parametric regions in which bubbles are productive: (i) $\Sigma_1 > 1$ if and only if $\alpha/\left(1-\alpha\right) > \left(\beta+\beta^2\right)/\left(1+\beta+\beta^2\right)$; (ii) $\overline{\Omega}\left(\Sigma'\right) > 1$ if and only if $\alpha/\left(1-\alpha\right) > \left(1-\beta^2\right)\left(\beta+2\beta^2\right)/\left[\left(1+\beta+\beta^2\right)\left(2+\beta\right)\right]$; (iii) $\Sigma_3 < \Sigma_{b_1}$ and $\Sigma_3 > \Sigma_2$ if and only if $\alpha/\left(1-\alpha\right) < \beta^2/\left(1+\beta+\beta^2\right)$; (iv) $\Sigma_3 < 1$ if and only if $\alpha/\left(1-\alpha\right) < \left(\beta/2+\beta^2\right)/\left(1+\beta+\beta^2\right)$; and (v) $\Sigma_2 > 1$ if and only if $\alpha/\left(1-\alpha\right) > \beta^2/\left(\beta^2+1\right)$. Using these relationships, we characterize the following parametric cases:

(i). If $\frac{\alpha}{1-\alpha} > \frac{\beta+\beta^2}{1+\beta+\beta^2}$ then $\Sigma_1 > 1$ and $\Sigma_{b_1} > 1$. Obviously, $\Sigma < \Sigma_1$ and $\Sigma < \Sigma_{b_1}$. Therefore, $b_1 < 0$ and, as shown in Proposition 4.4, the bubble is productive when $\Omega < \widetilde{\Omega}$ and the bubble does not exist otherwise. This parametric region is included in Case 1 of the proposition and it is displayed in Figure 1.

(ii). If $\frac{\alpha}{1-\alpha} < \frac{\beta+\beta^2}{1+\beta+\beta^2}$, $\frac{\alpha}{1-\alpha} > \frac{\left(1-\beta^2\right)\left(\beta+2\beta^2\right)}{\left(1+\beta+\beta^2\right)\left(2+\beta\right)}$, and $\frac{\alpha}{1-\alpha} \notin \left(\frac{\beta^2}{1+\beta+\beta^2}, \frac{\beta/2+\beta^2}{1+\beta+\beta^2}\right)$, then $\Sigma_1 < 1$, $\overline{\Omega}\left(\Sigma_1\right) > 1$, and either $\Sigma_3 \notin \left(\Sigma_{b_1}, \min\left\{1, \Sigma_2\right\}\right)$ or $\Sigma_3 > 1$. In this parametric region, bubbles are productive with $b_1 < 0$ and $b_2 > 0$ when $\Omega \in \left(\widetilde{\Omega}, \overline{\Omega}\right)$. This region is included in Case 1 of the proposition and it is displayed in Figure 2.

(iii). If $\frac{\alpha}{1-\alpha} < \frac{\beta+\beta^2}{1+\beta+\beta^2}$, $\frac{\alpha}{1-\alpha} > \frac{\left(1-\beta^2\right)\left(\beta+2\beta^2\right)}{\left(1+\beta+\beta^2\right)\left(2+\beta\right)}$, and $\frac{\alpha}{1-\alpha} \in \left(\frac{\beta^2}{1+\beta+\beta^2}, \frac{\beta/2+\beta^2}{1+\beta+\beta^2}\right)$, then $\Sigma_1 < 1$, $\overline{\Omega}\left(\Sigma_1\right) > 1$, $\Sigma_{b_1} \in \left(\Sigma_1, 1\right)$ and $\Sigma_3 \in \left(\Sigma_{b_1}, \min\left\{1, \Sigma_2\right\}\right)$. Bubbles can be productive in two different parametric regions: when $\Sigma < \Sigma_{b_1}$ and hence $b_1 < 0$ and $b_2 > 0$ and when $\Sigma > \Sigma_{b_1}$ and, hence, $b_1 > 0$ and $b_2 < 0$. These regions are included in Case 2 of the proposition and they are displayed in Figure 4.

