The Bargaining Set of an Exchange Economy with Discrete Resources*

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Abstract

A central notion for allocation problems when there are private endowments is *core*: no coalition should be able to block the allocation. However, for an exchange economy of discrete resources, *core* can be empty. An alternative core-type stability axiom is the *bargaining set* via Aumann and Maschler (1964): a blocking by a coalition is justified only if there is no counter-objection to it and an allocation is in the *bargaining set* if there does not exist a justified blocking. We prove that the *bargaining set* characterizes a well-known class of trading mechanisms, the top trading cycles.

Keywords : Assignment problem, core, bargaining set, Top Trading Cycles, entitlement

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1 Introduction

An exchange economy of discrete resources with private endowments is when each agent owns an indivisible good and these goods are to be allocated among agents without monetary transfers via direct mechanisms. The central notion when there are private endowments is *individual rationality* which requires that the assignment should be such that no agent is worse off than her endowment. There is another trademark property of this problem, *core*: no coalition of agents should be able to block the assignment; that is, they should not prefer reallocating their endowments among themselves (by leaving the economy) over the assignment. However, *core* is in general empty. An alternative (and weaker) notion is the *bargaining set* by Aumann and Maschler (1964): a blocking is justified only if there is no counter-objection to it and an allocation is in the *bargaining set* if there does not exist a justified blocking. We prove that, in the context of exchange economies of discrete resources, the well known Top Trading Cycles class is characterized by the *bargaining set*.

If preferences are strict, *core* is a singleton and it is the only solution which satisfies *individual rationality*, *Pareto efficiency* and *strategy-proofness* (Ma (1994), Sönmez (1999)). Also, core is equivalent to the outcome of the Gale's well-known **Top Trading Cycles (TTC)** algorithm (Shapley and Scarf, 1974). The TTC algorithm works as follows: Each agent points at her most preferred available object (all objects are available at the beginning) and each object points at its owner. Since all agents and objects point, there is at least one cycle where each agent owns the most preferred object of the previous agent in the cycle. The algorithm assigns to each agent in the cycle her most preferred available object (that is, the object she points at) and removes her with her assigned object. This continues until no one is left. The resulting mechanism is *group strategy-proof* and *Pareto efficient* (Roth, 1982). When an agent may be endowed with multiple objects or no object, the top trading cycles rule is generalized to the *hierarchical change* rule, which is characterized by *Pareto efficiency*, *group strategy-proofness* and *reallocation-proofness* (Pápai, 2000). A more general trading mechanism is *trading-cycles* and it is characterized by *group strategy-proofness* and *Pareto efficiency* (Pycia and Ünver, 2016).

While the extension of the TTC algorithm to the weak preferences domain is not trivial, such extensions satisfying *individual rationality*, *Pareto efficiency* and *strategy-proofness* are shown to exist (Jaramillo and Manjunath (2012), Alcalde-Unzu and Molis (2011), Saban and Sethuraman (2013)). *Strategy-proofness* characterizes a subclass of these generalized TTC class satisfying *Pareto efficiency*

(Saban and Sethuraman, 2013).

When the restrictive strict preferences assumption is removed, *core* can be empty (Shapley and Scarf, 1974). Actually, *core* is non-empty only for a very special preference and endowment structure (Quint and Wako, 2004). A weakening of *core* is *weak core*: blocking is allowed only if each agent in the blocking coalition is strictly better off than the assignment. The extensions of the TTC discussed in the previous paragraph are in the *weak core*. Our focus is on another notion, the *bargaining set*, which incorporates an important consideration into the process of blocking an assignment: when blocking, coalitions should consider possible counter-blockings of other coalitions. More precisely, an assignment is in the *bargaining set* if blocking by a coalition implies that there is another coalition blocking the assignment resulting from the initial blocking (Definition 1). This notion is formulated by Aumann and Maschler (1964) and later analysed for different economies. In the context of a market game with a continuum of players, the *bargaining set* is equivalent to the set of Walrasian allocations (Mas-Colell, 1989). For non-transferable utility games, the *bargaining set* is non-empty under certain conditions (Vohra, 1991).[1] For an exchange economy with differential information and a continuum of traders, the *bargaining set* and the set of Radner competitive equilibrium allocations are equivalent (Einy, Moreno, and Shitovitz, 2001). While the *bargaining set* takes into account only one step of counter-objection to a blocking coalition, the consideration of a chain of counter-objections implies a more refined axiom (Dutt, Ray, Sengupta, and Vohra, 1989).

The idea of *bargaining set* also inspires some works on allocation of discrete resources in school choice context in terms of relaxing *stability* notion, which is central to matching theory: If a student has an objection to an allocation because she claims an empty slot at a school, then there will be a counter-objection once she is assigned to that school since the priority of some other student will be violated at that school. Roughly speaking, an outcome is in the *bargaining set* if and only if for each objection to the outcome, there exists a counter-objection (Ehlers, Hafalir, Yenmez, and Yildirim, 2014).[2] Some other works refer to *bargaining set* in similar ways (see Ehlers (2010), Kesten (2010), Alcade and Romero-Medina (2015)).

The paper is organized as follows: Section 2 introduces the model and the graph theoretical frame-

[1] There are slight differences in the formulation of the bargaining set defined by Aumann and Maschler (1964) and Mas-Colell (1989). See Vohra (1991) for the differences between these two formulations and also other variants of the notion.

[2] Ehlers, Hafalir, Yenmez, and Yildirim (2014) refer to this property as constrained non-wastefulness in the school choice context.

work, on which the mechanisms and the proofs are built. Section 3 defines *core* and *bargaining set* notions. Section 4 defines the extensions of the top trading cycles rule (defined on the strict preferences) to the weak preferences domain. We introduce an alternative class of assignment rules in Section 5. We state and prove our main result in Section 6.

2 Model

2.1 Assignment Problem

A non-empty finite set of objects O has to be allocated to a non-empty finite set of agents N with $|N| = |O|$ in such a way that each agent receives exactly one object; monetary transfers between agents are not permitted.

An **assignment** is a bijection $\mu : N \to O$. An **endowment profile** is a bijection $\omega : N \to O$. Each agent i has a complete and transitive preference relation R_i on O; that is, we allow for indifferences. Let $b_i(O')$ be the set of agent i's best objects in $O' \subseteq O$. Let $\alpha_{i1} = b_i(O)$ and for each k, $\alpha_{ik} = b_i(O \setminus \underset{l=1,\ldots,k-1}{\cup} \alpha_{il})$ (note that for each k, α_{ik} is an indifference set). Let k_i be the number of agent i's indifference sets. Whenever convenient, agent i's preferences are represented as a sequence of her indifference sets in the associated rank order; that is, $R_i = \alpha_{i1},\ \alpha_{i2}, \ldots, \alpha_{ik_i}$. Let $R = (R_i)_{i\in N}$ be a preference profile. We fix O and N throughout the paper and denote an assignment problem by a pair (ω, R). An assignment μ is **individually rational** if for each i, $\mu(i)\ R_i\ \omega(i)$.

An **entitlement** is a pair: a set of agents $N' \subset N$ and a correspondence $\epsilon : N' \longmapsto O$ such that for each $i \in N'$, $\epsilon(i) \subseteq \alpha_{ik}$ for some k. An *entitlement* essentially maps each agent (in a given subset of agents) to a welfare level via an indifference class, rather than via a particular object. An entitlement (N', ϵ) is **feasible under** μ if μ is *individually rational* and for each $i \in N'$, $\mu(i) \in \epsilon(i)$. An entitlement (N', ϵ) is **feasible** if there exists an assignment μ such that it is *feasible under* μ.

2.2 Preliminaries on Graphs

Let $G = (V, E)$ be a directed graph, where V is the set of *vertices* and E is the set of *directed edges*, that is a family of ordered pairs from V. For each $U \subset V$, let $\delta^{in}(U)$ be the set of edges $(u, v) \in E$ such that $u \in V \setminus U$ and $v \in U$ (i.e. the set of edges **entering** U) and $\delta^{out}(U)$ be the set of edges $(u, v) \in E$ such that $u \in U$ and $v \in V \setminus U$ (i.e. the set of edges **leaving** U). If U

is a singleton, say $U = \{v\}$, then we use $\delta^{in}(v)$ (and $\delta^{out}(v)$) instead of $\delta^{in}(U)$ (and $\delta^{out}(U)$). A **subgraph** of G is any directed graph $G' = (V', E')$ with $\emptyset \neq V' \subseteq V$ and $E' \subseteq E$ and each edge in E' consisting of vertices in V'. For a set of vertices $T \subseteq V$, the **subgraph of** G **induced by** T is the subgraph (T, E') such that $E' = \{(u, v) \in E : u, v \in T\}$. A sequence of vertices $\{v_1, \dots\ v_m\}$ is a **path from** v_1 **to** v_m if (i) $m \geq 1$, (ii) $v_1, \dots, v_m$ are distinct (except for possibly $v_1 = v_m$), and (iii) for each $k = 1, \dots, m - 1$, $(v_k, v_{k+1}) \in E$. A **cycle** is a path $\{v_1, \dots\ v_m\}$ is a cycle if $m \geq 2$ and $v_1 = v_m$.

A set of vertices $T \subseteq V$ is **strongly connected** if the subgraph induced by T is such that for any $u, v \in T$, there is a path from u to v. A **minimal self-mapped set** is a set of vertices $S \subseteq V$ that satisfies two conditions: (i) $S = \bigcup_{v \in S} \delta^{out}(v)$[3] and (ii) $\nexists S'$ with $\emptyset \neq S' \subset S$ such that $S' = \bigcup_{v \in S'} \delta^{out}(v)$. The following follows from Proposition 2.2 by Quint and Wako (2004).

Remark 1 *Let $G = (V, E)$ be a directed graph. A set of vertices $S \subseteq V$ is non-empty and strongly connected such that $\delta^{out}(S) = \emptyset$ if and only if S is a minimal self-mapped set.*

Whenever convenient, we refer to this equivalence result and say that a set of vertices S is a *minimal self-mapped set* if (i) for any two vertices in S, there is a path from one to the other, and (ii) there is no path from any vertex $u \in S$ to any vertex $v \notin S$. The following follows directly from Remark 1 and the $\mathcal{MSMS}$ algorithm introduced by Quint and Wako (2004).

Remark 2 *Let $G = (V, E)$ be a directed graph. If for each $v \in V$, $\delta^{out}(v) \neq \emptyset$, then a minimal self-mapped set exists.*

Let $w : E \to \Re$ be a function. We denote $\sum_{e \in F \subseteq E} w(e)$ by $w(F)$. A function $f : E \to \Re$ is called a **circulation** if for each $v \in V$, $f(\delta^{in}(v)) = f(\delta^{out}(v))$. Let $d, c : E \to \Re$ with $d \leq c$. A circulation f **respects d and c** if for each edge e, $c(e) \geq f(e) \geq d(e)$. A minimal self-mapped set S is **covered** if there exists an integer-valued circulation f such that for each $v \in S$, $f(e) = 1$ for some edge e entering v.

[3] Note that $\bigcup_{v \in S} \delta^{out}(v)$ and $\delta^{out}(S)$ are different sets in general.

3 The bargaining set

Let S be a group of agents. When we say agents in S **allocate** (or **reallocate**) **their endowments**, we imply that they do it in a best possible way; that is, there does not exist another allocation (or reallocation) of these endowments such that no agent is worse off and at least one agent is better off than the original allocation (or reallocation). An assignment μ is **strictly blocked by** S if the agents in S can reallocate their endowments in a way that makes each of them better off than at μ; that is, there exists μ' such that $\mu'(S) = \omega(S)$ and for each $i \in S$, $\mu'(i)\ P_i\ \mu(i)$. An assignment μ is **blocked by** S if the agents in S can reallocate their endowments in a way that makes no agent worse off and at least one agent better off than at μ; that is, there exists μ' such that $\mu'(S) = \omega(S)$ and for each $i \in S$, $\mu'(i)\ R_i\ \mu(i)$, and for some $j \in S$, $\mu'(j)\ P_j\ \mu(j)$. An assignment μ is **weakly blocked by** S if the agents in S can reallocate their endowments in a way that makes no agent worse off; that is, there exists μ' such that $\mu'(S) = \omega(S)$ and for each $i \in S$, $\mu'(i)\ R_i\ \mu(i)$. The **weak core** is the set of assignments that are not strictly blocked by any coalition. The **core** is the set of assignments that are not blocked by any coalition.

