Jumping the welfare gap
in designing public transfers

by

András Simonovits

Institute of Economics, RCERS, Hungarian Academy of Sciences
also Mathematical Institute of Budapest University of Technology,
and Budapest, Budaörsi út 45, Hungary
March 4, 2017

Abstract*

We consider three transfer models with a representative individual who discounts the utility of the merit good with respect to the standard one's. In each model, a paternalistic government taxes the consumer and transfers him additional merit goods in return. The private purchase of the merit goods is cheaper than the transfer. Even if the optimal transfer system is welfare superior to the transfer-free system, a system with much lower transfer may be inferior, therefore this welfare gap should be jumped. Various pension modelers (e.g. Feldstein, 1985; van Groezen, Leers and Meijdam, 2003) overlooked this problem and drew wrong conclusions.

Keywords: transfers, pensions, taxes, social welfare, paternalism

JEL numbers: D10, H55, J13, J14, J18, J26

1. Introduction

In real life, mandatory transfers (e.g. public pensions, basic income, health care, public education etc.) have multiple functions: (i) fight the distortionary effect of erroneous private preferences, preferring standard good to merit goods; (ii) redistribute from the rich to the poor or (iii) supply insurance against bad luck in incomplete markets. We join that stream of the theoretical literature which neglected functions (ii) and (iii) and concentrated on function (i). Therefore it is sufficient to postulate a representative individual, who underestimates the real utility of merit goods with respect to the standard good. There are two ways to obtain merit goods: publicly and privately. We assume that buying one unit of merit good is cheaper than receive it as a transfer. The main result of the paper, proved for three different models, is as follows: even if the government can choose an optimal transfer, which yields higher social welfare than the private optimum; there is a welfare gap of low transfers, which yield lower social welfare.

This result has two implications: (a) because there are two (local) maxima: one private (S) and one public (T), we have to compare them to determine the global maximum; (b) even if the public optimum is superior to the private one, but the political opposition compels the government to choose a transfer well below the social optimum, then the society may end up with a lower welfare than not having contribution at all.

To obtain public pension models, we specify standard goods as young-age consumption and merit goods as old-age consumption, the transfer as public pension. Then the three functions are realized as follows: (i) myopic workers discount the utility of old-age consumption; (ii) the government ensures a minimum guaranteed pension; or (iii) unisex indexed life annuity defends the pensioners against the longevity risk (Barr and Diamond, 2008). In principle, this framework is suitable to characterize the socially optimal pension system but in practice, the neglected complications should not be forgotten.

Though the most obvious realization of these models are the public pension systems but other transfer systems like public spending on health or education can also be analyzed with their help. For example, Peter Diamond suggested the following example (personal communication): Assume that people dispose of garbage infrequently—bothering their neighbors. The government introduces a more expensive garbage collection system. If the government collects garbage only a little more frequently than the private equilibrium, then the public system only makes things worse. It takes a large enough effort to make things better.

We shall consider three models: Model 1 (outlined above) is the simplest transfer model with private saving when the individual discounts the merit good's utility. Model 2 extends Model 1 when the individual also underestimates the size of the transfer but the government does not. Model 3 is formulated as a pension model; we introduce endogenous fertility into Model 1 and break down the tax: one part finances the child benefits promoting fertility, the other part pays contributions for pensions. The new phenomenon, jumping the gap is present in each of them. We shall numerically illustrate our findings.

In the remaining part of the Introduction, a very short review of the related pension literature is given: Feldstein (1985) was the first studying the contradiction between myopia and efficiency of private life-cycle saving in designing public pension system with a simple model of representative agent (our Model 2). But due to his unrealistically

high annual real interest rate (11.4%) and extremely low (sometimes zero) presumed-to-actual benefit ratio, he 'could' neglect the whole problem of multiple maxima and the arising gap; and that way he 'proved' the inefficiency of public transfers. (Though he also analyzed workers of heterogeneous discount factors, we skip this complication. Note, however, that by introducing means-testing for myopes, Feldstein (1987) somehow improved his own evaluation of the public pension system.)

Since 1985, a lot of papers have been published on the optimal transfer systems, including very sophisticated ones (e.g. the pioneering work by Auerbach and Kotlikoff (1987), using dynamic general equilibrium models). For pedagogical reasons, however, we confine our attention to the very simple paternalistic models. (As Cremer and Pestieau's (2011) excellent survey emphasized, old paternalism advocates government action against childish decisions of the individuals, new paternalism justifies intervention in the name of the true self against the wrong self.)

In fact, it was van Groezen, Leers and Meijdam (2003) (for short, GLM) who formulated Model 3 without excluding negative saving. I demonstrated the existence of two optima and the phenomena of jumping the gap (Simonovits, 2013).

We continue our very selective survey with Cremer, De Donder, Maldonado and Pestieau (2008), who extended Feldstein's analysis to flexible labor supply, heterogeneous wages and redistributive pensions without having private life-cycle saving, analyzed the dependence of the socially optimal contribution rate on the model's parameters (e.g. the share of myopic workers, also discussed by Feldstein, 1985, Part II) and obtained much more favorable results concerning optimal public pension systems than Feldstein.

In the simplest pension model (Model 1, Simonovits, 2015a), I assumed homogeneous wages and discount factors, while allowed for moderately efficient private savings. In a further work (Simonovits, 2015b), I considered heterogeneous wages, and rather than following Cremer et al. (2008) in neglecting the correlation between wages and discount factors; the discount factor was made an increasing function of the wage and introduced cap on the contribution base. In that framework, I could only determine the socially optimal contribution rate and cap by numerical methods, therefore the issue of jumping the gap was not emphasized.