(iv). If $\frac{\alpha}{1-\alpha} < \frac{\beta+\beta^2}{1+\beta+\beta^2}$, $\frac{\alpha}{1-\alpha} < \frac{\left(1-\beta^2\right)\left(\beta+2\beta^2\right)}{\left(1+\beta+\beta^2\right)\left(2+\beta\right)}$, and $\frac{\alpha}{1-\alpha} \in \left(\frac{\beta^2}{1+\beta+\beta^2}, \frac{\beta/2+\beta^2}{1+\beta+\beta^2}\right)$ then $\Sigma_1 < 1$, $\overline{\Omega}\left(\Sigma_1\right) < 1$, and $\Sigma_3 \in \left(\Sigma_{b_1}, \min\left\{1, \Sigma_2\right\}\right)$. In this parametric region, bubbles are productive with $b_1 > 0$ and $b_2 < 0$. This region is included in Case 3 of the proposition and it is displayed in Figure 5.

(v). If $\frac{\alpha}{1-\alpha} < \frac{\beta+\beta^2}{1+\beta+\beta^2}$, $\frac{\alpha}{1-\alpha} < \frac{\left(1-\beta^2\right)\left(\beta+2\beta^2\right)}{\left(1+\beta+\beta^2\right)\left(2+\beta\right)}$, and $\frac{\alpha}{1-\alpha} \notin \left(\frac{\beta^2}{1+\beta+\beta^2}, \frac{\beta/2+\beta^2}{1+\beta+\beta^2}\right)$ then $\Sigma_1 < 1$, $\overline{\Omega}\left(\Sigma_1\right) < 1$, $\Sigma_{b_1} \in \left(\Sigma_1, 1\right)$ and $\Sigma_3 \in \left(\Sigma_2, \Sigma_{b_1}\right)$. In this parametric region, bubbles are productive with $b_1 < 0$ and $b_2 > 0$. This region in included in Case 1 of the proposition and it is displayed in Figure 3.

The parameter conditions in the Cases 2 and 3, shown in the proposition, are obtained once the following relations are taken into account: $\frac{\beta+\beta^2}{1+\beta+\beta^2} > \frac{\beta/2+\beta^2}{1+\beta+\beta^2} > \frac{\left(1-\beta^2\right)\left(\beta+2\beta^2\right)}{\left(1+\beta+\beta^2\right)\left(2+\beta\right)}$ and $\frac{\beta/2+\beta^2}{1+\beta+\beta^2} > \frac{\beta^2}{1+\beta+\beta^2}$. To obtain Case 1, note that the previous analysis implies that $b_1 < 0$ and $b_2 > 0$ when $\frac{\alpha}{1-\alpha} > \frac{\beta+\beta^2}{1+\beta+\beta^2}$ or $\frac{\alpha}{1-\alpha} < \frac{\beta+\beta^2}{1+\beta+\beta^2}$ and $\frac{\alpha}{1-\alpha} \notin \left(\frac{\beta^2}{1+\beta+\beta^2}, \frac{\beta/2+\beta^2}{1+\beta+\beta^2}\right)$. Taking into account that $\frac{\beta+\beta^2}{1+\beta+\beta^2} > \frac{\beta/2+\beta^2}{1+\beta+\beta^2}$ and that $\frac{\beta/2+\beta^2}{1+\beta+\beta^2} > \frac{\beta^2}{1+\beta+\beta^2}$, we obtain that $b_1 < 0$ and $b_2 > 0$ when $\frac{\alpha}{1-\alpha} \notin \left(\frac{\beta^2}{1+\beta+\beta^2}, \frac{\beta/2+\beta^2}{1+\beta+\beta^2}\right)$. This is the condition in Case 1.

D. Equilibrium with bubbles and taxes

We next use (5.1), (5.2) and (5.3) to rewrite equations (5.4) and (5.5) as

$$b_{2,t+1} = \frac{\beta^2\left(1-\tau_w^2\right)\xi_2 w_{t+1}+\beta^2\left(1-\tau_k^1\right)a_{t+1}q_{t+1}\phi_1+\beta^2 R_{t+1}\left[\left(1-\tau_w^1\right)\xi_1 w_t-a_{t+1}\right]}{1+\beta+\beta^2}- \tag{D.1}$$
$$-\frac{(1+\beta)\left(1-\tau_k^2\right)a_{t+1}\frac{q_{t+2}\phi_2}{R_{t+2}}+(1+\beta)\frac{p_{t+2}}{R_{t+2}}}{1+\beta+\beta^2},$$

$$b_{1,t} = \frac{\beta(1+\beta)\left[\left(1-\tau_w^1\right)\xi_1 w_t-a_{t+1}\right]-\frac{\left(1-\tau_w^2\right)\xi_2 w_{t+1}}{R_{t+1}}}{1+\beta+\beta^2}- \tag{D.2}$$
$$-\frac{\frac{a_{t+1}}{R_{t+1}}\left(\left(1-\tau_k^1\right)q_{t+1}\phi_1+\left(1-\tau_k^2\right)\frac{q_{t+2}\phi_2}{R_{t+2}}\right)+\frac{p_{t+2}}{R_{t+1}R_{t+2}}}{1+\beta+\beta^2}.$$