An assignment μ can be considered as a set of cycles, where each agent in a cycle is assigned to the object she points to in that cycle. When an assignment μ is blocked by a coalition S, we assume the least about the formation of the coalition and that the resulting assignment η is the following: each agent in the coalition S is assigned to the endowment of another agent in S; each agent in a cycle, which has an empty intersection with S, is assigned to the same object which she is assigned under μ; and every other agent is assigned to her endowment. Thus, a blocking coalition's effect is only through the cycles it breaks down. When we say that coalition S **blocks** μ **via** η, we mean that η is the assignment described above.

If the preferences are strict, then the *core* is non-empty (it is a singleton set). On the other hand, if we allow indifferences, the *core* might be an empty set, but as a weaker notion, the *weak core* is always non-empty.

A different notion is **bargaining set**: any blocking by a coalition S is deterred by another coalition, say $C(S)$, including agents in S. The agents in $S \setminus C(S)$ (the ones, who are better off in case blocking by S occurs) cannot convince the agents in $S \cap C(S)$ to take part in this blocking coalition. Next, we define this notion. Let M denote the set of *individually rational* assignments. For a given assignment

μ and a set of agents C, let $M_C(\mu) = \{\mu' \in M : \text{for each } i \in C : \mu'(i)\ I_i\ \mu(i)\}$.

Definition 1 *An assignment μ is in the* **bargaining set** *if and only if*

(i) it is not strictly blocked by any coalition, and

(ii) if S blocks μ via η, then there exists $C(S)$ with $S \cap C(S) \neq \emptyset$, such that $C(S)$ blocks η via some $\mu'' \in M_{C(S)}(\mu)$.

Bargaining set is strong in the sense that only a specific deterrence prevents blocking: the agents in $C(S)$ can deter blocking only via an assignment indifferent for themselves to the current assignment. Note that if this restriction on deterrence is removed, then, since deterring blocking coalitions is easier in this case, the notion will be weaker. Thus, this restriction actually allows less amount of blocking and makes the notion move away further from *weak core* to *core*. For an assignment problem (ω, R), we denote the *bargaining set* by $B(\omega, R)$.

Bargaining set is clearly stronger than *weak core*. There is another motivation for a stronger notion than *weak core*, which is, as the next example demonstrates, "too" weak.

Example 1 *Let $N = \{i_1, i_2, i_3, i_4, i_5, i_6\}$ and $O = \{o_1, o_2, o_3, o_4, o_5, o_6\}$ where $\omega(i_k) = o_k$. The preferences are given below with each set in the table being an indifference set:*

R_{i_1}	R_{i_2}	R_{i_3}	R_{i_4}	R_{i_5}	R_{i_6}
$\{o_2\}$	$\{o_3, o_4\}$	$\{o_1\}$	$\{o_2, o_6\}$	$\{o_6\}$	$\{o_5\}$
$\{o_1\}$	$\{o_2\}$	$\{o_3\}$	$\{o_4\}$	$\{o_4\}$	$\{o_6\}$
				$\{o_5\}$	

The assignment $\mu = (o_2, o_3, o_1, o_6, o_4, o_5)$ is in the weak core *and the* core *is empty. One can argue that agents i_5 and i_6 are endowed with each other's unique best objects and thus, they should be able to exchange their objects. Also, note that the coalition $S = \{i_5, i_6\}$ weakly dominates μ and any assignment, at which agent i_5 and i_6 are assigned objects o_6 and o_5, respectively, is not weakly blocked by a coalition including agent i_5 or i_6.*

Example 1 suggests that we need a stronger notion to account for such cycles $S = \{i_5, i_6\}$ above. This simple requirement can be captured via the following property which guarantees such trades.

Definition 2 *An assignment μ satisfies* **top-trade property** *if for any cycle $S = \{a_1, a_2, ..., a_K\}$ where $\omega(a_k)$ is agent a_{k-1}'s single best object for $k = 2, ..., K$ and $\omega(a_1)$ is agent a_K's single best object, $\mu(a_{k-1}) = \omega(a_k)$ for $k = 2, ..., K$ and $\mu(a_K) = \omega(a_1)$.*

Proposition 1 *Let (ω, R) be an assignment problem and let $\mu \in B(\omega, R)$. If A is a minimal self-mapped set which is covered, then μ allocates the objects in A to the agents in A such that each agent receives one of her top objects in A.*

Proof. Suppose not. Then, $S = A$ blocks μ, say via η, under which each agent in A receives a top object. Since $\mu \in B(\omega, R)$, there is a $C(S)$ with $C(S) \cap S \neq \emptyset$ such that $C(S)$ blocks η via some $\mu' \in M_{C(S)}(\mu)$. Thus each agent in $C(S) \cap S$ receives a top object under both μ and η. Note that $C(S) \nsubseteq S$, since otherwise $C(S)$ cannot block η via some $\mu' \in M_{C(S)}(\mu)$. There is an agent in $C(S) \cap S$ who receives, under η, an object owned by some agent in $C(S) \setminus S$. But since S is a minimal self-map set , this object cannot be a top object for this agent, contradicting with the fact that each agent in $C(S) \cap S$ receives a top object under η. ■

A direct corollary to the above proposition is the following.

Corollary 1 *If an assignment is in the bargaining set, then it satisfies the top-trade property.*

Proof. Since any cycle $S = \{a_1, a_2, ..., a_K\}$, where $\omega(a_k)$ is agent a_{k-1}'s single best object for $k = 2, ..., K$ and $\omega(a_1)$ is agent a_K's single best object, is a minimal self-mapped set that is covered, by Proposition 1, μ assigns each agent in S a top object from the endowment set of the agents in S. Since each agent has a single top object, the top-trade property is satisfied. ■

4 The class of the Top Trading Cycles (TTC) assignment rules

The TTC class is a set of assignment rules as an extension of the well-known TTC mechanism defined on the strict domain. Each rule in this class takes an assignment problem; that is, a preference profile and an endowment profile, (ω, R), as input and produces an allocation, $TTC(\omega, R)$, as an output.

Let F be a selection rule: for each minimal self-mapped set that is not covered, F selects one of the cycles in the minimal self-mapped sets. The TTC updates the endowment profile by assigning each agent in the cycle to the object that she points to in the same cycle. We call this endowment update as **top-trading the cycle**. Let $\omega_1 = \omega$ and for $k \geq 1$, the steps below are repeated until all agents and objects are removed.

Step k.1: Let each agent point to her maximal objects among the remaining objects and each remaining object points to its owner according to the endowment profile ω_k. Select a minimal self-mapped set T in this digraph.

Step k.2: (i) If T is covered, then each agent in T is removed by assigning her one of the best objects in T. (ii) Otherwise, select one of the cycles in the minimal self-mapped set using the selection rule F, and update the endowment profile by top-trading the cycle to obtain ω_{k+1}.

Lemma 1 *If an agent i's endowment is updated at some step before T_k is removed, then agent i gets a top object among the remaining objects after $\bigcup_{j=0}^{k-1} T_j$ and $\mu(\bigcup_{j=0}^{k-1} T_j)$ are removed.*

Proof. Once an agent's endowment is updated before T_k is removed the agent is tentatively endowed with a top object among the remaining objects after $\bigcup_{j=0}^{k-1} T_j$ and $\mu(\bigcup_{j=0}^{k-1} T_j)$ are removed. Since, at any step before the agent is assigned an object and removed, the agent always points to her top objects among the remaining objects, she is never degraded to an object that is not a top object among the remaining ones. Thus, she ends up with a top object among the remaining ones. ■

Lemma 2 *If an agent i's endowment is updated at some step before T_k is removed, then the original owner of agent i's new endowment receives a top object among the remaining objects after $\bigcup_{j=0}^{k-1} T_j$ and $\mu(\bigcup_{j=0}^{k-1} T_j)$ are removed.*

Proof. First note that any agent who originally owns a top object receives a top object, otherwise the mechanism would not be individually rational. Now, suppose at some step, say before T_k is removed, that agent i does not originally own a top object among the remaining objects after $\bigcup_{j=0}^{k-1} T_j$ is removed. And suppose her endowment is updated right before T_k is removed, such that she now owns (tentatively) a top object among the remaining objects after $\bigcup_{j=0}^{k-1} T_j$ and $\mu(\bigcup_{j=0}^{k-1} T_j)$ are removed. Then, in the endowment update that is carried out in this step, agent i points to an object, call it o_k, which is one of her top objects among the remaining objects and according to the prior it is the highest ranked among the remaining objects. Note that agent i does not own a top object among the remaining objects until this step, in the chain partition $i \in N_0$. So among her top objects she points to the one according to the priority ordering. Now there are two cases, either the current owner of o_k, call her j, is also the original owner of o_k, or not. If it is the first case, then j gets a top object by Lemma 1. But if it is the second case, then it means that the object o_k was in an earlier endowment update hence the original owner of o_k, say agent k, received (tentatively) a top object at some earlier step. Again, by Lemma 1, agent k who is the original owner of agent i's new endowment gets a top object. ■

5 A new class of assignment rules: Deferred Top Trading Cycles (DTTC)

We propose a new class of assignment rules, the Deferred Top Trading Cycles (DTTC). This is motivated by a very simple idea: At each step, a cycle is chosen to concede each agent in the cycle the object she points to. But different from the TTC, the endowment is not updated, instead, the agent is guaranteed the welfare level which corresponds to the object she points to. Thus, the Let (N', ϵ) be a feasible entitlement. Since agents in N' are entitled certain welfare levels, it could be that only some objects are available for a given agent. Let $O(i)$ be the set of *available* objects for agent i (the interpretation is that agent i can receive an object only from this set).[4] Let the triple $(N', \epsilon, (O(i))_{i\in N})$ denote a **partial assignment problem** where each $i \in N'$ is entitled one of the objects in $\epsilon(i)$ and only the objects in $O(i)$ are available for agent $i \in N$.

Given a *partial assignment problem* $(N', \epsilon, (O(i))_{i\in N})$, the graph $G(N', \epsilon, (O(i))_{i\in N})$ is the associated **directed graph** where each object points to its owner and each agent $i \in N$ points to the objects in $b_i(O(i))$. The partial assignment problem, when there is no entitlement, is denoted by $(\emptyset, \epsilon_0, O_0)$ where $O_0 = (O_0(i))_{i\in N}$ with $O_0(i) = O$ for each $i \in N$. The associated directed graph $G(\emptyset, \epsilon_0, O_0)$ is such that each object points to its owner and each agent points to her best objects.

We denote a **cycle** in the graph $G(N', \epsilon, (O(i))_{i\in N})$ by the set of agents included in the cycle unless otherwise noted and no confusion arises. An **improvement cycle** is a cycle including an agent in $N \setminus N'$. A **feasible improvement cycle** is an *improvement cycle* S such that there exists an *individually rational* assignment μ with the following properties: (1) the entitlement (N', ϵ) is *feasible under* μ, (2) each agent $i \in S \setminus N'$, $\mu(i) \in b_i(O(i))$.

The DTTC rule is based on creating new entitlements at each step while respecting existing ones and also *individual rationality*. The DTTC is similar to the Top Trading Cycles (TTC) algorithm since

[4] In the definition of the DTTC below, we specify how $(O(i))_{i\in N}$ is sequentially constructed.

a cycle is selected at each step. But there is an important difference: agents in the cycle do not trade their (current) endowments (as it is in the TTC algorithm), rather each agent in the cycle is reserved an object from the indifference class including the object that she points to and trading is deferred to the end. An implication is that the endowment profile does not change throughout the algorithm as opposed to the TTC, where endowments are updated by assigning each agent the object that she points to in the selected cycle at each step.

Deferred Top Trading Cycles Algorithm: Let $N_0 = \emptyset$. For each $k \geq 1$, let $G_{k-1} = G(N_{k-1}, \epsilon_{k-1}, O_{k-1})$.

