Mutual Insurance Networks and Unequal Resource Sharing in Communities[*], *First Draft, February 2015*

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Abstract

We study formation of mutual insurance networks in a model where agents who obtain more resources share a fixed amount of resources with all directly linked agents that obtain fewer resources. We identify the pairwise stable networks and efficient networks in a basic model where agents are identical. Then, we introduce in the model two types of heterogeneity: an exogenous one, where agents differs in their income or in their preferences over the transfer scheme, and an endogenous heterogeneity where the costs of linking to an agent depends on the number of links the latter has already formed in the network. We examine the impact of these heterogeneities on stability and efficiency.

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1 Introduction

A growing body of evidence has shown that while household income in developing countries varies greatly, consumption is remarkably smooth at a community level (e.g., Townsend, 1994, Paxson, 1992, Jacoby and Skouas, 1997). Given the lack of formal insurance especially in the rural areas, this suggests that informal institutions play a crucial role in helping households to counter the effects of income variation. In this paper we study the formation of these informal mutual insurance networks building on several stylized facts.

A first key feature is that informal insurance is not a village level phenomenon. Indeed, as has been well documented in the literature on social networks, in times of need individuals do not rely on the entire village, instead they seek help primarily through a network of mutual insurance relationships with friends and family (see Fafchamps and Lund, 2003, and Wellman and Currington, 1988). Another important feature is that mutual insurance networks are not complete within the observed set of individuals. That is, within any community, individuals do not enjoy the benefits of being insured by all others individuals of the village. Finally, the sharing of resources in times of need is not equal (Townsend, 1994). In fact to the best of our knowledge, this aspect of the formation of mutual insurance networks has not been addressed before.

Our goal is to provide a picture of the mutual insurance arrangements within the community from a networks perspective. We examine when symmetric and asymmetric network architectures can be stable among *ex ante* symmetric agents. We ask whether such arrangements may be locally complete, *i.e.*, involve every individual in a small group. We also study efficiency and the impact of agent heterogeneity in this problem.

In our model mutual insurance takes place between pairs of individuals in a village or a small community. A specific feature is the way agents "share" their resources (and hence the risk): individuals who draw high resources give a fixed amount of resources to individuals in their immediate neighborhood in the network who draw low resources. Thus, agents do not engage in equal sharing of resources. This type of sharing mechanism has two realistic features: *(i)* it ensures that individuals who draw high resources can always transfer resources to all their neighbors who draw low resources, and *(ii)* individuals who draw high resources always obtain higher benefits than individuals who draw low resources.

In our benchmark model, we assume that each agent obtains random resources which take on two values: high or low. If a person draws the high endowment state, then she gives an amount $\delta > 0$ of resources to each of her neighbors (agents with whom she has established a bilateral risk-sharing agreement) that has drawn the low endowment state. Conversely, if a person draws the low endowment state, then she obtains an amount δ of resources from each of her neighbors who has drawn the high endowment state.[1] Note that such a mutual insurance network exposes agents to the risk of their neighbors. Indeed, if two individuals decide to insure each other, then each of them increases her chances of obtaining a satisfying payoff when her own resources are low, but also increases her chances of reducing her payoffs when her own resources are high.

We also assume that informal agreements are not binding and hence to make them work agents need to invest time in their relationships. So, in our model, establishing such mutual insurance agreements is costly. More precisely, the cost of an agreement (a link) between two individuals depends on the number of agreements established by them. In particular, we assume that the marginal cost of individual i, when she forms a link with an individual j, is increasing with the number of links formed by i. This captures the idea that a mutual insurance agreement between two agents requires the agents to spend a minimal amount of time on it. Now the more links they have already formed, the less time they have to spend on any additional link, and so the higher is the cost of time. It follows that the cost of an additional link increases with the number of links.

Using this framework, we examine the formation of mutual insurance links and ask what structures will emerge when agents cannot coordinate link formation across the entire population. We use pairwise stability as the equilibrium concept (see Jackson and Wolinsky, 1996). In a pairwise stable network no pair of unlinked agents has an incentive to reach a mutual insurance agreement (add a link), and no individual has an incentive to break a mutual insurance agreement (remove one of her links). We contrast pairwise stable networks with the efficient networks for mutual insurance agreements. An efficient network is one which maximizes the

[1] In reality transfers can take a wide range of values depending on the incomes of the individuals and may be even the needs of the agents in a community. However, in our model individuals have either high or low incomes. Therefore it is reasonable to assume that in the high income state agents give a fixed amount to their neighbors who have drawn the low income state.

amount of total expected payoffs obtained by agents.

In the basic model, we have several findings.

- We establish that there exist pairwise stable networks, in which individuals are in asymmetric situations relative to the risk they support. More precisely, in stable pairwise networks, either all individuals form the same number of links, or there exist two types of individuals relative to the number of links they form. Thus, agents have different risk-sharing outcomes, despite that they have identical preferences and their incomes are identically distributed. Moreover, in pairwise stable networks where there exist two types of agents relative to the number of links they form, agents who obtain the smallest amount of insurance are always linked together.
- We show that in efficient networks agents always obtain similar amounts of insurance. In that case, an efficient network is a pairwise stable network (or a network very similar to a stable network), but the converse is not true. More precisely, we show that a non-efficient pairwise stable network is always under-connected with regard to efficiency. Thus there may exist a conflict between pairwise stable networks and efficient networks.

Then we extend the basic model by introducing two types of agents heterogeneity, an exogenous one and an endogenous one.

First, we consider situations where agents are exogenously heterogeneous: they do not obtain the same income when they draw the high income state. In the first case, we assume that there exist two types of agents concerning the income they get when they draw the high income state. In that case, we show that there exist situations where only people who get the highest income when they draw the high income state get access to insurance, while people who get the lowest income when they draw the high income state will never be insured. It follows that the insurance mechanism increases the gap between the expected well-being of high potential income people and the expected well-being of low potential income people. This kind of pairwise stable networks are compatible with a result stressed by several empirical studies: the majority of transfers takes place only between sub-groups of agents (see Rosenzweig, 1988, and Udry, 1994).

Second, we consider situations where the cost of linking to an agent is increasing in the number of links that this agent has already formed. This captures the fact that insurance agreements are informal and are honored if agents involved in a relationship invest time. In such situation, it is more difficult to establish a relationship with an agent who already has numerous links since she has less time available. Note that in this situation agents heterogeneity arises endogenously in the model: an agent who is involved in a lot of links is less valuable than an agent who is involved in few links. In this framework, pairwise stable networks contain between one and three groups of agents relative to the number of links they are involved in. Let us consider pairwise stable networks which contain three groups of agents. We show again that agents who obtain the smallest amount of insurance are always linked together. However, by contrast with the basic model, we show that agents who have the highest amount of insurance cannot be linked together. Concerning efficiency, we show that unlike in the basic model, if a network is efficient, then it is not always pairwise stable (or a network very similar to a pairwise stable network). More precisely, there exist situations where efficient networks and pairwise networks never coincide and the efficient network is under connected with respect to stability.

A recent theoretical literature about revenue sharing in developing economies examines the formation of risk-sharing networks. Bramoulle and Kranton (2006 and 2007) discuss the stability/efficiency dilemma of risk-sharing networks. A distinctive aspect of their work is that after the income realization occurs, linked pairs of agents meet (sequentially and randomly) and share their current money holding equally. The authors show that with many rounds of such meetings, an individual money holding converges to the mean of realized income in her group,[2] that is in a group there is always equal revenue sharing among individuals.

By contrast, in our paper we deal with situations where individuals do not engage in equal income sharing. In particular, after income sharing, an individual who has initially obtained high income always ends up better than an individual who has obtained low income. There is an interesting difference between our paper and the Bramoulle and Kranton papers concerning externalities generated by links.

In our paper, when an agent i forms a link with an agent j, this link may have a negative impact on the utility of i's neighbors (there is a negative externality), since agent i will now have less time to spend on relations with her neighbors (*idem* for j's neighbors). It follows

[2]In their paper, a group consists of agents who are directly or indirectly linked.

that when two agents have an incentive to form a link, this link may decrease social welfare. In the Bramoulle and Kranton model, when an agent i forms a link with an agent j, this link has a positive impact on the expected utility of i's neighbors. Indeed, in their model there is equal income sharing between all the agents of the groups. Therefore due to the additional link between i and j, i's neighbors will share their income with an additional agent (agent j) and their expected utility will increase. It follows that it can be that two agents have no incentive to form a link, and this link increases the social welfare.

Belhaj and Deroian (2012) also examine a situation where the bilateral partial risk-sharing rule is such that neighbors share equally a part of their revenue. However, they focus on the impact of informal risk-sharing on risk taking incentives when transfers are organized through a social network. Some papers explain partial risk-sharing by self-enforcing mechanisms in networks (Bloch et al., 2008).[3] These models consider that individuals can use their links to punish individuals who deviate from the insurance scheme. For instance, if an agent deviates from the insurance scheme (*i.e.* fails to transfer money to directly connected agents that have negative income shocks), the victim will communicate such behavior to other connected agents who in turn will terminate the insurance scheme with the deviating household as a punishment. In this paper, we do not deal with the self-enforcement mechanism problem. Instead, we assume that establishing a relationship is costly and it commits the parties to future resource sharing, say, due to a social norm or a social punishment in case of non-sharing.[4] More precisely, we assume that the self-enforcement mechanism problem is solved when agents invest enough time and resources in the informal insurance agreements.[5]

The rest of the paper is organized as follows. In section 2, we present the definitions and the basic model setup. In section 3, we provide the main properties of the payoff function of agents. In section 4, we examine pairwise stable networks and efficient networks in the basic model context. In section 5, we extend the basic model by introducing agents who do not

[3] This literature extends the literature about the robustness of mutual insurance (see for instance Genicot and Ray, 2003).

[4] This kind of relation can be illustrated with the marriage of daughters in India which are arranged to maximize gains from risk sharing, see Rosenzweig and Stark, 1989.

[5] The time an agent invests in the relationship and the social punishment in case of non-sharing are related since a bilateral relation in which agents have invested a lot of time can be more easily observed by the peers.

obtain the same income when they draw the high income state. In section 5, we assume that the cost of each link depends on both agents involved in. In section 6, we conclude.

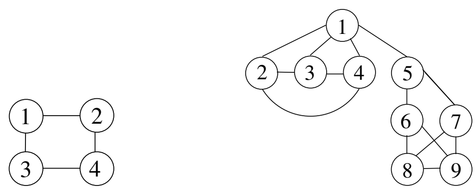
2 Basic model setup

Let $N = \{1, \ldots, n\}$ be a community of n, $n \geq 3$, *ex ante* identical agents. Agents receive an endowment and consume resources. Each agent's endowment is a random variable that takes two values. The low endowment state is called state 0 while the high endowment state is called state 1. Each agent i obtains an endowment 0 in state 0 while she obtains $\Theta > 0$ in state 1. State 1 occurs with probability $p > 0$ while the low endowment state occurs with probability $1 - p > 0$. The realizations of endowments are independent and identical across the agents.