From using (2.6) and (2.5), equations (D.1) and (D.2) can be rewritten as

$$b_{2,t+1} = \frac{\beta^2\left(1-\tau_w^2\right)\xi_2(1-\alpha)Ak_{t+1}^\alpha+\beta^2\left(1-\tau_k^1\right)a_{t+1}\alpha Ak_{t+1}^{\alpha-1}\phi_1+\beta^2 R_{t+1}\left[\left(1-\tau_w^1\right)\xi_1(1-\alpha)Ak_t^\alpha-a_{t+1}\right]}{1+\beta+\beta^2}-$$
$$-\frac{(1+\beta)\left(1-\tau_k^2\right)a_{t+1}\frac{\alpha Ak_{t+2}^{\alpha-1}\phi_2}{R_{t+2}}+(1+\beta)\frac{\xi_2 p(1-\alpha)Ak_{t+2}^\alpha}{R_{t+2}}}{1+\beta+\beta^2},$$

$$b_{1,t} = \frac{\beta(1+\beta)\left[\left(1-\tau_w^1\right)\xi_1(1-\alpha)Ak_t^\alpha-a_{t+1}\right]-\frac{\left(1-\tau_w^2\right)\xi_2(1-\alpha)Ak_{t+1}^\alpha}{R_{t+1}}}{1+\beta+\beta^2}-$$
$$-\frac{\frac{a_{t+1}}{R_{t+1}}\left(\alpha\left(1-\tau_k^1\right)Ak_{t+1}^{\alpha-1}\phi_1+\left(1-\tau_k^2\right)\frac{\alpha Ak_{t+2}^{\alpha-1}\phi_2}{R_{t+2}}\right)+\frac{p\xi_2(1-\alpha)Ak_{t+2}^\alpha}{R_{t+1}R_{t+2}}}{1+\beta+\beta^2}.$$

We use (5.7) to rewrite the previous two equations as follows

$$b_{1,t} = \frac{\beta(1+\beta)\left(1-\tau_w^1\right)\xi_1(1-\alpha)Ak_t^\alpha-\frac{\left(1-\tau_w^2\right)\xi_2(1-\alpha)Ak_{t+1}^\alpha}{R_{t+1}}-\frac{p\xi_2(1-\alpha)Ak_{t+2}^\alpha}{R_{t+1}R_{t+2}}}{1+\beta+\beta^2}-a_{t+1},$$

$$b_{2,t+1} = \frac{\beta^2\left(1-\tau_w^2\right)\xi_2(1-\alpha)Ak_{t+1}^\alpha+\beta^2 R_{t+1}\left(1-\tau_w^1\right)\xi_1(1-\alpha)Ak_t^\alpha-(1+\beta)\frac{p\xi_2(1-\alpha)Ak_{t+2}^\alpha}{R_{t+2}}}{1+\beta+\beta^2}+$$
$$+a_{t+1}\left[\left(1-\tau_k^1\right)\phi_1\alpha Ak_{t+1}^{\alpha-1}-R_{t+1}\right].$$

In a steady state, $R=n$ and we obtain

$$b_1 = (1-\alpha)Ak^\alpha\left[\frac{\beta(1+\beta)\left(1-\tau_w^1\right)\xi_1-\left(1-\tau_w^2\right)\frac{\xi_2}{n}-\frac{\xi_2 p}{n^2}}{1+\beta+\beta^2}-\frac{a}{(1-\alpha)Ak^\alpha}\right],$$

$$b_2 = (1-\alpha)Ak^\alpha\left[\frac{\beta^2\left[\left(1-\tau_w^2\right)\xi_2+\left(1-\tau_w^1\right)n\xi_1\right]-(1+\beta)\frac{\xi_2 p}{n}}{1+\beta+\beta^2}+\frac{a\left[\left(1-\tau_k^1\right)\phi_1\alpha Ak^{\alpha-1}-n\right]}{(1-\alpha)Ak^\alpha}\right].$$