(k.1) In the graph G_{k-1}, select a *feasible improvement cycle* N^k. Let $S^k = N^k \setminus N_{k-1}$. For each $N' \subseteq N_k = N_{k-1} \cup S^k$ such that $\left| \underset{i \in N'}{\cup} b_i(O_{k-1}(i)) \right| = |N'|$, the objects in $\underset{i \in N'}{\cup} b_i(O_{k-1}(i))$ are *unavailable* for the agents in $N \setminus N'$. For each $i \in N$, the set of available objects $O_k(i)$ is the set $O_{k-1}(i)$ minus *unavailable* objects for her. Let $O_k = (O_k(i))_{i \in N}$. The entitlement ϵ_k on N_k is defined as follows: for each $i \in N_{k-1}$, $\epsilon_k(i) = \epsilon_{k-1}(i)$ and for each $i \in S^k$, $\epsilon_k(i) = \alpha_{il}$ with $\alpha_{il} \supseteq b_i(O_{k-1}(i))$.

(k.2) If $N_k = N$, the algorithm terminates and it gives an assignment μ such that for each i, $\mu(i) \in \epsilon_k(i)$.

The DTTC is a class and each selection of cycles gives an assignment (or an *essentially single-valued* set of assignments: each agent is indifferent between any two assignments in this set). For each problem (ω, R), the set $DTTC(\omega, R)$ denotes the set of outcomes of the DTTC; that is, the assignments obtained by the DTTC via all possible selections of feasible improvement cycles in Step k.1.

6 A characterization of the bargaining set

Our main result is that the *bargaining set* is characterized by the TTC and also by the DTTC. We present this result, the proof of which relies on the graph theoretical framework introduced in Section 2.2 and **Hoffman's Circulation Theorem**, an important result from graph theory.

Theorem 1 *An assignment is in the bargaining set if and only if it is an outcome of the TTC.*

PROOF of THEOREM 1: We prove the theorem in parts: (I) the DTTC is well-defined, (II) each assignment in the *bargaining set* is an outcome of the DTTC, (III) each outcome of the TTC is in the *bargaining set*, and (IV) for each outcome μ of the DTTC, there exists a selection rule F such that μ is obtained by the TTC via F.

I. The DTTC is well-defined. We show that a feasible improvement cycle at Step k.1 exists and thus, the DTTC algorithm is well-defined.

Base case: At Step 1, there exists at least one improvement cycle and it is feasible to assign each agent in this cycle the object she points to. Thus, there is a feasible improvement cycle, which implies that the statement holds for Step 1.

Inductive step: We assume that the DTTC is well defined through steps 1 to $k-1$ and thus, there exists an assignment such that each agent $i \in N_{k-1}$ receives an object from the set $\epsilon_{k-1}(i)$. We use a graph theoretical notation to prove the inductive hypothesis that there exists a feasible improvement cycle in the graph G_{k-1}; that is, Step k.1 of the DTTC is well-defined.

In the DTTC algorithm, a cycle is chosen at each step, and each agent in this cycle is entitled one of the objects that she points to at that step. This corresponds to a set of new constraints along the existing ones such that each agent $i \in N_{k-1}$ is already entitled one of the objects from the set $\epsilon_{k-1}(i)$. These constraints possibly imply that certain objects are no longer available for other agents. We argue that given these existing constraints at each step, the existence of a feasible improvement cycle is implied by the existence of a *circulation respecting* properly defined bounds c, d. For this argument, we reconstruct the graph and bounds d and c at each step to be consistent with the existing constraints. The reason we refer to circulations is the difficulty due to the fact that not every improvement cycle is feasible, as shown in the following example.

Example 2 *In Figure 1 below, cycle C_1 is chosen and entitlement is such that agent a_i is entitled to $\{h_x, h_y\}$ and agent a_j is entitled to $\{h_u, h_z\}$. Note that non of the objects become unavailable but cycle C_2 is not feasible. But there is a feasible cycle for sure, which is C_3 in this case.*

For the graph G_{k-1}, define the functions d_{k-1}, c_{k-1} as follows: For each edge e in the graph G_{k-1}, we set $c_{k-1}(e) = 1$. Since the upper-bound function c is constant at each step, we suppress its subscript, and set $c = 1$ for each k. If an edge e is from an object to an agent in N_{k-1}, then we set $d_{k-1}(e) = 1$; otherwise, we set $d_{k-1}(e) = 0$. Since we suppress the upper-bound c, for convenience we say a circulation f in G_{k-1} *respects* d_{k-1} if for each edge e, $1 \geq f(e) \geq d_{k-1}(e)$. Also, if a circulation f is such that for an agent i and an object o, $f((i,o)) = 1$, we say that **object o is**

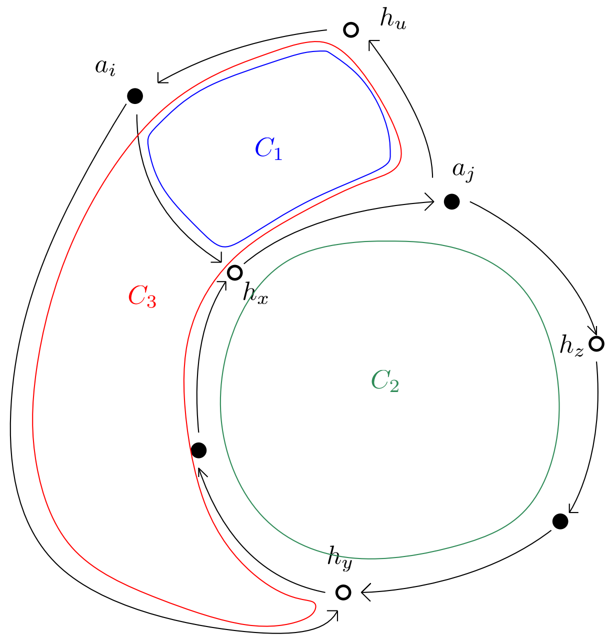

Figure 1: Not every improvement cycle is feasible.

assigned to agent i **under** f. As demonstrated next, the existence of a circulation respecting d_{k-1} implies the feasibility of the entitlement $(N_{k-1}, \epsilon_{k-1})$.

Lemma 3 *If there exists a circulation in* G_{k-1} *respecting* d_{k-1}*, then the entitlement* $(N_{k-1}, \epsilon_{k-1})$ *is feasible.*

Proof. Let f be a circulation in G_{k-1} respecting d_{k-1}. Since for each e from an object to an agent in N_{k-1}, $d_{k-1}(e) = 1$, and f respects d_{k-1}, for each agent $i \in N_{k-1}$, there is an edge with the $f-$value equivalent to one to an object in $b_i(O_{k-1}(i)) \subseteq \epsilon_i(k-1)$. Let μ be an assignment defined as follows: for each $i \in N$ such that $f(i, o) = 1$ for some $o \in O$, let $\mu(i) = o$; for any other agent j, let $\mu(j) = \omega(j)$. Since each other agent points to her best available objects in the graph G_{k-1}, this is an *individually rational* assignment and thus, the entitlement $(N_{k-1}, \epsilon_{k-1})$ is *feasible.* ■

Lemma 3 enables us to slightly modify the induction argument in the following way: The existence of a feasible improvement cycle in the base case is equivalent the existence of a circulation respecting d_1 such that, for each e from the object in the feasible improvement cycle to its owner, $d_1(e) = 1$ and for each other edge, it is equal to zero. Thus, inductive hypothesis is that there exists a circulation in G_{k-1} respecting d_{k-1}. Let d_k be a lower-bound function such that:

$$\text{for each } i \in N, d_k((\omega(i), i)) \geq d_{k-1}((\omega(i), i)), \tag{1}$$

$$\text{for some } j \in N \setminus N_{k-1}, d_k((\omega(j), j)) = 1 > d_{k-1}((\omega(j), j)) = 0. \tag{2}$$

Let S be the set of the agents satisfying property (2). Lemma 3 implies that, if there exists a circulation respecting d_k in the graph G_{k-1}, then there exists an assignment, under which the entitlement for the agents in $(N_{k-1}, \epsilon_{k-1})$ is *feasible* and each agent $i \in S$ is assigned one of her best *available* object, that is an object from the set $b_i(O_{k-1}(i))$, which is equivalent to the existence of a *feasible improvement cycle*. Thus, showing the existence of a circulation respecting a lower-bound function d_k with properties (1) and (2) as defined above (as the inductive step) is sufficient for the DTTC algorithm being well-defined. The existence of a circulation guarantees that *individual rationality* is maintained throughout the algorithm since by definition, a circulation implies that whenever an object becomes *unavailable*, its owner is assigned to an object under that circulation. Thus, we have the following remark:

Remark 3 *If an object o becomes unavailable for some agents at Step k, then $\omega^{-1}(o) \in S^{k'}$ for some $k' \leq k$.*

Our proof of the existence of a circulation respecting a lower-bound function d_k with properties (1) and (2) relies mostly on the following theorem which characterizes the conditions under which there exists a circulation respecting the bounds on the set of edges E.

Hoffman's Circulation Theorem *Let $G = (V, E)$ be a directed graph and let $d, c : E \to \Re$ with $d \leq c$. Then, there exists a circulation f satisfying $d \leq f \leq c$ if and only if for each $U \subseteq V$, $d(\delta^{in}(U)) \leq c(\delta^{out}(U))$. If moreover d and c are integers, f can be taken integer-valued.*

By our inductive hypothesis, Hoffman's Circulation Theorem implies that for each $U \subseteq N \cup O$, $d_{k-1}(\delta^{in}(U)) \leq \left|\delta^{out}(U)\right|$.[5] At Step k, there exists at least one agent whom is not entitled any object (note that otherwise, the algorithm terminates). This implies that in the graph G_{k-1} with the lower-bound function d_{k-1}, there exists at least one edge from an object to its owner with lower-bound

[5] Note that since the upper-bound (or capacity) function c is constant at one, we can rewrite the right-hand side of the condition in Hoffman's Circulation Theorem as the number of leaving edges.

zero.[6] If in the graph G_{k-1}, for each $U \subset N \cup O$, $d_{k-1}(\delta^{in}(U)) < |\delta^{out}(U)|$, then Hoffman's Circulation Theorem implies that by increasing the lower-bound of an edge e from an object to an agent who is not entitled any object,[7] a circulation exists respecting this new lower-bound and the inductive step follows trivially. Thus, we assume that there exists at least one set of vertices $U \subset N \cup O$, such that $d_{k-1}(\delta^{in}(U)) = |\delta^{out}(U)|$. Let A_X be the set of agents in the set X, and O_X the set of objects in the set X.

Lemma 4 *We can assume without loss of generality that* $O_U = \underset{i \in A_U}{\cup} b_i(O_{k-1}(i))$.

Proof. Let $i \in A_U$ and $o \in b_i(O_{k-1}(i))$. Suppose $o \notin O_U$. Thus, the edge (i, o) is in $\delta^{out}(U)$. Note that the inequality in Hoffman's Circulation Theorem is satisfied for any set of vertices, in particular for $U \cup \{o\}$. Moreover, $d_{k-1}(\delta^{in}(U \cup \{o\})) \geq d_{k-1}(\delta^{in}(U)) - 1$ and this holds with equality only if $\omega^{-1}(o) \in A_U$ and $d_{k-1}((o, \omega^{-1}(o))) = 1$ (that is, agent $\omega^{-1}(o)$ is in N_{k-1}). Moreover, $|\delta^{out}(U \cup \{o\})| \leq |\delta^{out}(U)| - 1$, and this holds with equality only if agent i is the only agent in A_U pointing to o. Thus, by these two inequalities, (1) the only agent in the set A_U pointing to object o is agent i, and (2) agent $\omega^{-1}(o)$ is in $A_U \cap N_{k-1}$. But then, $d_{k-1}(\delta^{in}(U \cup \{o\})) = |\delta^{out}(U \cup \{o\})|$. Thus, we can assume without loss of generality that $O_U \supseteq \underset{i \in A_U}{\cup} b_i(O_{k-1}(i))$.