Networks. To model bilateral mutual insurance agreements in a small population, we use tools from the theory of networks. Although the agreements themselves are bilateral, the amount of resources consumed by each agent depends on how many other agents she is connected with, and the endowments of these agents. Hence tools from network theory are useful for modeling such bilateral insurance networks. In the model, we assume that individuals i and j can have a mutual insurance agreement by forming a costly link between themselves. This assumption reflects the idea that there are always some costs (time at the least) to build a relationship.

We represent links and a network of links with the following notation: A network g is an $n \times n$ matrix, where $g_{ij} = 1$ when i and j are linked (*i.e.*, have established a risk-sharing agreement) and $g_{ij} = 0$ otherwise. We assume that risk-sharing relations are mutual, so that $g_{ij} = g_{ji}$. By convention, $g_{ii} = 0$. Let $g + g_{ij}$ denote the network obtained by replacing $g_{ij} = 0$ in g by $g_{ij} = 1$. Similarly, let $g - g_{ij}$ denote the network obtained by replacing $g_{ij} = 1$ in g by $g_{ij} = 0$. We say that there is a *chain* between two agents i and j in the network g if there exists a sequence of agents $i_1, \ldots, i_k$ such that $g_{ii_1} = g_{i_1 i_2} = g_{i_2 i_3} = \ldots = g_{i_k j} = 1$. A subset of agents is *connected* if there is a chain between any two agents in the subset. A *component* of the network g is a maximal connected subset of agents. Moreover, network $g[N']$ is a sub-network of g if it consists in the agents in $N' \subset N$, and $i \in N'$ and $j \in N'$ are linked in $g[N']$ *if and only if* they are linked in g.

The *empty network* is the network where all agents have formed no links. The *complete network* is the network where each agent has formed links with all the agents. A *k-regular* network is a network where *all* agents have formed exactly k links. A k_{-}*-regular network* is a network where all agents but one have formed k links; the agent who is the exception has formed $k-1$ links. A k_{+}*-regular network* is a network where all agents but one have formed k links; the agent who is the exception has formed $k+1$ links. An *almost-k-regular-network* is either a k_{-}-regular network, or a k_{+}-regular network. We illustrate the notions of *k-regular* network, and k_{+}*-regular network* in Figure 1.



2-regular network 3_{+}-regular network

Figure 1: Networks architectures

In the following, the neighbors of agent i, that is agents with whom i has formed a link, will play a crucial role. Hence we define $N_i(g) = \{j \in N \mid g_{ij} = 1\}$ as the set of the neighbors of i. Let $n_i(g) = |N_i(g)|$ be the degree of agent i. We let $\mathcal{N}_k(g) = \{i \in N \mid n_i(g) = k\}$ be the set of agents who form k links in g, $\mathcal{N}_k^{+}(g) = \{i \in N \mid n_i(g) > k\}$ and $\mathcal{N}_k^{-}(g) = \{i \in N \mid n_i(g) < k\}$ be the sets of agents who form more than k links, and less than k links in g respectively. Let $\mathcal{N}_k^{\neq}(g) = \mathcal{N}_k^{-}(g) \cup \mathcal{N}_k^{+}(g)$. Finally, we let $n^M(g) = \max_{i \in N} n_i(g)$ and $n^m(g) = \min_{i \in N} n_i(g)$.

Payoffs. Having described the set of players and their strategies, we now ask: Given a network g, how are expected payoffs determined under different endowment realizations in the network? We consider a benchmark model where agents are *ex ante* identical: they get the same resources Θ when they draw state 1 and the same resources 0 when they draw state 0. Moreover, if an agent draws state 1, then she always gives $\delta \in (0, 1)$ to each of her neighbors who has drawn state 0.[6] Conversely, if an agent draws state 0, then each of her neighbors

[6] Here, we assume that the transfer amount δ comes from some kind of social norm. Our goal in this paper is to study what is the architecture of the mutual insurance network within a community. Hence we do not explicitly

who has drawn state 1 always gives her δ. Note that in our model, agents may receive different amounts of transfers depending on the network architecture. Moreover, we assume that $\Theta > (n-1)\delta$, to ensure that players who draw state 1 always end better than agents who draw state 0, after transfers occur.

In this paper we assume that the payoff obtained by an agent, say i, can be divided into two parts.

1. The benefits part which involves uncertainty captures the fact that each additional link formed by i allows her to obtain additional insurance when she draws the bad state (0), and the fact that i has to insure more agents when she draws the good state (1).
2. The costs part which involves no uncertainty captures the fact that links are costly, with additional links being more costly.

We now present these two parts of an agent's payoff function.

Benefits. Working with a general payoff function in the context of a network formation problem poses tractability issues. Hence from now on we deal with the exponential utility function: $u_i(x) = 1 - \exp[-\rho x]$, where x is the income of agent i, and ρ is a positive parameter. Consequently, our model exhibits Constant Absolute Risk Aversion.[7] It follows that if agent i draws state 0 and k agents in her neighborhood draw state 1, then she obtains a benefit equal to $b^g(0,k) = u_i(k\delta) = 1 - \exp[-\rho k\delta]$. Conversely, if agent i draws state 1 and k agents in her neighborhood draw state 1, then she obtains a benefit equal to $b^g(1,k) = u_i(\Theta - (n_i(g) - k)\delta) = 1 - e^{[-\rho(\Theta-(n_i(g)-k)\delta)]}$.

We now define the expected neighborhood benefits (ENB) function, $B_i(g)$, which captures the expected benefits obtained by an agent i given her neighborhood $n_i(g)$. We have:

$$\begin{aligned} B_i(g) = \phi(n_i(g)) \quad &= \quad p \textstyle\sum_{k=0}^{n_i(g)} \binom{n_i(g)}{k} p^k (1-p)^{n_i(g)-k} b^g(1,k) \\ &\quad + (1-p) \textstyle\sum_{k=0}^{n_i(g)} \binom{n_i(g)}{k} p^k (1-p)^{n_i(g)-k} b^g(0,k), \end{aligned} \tag{1}$$

where $\binom{x}{y}$ is just the probability of y high resources out of x draws.

In the following, for each function f we set $\Delta f(x) = f(x) - f(x-1)$. Moreover, since we

model where δ comes from.

[7] Here we use an exponential function, but we obtain the same qualitative results for some other CARA functions.

use non-continuous function, we use a slightly modified version of convexity and concavity. We say that f is concave if for all x, $\Delta f(x+1) - \Delta f(x) \leq 0$ and f is convex if for all x, $\Delta f(x+1) - \Delta f(x) \geq 0$.

Costs of links. Informal insurance arrangements are potentially limited by the presence of various incentive constraints. As a first cut, it appears that the most important constraint arises from the fact that these arrangements are informal, *i.e.*, not written on legal paper. It follows that they will be honored only if agents involved in such a relationship invest time. Since each agent has a limited amount of time, the costs for agent i of forming an additional link with some agent j should increase with the number of links formed by agent i. We have:

$$C_i(g) = f_1(n_i(g)),$$

where f_1 is a strictly increasing and convex function. In addition, to simplify the analysis we assume that $f_1(0) = 0$ and $f(1) - f(0) \neq \phi(1) - \phi(0)$.

Expected payoffs function. The expected payoff function, $U_i(g)$, of each agent i, given the network g, is the difference between the ENB function and the cost function of forming links:

$$U_i(g) = B_i(g) - C_i(g) = \Phi(n_i(g)) = \phi(n_i(g)) - f_1(n_i(g)). \tag{2}$$

Pairwise stable networks and efficient networks. A network g is *pairwise stable* if no pair of unlinked agents would benefit by forming a link in g and if no agent would benefit from severing one of her existing links in g. Formally, following Jackson and Wolinsky (1996) we have (i) for all $g_{ij} = 1$, $U_i(g) \geq U_i(g - g_{ij})$ and $U_j(g) \geq U_j(g - g_{ij})$; and ($ii$) for all $g_{ij} = 0$, if $U_i(g) \leq U_i(g + g_{ij})$, then $U_j(g) > U_j(g + g_{ij})$.

An *efficient network* is one that maximizes the sum of the expected payoffs of the agents. Let $W(g) = \sum_{i \in N} U_i(g)$ be the total expected payoffs obtained in a network g. A network g^e is efficient if $W(g^e) \geq W(g)$ for all networks g.

3 Properties of the ENB function

Our first proposition provides some useful properties of the ENB function.

Proposition 1 *Suppose that the ENB function is given by equation 1. (a) Then, the ENB function is strictly increasing and strictly concave with the number of links formed. (b) Moreover, the marginal ENB function increases with* Θ*. (c) Finally, there exists* $\tilde{p} \in (1/2, 1)$ *such that the marginal ENB function increases with* p *iff* $p < \tilde{p}$*.*

Proof See Appendix in 8.1.

□

Proposition 1 (a) states that the ENB obtained by agent i is increasing. In other words, each agent i prefers to be more insured than less insured when the cost of insurance (the cost of forming links) is sufficiently low. Moreover, Proposition 1 implies that Φ is strictly concave: the marginal ENB that an agent i obtains from an additional link strictly decreases with the number of links she has formed. Consequently, if the cost of forming links is constant, then the incentive of an agent to form an additional link decreases with the number of links she has formed.

Since ϕ is strictly concave and $-f_1$ is concave, Φ is strictly concave. We have three possibilities. (i) Suppose $\Phi(1) - \Phi(0) < 0$. Then, $\Phi(k) - \Phi(k-1) < 0$, for all $k \in \{2, \ldots, n\}$: no agent has an incentive to form a link. (ii) Suppose $\Phi(1) - \Phi(0) = 0$. Again, $\Phi(k) - \Phi(k-1) < 0$, for all $k \in \{2, \ldots, n\}$. Consequently, each agent has a weak incentive to form one link instead of zero. (iii) Suppose $\Phi(n-1) - \Phi(n-2) \geq 0$. Then, $\Phi(k) - \Phi(k-1) > 0$, for all $k \in \{2, \ldots, n\}$: each agent has a weak incentive to form $n-1$ links. (iv) Suppose $\Phi(1) - \Phi(0) > 0$ and $\Phi(n-1) - \Phi(n-2) < 0$. Then, there exists $\tilde{k} \in \{1, \ldots, n-2\}$, such that $\Phi(\tilde{k}) - \Phi(\tilde{k}-1) \geq 0$ and $\Phi(\tilde{k}+1) - \Phi(\tilde{k}) < 0$. Let $k^\star$ be either the highest k such that $\Phi(k) - \Phi(k-1) \geq 0$, or 0 if for all $k \in \{1, \ldots, n-1\}$, $\Phi(k) - \Phi(k-1) < 0$. We have $\Phi(k^\star) \geq \Phi(k)$, for all $k \in \{0, \ldots, n-1\}$.

Proposition 1 (b) suggests that we should observe more insurance links in population with high incomes in state 1. In that case, the insurance mechanism increases the gap between the expected well-being of high potential income communities and the expected well-being of low potential income communities.