Using (3.7), we obtain $a=n\left[\left(n\xi_1+\xi_2\right)/\left(n\phi_1+\phi_2\right)\right]k$ and, hence,

$$b_1 = (1-\alpha)Ak^\alpha\left[\frac{\beta(1+\beta)\left(1-\tau_w^1\right)\xi_1-\left(1-\tau_w^2\right)\frac{\xi_2}{n}-\frac{\xi_2 p}{n^2}}{1+\beta+\beta^2}-\left(\frac{n\xi_1+\xi_2}{n\phi_1+\phi_2}\right)\frac{n}{(1-\alpha)Ak^{\alpha-1}}\right],$$

and

$$b_2 = (1 - \alpha) A k^\alpha \left[\frac{\beta^2 \left[\left(1 - \tau_w^2\right) \xi_2 + \left(1 - \tau_w^1\right) n \xi_1 \right] - (1+\beta) \frac{\xi_{2p}}{n}}{1 + \beta + \beta^2} + \frac{\frac{(n\xi_1 + \xi_2)}{(n\phi_1 + \phi_2)} n \left[\left(1 - \tau_k^1\right) \phi_1 \alpha A k^{\alpha - 1} - n \right]}{(1 - \alpha) A k^{\alpha - 1}} \right].$$

From (5.7), we obtain

$$\left(\frac{n^2}{n \left(1 - \tau_k^1\right) \phi_1 + \left(1 - \tau_k^2\right) \phi_2} \right) \frac{1}{\alpha} = A k^{\alpha - 1}.$$

We use this expression to obtain (5.8) and (5.9) in the main text.

E. Empirical strategy to obtain Σ and Ω

In this appendix we describe how the data in Table 1 on the distribution of gross labor and capital income by age has been obtained. The data sources used are the US census and the Eurostat. US government census provides average income and total population in 2015 for the following age groups: young (age 25-44), middle age (age 45-64) and old (65 and over). Eurostat provides the same data in 2015 for the different European economies shown in Table 1 and for the following age groups: young (age 25-49), middle age (age 50-64) and old (65 and over). As the number of years people belong to each age group is different with the Eurostat data, we divide total income of each age group by the number of years individuals belong to each age group. This normalization makes the different age groups comparable. From using these data, we obtain the total income of each age group and the total income of the economy is obtained as the sum of the income of each age group.

We next use the Penn World Table to obtain the labor income share in 2014. We use the labor income share and total income to obtain for each country the labor income and the capital income of the economy. Consistent with the assumptions in the model, we assume that (i) the young individuals do not obtain labor income and (ii) the old individuals do not obtain capital income. Based on these assumptions, we obtain Σ as the ratio between the income of the young and the total labor income in the economy and we obtain $\widehat{\Omega}$ as the difference between one and the ratio between the income of the old and the total capital income of the economy. The values of Σ and $\widehat{\Omega}$ are displayed in Table 1.

The value of Σ and $\widehat{\Omega}$ are obviously biased because the two aforementioned assumptions are not strictly true. To measure how problematic are these two assumptions, we use the US census data to obtain that the fraction of labor income obtained by the old individuals is only 4% and the net worth owned by the young is only 9.4%. These small numbers imply that the two assumptions are not too strong and, hence, the bias in the measures of Σ and $\widehat{\Omega}$ should be small.

A more serious problem with the data is that the income of the old also includes the pensions they receive. Using the notation introduced in Section 5 and the definition of $\widehat{\Omega}$, the expression of $\widehat{\Omega}$ is in fact

$$\widehat{\Omega}_t = 1 - \frac{q_{t+2} \phi_2 a_{t+1} + p_{t+2}}{q_{t+1} \phi_1 a_{t+1} n + q_{t+2} \phi_2 a_{t+1} + p_{t+2}},$$

where p_{t+2} are the pensions received by individuals when old. This expression at the steady state simplifies as

$$\widehat{\Omega} = \frac{n\phi_1}{n\phi_1 + \phi_2 + \frac{p}{qa}}.$$

Let us defined by σ the replacement rate of pensions and, hence, $p = \sigma\xi_2 w$. Using the replacement rate, (2.5), (2.6), and (3.7), we obtain

$$\Omega = \widehat{\Omega}\left[1 + (1 - \Sigma)\frac{\sigma(1 - \alpha)}{\alpha n}\right],$$

where $\Omega = n\phi_1 / (n\phi_1 + \phi_2)$ is the fraction of capital income obtained by the middle age individuals and that we have used in the main text of this paper. The previous equation clearly shows that $\widehat{\Omega}$ is a biased measure of the distribution of capital income by age group when pensions are introduced. In the last step of our empirical strategy, we use this equation to obtain the value of Ω. To this end, we must obtain the values of σ, α and n. The value of σ is obtained from OECD data set 2014, where the replacement rate is defined as the gross pension divided by the gross pre-retirement wage and, hence, it corresponds to our definition of σ. The value of α is obtained from the labor income share in the Penn World table 2014 and the value of n is obtained from OECD data as the ratio between total population age 45-64 divided by total population age 65 and over. The value of Ω obtained from this analysis is displayed in the last column of Table 1.