Suppose there exists $o \in O_U \setminus \underset{i \in A_U}{\cup} b_i(O_{k-1}(i))$. Suppose also $\omega^{-1}(o) \notin A_U$. Thus, there is an edge from the set U to the set $N \setminus U$. Since there is no vertex in U pointing to o, we have $|\delta^{out}(U \setminus \{o\})| = |\delta^{out}(U)| - 1$. Since the sets U and $U \setminus \{o\}$ have the same sets of agents (A_U) and also the same sets of objects pointing to these agents, $d_{k-1}(\delta^{in}(U \setminus \{o\})) = d_{k-1}(\delta^{in}(U))$. This implies that $d_{k-1}(\delta^{in}(U \setminus \{o\})) > |\delta^{out}(U \setminus \{o\})|$, which contradicts Hoffman's Circulation Theorem. Thus, $\omega^{-1}(o) \in A_U$. Suppose $d_{k-1}((o, \omega^{-1}(o))) = 1$. Since there is no edge from an agent in A_U to o, and also no edge from object o to an agent in $A \setminus A_U$, the sets U and $U \setminus \{o\}$ have the same set of leaving edges; that is $|\delta^{out}(U \setminus \{o\})| = |\delta^{out}(U)|$. But since object o points to an agent in A_U and this edge has a lower-bound one, $d_{k-1}(\delta^{in}(U \setminus \{o\})) = d_{k-1}(\delta^{in}(U)) + 1$. This implies that $d_{k-1}(\delta^{in}(U \setminus \{o\})) > |\delta^{out}(U \setminus \{o\})|$. This contradicts with Hoffman's Circulation Theorem. Thus, $d_{k-1}((o, \omega^{-1}(o))) = 0$. As in the previous case, the sets U and $U \setminus \{o\}$ have the same set of leaving edges; that is $|\delta^{out}(U \setminus \{o\})| = |\delta^{out}(U)|$. Moreover, since the edge from object o to its owner $\omega^{-1}(o)$ has a lower-bound zero,

[6] Note that by construction of the bounds, only edges from objects to their owners can have a lower-bound of one. All edges from agents to objects have lower-bound zero. This is sufficient for the underlying entitlement to be feasible (see Lemma 3).

[7] That is, define d_k as follows: let $d_k(e) = 1$ (note that $d_{k-1}(e) = 0$) and for all other edges, d_k and d_{k-1} coincide.

$d_{k-1}(\delta^{in}(U \setminus \{o\})) = d_{k-1}(\delta^{in}(U))$. Thus, $d_{k-1}(\delta^{in}(U \setminus \{o\})) = \left|\delta^{out}(U \setminus \{o\})\right|$, and we can assume without loss of generality that $O_U \subseteq \underset{i \in A_U}{\cup} b_i(O_{k-1}(i))$. ■

Let A_X^1 be the set of agents in A_X who are entitled an object (vertices with an entering edge of lower-bound one) and let $A_X^0 = A_X \setminus A_X^1$. For $n = 0, 1$, let $A_X^{n,Y}$ be the set of agents in A_X^n, the endowment of each of whom is in O_Y. The following lemma demonstrates that each edge from an object in $N \setminus U$ into U has a lower-bound one.

We consider a set of vertices $U \subset N \cup O$, such that $d_{k-1}(\delta^{in}(U)) = \left|\delta^{out}(U)\right|$ and assume by Lemma 4 that $O_U = \underset{i \in A_U}{\cup} b_i(O_{k-1}(i))$. We need to show that there exists a circulation f respecting d_{k-1} with $f(e) = 1$ for some $e = (o, \omega^{-1}(o))$ where $\omega^{-1}(o) \in N \setminus N_{k-1}$. This completes the proof (that the DTTC algorithm is well-defined) since by definition, this implies the existence of a cycle including o and $\omega^{-1}(o)$, which is a feasible improvement cycle. So, let f be a circulation such that for no edge including an agent in $N \setminus N_{k-1}$, $f(e) = 1$.

Lemma 5 *The set* $A_{N \setminus U}^{0,U}$ *is empty.*

Proof. Let $i \in A_U^0$ such that $\omega(i) \in N \setminus U$. By definition, the set O_U consists of objects endowed by the agents $A_U^{1,U}$ and $A_U^{0,U}$, and also the agents $A_{N \setminus U}^{1,U}$ and $A_{N \setminus U}^{0,U}$. Thus,

$$|O_U| = \left|A_U^{1,U}\right| + \left|A_U^{0,U}\right| + \left|A_{N \setminus U}^{1,U}\right| + \left|A_{N \setminus U}^{0,U}\right|. \tag{3}$$

Also, the set A_U consists of $A_U^{1,U}$, $A_U^{0,U}$, $A_U^{1,N \setminus U}$ and $A_U^{0,N \setminus U}$.

Since U is such that $d_{k-1}(\delta^{in}(U)) = \left|\delta^{out}(U)\right|$ and by Lemma 4, no agent in U points to an object in $N \setminus U$, we have that

$$\left|A_U^{1,N \setminus U}\right| = \left|\omega(A_{N \setminus U}^{1,U})\right| + \left|\omega(A_{N \setminus U}^{0,U})\right|, \tag{4}$$

note that the right-hand side is the number of edges with lower-bound one entering U, and the left-hand side is the number of all edges leaving U.

Since f is a circulation such that for no edge e including an agent in $N \setminus N_{k-1}$, $f(e) = 1$, only the edges entering and leaving $A_U^{1,U}$, $A_U^{1,N \setminus U}$, $\omega(A_U^{1,U})$ and $\omega(A_{N \setminus U}^{1,U})$ have an $f-$value equivalent to one (that is, agents in $A_U^{1,U}$, $A_U^{1,N \setminus U}$ are assigned to objects in $\omega(A_U^{1,U})$ and $\omega(A_{N \setminus U}^{1,U})$ under f). This implies that

$$\left|A_U^{1,U}\right| + \left|A_U^{1,N \setminus U}\right| = \left|\omega(A_U^{1,U})\right| + \left|\omega(A_{N \setminus U}^{1,U})\right|. \tag{5}$$

Since $\left|A_U^{1,U}\right| = \left|\omega(A_U^{1,U})\right|$, this implies that

$$\left|A_U^{1,N\setminus U}\right| = \left|\omega(A_{N\setminus U}^{1,U})\right|. \tag{6}$$

Thus, equalities (4) and (6) together imply that $\left|\omega(A_{N\setminus U}^{0,U})\right| = 0$ (that is, $A_{N\setminus U}^{0,U} = \emptyset$). ■

Lemma 6 *There exists a lower-bound function d_k, which satisfies properties (1) and (2), and a circulation respecting d_k. Thus, there exists a feasible improvement cycle.*

Proof. Let f be a circulation respecting d_{k-1} such that for no edge e including an agent in $N \setminus N_{k-1}$, $f(e) = 1$.

Case 1: The set $A_U^{0,U}$ is non-empty.

Suppose that there exists $A' \subseteq A_U^{0,U}$ such that for each $i \in A'$, $\omega(A') \cap b_i(O_{k-1}(i)) \neq \emptyset$. Since there is an edge from each vertex in $A' \cup \omega(A')$ to a vertex in the same set, there is a cycle C including only the vertices in $A' \cup \omega(A')$. Let d_k be such that for each object o in C, $d_k((o, \omega^{-1}(o))) = 1$, and for each other object o', $d_k((o', \omega^{-1}(o')) = d_{k-1}((o', \omega^{-1}(o')))$. Clearly, the lower-bound function d_k satisfies property (1). Moreover, since each object o in C is such that $\omega^{-1}(o) \in A' \subseteq A_U^{0,U}$, by definition $d_{k-1}((o, \omega^{-1}(o))) = 0$. Thus, d_k satisfies property (2) as well. Moreover, since by assumption, circulation f is such that for no edge e including an agent in $N \setminus N_{k-1}$, $f(e) = 1$, the following function f' is a circulation as well and respects d_k: for each edge e in the cycle, $f'(e) = 1$, and for each other edge e', $f'(e') = f(e')$. Thus, there exists a feasible improvement cycle.

Suppose that for each $A' \subseteq A_U^{0,U}$, there exists $i \in A'$ such that $\omega(A') \cap b_i(O_{k-1}(i)) = \emptyset$ (†). Suppose for each $i \in A_U^1$, $b_i(O_{k-1}(i)) \cap \omega(A_U^{0,U}) = \emptyset$. By Lemma 5, this implies that $\underset{i \in A_U^1}{\cup} b_i(O_{k-1}(i)) = \omega(A_U^{1,U}) \cup \omega(A_{N\setminus U}^{1,U})$. But then,

$$\left|\underset{i \in A_U^1}{\cup} b_i(O_{k-1}(i))\right| = \left|\omega(A_U^{1,U})\right| + \left|\omega(A_{N\setminus U}^{1,U})\right| = \left|A_U^{1,U}\right| + \left|A_U^{1,N\setminus U}\right| = \left|A_U^1\right|, \tag{7}$$

where the second equality follows from equality (5) in Lemma 5, and the third equality follows from the definition of A_U^1. By definition of the DTTC, this implies that the objects in $\omega(A_U^{1,U}) \cup \omega(A_{N\setminus U}^{1,U})$ are *unavailable* for the agents in $N \setminus A_U^1$, in particular for the agents in $A_U^{0,U}$. Thus, for each $i \in A_U^{0,U}$, we have that $O_{k-1}(i) \cap O_U = \emptyset$, which contradicts the assumption that $O_U = \underset{i \in A_U}{\cup} b_i(O_{k-1}(i))$ by Lemma 4. Thus, there exists $j \in A_U^1$ such that $b_j(O_{k-1}(j)) \cap \omega(A_U^{0,U}) \neq \emptyset$.

Let $i_1 \in A_U^{0,U}$ such that $\omega(i_1) \in b_j(O_{k-1}(j))$. By (†), $\omega(i_1) \notin b_{i_1}(O_{k-1}(i_1))$. Let $i_2 \in A_U^{0,U}$ such that $\omega(i_2) \in b_{i_1}(O_{k-1}(i_1))$. Let $A' = \{i_1, i_2\}$ in (†). Since $\omega(i_2) \in b_{i_1}(O_{k-1}(i_1))$, by (†), $\{\omega(i_1), \omega(i_2)\} \cap b_{i_2}(O_{k-1}(i_2)) = \emptyset$. Since the set $A_U^{0,U}$ is finite, by applying (†) repeatedly in this way, we obtain a sequence $i_1, \ldots, i_m$ in $A_U^{0,U}$ such that for $l = 1, \ldots, m-1$, $\omega(i_{l+1}) \in b_{i_l}(O_{k-1}(i_l))$, and for some $o \in \omega(A_U^{1,U}) \cup \omega(A_{N\setminus U}^{1,U})$, $o \in b_{i_m}(O_{k-1}(i_m))$ $(\star)$.

Let $S = \{i_1, \ldots, i_m\}$. Let d_k be such that for each object $o \in \omega(S)$, $d_k((o, \omega^{-1}(o))) = 1$, and for each other object o', $d_k((o', \omega^{-1}(o')) = d_{k-1}((o', \omega^{-1}(o')))$ $(\otimes)$. Clearly, the lower-bound function d_k satisfies property (1). Moreover, since each object $o \in \omega(S)$ is such that $\omega^{-1}(o) \in S \subseteq A_U^{0,U}$, by definition $d_{k-1}((o, \omega^{-1}(o))) = 0$. Thus, d_k satisfies property (2) as well.

We claim that there is bijection $\eta : S \cup A_U^1 \to \omega(S) \cup \omega(A_U^{1,U}) \cup \omega(A_{N\setminus U}^{1,U})$ such that for each $i \in S \cup A_U^1$, $\eta(i) \in b_i(O_{k-1}(i))$.[8] To prove the existence, we refer to the well-known **Hall's Marriage Theorem**.