At the end of the survey, it should be mentioned that both Feldstein (1985) and GLM embedded their static models into a dynamic model of infinite stream of overlapping generations, respectively. Furthermore, Feldstein chose a Lernerian rather than the standard Samuelsonian social welfare function, i.e. he defined the one-period social welfare function as the sum of the current young- and old-age consumption's utilities rather than the undiscounted lifetime utility function of a cohort. But to understand our problem of jumping the gap, these complications can be safely ignored.

Moreover, working now with static models, we could have reduced the length of the old-age period (or the weight of the merit good) to much less than 1, and make the description more realistic. For example, choosing a factor 1/2, the T-optimal contribution rate would drop from 1/2 to 1/3. To keep the distance between the old and the new models at a minimum, however, we have not followed Simonovits (2015b) and other papers.

The structure of the remainder of this paper is as follows: Sections 2-4 present models 1, 2 and 3, respectively; and Section 5 concludes.

2. Single discount

Model 1 (M1)

The quantities of the abstract transfer models are generally positive real numbers, except if stated otherwise. The whole society is represented by a single individual, consuming two goods in quantities c (ordinary good) and d (merit good). The individual's pretax earning is equal to 1. By paying s, he can buy ρs units of merit goods, where $\rho > 1$. For comparability with Feldstein (1985), we introduce population and real wage growth factors ν and γ, respectively, and their product, $g = \nu\gamma > 1$—being the growth factor of the total wage. In addition, he must pay tax to the government, defined by the tax rate t and he receives gt units of merit good as a benefit: $0 \leq s, t \leq 1$ and $0 \leq s+t \leq 1$. As is known, there is an important indicator, the *relative interest factor* which combines the impact of the two parameters g and ρ:

$$R = \frac{\rho}{g}.$$

First assumption—the private purchase is more efficient than the public one:

$$R > 1. \tag{A1}$$

We assume that the government and the representative individual play a Stackelberg-duopoly game: first the government (as the leader) chooses t and then the individual (as the follower) chooses s. Their objective functions $V(t,s)$ and $U(t,s)$ will be presented below but will be abbreviated as $V(t)$ and $U(s)$, respectively. Making its decision, the government anticipates the individual's reaction to the transfer.

Then in our mixed public–private system, the consumption pair are equal to

$$c = 1 - t - s \quad \text{and} \quad d = gt + \rho s = g(t + Rs). \tag{1}$$

The individual has an additively separable Cobb–Douglas utility function

$$U(c,d) = \log c + \delta \log d, \tag{2}$$

where δ is the discount factor of the utility of the merit good, $0 \leq \delta < 1$.

Inserting the consumption equations (1) into the individual utility function (2), for a given tax rate t, one obtains the reduced utility function:

$$U[s] = \log(1 - t - s) + \delta \log(t + Rs) + \delta \log g. \tag{3}$$

If the tax rate is low enough, then the optimality condition and the optimal s are respectively

$$U'[s] = -\frac{1}{1-t-s} + \frac{\delta R}{t + Rs} = 0, \quad \text{i.e.} \quad s = \frac{\delta(1-t) - R^{-1}t}{1+\delta} > 0;$$

otherwise $(U'[0] < 0)$ there is no purchase: $s = 0$.

We define the *separator* tax rate t_W as that tax rate at which the purchasing intention for merit goods becomes just zero, i.e. t_W is the unique root to the implicit equation

$$\delta R = \delta R t_W + t_W, \quad \text{i.e.} \quad t_W = \frac{\delta R}{1 + \delta R}. \tag{4}$$

We have now the following classification: for $0 \le t < t_W$, the purchasing intention is positive; for $t_W \le t < 1$, the purchasing intention is nonpositive. Because of the existence of the credit constraint (CR), positive intentions are preserved (Slack CR), while nonpositive intentions become zero purchase (Tight CR). The following branching equations give the optimal consumption pair, where subindices S and T refer to S- and T-branches, respectively.

Lemma 1. *For any tax rate t, the optimal consumption pair are respectively equal to*

$$c_S(t) = \frac{1 - t + R^{-1}t}{1 + \delta}, \qquad d_S(t) = \delta g R c_S(t) \quad \text{if} \quad 0 \le t < t_W \tag{5S}$$

and

$$c_T(t) = 1 - t, \qquad d_T(t) = gt \quad \text{if} \quad t_W \le t < 1. \tag{5T}$$

To fight myopia, the government evaluates the social welfare provided by a transfer financed by tax rate t, with the undiscounted (paternalistic) social welfare function

$$V(t) = \log c(t) + \log d(t). \tag{6}$$

To find its global maximum, we will separate the two cases of slack and tight credit constraints.

By A1, both $c_S(t)$ and $d_S(t)$ are decreasing functions in $0 < t < t_W$ (crowding-out effect), therefore the corresponding S-optimum is achieved at $t = 0$, a corner maximum. To obtain the T-optimum, take the derivative of $V(t)$ in the second interval,

$$V'(t) = -\frac{1}{1 - t} + \frac{1}{t}, \qquad t_W \le t < 1.$$

Then $V'(t)$ is positive in interval $[t_W, 1/2)$ and negative in interval $(1/2, 1)$. To make the first interval nonempty, we have to assume that $t_W < 1/2$, i.e. by (4),

$$\delta R < 1. \tag{A2}$$

(5S) implies that for $g = 1$, A2 is equivalent to $c(0) < d(0)$.