Let us now give the intuition behind this result. Obviously, when agent i draws the low income state, then her ENB is not affected by her income. Suppose now that agent i draws the high income state. Due to the strict concavity of the payoff function, when the income increases, the utility function of agent i is less affected by the loss of money she incurs when she has to help one of her neighbors. It follows that the marginal expected neighborhood benefits function of agent i increases with Θ.

Proposition 1 (c) suggests that we should observe more insurance links in a population with high probability of success than in a population with low probability of success when the probability of success is low. Conversely, we should observe less insurance links in a population with high probability of success than in a population with low probability of success when the probability of success is high.
The intuition of this result is easy to understand from the polar cases. When an agent has a zero probability of failure then she obtains no marginal benefits from forming an insurance links. So, these marginal benefits are lower than the marginal benefits obtained by an agent who has high probability of success. Similarly, when agents have a zero probability of success then they obtain no marginal benefits from forming insurance links. So, these marginal benefits are lower than the marginal benefits obtained by agents who have a low probability of success.

4 Pairwise stable and efficient networks analysis

We prove the existence and we characterize pairwise stable networks. To ensure the existence of pairwise stable networks, we use a theorem established by Erdös and Gallai. We need the following definition to present this theorem.

Definition 1 *A finite sequence $s = (d_1, d_2, \ldots, d_n)$, such that $d_1 \geq d_2 \geq \ldots \geq d_n$, of nonnegative integers is graphical if there exists a network g whose nodes have degrees $d_1, d_2, \ldots, d_n$.*

Theorem 1 *(Erdös and Gallai, 1960) A sequence $s = (d_1, d_2, \ldots, d_n)$ of nonnegative integers,* *such that $d_1 \geq d_2 \geq \ldots \geq d_n$, and whose sum is even is graphical if and only if*

$$\sum_{i=1}^{r} d_i \leq r(r-1) + \sum_{i=r+1}^{n} \min\{d_i, r\}, \text{ for every } r,\ 1 \leq r < n.^{8} \tag{3}$$

In the next proposition, we focus on the pairwise stable networks. We show that there always exists a pairwise stable network. We establish that no agent forms more than $k^\star$ links in pairwise stable networks and agents who form strictly less than $k^\star$ links are linked together. Moreover, we prove that networks where agents form $k^\star$ links, *i.e.*, agents are in symmetric position relative to the number of insurance links they form, are pairwise stable and efficient when n is even. Recall that in $k^\star$-regular networks, agents do not always obtain the same amount of benefits: in our model agents who draw state 0 always obtain a lower amount of benefits than the benefits obtained by agents who draw state 1. Finally, we establish that the complete network is the unique pairwise stable network when the cost of forming links is sufficiently low.

To present the next proposition, we need some additional definitions. Let $g[\mathcal{N}_k^{\neq}]$ be the sub-network associated with the set of agents $\mathcal{N}_k^{\neq}(g)$.

Proposition 2 *Suppose that the ENB function is given by equation 1.*

(a) There always exists a pairwise stable network.

(b) Network g is pairwise stable if and only if $\mathcal{N}_{k^\star}^{\neq}(g) = \mathcal{N}_{k^\star}^{-}(g)$ and $g[\mathcal{N}_{k^\star}^{\neq}]$ is complete.

(c) If n or $k^\star$ is even, then $k^\star$-regular networks are pairwise stable and efficient.

(d) If n and $k^\star$ are odd, then $k^\star_-$-regular networks are pairwise stable. Moreover, if $\Phi(k^\star+1) < \Phi(k^\star-1)$, then $k^\star_-$-regular networks are the unique efficient networks; and if $\Phi(k^\star+1) > \Phi(k^\star-1)$, then $k^\star_+$-regular networks are the unique efficient networks.

(e) If there is no costs of forming links, then the unique pairwise equilibrium network is the complete network.

Proof See Appendix in 8.2. □

Let us provide the intuition of Proposition 2. Part (b) follows the fact that no agent has an incentive to form strictly more than $k^\star$ links and that two unlinked agents, who have formed

[8] The theorem can also be found in Harary, 1969, Chapter 6 pp. 59-62 and the statement here is based on his presentation.

a number of links strictly smaller than $k^\star$, have an incentive to form a link together. We now deal with the pairwise stable networks described in parts (c) and (d). By construction, agents who form $k^\star$ links have no incentive to modify her strategy. Similarly, since Φ is strictly concave, if agent i cannot form $k^\star$ links, then she forms $k^\star - 1$ links in a pairwise equilibrium. By Theorem 1, we know that if n or $k^\star$ is even, then the sequence $(k^\star, \dots, k^\star)$ is graphical and when if n and $k^\star$ are odd the sequence $(k^\star - 1, k^\star, \dots, k^\star)$ is graphical. Consequently, If n or $k^\star$ is even, then $k^\star$-regular networks are pairwise stable and if n and $k^\star$ are odd then, $k^\star_-$-regular networks are pairwise stable. Part (a) is straightforward from parts (c) and (d). Part (e) follows the fact that the ENB is strictly increasing. Finally, we deal with the efficient networks described in parts (c) and (d). Due to the strict concavity of Φ in efficient networks every agent should form $k^\star$ links. Again, by Theorem 1 we know that the sequence $(k^\star, \dots, k^\star)$ is not always graphical. If this sequence is not graphical, then agents who do not form $k^\star$ links have to form either $k^\star - 1$ or $k^\star + 1$ links since Φ is strictly concave. Obviously, an agent forms $k^\star - 1$ links instead of $k^\star + 1$ links if and only if $\Phi(k^\star - 1) > \Phi(k^\star + 1)$. Finally, by Theorem 1, we know that the sequences $(k^\star - 1, k^\star, \dots, k^\star)$ and $(k^\star, \dots, k^\star, k^\star + 1)$ are graphical when the sequence $(k^\star, \dots, k^\star)$ is not graphical. Consequently, in an efficient network there is at most one agent who does not form $k^\star$ links; this agent forms either $k^\star - 1$, or $k^\star + 1$ links.

Let us now discuss Proposition 2. First, there is a kind of solidarity among the less insured agents: agents who obtain the smallest amount of insurance are always linked together. This property illustrates the fact that agents, who do not have a sufficient amount of insurance, will always reach mutual insurance agreements.

Second, we observe that in our setting agents can be partitioned into distinct components in a pairwise stable network. Moreover some of these stable insurance networks may also be locally complete as the network between agents 1, 2 and 3 in the following example.

Example 1 Suppose $N = \{1, \dots, 12\}$ and $k^\star = 2$. Then, network g shown in Figure 2 is a pairwise stable network.

Third, networks where agents are in symmetric positions and networks where agents are in asymmetric positions are candidates to be pairwise stable.

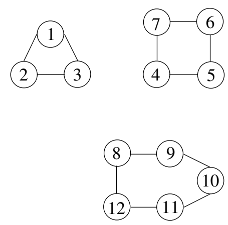


Figure 2: Network g

Fourth, results in Ben-Porath (1980), Platteau (1991), Fafchamps (1992), indicate that informal insurance of an agent does not involve all the village, but just a part of it (family or friends of this agent). Proposition 2 is in line with this result since the complete network is a pairwise stable only if the cost of forming links are sufficiently low.

Fifth, we note that if n or $k^{\star}$ are even, then an efficient network is always pairwise stable while there exist pairwise stable networks that are not efficient. More precisely, non-efficient pairwise stable networks are always under-connected with regard to efficiency. Indeed, we know that in an efficient network each agent forms $k^{\star}$ links. In a pairwise stable network no agent forms more than $k^{\star}$ links, but some agents can form less than $k^{\star}$ links. Suppose now that n and $k^{\star}$ are odd. If $\Phi(k^{\star}+1) < \Phi(k^{\star}-1)$, then $k^{\star}_{-}$-regular network are efficient and pairwise stable networks. If $\Phi(k^{\star}+1) > \Phi(k^{\star}-1)$, then $k^{\star}_{+}$-regular networks are efficient. These networks are very similar to pairwise stable $k^{\star}_{-}$-regular networks. However, as in the case where n or $k^{\star}$ are even, there exist pairwise stable networks that are significantly under connected with regard to efficiency. Recall that empirical results (see Lund and Fafchamps, 2003) refute the hypothesis that informal insurance system achieve an efficient risk sharing.

Let us now provide some some insights concerning the role played by the value of the gift δ. This parameter can be interpreted as a social norm: it is the value that a lucky agent should give to an unlucky agent. We establish through two examples that the social norm plays an ambiguous role on the number of links of the insurance networks.

Example 2 Suppose that $N = \{1, \ldots, 6\}$, $(\delta, p, \rho, \Theta) = (0.15, 0.3, 0.5, 15)$, and $C_i(g) = 0.05 \times n_i(g)$, then the complete network is a pairwise stable network, and the empty network is not a pairwise stable network. Suppose now that $\delta = 0.1$ instead of 0.15, then the empty network is pairwise stable. In other words, when p and δ are low, an increase of the social norm implies that agents should increase their number of links.

Example 3 Suppose that $N = \{1, \ldots, 6\}$, $(\delta, p, \rho, \Theta) = (0.95, 0.93, 0.95, 15)$, and $C_i(g) = 0.266 \times n_i(g)$, then the empty network is the unique pairwise stable network. Suppose now that $\delta = 0.91$ instead of 0.95, then the empty network is not pairwise stable. In other words, when p and δ are high, an increase of the social norm implies that agents should decrease their number of links.

5 Pairwise stable networks with heterogeneous agents

We examine a situation where agents do not obtain the same income when they draw state 1. More precisely, we assume that the population is partitioned into two sub-sets of agents: N^{Θ} and $N^{\Theta'}$. Agents in N^{Θ} obtain an income equal to Θ if they draw state 1, and agents in $N^{\Theta'}$ obtain an income equal to $\Theta' < \Theta$ if they draw state 1. In the following, we call the members of N^{Θ} *high potential agents* and the members of $N^{\Theta'}$ *low potential agents*.

We denote by $\Phi^x(n_i(g))$ the expected payoff of agent $i \in N^x$, $x \in \{\Theta, \Theta'\}$, when she forms $n_i(g)$ links. By using similar argument as in the previous section, we define k^x, $x \in \{\Theta, \Theta'\}$, either as the highest number such that $\Phi(k^x) - \Phi(k^x - 1) \geq 0$, or $k^x = 0$ if such a number does not exist. We observe that k^x maximizes $\{\Phi^x(k) : k \in \{0, \ldots n-1\}\}$. For $x \in \{\Theta, \Theta'\}$, let $\mathcal{N}_x^{\neq}(g) = \{i \in N^x : n_i(g) \neq k^x\}$ be the set of agents who belong to N^x and do not form an optimal number of links in g; and let $\mathcal{N}^{\neq}(g) = \mathcal{N}_{\Theta}^{\neq}(g) \cup \mathcal{N}_{\Theta'}^{\neq}(g)$ be the set of agents who do not form an optimal number of links in g. Similarly, we define for $x \in \{\Theta, \Theta'\}$, $\mathcal{N}_x^{-}(g) = \{i \in N^x : n_i(g) < k^x\}$ and $\mathcal{N}^{-}(g) = \mathcal{N}_{\Theta}^{-}(g) \cup \mathcal{N}_{\Theta'}^{-}(g)$. By Proposition 1, we know that $\partial \Delta B_i(g, ij)/\partial \Theta > 0$, so $k^{\Theta} \geq k^{\Theta'}$.