Hall's Marriage Theorem *Let* $\mathcal{A} = \{A_1, \ldots, A_n\}$ *be a family of subsets of some finite set* X*. There exists a bijection* $\pi : \{1, \ldots, n\} \to Y \subseteq X$ *such that for each* $i \in \{1, \ldots, n\}$, $\pi(i) \in A_i$ *if and only if for each subset* I *of* $\{1, \ldots, n\}$

$$\left| \bigcup_{i \in I} A_i \right| \geq |I| \,. \tag{8}$$

In our context, Hall's Marriage Theorem gives the necessary and sufficient conditions for the existence of such a bijection: there exists a bijection $\eta : S \cup A_U^1 \to \omega(S) \cup \omega(A_U^{1,U}) \cup \omega(A_{N\setminus U}^{1,U})$ such that for each $i \in S \cup A_U^1$, $\eta(i) \in b_i(O_{k-1}(i))$ if and only if for each subset A' of $S \cup A_U^1$,

$$\left| \bigcup_{i \in A'} b_i(O_{k-1}(i)) \right| \geq |A'| \,. \tag{9}$$

Suppose the claim does not hold. Then, by the condition (9), there exists a set $A' \subseteq S \cup A_U^1$ such that $\left| \bigcup_{i \in A'} b_i(O_{k-1}(i)) \right| < |A'|$. Since each agent in S has a different best available object, the set A' cannot be a subset of S. Also, f is a circulation respecting d_{k-1} such that for no edge e including an

[8] Since η is such a bijection and f is a circulation, the following function f' is also a circulation respecting d_k: for each edge e in U, $f'(e) = 1$ if and only if e is from an object in $\omega(S) \cup \omega(A_U^{1,U})$ to its owner, or from an agent $i \in S \cup A_U^1$ to the object $\eta(i)$, and for any other edge e', $f'(e') = f(e')$.

agent in $N \setminus N_{k-1}$, $f(e)=1$ and agents in A^1_U are assigned to objects in $\omega(A^{1,U}_U) \cup \omega(A^{1,U}_{N\setminus U})$ under f. Thus, the set A' cannot be a subset of A^1_U neither. Suppose the set A' does not include i_m. Each agent in $i_l \in A' \cap S$ can be assigned to object $\omega(i_{l+1})$ and each agent $i \in A' \cap A^1_U$ can be assigned to the object, which she is assigned under f. Hall's Marriage Theorem implies that condition (9) holds for the set A' and each subset of it, contradicting with the set A' violating (9). Thus, $i_m \in A'$.

Let A' be such that $i_m \in A'$ and $A' \cap A^1_U \neq \emptyset$. Then, since $\left| \underset{i \in A'}{\cup} b_i(O_{k-1}(i)) \right| < |A'|$, we have

$$\left| \underset{i \in A' \cap S}{\cup} b_i(O_{k-1}(i)) \right| + \left| \underset{i \in A' \cap A^1_U}{\cup} b_i(O_{k-1}(i)) \right| < \left| A' \cap S \right| + \left| A' \cap A^1_U \right|. \tag{10}$$

Since $\left| \underset{i \in A' \cap S}{\cup} b_i(O_{k-1}(i)) \right| \geq |A' \cap S|$ and $\left| \underset{i \in A' \cap A^1_U}{\cup} b_i(O_{k-1}(i)) \right| \geq \left| A' \cap A^1_U \right|$, the intersection of the two sets $\underset{i \in A' \cap S}{\cup} b_i(O_{k-1}(i))$ and $\underset{i \in A' \cap A^1_U}{\cup} b_i(O_{k-1}(i))$ cannot be empty. (Note that some of the objects in $\underset{i \in A' \cap A^1_U}{\cup} b_i(O_{k-1}(i))$ are possibly in the set $\omega(S)$, but since under f each agent in A^1_U can be assigned to an object in $\omega(A^{1,U}_U) \cup \omega(A^{1,U}_{N\setminus U})$, actually,

$$\left| \underset{i \in A' \cap A^1_U}{\cup} \left(b_i(O_{k-1}(i)) \cap \left(\omega(A^{1,U}_U) \cup \omega(A^{1,U}_{N\setminus U}) \right) \right) \right| \geq \left| A' \cap A^1_U \right|. \tag{11}$$

Thus, it must be that the set $\underset{i \in A' \cap S}{\cup} b_i(O_{k-1}(i))$ has a non-empty intersection with $\underset{i \in A' \cap A^1_U}{\cup} b_i(O_{k-1}(i))$ in $\omega(A^{1,U}_U) \cup \omega(A^{1,U}_{N\setminus U})$. Note that, by ($\star$), the only such object (which is possibly in this intersection) is object o ($\star\star$). First suppose that $\omega(A^0_U) \cap \left(\underset{i \in A' \cap A^1_U}{\cup} b_i(O_{k-1}(i)) \right) = \emptyset$. But then, condition (10) implies that $\left| \underset{i \in A' \cap A^1_U}{\cup} b_i(O_{k-1}(i)) \right| = \left| A' \cap A^1_U \right|$. But then, by definition of the DTTC, object o is not in the set $O_{k-1}(i_m)$) (that is, unavailable for agent i_m), which is a contradiction with ($\star$). Thus, $\omega(A^0_U) \cap \left(\underset{i \in A' \cap A^1_U}{\cup} b_i(O_{k-1}(i)) \right) \neq \emptyset$. Suppose there exists an object in the set $\omega(A^0_U) \setminus \omega(S)$. But then, condition (11) implies that $\left| \underset{i \in A' \cap A^1_U}{\cup} b_i(O_{k-1}(i)) \right| > \left| A' \cap A^1_U \right|$. But, since we have ($\star\star$) and also $\left| \underset{i \in A' \cap S}{\cup} b_i(O_{k-1}(i)) \right| \geq |A' \cap S|$, condition (10) cannot hold. Thus, the object in $\omega(A^0_U) \cap \left(\underset{i \in A' \cap A^1_U}{\cup} b_i(O_{k-1}(i)) \right)$ is in the set $\omega(S)$. We can assume without loss of generality that this object is $\omega(i_1)$.[9] But then, since $\omega(i_1) \notin \underset{i \in A' \cap S}{\cup} b_i(O_{k-1}(i))$, condition (10) cannot

[9]This is because otherwise we can redefine the set S such that the first agent in the sequence constructed ($\star$), is the owner of the object in $\omega(A^0_U) \cap \left(\underset{i \in A' \cap A^1_U}{\cup} b_i(O_{k-1}(i)) \right)$.

hold. Thus, we conclude a set A' satisfying condition (10) does not exist. Thus, by Hall's Marriage Theorem, there exists a bijection $\eta : S \cup A^1_U \to \omega(S) \cup \omega(A^{1,U}_U) \cup \omega(A^{1,U}_{N\setminus U})$ such that for each $i \in S \cup A^1_U$, $\eta(i) \in b_i(O_{k-1}(i))$. But then, by footnote 6, there exists a circulation f' which respects d_k, thus there exists a feasible improvement cycle.

Case 2: The set $A^{0,U}_U$ is empty.

We claim that $U = A^1_U \cup \omega(A^{1,U}_U) \cup \omega(A^{1,U}_{N\setminus U})$. Suppose this does not hold. Then, the set $A^{0,N\setminus U}_U$ is non-empty. Since the set $A^{0,U}_U$ is empty, for each agent $i \in A^1_U$, $b_i(O_{k-1}(i)) \subseteq \omega(A^{1,U}_U) \cup \omega(A^{1,U}_{N\setminus U})$. By equality (5), the objects in $\omega(A^{1,U}_U) \cup \omega(A^{1,U}_{N\setminus U})$ are available only for the agents in A^1_U. But then, by definition of the DTTC, for each agent $i \in A^{0,N\setminus U}_U$, $O_{k-1}(i) \cap \Big(\omega(A^{1,U}_U) \cup \omega(A^{1,U}_{N\setminus U})\Big) = \emptyset$. But this contradicts with $O_U = \underset{i \in A_U}{\cup} b_i(O_{k-1}(i))$ by Lemma 4. Thus, $U = A^1_U \cup \omega(A^{1,U}_U) \cup \omega(A^{1,U}_{N\setminus U})$.

The set $N \setminus U$ consists of the agents in $A^1_{N\setminus U} = A^{1,U}_{N\setminus U} \cup A^{1,N\setminus U}_{N\setminus U}$ and $A^0_{N\setminus U} = A^{0,N\setminus U}_{N\setminus U}$ (by Lemma 5, the set $A^{0,U}_{N\setminus U}$ is empty), and the objects in $\omega(A^{1,N\setminus U}_U) \cup \omega(A^{1,N\setminus U}_{N\setminus U}) \cup \omega(A^{0,N\setminus U}_{N\setminus U})$. Since f is a circulation respecting d_{k-1} such that for no edge e including an agent in $N \setminus N_{k-1}$, $f(e) = 1$, the agents in $A^1_{N\setminus U}$ are assigned to the objects in $\omega(A^{1,N\setminus U}_U) \cup \omega(A^{1,N\setminus U}_{N\setminus U})$. Note that this possible by equality (6). But then, this is the symmetric to Case 1 above, and the same argument applies: we construct a sequence of agents (the same way as in $(\star)$) and a lower-bound function d_k (the same way as in $(\otimes)$), then show that there exists a circulation f' respecting d_k. ■

An equivalent algorithm to the DTTC. Before we prove the equivalence between the DTTC and the *bargaining set*, we introduce a slight modification of the DTTC, which is without loss of generality. This equivalent version of the DTTC enables us to use a simpler exposition in the proof.

In the graph G_{k-1}, let T be a *minimal self-mapped set.*[10] Let $N(T)$ be the set of agents in T. Since each object points to its owner, by definition of a minimal self-mapped set, T consists of a set of agents and their endowments.[11] Thus, $T = N(T) \cup \omega(N(T))$. Suppose $N(T) \nsubseteq N_{k-1}$.[12] Since there is no edge leaving T, the circulation f respecting d_{k-1} is such that for each edge e entering T, $f(e) = 0$. Thus, the induction argument based on circulations constructed via Lemma 3 through 6 applies to the subgraph induced by T. Thus, there exists a feasible improvement cycle S in T. Moreover, suppose

[10]Please see Section 2.2 for the definition of a minimal self-mapped set and other graph theoretical concepts we use in what follows.

[11]This follows from the fact that a minimal self-mapped set is strongly connected (Remark 1) and each path to an agent includes her endowment.

[12]If $N(T) \subseteq N_{k-1}$, then since T is a minimal self-mapped set, no object in $\omega(N(T))$ is available for agents in $N \setminus N(T)$.

each feasible improvement cycle selected until Step $k' \geq k$ is disjoint with T. Then, at Step k', the set T is a minimal self-mapped set and S is a feasible improvement cycle.

Let S' be a feasible improvement cycle, which does not belong to a minimal self-mapped set in G_{k-1}. Suppose each feasible improvement cycle selected until Step $k' \geq k$ belongs to a minimal self-mapped set at that step. Then, S' is a feasible improvement cycle at Step k' as well. Thus, the DTTC can be redefined such that at each step, a feasible improvement cycle is selected from a minimal self-mapped set at that step.

Lemma 7 *Let G_{k-1} be the directed graph at Step k of the DTTC. Let T be a minimal self-mapped set in G_{k-1}. If T is covered, then the DTTC assigns each agent $i \in N(T)$ an object from the set $O_{k-1}(i)$ (that is, one of the objects which she points to in $\omega(N(T))$).*

Proof. Since T is covered in G_{k-1}, by definition, there is an integer-valued circulation f' such that for each $v \in T$, $f'(e) = 1$ for some edge e entering v. Thus, by definition of this circulation, it is possible to assign each agent $i \in N(T)$ to one of the objects in $O_{k-1}(i)$. This implies that each time a feasible improvement cycle is selected from the set T, for each agent $i \in N(T)$, the set of best available objects is still a subset of $O_{k-1}(i)$. Thus, the DTTC assigns agent $i \in N(T)$ one of the objects in $O_{k-1}(i) \subseteq \omega(N(T))$. ■

Thus, we modify the DTTC as follows: Let G_{k-1} be the graph at the end of Step $k-1$. At Step (k.1), (i) if there exists a minimal self-mapped set T in the graph G_{k-1}, which is covered, then each agent $i \in N(T)$ is assigned one of the objects in $O_{k-1}(i)$ and the objects $\omega(N(T))$ become *unavailable* for the agents in $N \setminus N(T)$,[13] otherwise (ii) a feasible improvement cycle is selected from a minimal self-mapped set in the graph G_{k-1}.[14]

II. Each assignment in the *bargaining set* is an outcome of the DTTC. Let (ω, R) be an assignment problem and μ be an assignment in the *bargaining set*. By induction, we assume that for each step $k' < k$, for at least one feasible improvement cycle, each agent in that cycle is assigned to one of her best available objects in graph $G_{k'}$. We prove that the inductive hypothesis holds for Step k. That is, given the entitlement defined by μ, say $(N_{k-1}, \epsilon_{k-1})$, and the graph G_{k-1}, and a minimal self-mapped set in this graph, there exists a cycle in this set such that each agent in this cycle

[13]This follows from the fact that $\left| \underset{i \in N(T)}{\cup} b_i(O_{k-1}(i)) \right| = |\omega(N(T))| = |N(T)|$ and Hall's Marriage Theorem.