Lemma 2. *Under assumptions A1–A2, the social welfare function V reaches its T-maximum at $t^* = 1/2$.*

From now on, we confine our attention to interval $[0, 1/2]$. Obviously, separator t_W is now the *worst* S-tax rate in interval $[0, 1/2)$ but it may not be so in other models (like M2 below).

Compare the two optimal pairs of consumption:

$$c(0) = \frac{1}{1 + \delta}, \qquad d(0) = \frac{\delta g R}{1 + \delta} \quad \text{(S − optimum)}$$

4

and
$$c(1/2) = \frac{1}{2}, \qquad d(1/2) = \frac{g}{2} \quad (\mathrm{T} - \text{optimum}).$$

To have a meaningful model, in addition to $c(0) > c(1/2)$ (due to myopia) we must assume that $d(0) < d(1/2)$ (the aim of coercion). Inserting the consumption values into the second inequality, we arrive to our third assumption:

$$\frac{\delta R}{1 + \delta} < \frac{1}{2}, \quad \text{i.e.} \quad \delta < \frac{1}{2R - 1}. \tag{A3}$$

Note that A1 and A3 imply A2.

We are interested in that case when the T-optimal tax rate system is superior to the S-optimum:
$$V(1/2) > V(0).$$

Inserting the corresponding formulas (5)–(6) into the last inequality results in the fourth assumption:
$$-2\log 2 > -2\log(1 + \delta) + \log \delta + \log R. \tag{A4}$$

It is easy to see that for any relative interest factor $R > 1$, there exists a *critical* discount factor $\delta_R < 1$, for which the two maximal welfares are equal to each other:

$$-2\log 2 = -2\log(1 + \delta_R) + \log \delta_R + \log R.$$

Solving the quadratic equation $1 + 2\delta + \delta^2 = 4\delta R$ yields

$$\delta_R = 2R - 1 - 2\sqrt{R^2 - R}.$$

Fixing the interest factor $R > 1$, for every subcritical discount factor δ, $0 < \delta < \delta_R$, there exists a *neutral* tax rate $t_N \in (t_W, 1/2)$ for which the social welfare is the same as at the transfer-free system. In formula:

$$\log(1 - t_N) + \log t_N = -2\log(1 + \delta) + \log \delta + \log R.$$

A second quadratic equation arises: $(1 + \delta)^2 (t_N - t_N^2) = \delta R$ yielding

$$t_N = \frac{1 + \delta - \sqrt{(1 + \delta)^2 - 4(1 + \delta)\delta R}}{2(1 + \delta)}.$$

We have arrived to

Theorem 1. *Even if in M1, the optimal transfer system is superior to the transfer-free system (A4), under A1 and A3, for any relative interest factor R and every subcritical discount factor δ, $0 < \delta < \delta_R$, there exists an interval $(0, t_N)$ (with $t_N < 1/2$) for which any tax rate t in this interval delivers lower welfare than the transfer-free. Any sensible tax rate should lie between t_N and $1/2$.*

Numerical illustrations

We shall now numerically illustrate our findings. Copying Feldstein's annual population growth factor $\nu[1] = 1.014$ and his annual real wage growth factor $\gamma[1] = 1.022$, yielding $g[1] = \nu[1]\gamma[1] \approx 1.036$. Considering 30-year-long life-stages, our illustration generates from *annual* factors the 30-years cumulated factors:

$$\delta = \delta[1]^{30}, \quad \nu = \nu[1]^{30}, \quad \gamma = \gamma[1]^{30}, \quad g = g[1]^{30}, \quad \rho = \rho[1]^{30}, \quad R = R[1]^{30}.$$

Our analysis only shows qualitative relations; for example, the critical δ_R is a decreasing function of R. Now Table 1 numerically demonstrates the strength of this dependence. For example, in the second row of Table 1, the annual interest rate of 2% defines an annual critical discount rate of cc. 5 percent. But increasing the relative interest rate to 8%, the critical discount rate drops to cc. 11% (last row). The transfer-free optimal young-age consumption grows from 0.754 to 0.975, while the corresponding old-age consumption decreases from 0.958 to 0.741.

Table 1. *Annual relative interest and critical discount factors: M1*

Annual relative interest factor	critical discount	No transfer Young-age consumption	Old-age consumption
$R[1]$	$\delta[1]$	$c(0)$	$d(0)$
1.01	0.963	0.754	0.958
1.02	0.947	0.835	0.865
1.03	0.935	0.883	0.818
1.04	0.924	0.916	0.789
1.05	0.913	0.938	0.770
1.06	0.904	0.954	0.757
1.07	0.894	0.966	0.748
1.08	0.886	0.975	0.741

$\tau_T^* = 0.5$, $c_T^* = 0.5$ and $d_T^* = 1.44$.

We continue the illustration of Theorem 1 in Table 2. To evaluate the welfare properties of the optimal transfer system, we define the relative efficiency as a scalar ε, by which multiplying the wage, and correspondingly the consumption pair of the private system, the two social welfare values are equal. By simple calculation,

$$V(0) + 2\log\varepsilon = V(t^*), \quad \text{i.e.} \quad \varepsilon = \exp[(V(t^*) - V(0))/2].$$

Let $\delta[1] = 0.9$ and run $R[1]$ between 1.01 and 1.06. Table 2 shows that the separator tax rate t_W is slowly rising from 0.054 to 0.196, while the neutral tax rate t_N—the minimum rational tax rate—is steeply rising from 0.056 to 0.339. The relative efficiency of the public system drops from 2.18 to 1.056.