In the next proposition, we establish that the results obtained in Proposition 2 are robust. Moreover, we show that there exist costs of forming links such that only high potential agents form links in pairwise stable networks.

Proposition 3 *Suppose that the benefits function is given by equation* (1)*,* $N^x \neq \emptyset$ *and* $k^x \leq |N^x|$ *for* $x \in \{\Theta, \Theta'\}$.

(a) There always exists a pairwise stable network.

(b) Network g is pairwise stable if and only if $\mathcal{N}^-(g) = \mathcal{N}^{\neq}(g)$ and $g[\mathcal{N}^{\neq}]$ is complete.

(c) If both N^{Θ} and $N^{\Theta'}$ are even, then networks g where $g[N^x]$ is k^x-regular, $x \in \{\Theta, \Theta'\}$, and there is no link between agents in N^{Θ} and agents in $N^{\Theta'}$ are pairwise stable and efficient.

(d) If both N^x, $x \in \{\Theta, \Theta'\}$, and N^y, $y \in \{\Theta, \Theta'\} \setminus \{x\}$, are odd, then either (i) network g where $g[N^x]$ is k^x_--regular and $g[N^y]$ is k^y-regular and there is no link between agents in N^{Θ} and agents in $N^{\Theta'}$ are pairwise stable networks, or (ii) networks g where each agent in N^x forms k^x links and each agent in N^y forms k^y links are pairwise stable networks and efficient networks.

(e) Suppose that $f_1(n_i(g)) = Fn_i(g)$, $F > 0$. Then, there exists F such that only agents in N^{Θ} have formed links.

Proof See Appendix in 8.3. □

Intuitions of Proposition 3 parts (a), (b), (c) and (d) are similar to the intuition provided for Proposition 2. Part (e) of Proposition 3 follows the fact that $\partial \Delta B_i(g, ij)/\partial \Theta > 0$. Due to this property, the marginal expected utility obtained by player $i \in N^{\Theta}$ due to the first link she forms is higher than the marginal expected utility obtained by player $i \in N^{\Theta'}$ due to the first link she forms. It follows that there exists a cost of forming the first link $F > 0$, such that agents in N^{Θ} have an incentive to form this first link while agents in $N^{\Theta'}$ have no incentive to form this first link. In Part (e) of Proposition 3, we establish that there exist situations where high potential agents have formed links together while low potential agents have formed no links. In other words, only high potential people get access to insurance. In that case, the insurance mechanism increases the gap between the expected well-being of high potential people and the expected well-being of low potential people. In Example 4, we exhibit a situation where $g[N^{\Theta}]$ is complete and $g[N^{\Theta'}]$ is empty. In that case, the insurance mechanism increases the gap between the expected well-being of high potential people and the expected well-being of low potential people.

Example 4 We assume $N = \{1, \ldots, 6\}$, $\Theta = 7$, $\Theta' = 6$, $N^{\Theta} = \{1, 2, 3\}$, $N^{\Theta'} = \{4, 5, 6\}$, $\rho = 0.2$, $\delta = 0.5$, $p = 0.13$ and $C_i(g) = 0.02 \times n_i(g)$. Then network g, where $g_{N^{\Theta}}$ is complete and $g_{N^{\Theta'}}$ is empty, is the unique pairwise stable network.

Empirical studies (see Rosenzweig, 1988, and Udry, 1994) stress the fact that the majority of transfers takes place only between sub-groups of agents. This evidence is in line with Part (e) of the Proposition and Example 4. In our model, it is likely to obtain situations where high potential agents are better insured than low potential agents.

It is worth noting that in parts (c) we highlight the fact that there exist pairwise equilibrium networks where agents in N^{Θ} and agents in $N^{\Theta'}$ form links only with agents that belong to the same group. Hence, we find a possible understanding of why an agent i forms insurance links only with a sub-set of the village population.

We have examined players who are heterogeneous with regard to their income in state 1. By using Proposition 1.c, it is also possible to examine players who are heterogeneous with regard to their probability to draw state 1. Note that if the population is partitioned into two sub-sets of agents: N^p and $N^{p'}$, where agents in N^p (resp. $N^{p'}$) has a probability p (resp. p') to draw state 1, with $p > p'$. We denote by $\Phi^x(n_i(g))$ the expected payoff of agent $i \in N^x$, $x \in \{p, p'\}$, when she forms $n_i(g)$ links.

In the next proposition, we focus on cases where the heterogeneity of probabilities leads to pairwise networks where members of a subset of the population has formed no insurance links, while members of the other subset of the population has formed insurance links.

Proposition 4 *Suppose that the benefits function is given by equation* (1)*,* $N^x \neq \emptyset$ *for* $x \in \{p, p'\}$*, and* $f_1(n_i(g)) = Fn_i(g)$*,* $F > 0$*.*

1. *Suppose that* $p' > \tilde{p}$*. Then, there exists* F *such that only agents in* $N^{p'}$ *have formed links.*
2. *Suppose that* $p < \tilde{p}$*. Then, there exists* F *such that only agents in* N^p *have formed links.*

Proof We show the first part of the proposition. By Proposition 1 (c), we know that the marginal expected payoff function associated with each link k is lower for agents in N^p than agents in $N^{p'}$. By Proposition 1 (a), the ENB function is concave: the maximal marginal payoff is associated with the first link formed by a player. Consequently, $(\Phi^{p'}(1) - \Phi^{p'}(0)) - (\Phi^p(k) - \Phi^p(k-1)) > 0$, for $k \in \{1, \ldots, n-1\}$ and there exists $F > 0$ such that $(\Phi^{p'}(1) - \Phi^{p'}(0)) > F$ and $\Phi^p(1) - \Phi^p(0) < F$.

We show the second part of the proposition by using the same arguments as in the previous part and the fact that the marginal expected payoff function associated with each link k is higher for agents in N^p than agents in $N^{p'}$. □

In this proposition, we establish that when the probability that state 1 occurs is sufficiently high, then there exist parameters such that agents with the highest probability of success does not want to form insurance links, while players with the highest probability of success form insurance links together. In that situation the insurance mechanism decreases the gap between the expected well-being between the two types of agents. Similarly, when the probability that state 1 occurs is sufficiently low, then there exist parameters such that agents with the lowest probability of success does not want to form insurance links, while players with the highest probability of success form insurance links together. In that situation the insurance mechanism increases the gap between the expected well-being between the two types of agents.

6 Costs of forming links depend on the neighborhood of agents

We have assumed that the costs for agent i of forming an additional link with some agent j should increase with the number of links formed by agent i. Here we take into account the fact that these costs should also increase with the number of links formed by agent j. Indeed, it is more difficult to establish a relationship with an agent who already has numerous links since she has less time available.[9]

6.1 Cost function and expected payoff function

We assume the following cost function for link formation:

$$C_i(g) = f_1(n_i(g)) + \sum_{\ell \in N_i(g)} f_2(n_\ell(g)),$$

[9] Another option that makes such informal arrangements feasible is the threat of punishment as in Bloch, Genicot and Ray, 2008.

where f_2 is strictly increasing and convex on $\{1, \ldots, n-1\}$. We observe that this cost function extends the cost function we used in the previous section; the two costs function coincide when $f_2(x) = 0$, for all $x \in \{1, \ldots, n-1\}$.

Given this cost function, an additional link formed with agent j induces a cost for agent i equal to

$$C_i(g + ij) - C_i(g) = \Delta f_1(n_i(g) + 1) + f_2(n_j(g) + 1).$$

Since f_1 is strictly increasing and f_2 is a strictly positive valued function, $C_i(g+ij)-C_i(g) > 0$. The expected payoff function, $U_i(g)$, of each agent i, given the network g, is the difference between the ENB function and the cost function of forming links. We assume the same ENB as in section 2. We have:

$$U_i(g) = B_i(g) - C_i(g) = \phi(n_i(g)) - \left(f_1(n_i(g)) + \sum_{\ell \in N_i(g)} f_2(n_\ell(g)) \right). \tag{4}$$

To simplify the analysis, we assume that $\phi(1) - [f_1 + f_2](1) - (\phi(0) - f_1(0)) \neq 0$ and $\phi(n) - [f_1 + f_2](n) - (\phi(n-1) - [f_1 + f_2](n-1)) \neq 0$.

Proposition 1 allows us to characterize some properties of the marginal payoffs, $\Delta U_i(g, ij) = B_i(g + ij) - C_i(g + ij) - (B_i(g) - C_i(g))$, obtained by agent i in a network g when she forms an additional link with agent j. We have $\Delta U_i(g, ij) = \gamma(n_i(g) + 1, n_j(g) + 1)$ and

$$\gamma(n_i(g) + 1, n_j(g) + 1) = \Delta\phi(n_i(g) + 1) - \Delta f_1(n_i(g) + 1) - f_2(n_j(g) + 1),$$

Clearly, γ is strictly decreasing in its first argument since $\Delta\phi$ is strictly decreasing by Proposition 1 and Δf_1 is strictly increasing. Similarly, γ is strictly decreasing in its second argument since f_2 is strictly increasing.[10] Let $k^\star$ be either the highest k such that $\gamma(k^\star, k^\star) \geq 0$, or 0 if for all $k \in \{1, \ldots, n-1\}$, $\Phi(k) - \Phi(k-1) < 0$. Recall that γ is strictly decreasing in its two arguments. Hence, if $k^\star > 0$, then $\gamma(k, k) > 0$ for all $k < k^\star$.

6.2 Pairwise stable networks and efficient networks

The next proposition imposes conditions that a pairwise stable network must satisfy and sheds light on when insurance arrangements will exhibit symmetric and asymmetric structures. In

[10] T. Morrill (2011) has also defined a class of network formation game with degree-base utility function under negative externalities. However in order to deal with situations where transfers between agents are allowed, the author uses a simplified payoff function where the payoffs of an agent are linear in her number of links.

particular, this proposition does not exclude from the set of pairwise stable networks those networks where agents are in asymmetric positions relative to the amount of insurance they receive. Moreover, we establish that networks where some agents form strictly more than $k^\star$ links are candidate to be pairwise equilibrium networks. Finally, in this proposition, we bound the difference in the number of links formed by the agent who has formed the highest number of links and the agent who has formed the highest number of links in a pairwise stable network.

Proposition 5 *Suppose that the payoff function satisfies equation* (4)*.*

(a) There always exists a pairwise stable network.

(b) Let g be a pairwise stable network. Then, $g[\mathcal{N}^+_{k^\star}]$ is empty and $g[\mathcal{N}^-_{k^\star}]$ is complete. Moreover, if $\gamma(k^\star, k^\star+1) < 0$, then $\mathcal{N}^-_{k^\star}(g) = \mathcal{N}^{\neq}_{k^\star}(g)$.