[14]By Remark 2, part (ii) is well-defined.

is assigned a best-available object in this graph. This complete the proof since this is equivalent to Step $k.1$, where a feasible improvement cycle is chosen and μ is consistent with the new entitlement as defined in the DTTC. Suppose that in the graph G_{k-1}, there exists a minimal self-mapped set which is covered. Then, by Proposition 1, under μ, each agent in this minimal self-mapped set is assigned to one of the objects she points to. By Lemma 7, this is equivalent to the DTTC at Step $k.1$. Thus, let T be a minimal self-mapped set which is not covered, that is, it is not feasible to assign each agent in $N(T)$ one of her best available objects. As argued above (in the part where we discuss the modified version of the DTTC), there exists at least one feasible improvement cycle in T. We prove our result towards a contradiction by supposing that there exists at least one agent in each of these feasible improvement cycles, who is not assigned one of the best available objects in the graph G_{k-1}.

Let S be such a feasible improvement cycle. Since at least one of the agents in this set is not assigned one of her best available objects, the assignment μ is blocked by S via η. Let η be an assignment such that each agent in S is assigned one of her best available objects and the entitlement $(N_{k-1}, \epsilon_{k-1})$ is feasible. Without loss of generality, suppose that under η, for each feasible improvement cycle in T, it is not possible to assign each agent in this cycle a best available object. It is without loss of generality, since, if there exists such a cycle S', then we consider the blocking coalition as $S \cup S'$ instead of S. Now consider the coalition $N_{k-1} \cup S$ as a blocking coalition via η. Note that we can assume that agents in this blocking coalition reallocate their endowments according to η, since it is consistent with the definition of a blocking coalition reallocating their endowments by the definition of the DTTC up to Step k and also that each agent in S is assigned one of her best available objects in the graph G_{k-1} (note that these imply that there is no other reallocation such that no agent is worse off and at least one agent is better off than the assignment η).

We claim that there does not exist a coalition $C(S)$ in T disjoint with $N_{k-1} \cup S$, such that $C(S)$ blocks η via some $\mu'' \in M_{C(S)}(\mu)$. First, note that $C(S)$ cannot be a feasible improvement cycle. The reason is the following: if $C(S)$ is a feasible improvement cycle, then by reallocating their endowments, each agent in $C(S)$ is assigned one her best available objects in the graph G_{k-1}. But this contradicts with that, under μ, no feasible improvement cycle is such that each agent in this cycle is assigned one of her best available objects in the graph G_{k-1} (note that by definition, each agent i in the cycle $C(S)$ is indifferent between $\mu(i)$ and $\mu''(i)$). Similarly, $C(S)$ cannot be an improvement cycle which is not feasible: because the agents in such a set reallocates its endowments under μ'' such that each of them is assigned one of the best available objects in the graph G_{k-1}, but this is not possible under μ since this is not a feasible improvement cycle. Also, it is clear that $C(S)$ cannot be a non-improvement cycle since by definition, such a cycle is in the set N_{k-1} and cannot block η by μ'' since each agent i in this cycle is indifferent between $\eta(i)$ and $\mu''(i)$. Thus, there does not exist a coalition $C(S)$ in T disjoint with $N_{k-1} \cup S$, such that $C(S)$ blocks η via some $\mu'' \in M_{C(S)}(\mu)$. Thus, if we show that $C(S)$ must be in the set T, then the proof is complete.

Suppose there exists a coalition $C(S) \nsubseteq N(T)$ which blocks η via some $\mu'' \in M_{C(S)}(\mu)$. Since $C(S) \nsubseteq N(T)$ and T is a minimal self-mapped set, by definition, there is an agent, say i in $C(S)$, who is not assigned a best available object under $\mu'' \in M_{C(S)}(\mu)$ with coalition $C(S)$. Then, i does not get a best available object under μ. Also, i cannot be in $(N_{k-1} \cup S) \cap C(S)$, otherwise, she would get a best available object under η via $N_{k-1} \cup S$ but not under μ, contradicting $C(S)$ blocking η. Thus, i must be in another cycle in T, say cycle $S' \subseteq N(T)$ with $S' \neq S$. Now since agent $i \in S'$ is not assigned a best available object under μ, S' blocks μ, say via η', but there would be no $C(S')$ (with a non-empty intersection with S'), which would block η'. This is because agent $i \in S'$ is assigned a best available object under η' but not under μ, thus there is no $C(S')$ such that $\mu' \in M_{C(S')}(\mu)$ would block η'. This contradicts with μ being an assignment in the *bargaining set*. Thus, each agent in the coalition $C(S)$ must be assigned a best available object, which implies that $C(S)$ is a cycle in $N(T)$.

Remark 4 *Each outcome of the DTTC is in the bargaining set.*

Although this result derives from the implications below, we think it is better to include a simple proof of this result to demonstrate better the relationship between the DTTC and the *bargaining set*. Let (ω, R) be an assignment problem and μ be an outcome of the DTTC. Suppose μ is (strictly) blocked by a coalition S via ν. Let $S = S^B \cup S^I$ where S^B is the set of agents in S who are strictly better off under ν than under μ and S^I is the set of agents in S who are indifferent between ν and μ. Let $k^\star = \min\{k : S \cap S^k \neq \emptyset\}$.

Lemma 8 $S^B \cap S^{k^\star} = \emptyset$.

Proof. Suppose $S^B \cap S^{k^\star} \neq \emptyset$ and let $i \in S^B \cap S^{k^\star}$. By definition, agent $i \in S \cap S^{k^\star}$ is assigned under ν an object of another agent in $j \in S$, which is strictly better than $\mu(i)$ and *unavailable* for i at the beginning of Step $k^\star$. That is, $\omega(j)\ P_i\ \mu(i)$ and $\omega(j)$ becomes *unavailable* at some step $k < k^\star$. By Remark 3, this implies that $j \in S^{k'}$ for some $k' \leq k$. Since $j \in S$, this contradicts with $k^\star = \min\{k : S \cap S^k \neq \emptyset\}$. ■

By Lemma 8, $S = S^B$ cannot hold, which implies that S cannot strictly block μ, thus part (i) of Definition 1 follows immediately. Now, suppose S blocks μ via ν. Lemma 8 also implies that it cannot be $S \subseteq S^{k^\star}$. Thus, S is such that $S \not\subseteq S^{k^\star}$ and $S^B \cap S^{k^\star} = \emptyset$. Let $k^{\star\star} = \min\{k : S^B \cap S^k \neq \emptyset\}$ (note that $k^{\star\star} > k^\star$). Let $i \in S^B$. By definition of the set S^B, the object $\nu(i)$ is *unavailable* for her at Step $k^{\star\star}$. Suppose $\nu(i) \in O_{k'-1}(i) \setminus O_{k'}(i)$, that is, it becomes *unavailable* for her at Step k'. Let Step k be the first step at which for some agent in S^B, her assigned object under ν becomes *unavailable*. By definition, $k < k^{\star\star}$.

Case 1: Suppose a covered minimal self-mapped set T in the graph G_{k-1} is selected at Step k. Thus, $S^k = N(T)$ and the objects $\omega(N(T))$ become *unavailable* for the agents in the set $N \setminus T$. Since, by definition, ν is such that each agent in S is assigned to an endowment of another agent in S, and also T is a covered minimal self-mapped set,[15] an agent $i \in S \cap N(T)$ must be assigned to an endowment of an agent in $N \setminus N(T)$. But then, since T is covered, by definition, $O_{k-1}(i) \subseteq O(T)$ and $\mu(i) \in O_{k-1}(i)$, and we have $\mu(i)\ P_i\ \nu(i)$. This contradicts that S blocks μ.

Case 2: Suppose that S^k is a feasible improvement cycle in a minimal self-mapped set which is not covered. By definition of the DTTC algorithm, the entitlement (N_k, ϵ_k) is feasible only if $\nu(i)$ is assigned to one of the agents $j \in N_k$ and, if the object $\nu(i)$ is assigned to agent $i \in N \setminus N_k$, agent j is worse off under ν than under μ. Thus, there is a cycle C including $\nu(i)$ and agent j. For each agent i' in C, let μ' be such that $\mu'(i') = o'$, where o' is the object she points to in cycle C. Note that, by definition of the entitlement and the DTTC, $o'\ I_{i'}\ \mu(i')$. Also, since, under ν, agent j cannot be assigned to an object from the indifference set including object $\nu(i)$, by definition of ν, $\nu(j) = \omega(j)$.[16] Both S and cycle C include agent $\omega^{-1}(\nu(i))$, thus they intersect. We claim that cycle C does contain any agent in S^B. (Note that agent i is not in the cycle C.) Let $i' \in S^B \cap C$. In the graph G_{k-1}, each agent points to her best available objects and for agent i', the object $\nu(i')$ is available at the beginning of Step k, since by definition of k, it did not become *unavailable* before this step. Thus, agent i' points to her best available objects, that is object $\nu(i')$ and other available objects indifferent to it, if any.

[15] That T is minimal self-mapped set implies that the endowment of an agent in $N \setminus N(T)$ is not in T.

[16] See the the definition of blocking via an assignment in Section 3: when a coalition blocks an assignment, the resulting assignment is such that, if a cycle defined by the original assignment is broken because of a departing coalition, any agent in that cycle not included in the coalition gets her endowment.

Since $i' \in C$, by definition of the DTTC, it must be that $\nu(i')\ I_{i'}\ \mu(i')$, contradicting that $i' \in S^B$. It can be that an agent in S^B points to the endowment of an agent in C (agent i is such an agent pointing to object $\nu(i)$) but it cannot be that an agent in S^B is also in C and points to an object, which is the endowment of an agent not included in C (note that, by definition of S^B, it can not point to the endowment of an agent in C). Thus, $C \cap S \subseteq S^I$. Now consider ν and the coalition C. Each agent in $C \cap S$ is indifferent between μ' and ν and each other agent in $C \setminus S$ is either indifferent between μ' and ν (because the coalition S does not break the cycle including that agent) or better off under μ' than under ν (there exists at least one such agent, agent j). Thus, C blocks ν via μ' such that $\mu' \in M_C(\mu)$.

III. Each outcome of the TTC is in the *bargaining set*. Let (ω, R) be an assignment problem and let $\mu \in TTC(\omega, R)$. We need to show that μ satisfies Definition 1. We begin with part (ii) of Definition 1.

Part (ii) **of Definition 1:** Let T_0 be the first set of agents who leave the mechanism before any endowment update is made. Note that T_0 may be empty. Let V_1 be the set of all agents whose endowments are updated after the agents in T_0 leave the mechanism. Let V_t be the set of agents whose endowment has been updated after the agents in T_{t-1} and before the agents in T_t leave the mechanism. Let $T_t \neq \emptyset$ be the set of all agents who leave the mechanism after the endowment update for the agents in V_t and before the endowment update for the agents in V_{t+1} take place. Let T_K be the last set of agents who leave the mechanism. Thus $N = \bigcup_{t=0}^{K} T_t$. Note that if V_τ is empty for some τ, then $T_{\tau-1}$ would be the last set of agents who leave the mechanism, that is, $N = \bigcup_{t=0}^{\tau-1} T_t$.