Table 2. *The impact of interest factor: M1*

| Annual relative interest factor $R[1]$ | Separator | Neutral | Optimal public system's relative efficiency |
| | t a x   r a t e | | |
	t_W	t_N	ε
1.01	0.054	0.056	2.180
1.02	0.071	0.077	1.881
1.03	0.093	0.106	1.625
1.04	0.121	0.149	1.406
1.05	0.155	0.215	1.218
1.06	0.196	0.339	1.056

Remark. Annual discount factor: $\delta[1] = 0.9$.

3. Double discount

Model 2 (M2)

Following Feldstein (1985), in Model 2 we shall consider the case of double discounting. (For added generality, we keep considering transfers not just pensions.) Let the real $\alpha \in [0, 1]$ be the factor of underestimation of the transfer received, i.e. $\tilde{t} = \alpha g t$. Then the individual (wrongly) expects to consume merit goods $\tilde{d} = \alpha g t + \rho s$, his utility function implies an elevated optimal purchase:

$$\tilde{s} = \frac{[\delta(1 - t) - \alpha R^{-1}t]_+}{1 + \delta}, \tag{7}$$

where x_+ stands for the positive part of the real number x.

Note that for any tax rate t and for low enough α (e.g. $\alpha = 0$), $\tilde{s} > 0$ holds. Then the first branch is

$$c_\mathrm{S}(t) = \frac{1 - (1 - \alpha R^{-1})t}{1 + \delta} \quad \text{and} \quad d_\mathrm{S}(t) = g\,\frac{\delta R + [1 - \delta(R - 1) - \alpha]t}{1 + \delta} \tag{8S}$$

and the second branch (5T) remains as before. Note, however, that the branching point

$$t_\mathrm{W}(\alpha) = \frac{\delta R}{\delta R + \alpha} \tag{9}$$

is a decreasing function of α, and at the extreme, $t_\mathrm{W}(0) = 1$, i.e. the T-interval becomes empty. To have a local T-optimum, $t_\mathrm{W}(\alpha) < 1/2$ must hold, i.e. we assume

$$\alpha > \delta R \tag{A5}$$

7

(the fifth assumption). Due to A2, the feasibility interval $(\delta R, 1)$ for α is not empty.

As an aside, note that $c_{\mathrm{S}}(t)$ keeps decreasing but for low enough α, $d_{\mathrm{S}}(t)$ can be increasing for a while, namely if an extra assumption holds:

$$\alpha < 1 - \delta(R - 1). \tag{E}$$

A3 makes A5 and E compatible.

If assumption E holds, then the slack optimum t_{S} can be interior rather than the corner solution 0, therefore introducing notations $\chi = 1 - \alpha R^{-1}$ and $\kappa = 1 - \delta(R-1) - \alpha$, t_{S} satisfies

$$\tilde{V}'(t) = \frac{-\chi}{1 - \chi t} + \frac{\kappa}{\delta R + \kappa t} = 0. \tag{10}$$

Solving a linear equation yields

$$t_{\mathrm{S}} = \frac{\kappa - \delta R \chi}{2\kappa\chi}.$$

Any negative root should be replaced by zero.

Returning to the core of the model, we have

Theorem 2. *Under assumptions A1–A5, there is a narrower welfare gap* $(t_{\mathrm{S}}(\alpha), t_{\mathrm{N}}(\alpha))$ *in M2 which must be jumped.*

It is difficult to argue that in real life, all the assumptions A1–A5 plus E hold. But Feldstein's extreme assumption of $\alpha \approx 0$ is not only *ad hoc* but is definitely unrealistic and leads to the neglect of the T-optimum, which may be the true global optimum as well.

Numerical illustration

We shall use the underestimation coefficient $\alpha = 0.5$ in our numerical simulation. In Table 3, the annual relative interest factor $R[1]$ runs between 1.02 and 1.06, while the annual discount factor $\delta[1]$ varies between 0.9 and 0.94. We only depict 3×3 pairs. Obviously, occasionally we surpass the limits set by A4–A5, and we observe not only superior but also inferior T-optima. Note that for low relative annual interest and discount factors (1.02–1.04 and 0.9–0.92), the S-optimal tax rate is often equal to the separator (suggesting that even the government preferred the worker saved a negative amount). For higher relative interest factors (1.04–1.06) and high discount factor (0.94), the separator tax rate is often higher than the T-optimal one, making the T-system uniformly inferior to the S-system. Apart from these extremes, the picture is quite clear: the T-optimum cannot be neglected.

Table 3. *Dependence of optima on the parameter values: M2*

Annual relative interest factor $R[1]$	discount factor $\delta[1]$	S-optimal tax rates t_S	Separator t_W	Worker consumption c_S	Pensioner d_S	Relative efficiency of T w.r.t. S $\varepsilon_{T|S}$
1.02	0.90	0.133	0.133	0.867	0.385	1.472
	0.92	0.229	0.229	0.771	0.662	1.190
	0.94	0.311	0.361	0.670	0.998	1.040
1.04	0.90	0.216	0.216	0.784	0.623	1.216
	0.92	0.171	0.347	0.791	0.854	1.034
	0.94	0.000	0.503	0.865	1.266	0.812
1.06	0.90	0.140	0.327	0.836	0.791	1.045
	0.92	0.000	0.485	0.924	1.257	0.788
	0.94	0.000	0.642	0.865	2.243	0.610

Remark. $\alpha = 0.5$

In summary, except for the extreme case of zero expected benefit ($\alpha = 0$), Feldstein's analysis is incomplete and misleading. Working with realistic parameter values of α, R, δ, the optimal slack contribution rate may be much lower than the tight optimum and may yield much lower welfare.

