

Assets with possibly negative dividends*

(Preliminary and incomplete. Comments welcome.)

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Abstract

The paper introduces assets whose dividends can take any value (positive, negative or zero) in a dynamic general equilibrium model with financial market imperfections. We investigate the interplay between the asset markets and the production sector. The behavior of asset price and value is also studied.

Keywords: Infinite-horizon, general equilibrium, productivity, asset price, negative dividend

JEL Classification Numbers: D5, D90, E44, G12.

1 Introduction

The standard literature of asset pricing (Lucas, 1978; Ljungqvist and Sargent, 2012) assumes that dividends of assets are positive. However, recently some central banks and governments issues assets with negative nominal interest rates (see Figure 1 below). Such these assets may have interpretation: once we buy an asset (money, for example), we will (1) be able to resell it, and (2) have to pay an amount (instead of receive an amount as with the case of positive dividend). Motivated by this fact, our paper investigates the behavior of prices and values of assets whose dividends may take any value (negative, positive or zero), and the interplay between the asset markets and the production sector.

To do so, we build an infinite-horizon general equilibrium model with a production sector and an imperfect financial market. There are a finite number of heterogeneous consumers and one representative firm (without market power). Consumers have two choices for investing: buy physical capital and buy a long-lived asset (whose initial supply is exogenous and positive) which brings dividends in the future (as the Lucas tree). The novelty is that asset dividends may take any value (positive, negative or zero).

When asset dividends may be negative, it is not trivial that asset prices are positive because it is possible that nobody buys this asset. Hence, we may interpret: one can run negative dividend policy if there exists an equilibrium where asset prices are positive at any date. We focus our analysis on equilibria where asset prices are positive.

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Figure 1: Policy rate on excess reserves, percent

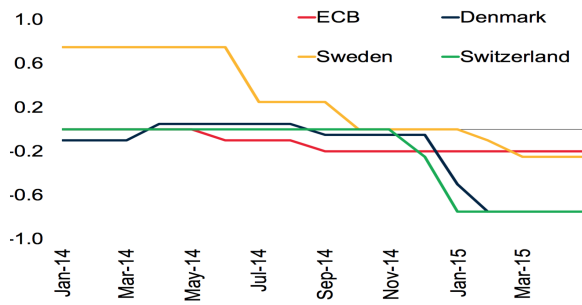
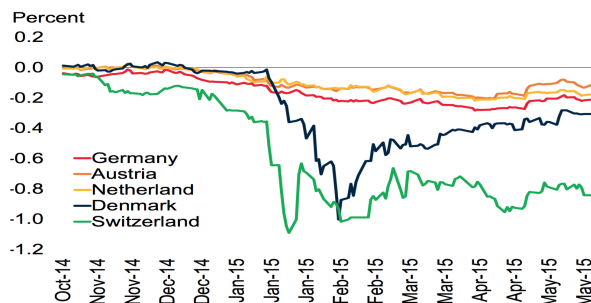


Figure 2: Two year government bond yields



Source: The World Bank, Global Economic Prospects, June 2015.

The first set of our contribution is to find out conditions under which we can/should run negative dividend policy.

We show that negative dividend policies cannot be sustained without a strong production sector. The idea behind is the following. If one agent buys asset whose future dividends are negative, she will be able resell this asset but have to pay an amount at the same time. If this amount is so high (i.e., negative dividend is so low), her income including that from capital may be not enough to cover this amount; this can happen if the production sector is weak. In this case, no one wants to buy assets with negative dividends, which implies in turn that asset prices must be zero.

We also prove that when asset dividends are negative at any date, there is no equilibrium with positive asset prices and borrowing constraints are not binding. We then provide examples where agents cannot borrow and asset dividends are negative at any date but assets prices are strictly positive. Let us explain the intuition of this example where we assume that there is a fluctuation on endowments, which in turn creates a fluctuation on agents' income. Consider a date. Agents, who have low endowment at the next date and cannot borrow, have to transfer their wealth from the present date to the next date. Hence they accept to buy financial asset at the present date with positive prices even this asset brings negative dividends in the future. The same argument is applied for other dates and agents. Therefore, asset prices are positive at any date.

The second set of our contribution concerns the behavior of asset price and value. Let us denote q_t and ξ_t the equilibrium asset price (in terms of consumption good) and asset dividend at date t . Given that the asset supply is positive at any date, we have so-called *no-arbitrage condition*, as in Santos and Woodford (1997),

$$q_t = \gamma_{t+1}(q_{t+1} + \xi_{t+1}) \quad (1)$$

where γ_{t+1} is the endogenous discount factor of the economy from date t to $t + 1$. By

iterating (1), we get the following decomposition.

$$q_0 = \left(\sum_{t=1}^T Q_t \xi_t \right) + Q_T q_T. \quad (2)$$

where $Q_t \equiv \gamma_1 \cdots \gamma_t$ is the endogenous discount factor of the economy from date 0 to t .