Suppose μ does not satisfy part (ii) of Definition 1. Then, there exists an S_0 which blocks μ via η_0, for which there is no $C(S_0)$, with a non-empty intersection with S_0, such that $C(S_0)$ blocks η_0 via some $\mu' \in M_{C(S_0)}(\mu)$. Under this supposition, denote it with $(\star\star)$, we show that $S_0 \cap \bigcup_{t=0}^{K} V_t \cup T_t = \emptyset$, through an induction argument. This will give us a contradiction since $\bigcup_{t=0}^{K} V_t \cup T_t = N$ and $S_0 \subseteq N$.

Induction Step t=1: $S_0 \cap (V_1 \cup T_1) = \emptyset$.

First note that if $V_1 = \emptyset$, then by definition $T_0 = N$, that is, every agent leaves the mechanism and no endowment update takes place. Thus, every agent receives a top object (among all objects) under μ. Thus, S_0 cannot block μ. Thus, consider the case where there is some endowment update, that is, $V_1 \neq \emptyset$. Also note that by definition, $T_1 \cap V_1 \neq \emptyset$, since otherwise no agent in T_1 would have an endowment update. Then, T_1 would essentially be removed before the endowment update of V_1, contradicting with the definitions of V_1, T_1 and T_0.

Now, suppose $S_0 \cap (V_1 \cup T_1) \neq \emptyset$, that is, $(S_0 \cap V_1) \cup (S_0 \cap T_1) \neq \emptyset$. Thus, at least one of the two intersections $S_0 \cap V_1$ or $S_0 \cap T_1$ is non-empty. Note that $S_0 \cap T_0 = \emptyset$, since otherwise there would be one agent in the intersection that would have a top object that is owned by some agent outside of T_0, which would contradict with T_0 being a covered minimal self-mapped set. Note also that each agent in $V_1 \cup T_1$ receives, under μ, a top object among the remaining objects after T_0 and objects in $\mu(T_0)$ are removed, by Lemma 1. Thus, the agent in S_0 who is strictly better off under η_0 than under μ must be in $S_0 \setminus (V_1 \cup T_1)$, that is, $S_0 \setminus (V_1 \cup T_1) \neq \emptyset$.[17] Now, we prove that there is a $C(S_0)$, which blocks η_0 via some $\mu' \in M_{C(S_0)}(\mu)$ with $C(S_0) \cap S_0 \neq \emptyset$ through the following lemma.

Lemma 9 *If $S_0 \cap (V_1 \cup T_1) \neq \emptyset$, then there is a $C(S_0)$, which blocks η_0 via some $\mu' \in M_{C(S_0)}(\mu)$.*

Proof. We consider two cases: one with $S_0 \cap V_1 \neq \emptyset$ and the other with $S_0 \cap V_1 = \emptyset$. For each case we find a blocking $C(S_0)$ with the desired properties.

Case1. Suppose $S_0 \cap V_1 \neq \emptyset$. In this case, consider $C(S_0) = V_1$.[18] Define $\mu' \in M_{C(S_0)}(\mu)$, where $\mu'(a) = u^{V_1}(a)$ for each $a \in V_1$, where $u^{V_1}(a)$ is the updated endowment of agent a at the end of the 1^{st} endowment update, that is, μ' carries out the cycle V_1. No agent in $C(S_0) = V_1$ is better off under η_0 than under μ'. This is because $\mu' \in M_{C(S_0)}(\mu)$ and that each agent in V_1 receives a top object under μ among the available objects, $\omega(N) \setminus \omega(T_0)$. Thus, $\mu'(a) R_a \eta_0(a)$ for each $a \in C(S_0)$. Also note that $\omega(V_1) = \mu'(V_1)$ by the definition of μ'. Now, we show that there is at least one agent in $C(S_0) = V_1$ who is worse off under η_0 than under μ. Since S_0 blocks μ as a cycle and since $S_0 \cap V_1 \neq \emptyset$, there is an agent $a \in S_0 \setminus V_1$ who receives an object, under η_0, say h, which is the original endowment of some agent in $S_0 \cap V_1$. There is also another agent $\hat{a} \in V_1 \setminus S_0$ for whom h is one of her top objects among $\omega(N) \setminus \omega(T_0)$, which may be the only top object or among her multiple top objects. Thus, under μ this agent $\hat{a}$ gets an object that is either h or some other top object among $\omega(N) \setminus \omega(T_0)$ (see Figure 2).

Under η_0 this agent $\hat{a} \in V_1 \setminus S_0$, gets her original endowment. Thus, if $h \; P_{\hat{a}} \; \omega(\hat{a})$, then $\hat{a}$ is worse off under η_0 than under μ. Suppose $\omega(\hat{a})$ is also a top object for $\hat{a}$ among $\omega(N) \setminus \omega(T_0)$. In the endowment

[17] Clearly, there may be more than one such agent.

[18] Set V_1 may consist of more than one cycles, in which case S_0 intersects with at least one of them, and we would take $C(S_0)$ to be that particular cycle. We treat V_1 as one single cycle. This is without loss of generality since the agents in those cycles in V_1 that do not intersect with S_0 receive, under η_0, the same object they receive under μ, by definition of blocking allocation η_0.

update $\omega(\hat{a})$ will be the new endowment of some other agent, say $\tilde{a}$, and this agent will receive a top object among $\omega(N)\backslash\omega(T_0)$ under μ, and $\omega(\hat{a})$ is a top object for her among $\omega(N)\backslash\omega(T_0)$ (see Figure 3). Then, if for $\tilde{a}$, $\omega(\hat{a})$ is a better object than her original endowment, $\tilde{a}$ would be worse off under η_0. Continuing in this fashion we find that there would be another agent who would be worse off under η_0 than under μ. To see this, suppose every agent in $V_1 \setminus S_0$ has her original endowment among her top objects among $\omega(N) \setminus \omega(T_0)$ (see Figure 4). Then, $V_1 \cup S_0$ would Pareto dominate μ, contradicting *Pareto efficiency* of TTC mechanism.[19] Thus, there is at least one agent in V_1 who is worse off under η_0 than under μ, and $C(S_0) = V_1$ blocks η_0 via $\mu' \in M_{C(S_0)}(\mu)$.

Case 2. Suppose $S_0 \cap V_1 = \emptyset$ and $S_0 \cap T_1 \neq \emptyset$. In this case, consider $C(S_0) = V_1 \cup T_1$ and $\mu' \in M_{C(S_0)}(\mu)$, where $\mu'(a) = \mu(a)$ for each $a \in T_1$ and $\mu'(a) = u^{V_1}(a)$ for each $a \in V_1 \setminus T_1$, where $u^{V_1}(a)$ is the updated endowment of agent a at the end of the 1^{st} endowment update. Note that $\omega(T_1 \cup V_1) = \mu'(T_1 \cup V_1)$. Now, the same argument in Case 1 above applies here as well and there is at least one agent in T_1 who is worse off under η_0 than under μ. Thus, $C(S_0) = V_1 \cup T_1$ blocks η_0 via $\mu' \in M_{C(S_0)}(\mu)$. ■

Thus, by Lemma 9, we get a contradiction with our supposition $(\star\star)$. Thus, $S_0 \cap (V_1 \cup T_1) = \emptyset$.

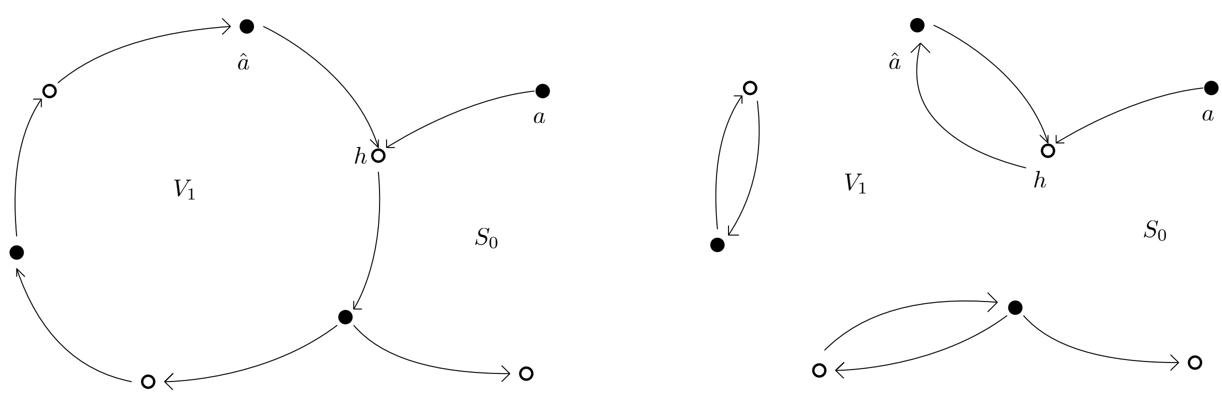

Figure 2: $\hat{a}$'s original endowment is not one of her top objects.

Induction Step t+1: If $S_0 \cap (V_t \cup T_t) = \emptyset$, then $S_0 \cap (V_{t+1} \cup T_{t+1}) = \emptyset$.

Suppose $S_0 \cap (V_t \cup T_t) = \emptyset$ for all t, and $S_0 \cap (V_{t+1} \cup T_{t+1}) \neq \emptyset$. First note that $S_0 \subseteq (V_{t+1} \cup T_{t+1}) \setminus \bigcup_{\tau=0}^{t} V_\tau \cup T_\tau$ is not possible. This is because each agent in $V_{t+1} \cup T_{t+1}$ receives an object that is top among the remaining objects after T_t and $\mu(T_t)$ are removed, by Lemma 1. If $S_0 \subseteq (V_{t+1} \cup T_{t+1}) \setminus \bigcup_{\tau=0}^{t} V_\tau \cup T_\tau$, then no agent in S_0 is strictly better off under η_0 than under μ.

[19] Pareto efficiency is shown in Saban and Sethuraman (2013).

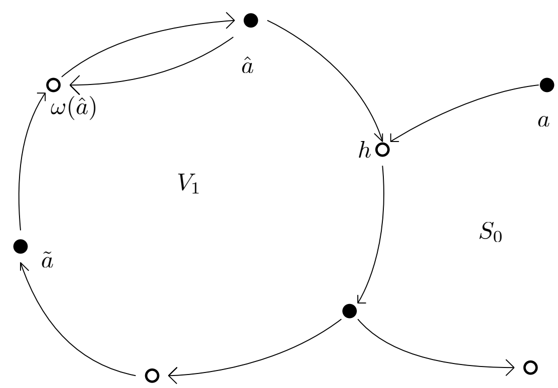

Figure 3: $\hat{a}$'s original endowment is one of her top objects.

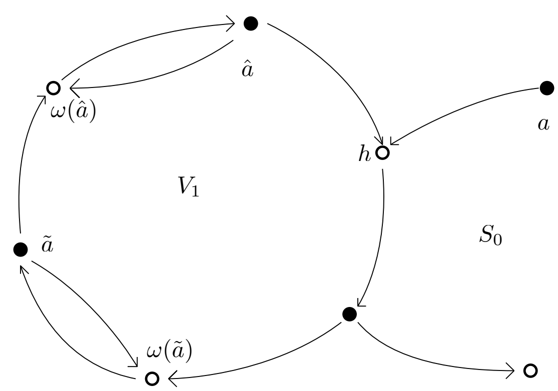

Figure 4: For both $\hat{a}$ and $\tilde{a}$ original endowment is a top object.

Thus, it must be the case that $S_0 \setminus (V_{t+1} \cup T_{t+1}) \neq \emptyset$. In this case, there is an agent $a \in S_0 \setminus (V_{t+1} \cup T_{t+1})$ and another agent $\hat{a} \in S_0 \cap (V_{t+1} \cup T_{t+1})$. The agent $\hat{a}$ cannot be strictly better off under η_0 than under μ. To see this, suppose she is better off. By Lemma 1, she receives a top object among the remaining objects after T_t is removed, since $\hat{a} \in V_{t+1} \cup T_{t+1}$. Thus, she must receive, under η_0, an object that is allocated (under μ) to some agent in T_t, say agent $\tilde{a}$. Then, $\tilde{a} \in S_0$, since $\eta_0(S_0) = w(S_0)$, where $w(S_0)$ is the set of original endowments of agents in S_0. This contradicts $S_0 \cap (V_t \cup T_t) = \emptyset$. Thus, each agent in $S_0 \cap (V_{t+1} \cup T_{t+1})$ is indifferent between μ and η_0. Thus, no agent in $\bigcup_{\tau=0}^{t+1} V_\tau \cup T_\tau$ is strictly better off under η_0 than under μ, that is, $\mu(a) R_a \eta_0(a)$ for each $a \in \bigcup_{\tau=0}^{t+1} V_\tau \cup T_\tau$.