4. Pension with endogenous fertility

Model 3 (M3)

Our third model is based on Simonovits' (2013) which in turn revised GLM. It is assumed that one part of the tax (θ) supports child raising, and the other part of the tax (τ) goes to the public pension system. With the extension to pensions, the fertility model becomes quite complex, especially taking account of the existence of credit constraint. We assume there is no real growth wage, i.e. relative and absolute interest factors coincide and we replace parameter ν by a choice variable n.

We add new notations and equations to the old ones: pension contribution rate: τ, tax rate: θ, saving: s. We take the cost of raising a child independent of the net wage: p, per child benefit: φ, fertility rate, i.e. the number of children per parent (half the total fertility rate): n, the pay-as-you-go benefit: b; all positive real numbers. We retain the heroic assumption of the literature: the number of children can be any positive real!

Young-age adult consumption (c) and old-age consumptions (d) are respectively equal to

$$c = 1 - \tau - \theta - s - (p - \varphi)n \quad \text{and} \quad d = \rho s + b. \tag{11}$$

The new lifetime utility function is equal to the old utility plus the utility of having children:

$$U(c, n, d) = \log c + \zeta \log n + \delta \log d, \tag{12}$$

where ζ is the coefficient of the relative utility of having children and δ is the discount factor; $\zeta > 0$ and $0 \leq \delta \leq 1$.

Introducing the net-of-contribution wage $\hat{\tau} = 1 - \tau$ and inserting (11) into (12) yield the reduced utility function:

$$U[s, n] = \log(\hat{\tau} - \theta - s - (p - \varphi)n) + \zeta \log n + \delta \log(\rho s + b). \tag{13}$$

Denoting the individual optimum by $c(\tau, \theta), n(\tau, \theta), d(\tau, \theta)$ we define a *paternalistic* social welfare function by replacing the discount factor δ with 1 in (12). Then our social welfare function is an undiscounted utility:

$$V(\tau, \theta) = \log c(\tau, \theta) + \zeta \log n(\tau, \theta) + \log d(\tau, \theta). \tag{14}$$

To learn if the credit constraint is slack or tight, we have to determine the *separatrix curve* $\theta(\tau)$ which separates the two domains in the (τ, θ)-plane. Delaying the proof, we now only present

Lemma 3. a) *The separatrix curve $\theta(\tau)$ is given by*

$$\theta(\tau) = \frac{[\delta p - \zeta \rho^{-1} \tau]\hat{\tau}}{\delta p + \rho^{-1} \tau}, \quad \text{where} \quad 0 \leq \tau < \zeta^{-1} \delta p \rho.$$

b) *The separatrix curve starts from $\theta(0) = 1$ and ends at $\theta(\tau_{\mathrm{M}}) = 0$, where $\tau_{\mathrm{M}} = \zeta^{-1} \delta p \rho$; and $\theta(\tau)$ is declining in $[0, \tau_{\mathrm{M}}]$.*
c) *If $0 \leq \tau < \tau_{\mathrm{M}}$ and $0 < \theta < \theta(\tau)$, then $s(\tau, \theta) > 0$: slack.*
d) *If $0 \leq \tau \leq \tau_{\mathrm{M}}$ and $\theta(\tau) \leq \theta \leq \hat{\tau}$, then $s(\tau, \theta) = 0$: tight.*
e) *If $\tau_{\mathrm{M}} < \tau \leq 1 - \theta$, then $s(\tau, \theta) = 0$: tight.*

Slack credit constraint (S)

We continue the analysis with the slack credit constraint. Copying GLM, we take the partial derivatives of (13) with respect to s and n yielding the first-order necessary conditions for optimum:

$$0 = U'_s[s, n] = \frac{-1}{\hat{\tau} - \theta - s - (p - \varphi)n} + \frac{\delta \rho}{\rho s + b} \tag{15a}$$

and

$$0 = U'_n[s, n] = \frac{-(p - \varphi)}{\hat{\tau} - \theta - s - (p - \varphi)n} + \frac{\zeta}{n}. \tag{15b}$$

For the time being, we do not exclude negative savings.

10

Lemma 4. *The conditional optima are*

$$c(\tau, \theta) = \frac{\hat{\tau} - \theta + \rho^{-1}b}{1 + \delta + \zeta}, \quad d(\tau, \theta) = \delta\rho c(\tau, \theta) \quad \text{and} \quad n(\tau, \theta) = \frac{\zeta}{p - \varphi}c(\tau, \theta).$$

In this model, the tax rate θ and pension b depend on fertility rate n, therefore we introduce the transfer equations:

$$\theta = \varphi n \tag{16a}$$

and

$$b = \tau n. \tag{16b}$$

In words: a) the tax rate is equal to the product of the child benefit rate and fertility rate; b) the pension benefit is equal to the product of the contribution and fertility.

The introduction of per-child support φ reduces the *private* cost of raising a child from p to $p - \varphi$. If (16a) holds, there is no income effect. The introduction of the contribution rate τ diminishes the young-age consumption by the same amount and increases the old-age consumption according to $b = \tau n$ [(16b)]. In a stationary economy with $n = 1$, the two changes cancel each other; for falling/growing population, the reduction of young-age consumption is greater/lower than the increase of the old-age consumption. Using the concept of dynamic efficiency: $\rho > n$, the comparison above changes. As is known, in a dynamically efficient economy, the introduction of a pay-as-you-go pension is suboptimal etc, except for excessive myopia.