(c) Suppose g contains agents i, i', j and j' such that $n_{i'}(g) \leq n_i(g) < k^\star < n_{j'}(g) \leq n_j(g)$. If there is a link between agents i and j in g, then there is a link between agents i' and j' in g.

(d) We have $|\mathcal{N}^+_{k^\star}(g)| < n/2$ and $n^M(g) - n^m(g) \leq |\mathcal{N}_{k^\star}(g)|$.

Proof See Appendix in 8.4. □

We now graphically illustrate Proposition 5. In Figure 3, network g satisfies the properties given in (b). Indeed, if we assume that $k^\star = 4$, we observe that agents 1 and 2 are involved in $k^\star + 1$ links, agents 3, 4 and 5 are involved in $k^\star$ links and agents 6 and 7 are involved in $k^\star - 1$ links, and they are linked.

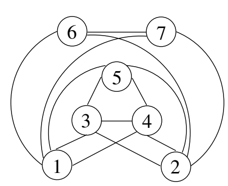


Figure 3: Network g satisfying (b) and (c)

Note that in Proposition 5 we do not examine agents who have formed exactly $k^\star$ links. Indeed, these agents can form links both with agents who have formed $x > k^\star$ links and with agents who have formed $y < k^\star$ links.

In Proposition 5, we highlight several properties of pairwise stable networks in a community of *ex ante* identical agents when we assume that the cost of forming an insurance link with an agent depends on the number of links formed by both agents involved in. First, there now exist agents who obtain insurance from strictly more than $k^\star$ agents (they are the most insured agents). These agents are never linked together. In other words, when insurance links require time to be maintained, some agents play a specific role in the provision of mutual insurance in the pairwise stable networks. These agents insure (and are insured by) a large part of the population. But an agent of this type does not interact with other agents of this type. In some sense there may exist "some insurance leaders" but these leaders themselves are not linked by mutual insurance agreements. Furthermore, we find again that there is a kind of solidarity among the less insured agents: agents who obtain the smallest amount of insurance are always linked together. This property illustrates the fact that agents, who do not have a sufficient amount of insurance, will always reach mutual insurance agreements. Pairwise stable networks may divide the population into sets of agents who are in asymmetric positions relative to their risk exposure. In other words, in a pairwise stable network, some agents are better off since they obtain insurance from others agents, who are involved in few mutual insurance agreements themselves. However, even if agents can have different numbers of bilateral insurance agreements, this difference is bounded. This result is in line with the evidence (see Lund and Fafchamps, 2003, who establish that certain categories of household are better insured than others).

We now deal with efficient networks. In this section, we always use the payoff function given by equation (4) and we let $\eta(k) = \phi(k) - f_1(k) - kf_2(k)$ for $k \neq 0$ and $\eta(0) = 0$. By Proposition 1, ϕ is strictly concave, and f_1 is convex. Moreover, kf_2 is convex, since f_2 is increasing and convex. It follows that η is strictly concave. Recall that $\phi(1) - [f_1 + f_2](1) - (\phi(0) - f_1(0)) \neq 0$ and $\phi(n-1) - [f_1 + f_2](n-1) - (\phi(n-2) - [f_1 + f_2](n-2)) \neq 0$. There are three possibilities (*a*) $\eta(1) - \eta(0) < 0$. In that case we have $\eta(0) > \eta(k)$, for all $k \in \{1, \ldots, n-1\}$, (*b*) $\eta(n-1) - \eta(n-2) > 0$. In that case we have $\eta(n) > \eta(k)$, for all $k \in \{0, \ldots, n-2\}$, (*c*) $\eta(1) - \eta(0) > 0$ and $\eta(n-1) - \eta(n-2) < 0$. In that case, there exists $\hat{k}$ such that $\eta(\hat{k}) > \eta(k)$, for all $k \neq \hat{k}$. It follows that η always admits a unique maximum, k^e. To simplify the analysis, we extend η to $\{-1, \ldots, n\}$ and we let $\eta(-1) = \eta(n) = -\infty$.

Note that the neighbors of agent i are connected to an agent with $n_i(g)$ neighbors. Consequently, we have $W(g) = \sum_{i\in N}[\phi(n_i(g)) - f_1(n_i(g)) - n_i(g)f_2(n_i(g))] = \sum_{i\in N}\eta(n_i(g))$. We know that there exists $k^e \in \{0,\dots,n-1\}$ such that $\eta(k^e)$ is maximal. Therefore $\sum_{i\in N}\eta(k^e) \geq \sum_{i\in N}\eta(n_i(g))$ for all $n_i(g) \in \{0,\dots,n-1\}$. Moreover, we have $\arg\max_{k\neq k^e}\eta(k) \subset \{k^e-1, k^e+1\}$, since η is concave and maximum for $k = k^e$. Therefore, we have $\sum_{i=1}^{n-1}\eta(k^e) + \max\{\eta(k^e-1), \eta(k^e+1)\} \geq \sum_{i=1}^{n-1}\eta(k^e) + \eta(n_n(g))$ for $n_n(g) \neq k^e$. These observations are summarized in the Part (a) of the next proposition.

Proposition 6 *Suppose that the payoff function satisfies equation* (4) *and let g^e be a non-empty efficient network.*

(a) Then, g^e is either a k^e-regular network, or an almost-k^e-regular network.

(b) Suppose n is even. Let $g^\star$ be a regular non-empty pairwise stable network. Then agents in $g^\star$ form at least the same number of links as in an efficient network.

(c) If $\gamma(k^e+1, k^e+1) - \gamma(k^e, k^e) > (k^e-1)(f_2(k^e-1) - f_2(k^e))$, then a non-complete efficient network is not pairwise stable.

Proof Part (a) of the proposition follows Lemma 4 for the existence and the arguments given above the proposition. We now show part (b) of the Proposition. First, suppose n is even and let $g^\star$ be a regular pairwise stable network. By Proposition 4 (b), we know that $g[\mathcal{N}^-_{k^\star}]$ is complete. It follows that $g^\star$ is a $k^\star$-regular network. Second, by the arguments presented before the proposition, an efficient network is a k^e-regular network. Moreover, we have for $0 < k < n$, $\gamma(k,k) - (\eta(k) - \eta(k-1)) = (k-1)(f_2(k) - f_2(k-1))$. We have $\gamma(k,k) - (\eta(k) - \eta(k-1)) \geq 0$, since f_2 is strictly increasing. Moreover, by definition of k^e, we have $\eta(k^e) - \eta(k^e-1) \geq 0$. It follows that we have $\gamma(k^e, k^e) \geq \eta(k^e) - \eta(k^e-1) \geq 0$. We know that by definition of $k^\star$, we have $\gamma(k,k) < 0$, for all $k > k^\star$. It follows that $k^\star$ is at least equal to k^e. Consequently, agents in a k^e-regular network have no incentive to remove a link.

In Part (c) we establish that if $\gamma(k^e, k^e) - \gamma(k^e+1, k^e+1) < (k^e-1)(f_2(k^e) - f_2(k^e-1))$, then the efficient network is not pairwise stable. We have $\gamma(k^e+1, k^e+1) - (\eta(k^e) - \eta(k^e-1)) = \gamma(k^e+1, k^e+1) - \gamma(k^e, k^e) + \gamma(k^e, k^e) - (\eta(k^e) - \eta(k^e-1)) > (k^e-1)(f_2(k^e-1) - f_2(k^e)) - (k^e-1)(f_2(k^e-1) - f_2(k^e)) = 0$. By construction, $\eta(k^e) - \eta(k^e-1) \geq 0$. It follows that $\gamma(k^e+1, k^e+1) - (\eta(k^e) - \eta(k^e-1)) > 0$ implies $\gamma(k^e+1, k^e+1) > 0$. Each agent in g^e has an incentive to form an additional link. It follows that g^e is not pairwise stable since it is not complete. □

The intuition behind Part (b) is as follows. Let $g^\star$ be the $k^\star$-regular network, and let i, j, k be three agents such that $g^\star_{ij} = g^\star_{ik} = 1$ and $g^\star_{kj} = 0$. If agents i deletes her link with agent j, then agent k will benefit from this deletion, since her payoffs will increase by $f_2(n_i(g^\star)-1)-f_2(n_i(g^\star))$. However, when i decides whether to delete her link with j in $g^\star$, she does not take into account the positive externality that would accrue to k from the deletion of this link.

7 Conclusion

In this paper, we study situations where agents form some mutual informal insurance arrangements on their own. More precisely, we examine when agents will create agreements with their neighbors concerning the following transfer scheme: each agent helps her neighbors who draw the low income state when she herself draws the high income state and each agent is helped by her neighbors who draw the high income state when she herself draws the low income state.

We find that efficient networks are either k-regular networks, or almost-k-regular networks. In other words, only networks where agents obtained a very similar level of insurance are efficient networks. By contrast, this is not always true for pairwise stable networks. Under certain conditions they can be asymmetric, with those having the lowest levels of insurance always connecting to each other, while those with the highest levels of insurance never forming any links with each other.

Then, we introduce in the model two types of heterogeneity: an exogenous one, where agents differ in their income and an endogenous heterogeneity where the costs of linking to an agent depends on the number of links the latter has already formed in the network. We examine the impact of these heterogeneities on stability and efficiency. We obtain several distinctive results. In particular, when agents do not obtain the same income when they draw the high income state, we show that the insurance mechanism may increase the gap between the expected well-being of high income potential people and the expected well-being of low income potential people. Indeed, in some situation only agents who obtain the highest income when they draw the high income form insurance links. Finally, we assume that the cost of each link depends on both agents involved in. This assumption implies that there exist three types of agents, with regard to the number of links they form, in a pairwise network. Agents, who form the highest number of links, do not form links together and agents, who form the smallest number of links, are all linked together. In this context, we provide condition under which an efficient network is not pairwise stable.

References

[1] M. Belhaj and F. Deroian. Risk taking under heterogenous revenue sharing. *Journal of Development Economics*, 98(2):192–202, 2012.

[2] F. Bloch, G. Genicot, and D. Ray. Informal insurance in social networks. *Journal of Economic Theory*, 143:36–58, 2008.

[3] Y. A. Bramoullé and R. Kranton. Risk sharing networks. *Journal of Economic Behavior and Organization*, 64:275–294, 2006.

[4] Y. A. Bramoullé and R. Kranton. Risk sharing across communities. *The American Economic Review*, 97(2):70–74, 2007.

[5] P. Erdős and T. Gallai. Gráfok előírt fokszámú pontokkal. *Matematikai Lapok*, 11:264–274, 1960.

[6] Lund S. Fafchamps, M. and. Risk sharing networks in rural philippines. *Journal of Development Economics*, (71):261287, 2003.

[7] G. Genicot and D. Ray. Group formation in risk-sharing arrangements. *Review of Economic Studies*, (70):87–113, 2003.

[8] F. Harary. *Graph Theory.* MA: Addison-Wesley, 1969.

[9] M. Jackson and A. Wolinsky. A Strategic Model of Social and Economic Networks. *Journal of Economic Theory*, 71(1):355–365, 1996.