F. Figures and Tables

Table 1. Income distribution by age group[12]

	Σ	$\widehat{\Omega}$	Ω
Austria	0,45	0,54	0,75
Belgium	0,45	0,58	0,71
Czech Republic	0,60	0,39	0,43
Denmark	0,41	0,60	0,87
Finland	0,43	0,61	0,80
France	0,39	0,73	0,96
Germany	0,45	0,67	0,80
Greece	0,56	0,50	0,58
Hungary	0,48	0,58	0,73
Italy	0,46	0,60	0,78
Luxembourg	0,44	0,35	0,43
Netherlands	0,46	0,57	0,80
Norway	0,48	0,41	0,49
Poland	0,54	0,44	0,48
Portugal	0,47	0,61	0,82
Spain	0,46	0,54	0,74
Sweden	0,44	0,51	0,67
United Kingdom	0,47	0,54	0,60
United States	0,59	0,52	0,58

Source. See Appendix E.

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[12]Σ is the fraction of labor income obtained by the young individuals and Ω is the fraction of capital income obtained by middle age individuals.

Table 2. Taxes and population growth[13]

	τ_w^1	τ_w^2	τ_k	n
Austria	0, 45	0, 52	0, 25	0, 96
Belgium	0, , 49	0, 60	0, 34	0, 98
Czech Republic	0, 39	0, 45	0, 19	1, 14
Denmark	0, 34	0, 42	0, 23	0, 94
Finland	0, 38	0, 49	0, 20	0, 89
France	0, 43	0, 54	0, 38	0, 97
Germany	0, 45	0, 51	0, 30	0, 84
Greece	0, 34	0, 46	0, 26	1, 13
Hungary	0, 49	0, 49	0, 19	1, 07
Italy	0, 40	0, 54	0, 31	0, 99
Luxembourg	0, 31	0, 45	0, 29	1, 14
Netherlands	0, 31	0, 41	0, 25	0, 91
Norway	0, 33	0, 42	0, 27	1, 08
Poland	0, 33	0, 35	0, 19	1, 10
Portugal	0, 36	0, 48	0, 29	1, 02
Spain	0, 36	0, 44	0, 28	1, 16
Sweden	0, 41	0, 51	0, 22	1, 01
United Kingdom	0, 26	0, 37	0, 20	1, 05
United States	0, 29	0, 36	0, 39	1, 00

Source. OECD Data base.

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[13]τ_w^1 (τ_w^2) is the labor income tax rate paid by a single person (no child) with an income equal to 67% (167%) of the average earnings in the year 2015. We assume that young (middle age) individuals obtain lower (higher) labor earnings and, hence, they are taxed with the low (high) tax rate. τ_k is the coorporate tax rate in the year 2015. We assume that $\tau_k^1 = \tau_k^2 = \tau_k$. Finally, the population growth rate is obtained from the ratio between the population in the interval 25-44 years and the population in the interval 45 to 64. The population growth rate is obtained for all countries in the year 2013, except for Belgium, France, Greece, Netherlands and Poland that it is obtained in the year 2012.

Table 3. Results from the simulation

	k^*	k	F
Austria	0, 06	– – –	−0, 60
Belgium	0, 05	– – –	−0, 60
Czech Republic	0, 16	– – –	−0, 64
Denmark	0, 05	– – –	−0, 45
Finland	0, 06	– – –	−0, 53
France	0, 03	– – –	−0, 34
Germany	0, 06	– – –	−0, 49
Greece	0, 11	– – –	−0, 40
Hungary	0, 05	– – –	−0, 66
Italy	0, 06	– – –	−0, 41
Luxembourg	0, 12	– – –	−0, 70
Netherlands	0, 08	– – –	−0, 34
Norway	0, 12	– – –	−0, 62
Poland	0, 14	– – –	−0, 60
Portugal	0, 06	– – –	−0, 31
Spain	0, 05	– – –	−0, 43
Sweden	0, 07	– – –	−0, 66
United Kingdom	0, 11	– – –	−0, 52
United States	0, 15	0, 19	0, 01