Inserting the transfer equations (16a)–(16b) into (15a)–(15b) yields the final optima (see GLM):

Theorem 3. *When the credit constraint is slack in M3 (Lemma 3c), then the individually optimal young-age consumption and fertility rate are given respectively:*

$$c_{\mathrm{S}}^* = \frac{\hat{\tau} - \theta[1 - \tau/(p\rho)]}{1 + \delta + \zeta - \zeta\tau/(p\rho)} \tag{17a}$$

and

$$n_{\mathrm{S}}^* = \frac{\zeta\hat{\tau} + (1 + \delta)\theta}{(1 + \delta + \zeta)p - \zeta\rho^{-1}\tau}. \tag{17b}$$

Proof. Combining (15a) and (15b) leads to

$$\frac{p\delta\rho}{\rho s + \tau n} = \frac{p}{\hat{\tau} - s - pn} = \frac{\zeta}{n} + \frac{\delta\tau}{\rho s + \tau n}.$$

With rearrangement,

$$s = \zeta^{-1}\rho^{-1}[p\delta\rho - (\delta + \zeta)\tau].$$

Rewrite (15a) as $(\rho s + \tau n) = \delta\rho(\hat{\tau} - s - pn)$ or

$$\rho(1 + \delta)s = \delta\rho\hat{\tau} - (p\delta\rho + \tau)n.$$

Inserting s into our last equation leads to (17a) and via $n(\tau, \theta)$ to (17b). ∎

Though we are unable to give a full picture of the dependence of social welfare on the transfer rates in the slack region, we prove that the transfer-free system is a (local) maximum for a large domain of the parameter space.

Theorem 4. *If $\zeta < 2$, then the transfer-free system $(\tau_S^*, \theta_S^*) = (0,0)$ is a local maximum, where the optimal outcomes are*

$$c_S^{**} = \frac{1}{1+\delta+\zeta}, \qquad d_S^{**} = \frac{\delta\rho}{1+\delta+\zeta} \quad \text{and} \quad n_S^{**} = \frac{\zeta}{(1+\delta+\zeta)p}. \qquad (17^*)$$

Remarks. 1. The assumption $\zeta < 2$ is quite mild. In fact, for $\tau = 0 = \theta$, the locally optimal fertility rate is

$$n_S^* = \frac{\zeta}{(1+\delta+\zeta)p}.$$

For example, for $\zeta = 2$, $p = 1/3$ and $\delta = 1$, the slack-optimal fertility rate is too high: $n_S^* = 1.5$.

2. Table 3 in Simonovits (2013) (omitted here) suggests that probably our local maximum is also a global one in the slack region.

Proof. In this proof, we shall drop the subindex S and study the behavior of the social welfare function (14). Take the partial derivatives with respect to the two transfer rates:

$$V_\tau'(\tau,\theta) = \frac{c_\tau'(\tau,\theta)}{c(\tau,\theta)} + \zeta\frac{n_\tau'(\tau,\theta)}{n(\tau,\theta)} + \frac{d_\tau'(\tau,\theta)}{d(\tau,\theta)}$$

and

$$V_\theta'(\tau,\theta) = \frac{c_\theta'(\tau,\theta)}{c(\tau,\theta)} + \zeta\frac{n_\theta'(\tau,\theta)}{n(\tau,\theta)} + \frac{d_\theta'(\tau,\theta)}{d(\tau,\theta)}.$$

Using $d = \delta\rho c$, the first and the third terms are equal in both equations. Moreover, using (17) at $\tau = 0 = \theta$

$$\frac{c_\tau'(0,0)}{c(0,0)} = \frac{\zeta p^{-1}\rho^{-1} - (\bar\zeta + \delta)}{\bar\zeta + \delta} < 0 \quad \text{and} \quad \frac{n_\tau'(0,0)}{n(0,0)} = \zeta^{-1}\frac{c_\tau'(0,0)}{c(0,0)}.$$

Hence

$$V_\tau'(\tau,\theta) = 3\frac{c_\tau'(0,0)}{c(0,0)} < 0.$$

By the linearity of $c(0,\theta)$ and $n(0,\theta)$, the elasticities are equal to 1 and –1, respectively, i.e. $V_\theta'(0,0) = -2 + \zeta < 0$ etc.

The second-order conditions are omitted. ∎

Tight credit constraint (T)

We turn now to the tight credit constraint: $s = 0$. We have to replace condition $U_s'[s,n] = 0$ [(15a)] by $U_s'[0,n] < 0$ and then (15b) becomes $U_n'[0,n] = 0$. By easy calculation we obtain

$$(p - \varphi)n = \zeta c = \zeta[\hat\tau - \theta - (p-\varphi)n].$$

Inserting the budget condition $\theta = \varphi n$, and notation $\bar\zeta = 1 + \zeta$, we end-up with

Theorem 5. a) *When the credit constraint is tight (Lemma 4d and e), the optimal fertility rate and the corresponding young-age consumption are respectively equal to*

$$n_T^* = \frac{\zeta\hat{\tau} + \theta}{\bar{\zeta}p} \quad \text{and} \quad c_T^* = \frac{\hat{\tau} - \theta}{\bar{\zeta}}. \tag{18}$$

b) *The optimal fertility rate is a decreasing function of the contribution rate τ and an increasing function of the tax rate θ.*

Finally we are able to give the

Proof of Lemma 3. Inserting the formulas (17)–(18) respectively for n_S^* and n_T^* into the separatrix's equation, $n_S^* = n_T^*$ results in

$$\frac{\zeta\hat{\tau} + (1+\delta)\theta}{(1+\delta+\zeta)p - \zeta\rho^{-1}\tau} = \frac{\zeta\hat{\tau} + \theta}{\bar{\zeta}p}.$$

With rearrangement, $[\zeta\hat{\tau}+(1+\delta)\theta]\bar{\zeta}p = [\zeta\hat{\tau}+\theta][(1+\delta+\zeta)p-\zeta\rho^{-1}\tau]$. After simplification, $\theta(\tau)$ is obtained. For $\theta > \theta(\tau)$, $s(\tau, \theta) = 0$; for $\theta < \theta(\tau)$, $s(\tau, \theta) > 0$.