[10] H. Jacoby and E. Skouas. Financial markets, and human capital in a developing country. *Review of Economic Studies*, 3(64):311–335, 1997.

[11] T. Morill. Network formation under negative degree-based externalities. *International Journal of Game Theory*, 40(2):367–385, 2011.

[12] C. H. Paxson. Using weather variability to estimate the response of savings to transitory income in thailand. *The American Economic Review*, 1(82):15–33, 1992.

[13] M. R. Rosenzweig. Risk, implicit contracts and the family in rural areas of low income countries. *Economic Journal*, 98:1148–1170, 1988.

[14] M. R. Rosenzweig and O. Stark. Consumption smoothing, migration and marriage: evidence from rural India. *Journal of Political Economy*, 97(4):905–926, 1989.

[15] R. Townsend. Risk and insurance in village india. *Econometrica*, 62:539–559, 1994.

[16] C. Udry. Risk and Insurance in a Rural Credit Market: An Empirical Investigation in Northern Nigeria. *Review of Economic Studies*, 61(3):495–526, 1994.

[17] B. Wellman, P. Carrington, and A. Hall. Networks as personal communities. Social structures: A network approach. 1988.

8 Appendix

8.1 Proof of Proposition 1

To simplify notation, we extend $b^g(1,\cdot)$ to $b^g(1,-1) = 1 - \exp[-\rho(\Theta - (n_i(g)+1)\delta)]$. To demonstrate Proposition 1, we need to compute the difference between $B_i(g+ij)$ and $B_i(g)$, called $\Delta B_i(g,ij)$. We set $P(n_i(g),k) = \binom{n_i(g)}{k} p^k (1-p)^{n_i(g)-k}$. Proposition 1 follows Lemmas 1, 2 and 3.

Lemma 1 *Suppose that the ENB function is given by equation* (1)*. Then, the ENB function is strictly increasing and strictly concave with the number of links formed.*

Proof 1. We calculate the marginal expected benefits associated with the addition of a link. We present successively the two situations which can arise when agent i forms an additional link with agent j in g.

Suppose j draws state 1. This occurs with probability p. Then the benefits obtained by agent i when she forms a link with agent j is

$$\Big(p\sum_{k=0}^{n_i(g)} P(n_i(g),k) b^g(1,k)\Big) + \Big((1-p)\sum_{k=0}^{n_i(g)} P(n_i(g),k) b^g(0,k+1)\Big)$$

Suppose j draws state 0. This occurs with probability $1-p$. Then the benefits obtained by agent i when she forms a link with j is

$$\left(p\sum_{k=0}^{n_i(g)} P(n_i(g),k)b^g(1,k-1)\right)+\left((1-p)\sum_{k=0}^{n_i(g)} P(n_i(g),k)b^g(0,k)\right)$$

We set $\Delta b^g(0,k+1) = (\exp[-\rho\delta])^k(1-\exp[-\rho\delta]) > 0$, and $\Delta b^g(1,k) = (\exp[\rho\delta])^{n_i(g)-k}$ $\exp[-\rho\Theta](\exp[\rho\delta]-1)$.

By using the binomial theorem and straightforward computations we obtain

$$\begin{aligned}\frac{\Delta B_i(g,ij)}{p(1-p)} &= \sum_{k=0}^{n_i(g)} P(n_i(g),k)\Delta b^g(0,k+1)-\sum_{k=0}^{n_i(g)} P(n_i(g),k)\Delta b^g(1,k)\\ &= (1-\exp[-\rho\delta])(1+p(\exp[-\rho\delta]-1))^{n_i(g)}\\ &\quad -(\exp[\rho\delta]-1)\exp[-\rho\Theta](\exp[\rho\delta]+p(1-\exp[\rho\delta]))^{n_i(g)}\\ &= (1+p(\exp(-\rho\delta)-1))^{n_i(g)}(1-\exp(-\rho\delta))(1-\exp(\rho(\delta(n_i(g)+1)-\Theta)).\end{aligned} \tag{5}$$

We let

$$u=(1+p(\exp(-\rho\delta)-1))^{n_i(g)};\ v=(1-\exp(-\rho\delta));\ w=(1-\exp(\rho(\delta(n_i(g)+1)-\Theta)).$$

We have $v>0$, $w>0$ since $(n-1)\delta<\Theta$, and $u>0$ since

$$1+p(e^{-\rho\delta}-1)>0 \iff p<\frac{-1}{e^{-\rho\delta}-1}=\frac{1}{1-e^{-\rho\delta}}=\frac{e^{\rho\delta}}{e^{\rho\delta}-1}=1+\frac{1}{e^{\rho\delta}-1},$$

which is always true. It follows that $\Delta B_i(g,ij)>0$. In other words, the expected neighborhood benefits function of agent i is strictly increasing in the number of links formed by agent i.

2. We show that the expected neighborhood benefits function of agent i is strictly concave. In order to prove this statement we need to assign a sign to the difference between the marginal benefits. To obtain this result, we assume that agent i adds the link $g_{ik}=1$ to the network $g+g_{ij}$. Following the same steps as in Point 1. we have:

$$\begin{aligned}\frac{\Delta B_i(g+ij,ik)}{p(1-p)} &= (1-\exp[-\rho\delta])(1+p(\exp[-\rho\delta]-1))^{n_i(g)+1}\\ &\quad -(\exp[\rho\delta]-1)\exp[-\rho\Theta](\exp[\rho\delta]+p(1-\exp[\rho\delta]))^{n_i(g)+1}.\end{aligned}$$

We now determine the sign of the difference between $\Delta B_i(g+ij, ik)$ and $\Delta B_i(g, ij)$. We observe that

$$\frac{(1+p(\exp[-\alpha\delta]-1))^{n_i(g)+1}}{(1+p(\exp[-\alpha\delta]-1))^{n_i(g)}} = 1+p(\exp[-\alpha\delta]-1 < 1,$$

and

$$\frac{(\exp[\alpha\delta]+p(1-\exp[\alpha\delta]))^{n_i(g)+1}}{(\exp[\alpha\delta]+p(1-\exp[\alpha\delta]))^{n_i(g)}} = (1-p)\exp[\alpha\delta]+p > 1.$$

It follows that $\Delta B_i(g+ij, ik) - \Delta B_i(g, ij) < 0$.

□

Lemma 2 *Suppose that the ENB function is given by equation* (1)*. The marginal ENB function of agent i increases with Θ.*

Proof By inspecting the proof of Lemma 1, we know that $\Delta B_i(g, ij)$ is equal to

$$\begin{aligned}\Delta B_i(g, ij) &= p(1-p)(1-\exp[-\rho\delta])(1+p(\exp[-\rho\delta]-1))^{n_i(g)} \\ &\quad -p(1-p)(\exp[\rho\delta]-1)\exp[-\rho\Theta](\exp[\rho\delta]+p(1-\exp[\rho\delta]))^{n_i(g)}.\end{aligned} \tag{6}$$

From this we have:

$$\frac{\partial \Delta B_i(g, ij)}{\partial \Theta} = \rho p(1-p)(\exp[\rho\delta]-1)\exp[-\rho\Theta](\exp[\rho\delta]+p(1-\exp[\rho\delta]))^{n_i(g)-1} > 0.$$

This completes the proof. □

Lemma 3 *Suppose that the ENB function is given by equation* (1)*. The marginal ENB function increases with p iff $p > \tilde{p}$.*

Proof By inspecting the proof of Lemma 1, we have $\Delta B_i(g, ij) = p(1-p)(1+p(\exp(-\rho\delta)-1))^{n_i(g)}vw$. Let $\varphi : p \mapsto p(1-p)(1+p(\exp(-\rho\delta)-1))^{n_i(g)}vw$. We have

$$\varphi'(p) = vw\left((1-2p)u + p(1-p)n_i(g)\left(1+p(\exp(-\rho\delta-1))\right)^{n_i(g)-1}(\exp(-\rho\delta-1))\right).$$

The sign of $\varphi'(p)$ is the same as the sign of

$$(1-2p)\left(1+p(e^{-\rho\delta}-1)\right)^{n_i(g)} + p(1-p)n_i(g)\left(1+p(\exp(-\rho\delta-1))\right)^{n_i(g)-1}(\exp(-\rho\delta-1)),$$

which is quadratic in p, and therefore changes of sign at most 2 times between $p=0$ and $p=1$. It is equal to 1, hence positive, at $p=0$, and equal to $-\exp(-\rho\delta)$, hence negative, at $p=1$.

Then it changes of sign exactly one time between $p = 0$ and $p = 1$ (otherwise it would change of sign three times), and there is a threshold $\tilde{p}$ such that $p(1-p)F$ is increasing on $[0, \tilde{p})$ and decreasing on $(\tilde{p}, 1]$. Note that at $p = \frac{1}{2}$, it is negative, so that $\tilde{p} < \frac{1}{2}$. □

8.2 Proof of Proposition 2

To complete the proof, we need the following lemma. In this lemma we provide conditions that ensure the existence of three kinds of networks that turn out to be quite useful subsequently.

Lemma 4 *Let n and k be nonnegative integers with $n > k$.*

1. *Let n or k be even. Then, the sequence $s = (k, \ldots, k)$ is graphical.*
2. *Let n and k be odds. Then, the sequences $s = (k, k, \ldots, k, k+1)$, $s' = (k, k, \ldots, k, k-1)$ are graphical.*

Proof We prove successively that the three sequences are graphical.

1. Suppose n or k is even. Let $n > k > 0$. Since either n, or k is even, the sum of the sequence $s = (k, k \ldots, k)$ is even. Equation 3 can be written as

$$rk \leq r(r-1) + \sum_{i=r+1}^{n} \min\{k, r\}, \text{ for every } r,\ 1 \leq r < n. \tag{7}$$

There are two cases. Suppose $r \leq k$. Then equation (7) is satisfied if

$$rk \leq r(r-1) + (n-r)r \Rightarrow k \leq (r-1) + (n-r) \Rightarrow k \leq n-1.$$

This equation is always satisfied. Suppose $r > k$. Then equation (7) is

$$rk \leq r(r-1) + (n-r)k \tag{8}$$

If $k = n-1$, then $s = (n-1, \ldots, n-1)$ is a graphical sequence since the complete network supports this sequence. Similarly, if $k = 0$, then $s = (0, \ldots, 0)$ is a graphical sequence since the empty network supports this sequence. We now deal with k, $0 < k < n-1$. We have

$$r(r-1) + (n-r)k - rk = r^2 - r(1+2k) + nk = \left(r - \frac{2k+1}{2}\right)^2 + nk - \left(\frac{2k+1}{2}\right)^2.$$

Since $nk \geq (k+2)k = (k+1)k+k$ and $\left(\frac{2k+1}{2}\right)^2 = (k+1)k+1/4$, we have $nk-\left(\frac{2k+1}{2}\right)^2 \geq 0$, for $0 < k < n-1$.

2. Suppose n and k are odd, with $n-1 > k > 0$ ($k \neq n-1$ since k and n are odd). Since n is odd, $n-1$ is even and since k is odd, $k+1$ is even. Consequently, the sum of the sequence $s = (k+1, k \ldots, k)$ is even.