Table 4. Results from the simulation

	τ_w^1	k^*	k	F
Austria	0,15	0,10	0,14	0,01
Belgium	0,13	0,10	0,15	0,01
Czech Republic	0,23	0,21	0,24	0,01
Denmark	0,14	0,07	0,12	0,01
Finland	0,15	0,09	0,14	0,01
France	0,17	0,05	0,10	0,01
Germany	0,19	0,10	0,15	0,01
Greece	0,24	0,13	0,16	0,01
Hungary	0,20	0,09	0,12	0,01
Italy	0,20	0,08	0,13	0,01
Luxembourg	0,00	0,20	$---$	-0,07
Netherlands	0,21	0,09	0,14	0,01
Norway	0,07	0,18	0,22	0,01
Poland	0,17	0,18	0,21	0,01
Portugal	0,24	0,07	0,11	0,01
Spain	0,18	0,08	0,11	0,01
Sweden	0,09	0,11	0,15	0,01
United Kingdom	0,11	0,13	0,17	0,01
United States	0,29	0,15	0,19	0,01

We assume that $\tau_k^1 = \tau_k^2 = 0,39$ and $\tau_w^2 = 0,36$

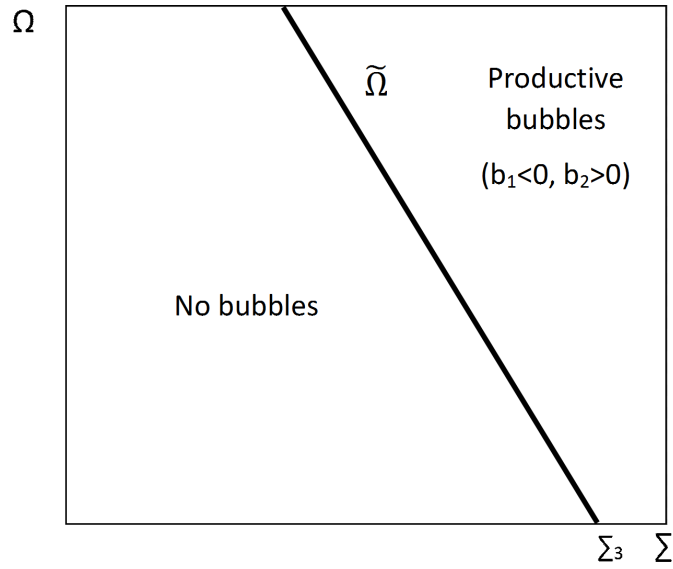## Figure 1. Case 1: bubbles and the distribution of income

Ω

$\widetilde{\Omega}$

Productive
bubbles

$(b_1<0, b_2>0)$

No bubbles

Σ_3 Σ

Figure 2. Case 1: bubbles and the distribution of income

Ω

Productive
bubbles

$(b_1<0, b_2>0)$

$\widetilde{\Omega}$

Unproductive
bubbles

No bubbles

$\overline{\Omega}$

Σ_1 Σ_2 Σ_3 Σ_{b1} Σ

Ω

Productive
bubbles

$(b_1<0, b_2>0)$

Unproductive
bubbles

$\widetilde{\Omega}$

No bubbles

$\overline{\Omega}$

Σ_1 Σ_2 Σ_{b1} Σ Σ_3
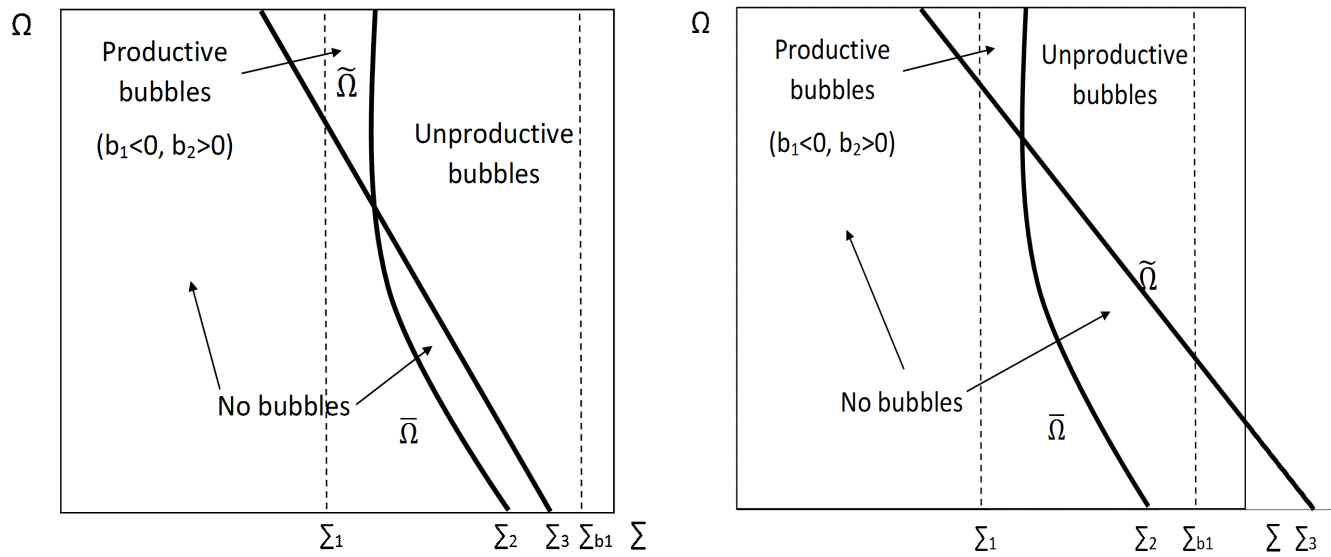
Figure 3. Case 1: bubbles and the distribution of income