The separatrix in Lemma 3 is a fraction: its numerator is the product of two decreasing positive functions and its denominator is an increasing positive function, therefore the fraction is also declining. ∎

Next we determine the socially optimal pair of contribution and tax rates in the tight region.

Theorem 6. *In the region of a tight credit constraint, the socially optimal contribution and tax rates are equal respectively to*

$$\tau_T^* = \theta_T^* = \frac{1}{3 + \zeta} \tag{19}$$

and the corresponding outcomes are

$$n_T^{**} = \frac{\zeta}{(1+\zeta)p}, \qquad c_T^{**} = \frac{1}{3+\zeta} \quad \text{and} \quad d_T^{**} = \frac{1}{(3+\zeta)^2 p}. \tag{18*}$$

Remarks. 1. It is easy to show that the optimal pair of transfer rates in (19) generate tight credit constraint.

2. It is natural that the optimal fertility rate in (18*) is a decreasing function of the relative utility of children but it is surprising that the two optimal transfer rates are equal to each other and are independent of any other parameter value.

3. In both regions, the socially optimal child support is always lower than the expenditure on children: $0 \le \varphi < p$.

Proof. Our starting point is as follows:

$$V\{\tau, \theta\} = \log\frac{\hat{\tau} - \theta}{\bar{\zeta}} + \zeta\log\frac{\zeta\hat{\tau} + \theta}{\bar{\zeta}p} + \log\frac{\tau(\zeta\hat{\tau} + \theta)}{\bar{\zeta}p} \to \max.$$

Using the identity $\log(x/y) = \log x - \log y$, the constant denominators can be dropped. Returning to $\hat{\tau} = 1 - \tau$, we have then an equivalent problem:

$$\tilde{V}\{\tau, \theta\} = \log(1 - \tau - \theta) + \bar{\zeta}\log(\zeta(1 - \tau) + \theta) + \log\tau \to \max.$$

Taking the partial derivatives of $\tilde{V}$ with respect to τ and θ and equate the derivatives to zero yield the first-order necessary conditions:

$$0 = \tilde{V}'_\tau = \frac{-1}{1-\tau-\theta} - \frac{\bar{\zeta}\zeta}{\zeta(1-\tau)+\theta} + \frac{1}{\tau} \tag{20}$$

and

$$0 = \tilde{V}'_\theta = \frac{-1}{1-\tau-\theta} + \frac{\bar{\zeta}}{\zeta(1-\tau)+\theta}. \tag{21}$$

From (21),

$$\theta = \frac{1-\tau}{2+\zeta} = \kappa(1-\tau). \tag{22}$$

Inserting back (22) into (20),

$$\frac{-1}{(1-\kappa)(1-\tau)} - \frac{\bar{\zeta}\zeta}{(\zeta+\kappa)(1-\tau)} + \frac{1}{\tau} = 0.$$

Hence Theorem 6 is obtained. ∎

Comparing the presumed welfare maxima of the two regions, the following quantity plays an important role. The *critical* discount factor $\delta^* = \delta(p,\zeta,\rho)$ is defined as the discount factor (depending on the raising cost, on the relative child utility and on the interest factor) for which the two systems achieve the same social welfare (cf. (17*) and (18*)). The function $\delta(p,\zeta,\rho)$ is to be determined from the following implicit equation:

$$V(0,0) = V(\tau_{\mathrm{T}}^*, \theta_{\mathrm{T}}^*)$$

but it would not be helpful.

Obviously, if $0 < \delta < \delta(p,\zeta,\rho)$, then the T-optimum is better than the S-optimum, and vice versa. From now on we shall confine our attention to the T-case, which is probably the more relevant and definitely the simpler case.

As before, we shall compare the welfare provided by a (τ,θ)-system with the transfer-free system's $(0,0)$ as follows. Let us define the *relative efficiency* of the former with respect to the latter by the positive number $\varepsilon(\tau,\theta)$ if multiplying the unit wage by ε in the transfer-free system, the welfare would reach that value provided by the transfer system with unitary wages. In formula:

$$V[\varepsilon,0,0] = V[1,\tau,\theta].$$

Due to the simple utility function (12), the optimal fertility rate is independent of the wage and the optimal consumption pair are homogeneous linear functions of the wage. Therefore

$$2\log\varepsilon(\tau,\theta)+\log c(0,0))+\zeta\log n(0,0)+\log d(0,0) = \log c(\tau,\theta)+\zeta\log n(\tau,\theta)+\log d(\tau,\theta).$$

Hence $\log\varepsilon(\tau,\theta)$ or $\varepsilon(\tau,\theta)$ can simply be determined:

$$2\log\varepsilon(\tau,\theta) + V[1,0,0] = V[1,\tau,\theta] \quad \text{i.e.} \quad \varepsilon(\tau,\theta) = \exp[0.5(V[1,\tau,\theta] - V[1,0,0])].$$

Numerical illustration

Here we numerically illustrate the simplest case of homogeneous fertility rate and proportional pensions. Assume a 30-year long period between accumulation and decumulation. To give room for an unfunded pension system, we assume $\delta\rho < 1$.