Now, we prove that there is a $C(S_0)$, which blocks η_0 via some $\mu' \in M_{C(S_0)}(\mu)$ with $C(S_0) \cap S_0 \neq \emptyset$ through the following lemma.

Lemma 10 *If $S_0 \cap (V_{t+1} \cup T_{t+1}) \neq \emptyset$, then there is a $C(S_0)$, which blocks η_0 via some $\mu' \in M_{C(S_0)}(\mu)$.*

Proof. We consider two cases: one with $S_0 \cap V_{t+1} \neq \emptyset$ and the other with $S_0 \cap V_{t+1} = \emptyset$. For each case we find a blocking $C(S_0)$ with the desired properties.

Case1. Suppose $S_0 \cap V_{t+1} \neq \emptyset$. In this case, consider $C(S_0) = V_{t+1} \cup \left(\bigcup_{\tau=0}^{t} V_\tau \cup T_\tau\right)$. Define $\mu' \in M_{C(S_0)}(\mu)$ such that $\mu'(a) = \mu(a)$ for each $a \in T_\tau$ for any $\tau \leq t$ and $\mu'(a) = u^{V_\tau}(a)$ for each $a \in C(S_0) \setminus \bigcup_{\tau=0}^{t} T_\tau$, where $u^{V_\tau}(a)$ is the updated endowment of agent a at the end of the τ^{st} endowment update. Thus, under μ', each agent who has already left the mechanism on or before t receives her assignment under μ, and each agent who has not left yet by $t+1$ but has an updated endowment, receives her last updated endowment. By Lemma 1, we have $\mu' \in M_{C(S_0)}(\mu)$. Note that by definition of μ' we also have $\omega(C(S_0)) = \mu'(C(S_0))$. Also note that no agent in $C(S_0)$ is better off under η_0 than under μ'. This is because, under η_0, any agent in $C(S_0) \setminus S_0$ either receives her allocation under μ or her endowment, and because $\mu' \in M_{C(S_0)}(\mu)$.[20] Thus, we have $\mu'(a) R_a \eta_0(a)$ for each $a \in C(S_0)$. Now, we show that there is at least one agent in $C(S_0)$ who is worse off under η_0 than under μ.

Note that $S_0 \cap (V_t \cup T_t) = \emptyset$ for all t, and $S_0 \cap V_{t+1} \neq \emptyset$. Now, since S_0 blocks μ as a cycle and since $S_0 \cap V_{t+1} \neq \emptyset$, there is an agent $a \in S_0$ who receives an object under η_0, say h, which is the (potentially updated) endowment of some agent in $S_0 \cap V_{t+1}$. There is also another agent $\hat{a} \in V_{t+1} \setminus S_0$ for whom h is one of her top objects among the remaining objects after T_t is removed, which may be the only top object or among her multiple top objects. Thus, under μ this agent $\hat{a}$ gets an object that is either h or some other top object among the remaining objects after T_t is removed. Under η_0 this agent $\hat{a} \in V_{t+1} \setminus S_0$, gets her original endowment. Thus, if $h \; P_{\hat{a}} \; \omega(\hat{a})$, then $\hat{a}$ is worse off under η_0 than under μ. Suppose $\omega(\hat{a})$ is also a top object for $\hat{a}$ among the remaining objects after T_t is removed. In the endowment update $\omega(\hat{a})$ will be the new endowment of some other agent, say $\tilde{a}$, and this agent will receive a top object among the remaining objects after T_t is removed under μ, and $\omega(\hat{a})$ is a top object for her among the remaining objects after T_t is removed. Then, if for $\tilde{a}$, $\omega(\hat{a})$ is a better object than her original endowment, $\tilde{a}$ would be worse off under η_0. Note that since $S_0 \cap V_{t+1} \neq \emptyset$ and η_0 gives each agent in $V_{t+1} \setminus S_0$ her original endowment, we can continue in this fashion and find that there would be another agent who would be worse off under η_0 than under μ. To see this, suppose every agent in $V_{t+1} \setminus S_0$ has her original endowment among her top objects among the remaining objects after T_t is removed. Then, $V_{t+1} \cup \left(\bigcup_{\tau=0}^{t} V_\tau \cup T_\tau\right) \cup S_0$ would Pareto dominate μ, contradicting *Pareto efficiency*

[20] Lemma 1 says that if an agent a's endowment is updated at stage t, then she gets a top object, under μ, among the remaining objects after agents in T_{t-1} and the objects allocated to them are removed. Agent a's updated endowment at stage t is also a a top object among those remaining objects after agents in T_{t-1} and the objects allocated to them are removed. Thus, an agent's updated endowment cannot be worse than the object she receives under μ.

of TTC mechanism. Thus, there is at least one agent in V_{t+1} who is worse off under η_0 than under μ, and $C(S_0)$ blocks η_0 via $\mu' \in M_{C(S_0)}(\mu)$.

Case2. Suppose $S_0 \cap V_{t+1} = \emptyset$ and $S_0 \cap T_{t+1} \neq \emptyset$. In this case, consider $C(S_0) = \bigcup_{\tau=0}^{t+1} V_\tau \cup T_\tau$. Define $\mu' \in M_{C(S_0)}(\mu)$ such that $\mu'(a) = \mu(a)$ for each $a \in T_\tau$ for any $\tau \leq t+1$ and $\mu'(a) = u^{V_\tau}(a)$ for each $a \in C(S_0) \setminus \bigcup_{\tau=0}^{t+1} T_\tau$, where $u^{V_\tau}(a)$ is the updated endowment of agent a at the end of the τ^{st} endowment update. Note that $\omega(C(S_0)) = \mu'(C(S_0)$. Now, the same argument in Case 1 above applies here as well and there is at least one agent in T_{t+1} who is worse off under η_0 than under μ. Thus, $C(S_0)$ blocks η_0 via $\mu' \in M_{C(S_0)}(\mu)$. ■

Thus, by Lemma 10, we get a contradiction with our supposition $(\star\star)$. Thus, $S_0 \cap (V_{t+1} \cup T_{t+1}) = \emptyset$. Induction argument implies that $S_0 \cap \bigcup_{\tau=0}^{K} V_\tau \cup T_\tau = \emptyset$. This is a contradiction since $\bigcup_{\tau=0}^{K} V_\tau \cup T_\tau = N$. Thus, our initial supposition must not hold, that is, μ satisfies part (ii) of Definition 1.

Part (i) **of Definition 1:** Suppose $\mu \in TTC(\omega, R)$ is strictly blocked by some coalition S. Then, all agents in S get a strictly better object within the coalition than under μ. $S \cap T_0 = \emptyset$, because otherwise S cannot strictly block μ since all agents in T_0 get a top object under μ. So there exits a $t \geq 1$ such that $S \cap \bigcup_{\tau=0}^{t-1} T_\tau = \emptyset$ and $S \cap T_t \neq \emptyset$. Now, consider the following two cases:

Case 1. No agent in $\bigcup_{\tau=0}^{t-1} T_\tau$ has an updated endowment (before T_t is removed) which is originally endowed by some agent in S. Thus, the original endowment of any agent in $S \cap T_t$ is not removed yet, before T_t is removed. Note that each agent in T_t, thus each agent in $S \cap T_t$, receives a top object (under μ) among the remaining objects after $\bigcup_{\tau=0}^{t-1} T_\tau$ and $\mu(\bigcup_{\tau=0}^{t-1} T_\tau)$ are removed. Note also that each agent in S, thus those in $S \cap T_t$, receives an object (under μ') which is an original endowment of some agent in S, since S is a coalition. Thus, for an agent $i \in S \cap T_t$, $\mu'(i)\ P_i\ \mu(i)$ is not possible.

Case 2. There is an agent in $\bigcup_{\tau=0}^{t-1} T_\tau$ who has an updated endowment (before T_t is removed) which is originally endowed by some agent in S. Denote the set of such agents by $\hat{S}$. By Lemma 2, each agent in $\hat{S}$ receives a top object (under μ) among the remaining objects after $\bigcup_{\tau=0}^{t-1} T_\tau$ and $\mu(\bigcup_{\tau=0}^{t-1} T_\tau)$ are removed. Since the coalition S via μ' creates a cycle, there is an agent $i \in \hat{S}$ who receives an object (under μ') which is the original endowment of some agent $j \in S \setminus \hat{S}$. By the definition of $\hat{S}$, agent j's original endowment is among the remaining objects after $\bigcup_{\tau=0}^{t-1} T_\tau$ and $\mu(\bigcup_{\tau=0}^{t-1} T_\tau)$ are removed. Since agent i receives (under μ) a top object among the remaining ones after $\bigcup_{\tau=0}^{t-1} T_\tau$ and $\mu(\bigcup_{\tau=0}^{t-1} T_\tau)$ are removed, $\mu'(i)\ P_i\ \mu(i)$ is not possible.

Thus, $\mu \in B(\omega, R)$, which finishes the proof of part **III**. ■

IV. For each outcome μ of the DTTC, there exists a selection rule F such that μ is obtained by the TTC via F. In the first step of the DTTC, a cycle is chosen and each agent in the cycle is entitled one of her best objects. The TTC obtains the same welfare level for these agents by top-trading the chosen cycle. Suppose, by an inductive argument, that at Step $k-1$, the entitlement given by the DTTC is obtained by top-trading a sequence of cycles: the TTC gives an endowment profile such that, given the entitlement $(N_{k-1}, \epsilon_{k-1})$ obtained by the DTTC, for each $i \in N_{k-1}$, her endowment at Step $k-1$, that is, $\omega_{k-1}(i)$, is in the set $\epsilon_{k-1}(i)$. Let N^k be the feasible improvement cycle chosen by the DTTC at Step k. By definition of a feasible improvement cycle (see Section 5), there exists an *individually rational* matching μ such that the entitlement $(N_{k-1}, \epsilon_{k-1})$ is feasible under μ and each agent in N^k is assigned one of the objects she points to. Suppose the sets N^k and N_{k-1} are disjoint. Since N^k is a *feasible improvement cycle* in the graph G_{k-1} and does not include any agent, whose endowment is updated, by top-trading N^k, the inductive hypothesis is satisfied. Suppose now that the intersection of N_{k-1} and N^k is non-empty.

Suppose that all the matchings such the entitlement $(N_{k-1}, \epsilon_{k-1})$ is feasible under μ and each agent in N^k is assigned one of the objects she points to, induce the same set of cycles including the agents in N_{k-1}. Then, the set N^k must be in this set of cycles, since otherwise by selecting N^k in this step would violate the existing entitlement, thus a contradiction that N^k is a feasible improvement cycle. Since the updated endowments of N_{k-1} are in this set of cycles, by making each agent in $N_{k-1} \cup N^k$ point to the object that she is assigned to under this set of cycles, and each object point to its updated owner in N_{k-1}, we obtain a sequence of top trading cycles and the update after top-trading this sequence of cycles gives the inductive hypothesis. Suppose that the matchings such the entitlement $(N_{k-1}, \epsilon_{k-1})$ is feasible under μ and each agent in N^k is assigned one of the objects she points to, do not induce the same set of cycles including the agents in N_{k-1}. But, after a number of steps, say at Step k', since the problem is finite, they should induce only one set of cycles including N_{k-1}. At that step, the same argument above applies. The only question remains is that whether one can change the ordering of the selection of the feasible improvement cycles in the DTTC, to achieve the same matching. While the answer is clearly negative in general, it is affirmative as long as the same entitlements are created. Since the same entitlements are created when a feasible improvement cycle is chosen at Step k' or at

Step k, the feasible improvement cycles chosen between steps k and k' are not affected by this change in the order of selection. Thus, the inductive hypothesis is satisfied in this case as well.

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