$b_1=0$

Ω

Productive
bubbles
$(b_1<0, b_2>0)$

Unproductive
bubbles

$b_2=0$

Productive
bubbles
$(b_1>0, b_2<0)$

No bubbles

$\widetilde{\Omega}$

$\overline{\Omega}$

Σ_1 Σ_{b1} Σ_3 Σ_2 Σ
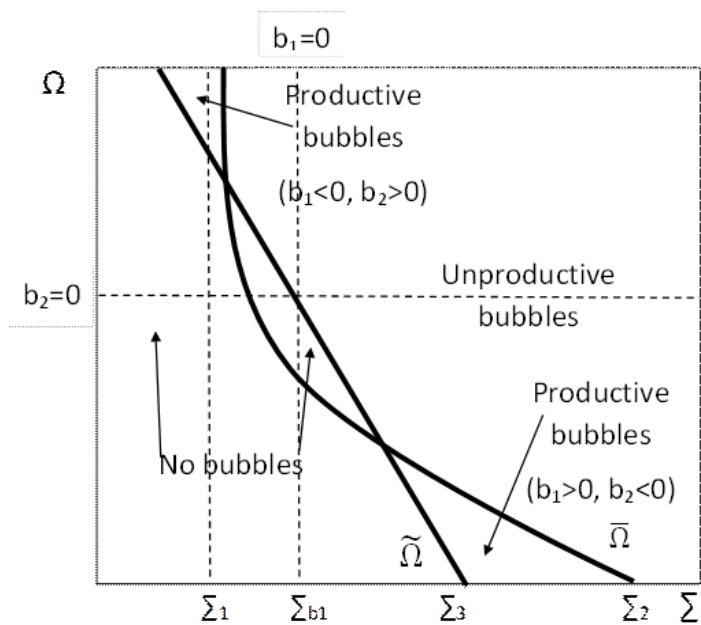
Figure 4. Case 2: bubbles and the distribution of income

$b_1=0$

Ω

Productive
bubbles
$(b_1<0, b_2>0)$

Unproductive
bubbles

$b_2=0$

Productive
bubbles
$(b_1>0, b_2<0)$

No bubbles

$\widetilde{\Omega}$

$\overline{\Omega}$

Σ_1 Σ_{b1} Σ_3 Σ_2 Σ
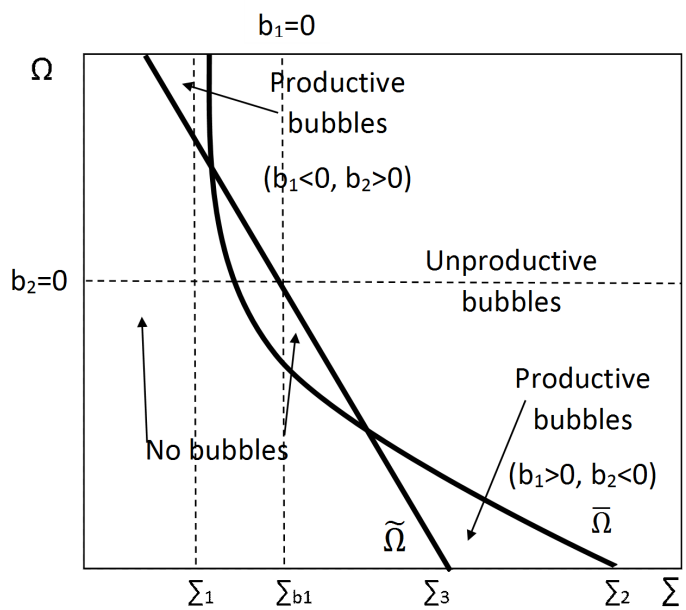
Figure 5. Case 3: bubbles and the distribution of income