   For $r = 1$, equation (3) is satisfied since $k+1 \leq (n-1)$ for $0 < k < n-1$. For $r \geq 2$, equation (3) is equal to

$$k + 1 + (r-1)k \leq r(r-1) + \sum_{i=r+1}^{n} \min\{k, r\}, \text{ for every } r,\ 2 \leq r < n. \tag{9}$$

   There are two cases. (1) Suppose $r \leq k$, with $k \leq n-2$, and $r \geq 2$. Then equation (9) is

$$k+1+(r-1)k \leq r(r-1) + (n-r)r \Rightarrow k \leq (r-1) + (n-r) - \frac{1}{r} \Rightarrow k \leq (n-1) - \frac{1}{r}.$$

   This equation is always satisfied since $k < n-1$ and $1/r < 1$ for $r > 2$. (2) Suppose $r > k$. Then equation (9) is

$$rk + 1 \leq r(r-1) + (n-r)k. \tag{10}$$

   We first deal with the case where $k = n-2$. In that case $r = n-1$. Therefore, we have:

$$(n-1)(n-2) + 1 \leq (n-1)(n-2) + n - 2,$$

   and since $n \geq 3$, this equation is always satisfied. We now deal with $k < n-2$, we have

$$r(r-1)+(n-r)k-rk-1 = r^2-r(1+2k)+nk-1 = \left(r - \frac{2k+1}{2}\right)^2 + nk - \left(\frac{2k+1}{2}\right)^2 - 1.$$

   Since $nk \geq (k+3)k = (k+1)k + 2k$ and $\left(\frac{2k+1}{2}\right)^2 + 1 = (k+1)k + 5/4$, we have $nk - \left(\frac{2k+1}{2}\right)^2 - 1 > 0$, for $0 < k < n-2$.

3. Suppose n and k are odd. Since n is odd, $n-1$ is even and since k is odd, $k-1$ is even. Consequently, the sum of the sequence $s = (k, \ldots, k, k-1)$ is even. equation (3) is equal to

$$rk \leq r(r-1) + \sum_{i=r+1}^{n-1} \min\{k, r\} + \min\{k-1, r\}, \text{ for every } r,\ 1 \leq r < n-1. \tag{11}$$

There are two cases. (1) Suppose $r \leq k-1$, with $k < n-1$ ($k \neq n-1$ since n and k are odd). Then equation (11) becomes

$$rk \leq r(r-1) + (n-r)r, \text{ for every } r,\ 1 \leq r < n.$$

We have already shown in point 1., equation (8), that this equation is always satisfied. (2) Suppose $r > k-1$. Then equation (11) becomes

$$rk \leq r(r-1) + (n-r)k - 1 \Rightarrow rk + 1 \leq r(r-1) + (n-r)k. \tag{12}$$

We first deal with the case where $k = n-2$. In that case either $r = n-1$, or $r = n-2$. We have shown in point 2., equation (10), that the previous equation is satisfied when $r = n-1$ and $k = n-2$. If $r = n-2$ and $k = n-2$, we have

$$(n-2)(n-2)+1 \leq (n-2)(n-3)+2(n-2) \Rightarrow (n-2)(n-2)+1 \leq (n-2)(n-2)+(n-2).$$

This equation is always satisfied since $n \geq 3$. Finally, we have shown in point 2. equation (10), that equation (12) is satisfied when $0 < k < n-2$.

□

Lemma 5 *Network g is pairwise stable if and only if $\mathcal{N}^{\neq}_{k^\star}(g) = \mathcal{N}^{-}_{k^\star}(g)$ and $g[\mathcal{N}^{\neq}_{k^\star}]$ is complete.*

Proof First, we show the *only if* part. Let g be a pairwise stable network. If an agent i has formed more than $k^\star$ links in g, then g is not a pairwise stable since $k^\star \in \arg\max\{\Phi(k) : k \in \{0, \ldots n-1\}\}$, and agent i has an incentive to remove a link. Similarly, if there exist two unlinked agents i and j who have formed less than $k^\star$ links in g, then g is not a pairwise stable network since i and j have an incentive to form a link together. It follows that $\mathcal{N}^{\neq}_{k^\star}(g) = \mathcal{N}^{-}_{k^\star}(g)$ and $g[\mathcal{N}^{\neq}_{k^\star}]$ is complete.

Second, we show the *if* part. Suppose that $\mathcal{N}^{\neq}_{k^\star}(g) = \mathcal{N}^{-}_{k^\star}(g)$ and $g[\mathcal{N}^{\neq}_{k^\star}]$ is complete. Since agents $j \in \mathcal{N}^{-}_{k^\star}(g)$ form $k^\star$ links and $k^\star \in \arg\max\{\Phi(k) : k \in \{0, \ldots, n-1\}\}$, they do not have any incentive to modify their strategy. Agents $j \in \mathcal{N}_{k^\star}(g)$ have no incentive to remove links since ENB is strictly concave by Proposition 1. Agents $j \in \mathcal{N}^{\neq}_{k^\star}(g)$ cannot form additional links since $g[\mathcal{N}^{\neq}_{k^\star}]$ is complete and agents $j' \in \mathcal{N}_{k^\star}(g)$ have no incentive to form additional links. It follows that g is pairwise stable. □

Lemma 6 *Suppose that the benefits function is given by equation 1. If n or $k^\star$ are even, then $k^\star$-regular networks are pairwise stable. If n and $k^\star$ are odd then, $k^\star_-$-regular networks are pairwise stable.*

Proof First, we assume that n or $k^\star$ are even. By Lemma 4, the sequence of degrees $s = (k^\star, k^\star \ldots, k^\star)$ is graphical: we can build a $k^\star$-regular-network, g. In $k^\star$-regular-networks every agent $i \in N$ obtains the maximum of $\{\Phi(k) : k \in \{0, \ldots, n-1\}\}$: she has no incentive to modify its number of links. Therefore, g is pairwise stable.

Second, assume that n and $k^\star$ are odd. By Lemma 4, the sequence of degrees $s = (k^\star - 1, k^\star, \ldots, k^\star)$ is graphical: we can build a $k^\star_-$-regular-network, g. In g every agent $i \in \mathcal{N}_{k^\star}(g)$ obtains the maximum of $\{\Phi(k) : k \in \{0, \ldots, n-1\}\}$ and agent i, who forms $k^\star - 1$ links in g, cannot find a partner to form an additional link. Moreover due to the strict concavity of Φ, agent i has no incentive to remove a link. It follows that g is pairwise stable. □

Lemma 7 *Suppose that the ENB is given by equation 1.*

(a) Suppose n or $k^\star$ are even, then $k^\star$-regular networks are the unique efficient networks.

(b) Suppose n and $k^\star$ are odd.

(i) If $\Phi(k^\star+1) < \Phi(k^\star-1)$, then $k^\star{}_-$-regular networks are the unique efficient networks.

(ii) If $\Phi(k^\star+1) > \Phi(k^\star-1)$, then $k^\star{}_+$-regular networks are the unique efficient networks.

Proof Suppose that n or $k^\star$ are even. By Lemma 4, $k^\star$-regular networks exist. In a $k^\star$-regular network g, each agent maximizes its expected payoffs. It follows that g is efficient. Suppose that n and $k^\star$ are odd. By Theorem 1, it is not possible to build a $k^\star$-regular network. By Lemma 4, $k^\star_+$-regular networks and $k^\star_-$-regular exist. In an almost-$k^\star$-regular network g, each agent except one, say i, maximizes its expected payoffs. Since i cannot form $k^\star$ links and Φ is strictly concave, she maximizes her payoffs when she forms $k^\star + 1$ or $k^\star - 1$ links. If $\Phi(k^\star + 1) < \Phi(k^\star - 1)$, then the agent, who has not formed $k^\star$ links, forms $k^\star - 1$ links in an efficient network; this agent forms $k^\star + 1$ links in an efficient network if $\Phi(k^\star + 1) > \Phi(k^\star - 1)$. □

Proof of Proposition 2 We prove successively the different parts of the Proposition. First, we show Part (a) of the proposition. By Lemma 6, we know that there always exists a pairwise stable network. Part (b) of the proposition follows Lemma 5. Parts (c) and (d) of the proposition follow Lemmas 6 and 7. Part (e) of the proposition follows Proposition 1 and the proof of Lemma 1. Indeed, we know by Proposition 1 that the ENB function is strictly increasing. By inspecting the proof of Lemma 1, we observe that the inequality $\Delta B_i(g, ij) > 0$ is always true. It follows that if $F = 0$, then the complete network is the unique pairwise stable network.

□

8.3 Proof of Proposition 3

We establish part (b) of the proposition by using similar arguments as in Lemma 5. We now show part (c) and (d).

Part (c). Suppose that N^x is even and $k^x < |N^x|$ for $x \in \{\Theta, \Theta'\}$. Then, by Lemma 4, for all k^Θ and $k^{\Theta'}$ it is possible to build two sub-networks $g[N^\Theta]$ and $g[N^{\Theta'}]$ in g that are k^Θ-regular and $k^{\Theta'}$-regular respectively. Hence, we build network g such that $g[N^\Theta]$ and $g[N^{\Theta'}]$ in g are k^Θ-regular and $k^{\Theta'}$-regular respectively and where there is no link between agents in N^Θ and agents in $N^{\Theta'}$ in g. Since each agent $i \in N$ maximizes $\{\Phi^x(k) : k \in \{0, \dots n-1\}\}$, $x \in \{\Theta, \Theta'\}$ in g, i has no incentive to modify her strategy and no pair of agents has an incentive to add a link: g is pairwise stable. Moreover, since each agent maximizes $\{\Phi^x(k) : k \in \{0, \dots n-1\}\}$, $x \in \{\Theta, \Theta'\}$, g is an efficient network.

Part (d). Suppose that N^x is odd and $k^x < |N^x|$ for $x \in \{\Theta, \Theta'\}$. We deal with three possibilities successively.

(i) Suppose that k^Θ and $k^{\Theta'}$ are even. By Lemma 4, it is possible to build two sub-networks $g[N^\Theta]$ and $g[N^{\Theta'}]$ in g that are k^Θ-regular and $k^{\Theta'}$-regular respectively. Again, we build network g such that $g[N^{\Theta'}]$ in g that are k^Θ-regular and $k^{\Theta'}$-regular respectively and where there is no link between agents in N^Θ and agents in $N^{\Theta'}$ in g. Since each agent $i \in N$ obtains the maximum of $\{\Phi^x(k) : k \in \{0, \dots n-1\}\}$, $x \in \{\Theta, \Theta'\}$, i has no incentive to modify her strategy and no pair of agents has an incentive to add a link: g is pairwise stable. Moreover, since each agent obtains the maximum of $\{\Phi^x(k) : k \in \{0, \dots n-1\}\}$, $x \in \{\Theta, \Theta'\}$, g is an efficient network.