Table 4 displays the critical discount factor for a fixed raising cost $p = 0.35$ for selected points of a grid. For a fixed coefficient ζ, the higher the interest factor, the lower the critical value. For example, for $\zeta = 0.4$, as the annual interest factor $\rho[1]$ rises from 1.02 to 1.04, the annual critical factor $\delta^*[1]$ drops from 0.943 to 0.920. In parallel, the steady state slack fertility rate rises from $n_S^* = 0.727$ to 0.771, while $n_T^* = 1.176$ remains invariant. Similarly, fixing $\rho[1] = 1.03$, as ζ rises from 0.3 to 0.5, the critical $\delta^*[1]$ rises from 0.927 to 0.934, basically invariant.

Table 4. *Critical discount factor function $\delta(\zeta, \rho)$: M3*

Coefficient of relative utility ζ	Interest factor $\rho[1]$	Compounded critical discount factor $\delta^*[1]$	Optimal steady state fertility rate slack n_S^*	tight n_T^*
0.3	1.02	0.938	0.593	1.126
	1.03	0.927	0.611	
	1.04	0.916	0.625	
0.4	1.02	0.943	0.727	1.176
	1.03	0.931	0.753	
	1.04	0.920	0.771	
0.5	1.02	0.946	0.846	1.224
	1.03	0.934	0.877	
	1.04	0.924	0.897	

Remark. $p = 0.35$, $\zeta = 0.4$.

Choose the middle (italicized) row in Table 4: $\zeta = 0.4$, but with some slight modification: $\rho[1] = 1.03$ (the annual interest rate being 3%), and $\delta[1] = 0.92$ (i.e. the annual discount rate of 8%), well below the critical value 0.931.

Changing the contribution and the tax rates, their impact can be studied also numerically. First we present the separatrix (discussed in Lemma 3). It declines from 1 to 0 while the contribution rate τ rises from 0 to 0.17. Second, from Theorem 3, $\tau^* = \theta^* = 0.294$, close 0.3.

Table 5 displays the characteristics of the model along the diagonal with $\tau = \theta$. First of all, note that the optimal saving is only positive for low enough contribution and tax rates, namely until reaching $\tau = \theta = 0.1$. The relative efficiency reaches its S-maximum between 0.15 and 0.2 (the neutral rate being close to 0.17) and then achieves the true maximum close to 0.3.

Table 5. *The impact of the equal transfer rates: M3*

Transfer rates $\tau = \theta$	Fertility rate n^*	Saving s^*	Young-age consumption c^*	Old-age d^*	Relative efficiency ε
0.00	0.771	0.055	0.675	0.134	1.000
0.05	0.850	0.033	0.619	0.123	0.936
0.10	0.932	0.008	0.566	0.113	0.871
0.15	1.000	0	0.500	0.150	0.958
0.20	1.061	0	0.429	0.212	1.068
0.25	1.122	0	0.357	0.281	1.134
0.30	1.184	0	0.286	0.355	1.153
0.35	1.245	0	0.214	0.436	1.117
0.40	1.306	0	0.143	0.522	1.009

5. Conclusions

We have revisited three models of public pension systems, where a myopic representative individual and a paternalistic government play a Stackelberg-game. These models (especially models 1 and 2) can easily be interpreted also as abstract transfer models. In our models, we assumed that the private purchase is cheaper than the public transfer. We were able to show or at least conjecture that even if the socially optimal tax yields higher social welfare than the private system does, there is a welfare gap for lower taxes which should be jumped. Model 1 (initiated in Simonovits (2015a)) only served as a stepping stone to Models 2 and 3 (Simonovits, 2013). We have dissected two very influential models: Feldstein (1985) and GLM (2003). Their authors overlooked the *coexistence* of the S- and the T-optima and the welfare gap spanned between them. I express my hope that the approach of the present paper can be applied to other transfer models as well.

References

Auerbach, A. J. and Kotlikoff, L. J. (1987): *Dynamic Fiscal Policy*, Cambridge, Cambridge University Press.

Barr, N. and Diamond, P. (2008): *Reforming Pensions: Principles and Policy Choices*, Oxford, Oxford University Press.

Cremer, H.; De Donder, Ph.; Maldonado, D. and Pestieau, P. (2008): "Designing a Linear Pension Scheme with Forced Savings and Wage Heterogeneity", *Journal of Economic Surveys 22*, 213–233.

Cremer, H. and Pestieau, P. (2011): "Myopia, Redistribution and Pensions", *European Economic Review 55*, 165–175.

Feldstein, M. S. (1985): "The Optimal Level of Social Benefits", *Quarterly Journal of Economics 100*, 302–320.

Feldstein, M. S. (1987): "Should Social Security Means be Tested?", *Journal of Political Economy 95*, 468–484.

van Groezen, B.; Leers, Th. and Meijdam, L. (2003): "Social Security and Endogenous Fertility: Pensions and Child Allowances as Siamese Twins", *Journal of Public Economics 87*, 233–251.

Samuelson, P. A. (1958): "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money", *Journal of Political Economy 66*, 467–482.

Simonovits, A (2013): Savings, Child Support, Pensions and Endogenous (and Heterogeneous) Fertility, IE-CERS-HAS Working Paper 35.

Simonovits, A. (2015a): "Paternalism in Pension Systems", *Fecundating Thoughts: Studies in Honor of Eighty-Fifth Birthday of János Kornai* eds. B. Hámori and M. Rosta, Cambridge, Cambridge Scholar Publishers.

Simonovits, A. (2015b): "Socially Optimal Contribution Rate and Cap in a Proportional Pension System", *Portuguese Economic Journal 14*, 45–63.