$b_1=0$

Ω

Unproductive
bubbles

$b_2=0$

Productive
bubbles

$(b_1>0, b_2<0)$

No bubbles

$\widetilde{\Omega}$

$\overline{\Omega}$

Σ_1 Σ_{b1} Σ_3 Σ_2 Σ
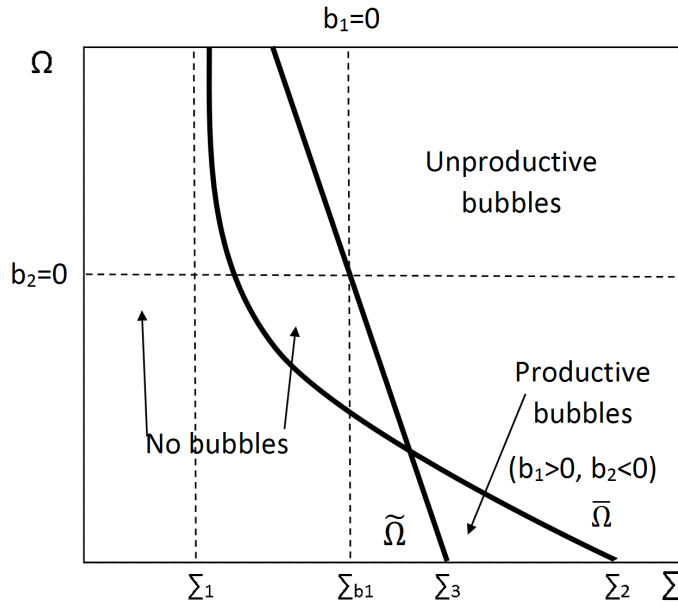
Figure 6. Productive bubbles

$\alpha/(1-\alpha)$

0.66

Case 1: $b_1<0$ and $b_2>0$

$\dfrac{\frac{\beta}{2}+\beta^2}{1+\beta+\beta^2}$

0.42

Case 2: $b_1<0$ and $b_2>0$ or

$\dfrac{\beta^2}{1+\beta+\beta^2}$

$b_1>0$ and $b_2<0$

Case 3: $b_1>0$ and $b_2<0$

$\dfrac{(1-\beta^2)(\beta+2\beta^2)}{(1+\beta+\beta^2)(2+\beta)}$

Case 1: $b_1<0$ and $b_2>0$

β

36

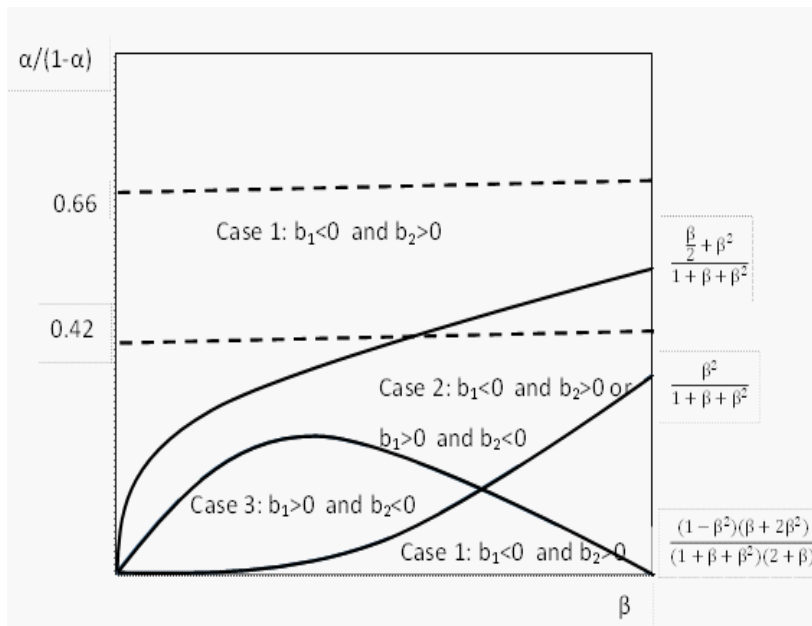**Figure 7. The effect of fiscal policies on capital**

Panel a   Panel b   Panel c