(ii) Suppose that k^{Θ} and $k^{\Theta'}$ are odd. Then by Lemma 4 it is possible to build two sub-networks $g[N^{\Theta}]$ and $g[N^{\Theta'}]$ in g that are k^{Θ}_{-}-regular and $k^{\Theta'}_{-}$-regular respectively. Let i_x, $x \in \{\Theta, \Theta'\}$, be the agent who forms $k^x - 1$ links in $g[N^x]$. We build the network g such that $g[N^{\Theta}]$ and $g[N^{\Theta'}]$ in g that are k^{Θ}_{-}-regular and $k^{\Theta'}_{-}$-regular respectively and where there is a link between agents i_{Θ} and $i_{\Theta'}$. Each agent $i \in N$ obtains the maximum of $\{\Phi^x(k) : k \in \{0, \dots n-1\}\}$, $x \in \{\Theta, \Theta'\}$. Therefore g is pairwise stable and efficient.

(iii) Suppose that k^x, $x \in \{\Theta, \Theta'\}$, is odd and k^y, $y \in \{\Theta, \Theta'\} \setminus \{x\}$, is even. Then by Lemma 4 it is possible to build network g which contains two sub-networks $g[N^x]$ and $g[N^y]$ in g that are k^x_{-}-regular and k^y-regular respectively. By using the same argument as in Lemma 7, we establish that g is pairwise stable.

Part (a) of the proposition follows parts (c) and (d).

We now establish part (e) of the proposition. By Proposition 1 (b), we know that the marginal expected payoff function associated with each link k is higher for agents in N^{Θ} than agents in $N^{\Theta'}$. By Proposition 1 (a), the ENB function is concave: the maximal marginal payoff is associated with the first link formed by a player. Consequently, $(\Phi^{\Theta}(1) - \Phi^{\Theta}(0)) - (\Phi^{\Theta'}(k) - \Phi^{\Theta'}(k-1)) > 0$, for $k \in \{1, \dots, n-1\}$ and there exists F such that $(\Phi^{\Theta}(1) - \Phi^{\Theta}(0)) > F$ and $\Phi^{\Theta'}(1) - \Phi^{\Theta'}(0) < F$.

8.4 Proof of Proposition 5

Lemma 8 *Suppose that the payoff function satisfies equation 4. There always exists a pairwise stable network.*

Proof In the following, we assume that $k^\star \neq 0$, otherwise the empty network is pairwise stable. Let n or $k^\star$ be even. We build a $k^\star$-regular network, $g^{k^\star}$ where all agents form $k^\star$ links. We know by Lemma 4 that $g^{k^\star}$ exists. We now show that $g^{k^\star}$ is pairwise stable. First, no agent has a strict incentive to remove a link since by construction $\gamma(k^\star, k^\star) \geq 0$ in $g^{k^\star}$. Second, no agent has an incentive to add a link since $\gamma(k^\star + 1, k^\star + 1) < 0$ in $g^{k^\star}$. Therefore $g^{k^\star}$ is a pairwise stable network.

Suppose now that n and $k^\star$ are odd. There are two possibilities

(i) Suppose $\gamma(k^\star+1, k^\star) \geq 0$ and $\gamma(k^\star, k^\star+1) \geq 0$. By Lemma 4, the sequence $(k^\star, \dots, k^\star, k^\star + 1)$ is graphical. We build the network g where one agent, say i, forms $k^\star + 1$ links and all other agents form $k^\star$ links. By Lemma 4, the network g exists. First, we establish that there is no pair of agents who have simultaneously an incentive to add a link. More precisely, we establish that the $(n-1)$ agents in $\mathcal{N}_{k^\star}(g)$ have no incentive to form an additional link. By definition no agent $i \in \mathcal{N}_{k^\star}(g)$ has an incentive to form a link with an agent $j \in \mathcal{N}_{k^\star}(g)$ since $\gamma(k^\star+1, k^\star+1) < 0$. Similarly, no agent $j' \in \mathcal{N}_{k^\star}(g)$ has an incentive to form a link with agent $i \in \mathcal{N}^+_{k^\star}(g)$ since $\gamma(k+1, k+2) < \gamma(k+1, k+1) < 0$; the first inequality follows the fact that γ is strictly decreasing in its two arguments. Second, we establish that no agent has an incentive to remove a link. By using the same arguments as in the case where n or k are even, we know that no agent j in $\mathcal{N}_{k^\star}(g)$ has an incentive to delete a link she forms with a member of $\mathcal{N}_{k^\star}(g)$. Similarly, since $\gamma(k^\star+1, k^\star) \geq 0$ agent $i \in \mathcal{N}^+_{k^\star}(g)$ has no incentive to remove one of her links. Finally, since $\gamma(k^\star, k^\star+1) \geq 0$ no agent $j \in \mathcal{N}_{k^\star}(g)$ has an incentive to remove the link she forms with $i \in \mathcal{N}^+_{k^\star}(g)$ in g. It follows that g is pairwise stable.

(ii) Suppose $\gamma(k^\star+1, k^\star) < 0$ or $\gamma(k^\star, k^\star+1) < 0$. We consider network g where one agent, say i, forms $k^\star - 1$ links and all other agents form $k^\star$ links. By Lemma 4, the sequence $(k^\star - 1, k^\star, \dots, k^\star)$ is graphical. Since $\gamma(k^\star+1, k^\star+1) < 0$, no unlinked agents $j \in \mathcal{N}_{k^\star}(g)$ and $j' \in \mathcal{N}_{k^\star}(g)$ have an incentive to form a link together. If agent $i \in \mathcal{N}^-_{k^\star}(g)$ forms a link with agent $j \in \mathcal{N}_{k^\star}(g)$, then i obtains a marginal payoff associated with this link equal to $\gamma(k^\star, k^\star+1)$, while j obtains a marginal payoff associated with this link equal to $\gamma(k^\star+1, k^\star)$. By assumption, $\min\{\gamma(k^\star, k^\star+1), \gamma(k^\star+1, k^\star)\} < 0$. Therefore agent i or agent j has no incentive to form this link. Moreover, by using similar arguments as in (i) no agent has an incentive to remove one of her links. It follows that g is pairwise stable.

Since there exists a pairwise equilibrium network when n or $k^\star$ is even, and there exists a pairwise equilibrium network when n and $k^\star$ are odd, there always exists a pairwise equilibrium network. □

Proof of Proposition 5. By Lemma 8, we establish Part (a) of the proposition. We show part (b) of the proposition. Let g be a pairwise stable network. To introduce a contradiction, suppose that g does not satisfy $g[\mathcal{N}^+_{k^\star}]$ is empty and $g[\mathcal{N}^-_{k^\star}]$ is complete. First, suppose that $g[\mathcal{N}^+_{k^\star}]$ is not empty. Then, there exist agents $i, j \in \mathcal{N}^+_{k^\star}(g)$ who have formed a link together. We have $\gamma(n_i(g), n_j(g)) \leq \gamma(k^\star + 1, n_j(g)) \leq \gamma(k^\star + 1, k^\star + 1) < 0$ since γ is decreasing in its two arguments and $n_i(g), n_j(g) \geq k^\star$. Consequently, agents i and j have an incentive to remove the link they have formed together and g is not pairwise stable, a contradiction. Suppose that $g[\mathcal{N}^-_{k^\star}]$ is not complete. Then there exist $i, j \in \mathcal{N}^-_{k^\star}(g)$ who have not formed a link together. Let k_i and k_j be the number of links formed by i and j respectively in g. We have $\gamma(k_i, k_j) \geq \gamma(k^\star + 1, k_j) \geq \gamma(k^\star, k^\star) > 0$ since γ is decreasing in its two arguments and $k_i, k_j \leq k^\star$. Consequently, agents i and j have an incentive to form a link together and g is not pairwise stable, a contradiction. Suppose now that $\gamma(k^\star, k^\star + 1) < 0$. Toward a contradiction, suppose $g[\mathcal{N}^+_{k^\star}] \neq \emptyset$, and let agent i belong to $g[\mathcal{N}^+_{k^\star}]$. Since $\gamma(k^\star, k^\star + 1) < 0$ and γ is decreasing in its second argument, agents who have formed $k^\star$ links have not formed a link with agent i in g. Moreover, by definition of $k^\star$, we know that agent i has not formed links with agents who have formed $k > k^\star$ links in g. Therefore, all the neighbors of agent i in g have formed $k < k^\star$ links. Moreover, we know that agents, who have formed $k < k^\star$ links in g, have formed links together. Therefore, these agents are less than $k^\star$. Consequently the number of neighbors of agent i is $k < k^\star < n_i(g)$, a contradiction.

We now show part (c). We consider agents i, i', j and j' such that $n_{i'}(g) \leq n_i(g) < k^\star < n_{j'}(g) \leq n_j(g)$. Suppose that there is a link between agents i and j. We have $\gamma(n_i(g), n_j(g)) \geq 0$ and $\gamma(n_j(g), n_i(g)) \geq 0$. We have $\gamma(n_{i'}(g), n_{j'}(g)) \geq \gamma(n_i(g), n_j(g)) \geq 0$ and $\gamma(n_{j'}(g), n_{i'}(g)) \geq \gamma(n_j(g), n_i(g)) \geq 0$ since γ is strictly decreasing in its two arguments. It follows that the link between i' and j' belong to g.

We now establish part (d) of the proposition. Suppose that $|\mathcal{N}^+_{k^\star}(g)| \geq n/2$. Since agents in $\mathcal{N}^+_{k^\star}(g)$ have not formed links together, agents in $N \setminus \mathcal{N}^+_{k^\star}(g)$ have formed at least $(n/2)(k^\star + 1)$ links with agents in $\mathcal{N}^+_{k^\star}(g)$. Then there is at least one agent in $\mathcal{N}^+_{k^\star}$ who has formed $(|\mathcal{N}^+_{k^\star}(g)|/|N \setminus \mathcal{N}^+_{k^\star}(g)|)(k^\star + 1)$ links. Since $(|\mathcal{N}^+_{k^\star}(g)|/|N \setminus \mathcal{N}^+_{k^\star}(g)|)(k^\star + 1) > k^\star$, we obtain a contradiction. We now establish that $n^M(g^\star) - n^m(g^\star) \leq |\mathcal{N}_{k^\star}(g)|$. By part ($b$) of the proposition, we have $n^M(g) \leq |\mathcal{N}_{k^\star}(g) \cup \mathcal{N}^-_{k^\star}(g)|$ and $n^m(g) \geq |\mathcal{N}^-_{k^\star}(g)|$. Since $\mathcal{N}_{k^\star}(g) \cap \mathcal{N}^-_{k^\star}(g) = \emptyset$, $|\mathcal{N}_{k^\star}(g) \cup \mathcal{N}^-_{k^\star}(g)| = |\mathcal{N}_{k^\star}(g)| + |\mathcal{N}^-_{k^\star}(g)|$. It follows that $n^M(g) - n^m(g) \leq |\mathcal{N}_{k^\star}(g)| + |\mathcal{N}^-_{k^\star}(g)| - |\mathcal{N}^-_{k^\star}(g)| = |\mathcal{N}_{k^\star}(g)|$.

□