In the standard theory,¹ ξ_t is assumed to be positive for any t . So, $\sum_{t=1}^T Q_t \xi_t$ is increasing in T , which implies that the discounted asset value $Q_T q_T$ converges to some value. When $Q_T q_T$ converges to zero, we can compute the asset price by $q_0 = \sum_{t=1}^{\infty} Q_t \xi_t$ (this kind of equilibrium is referred to no-bubble equilibrium).

Our contribution is to point out, by some examples, that when asset dividends (ξ_t) may be negative, the sum $\sum_{t=1}^T Q_t \xi_t$ may diverge, and the discounted asset value $Q_T q_T$ may diverge or converge to any value (even converge to infinity). We also show that asset prices (q_t) may fluctuate over time. Interestingly, there are some cases where asset prices are zero at infinitely many dates and they are positive at other dates. These show how hard is the researching for a robust result on prices and values of assets who dividends may be negative.

The remainder of the paper is organized as follows. Section 2 introduces our framework and presents some basic properties of equilibria. Section 3 provides analyses of equilibrium with positive asset prices. In Section 4, we give some examples illustrating and complementing our theoretical results. Section 5 concludes. All formal proofs are gathered in Appendix.

2 Framework

Our model is based on Lucas (1978), Santos and Woodford (1997) and Le Van and Pham (2016). The novelty is that we do not require the positivity of dividends. In addition, different from Lucas (1978), Santos and Woodford (1997), we introduce physical capital. However, for simplicity, we assume that consumers are prevented from borrowing.

Time is discrete and runs from 0 to \bar{T} , where \bar{T} may be finite or infinite. There are a finite number of households. Let us denote $I := \{1, 2, \dots, m\}$ the set of households.

Consumption good. There is a single consumption good at each date. At period t , the price of consumption good is denoted by p_t and agent i consumes $c_{i,t}$ units of consumption good.

Physical capital. Let us denote r_t the capital return at date t and δ the depreciation rate, $k_{i,t+1}$ the quantity of physical capital bought by agent i at date t .

Financial asset. At period t , if agent i buys $a_{i,t}$ units of financial asset with price q_t , she will receive ξ_{t+1} units of consumption good as dividend and she will be able to resell $a_{i,t}$ units of financial asset with price q_{t+1} . Note that ξ_t may take any value (negative, positive or zero).

Each household i takes the sequence of prices $(p, q, r) = (p_t, q_t, r_t)_{t=0}^{\bar{T}}$ as given and choose allocation sequences $(c_{i,t}, k_{i,t+1}, a_{i,t})_{t=0}^{\bar{T}}$ to maximize her intertemporal utility. The utility maximization problem of agent i is the following:

$$(P_i(p, q, r)) : \max_{(c_{i,t}, k_{i,t+1}, a_{i,t})_{t=0}^{\bar{T}}} \left[\sum_{t=0}^{\bar{T}} \beta_i^t u_i(c_{i,t}) \right] \quad (3)$$

¹See Tirole (1982), Kocherlakota (1992), Santos and Woodford (1997), Le Van and Pham (2016), ..., among others.

subject to

$$k_{i,t+1} \geq 0, \quad a_{i,t} \geq 0, \quad (4)$$

$$p_t(c_{i,t} + k_{i,t+1} - (1 - \delta)k_{i,t}) + q_t a_{i,t} \leq r_t k_{i,t} + (q_t + p_t \xi_t) a_{i,t-1} + p_t e_{i,t} + \theta_t^i \pi_t, \quad (5)$$

where $e_i \equiv (e_{i,t})$ is the sequence of endowment of agent i while π_t is the profit of the firm at date t (see below). $(\theta_t^i)_{i=1}^m$ is the share of profit at date t . $\theta_i := (\theta_t^i)_t$ is exogenous, $\theta_t^i \geq 0$ for all i and $\sum_{i=1}^m \theta_t^i = 1$.

For each period t , there is a representative firm which takes prices (p_t, r_t) as given and maximizes its profit by choosing physical capital amount K_t .

$$(P(p_t, r_t)) : \quad \pi_t \equiv \max_{K_t \geq 0} [p_t F(K_t) - r_t K_t] \quad (6)$$

Denote \mathcal{E} the economy which is characterized by a list

$$\left((u_i, \beta_i, e_i, k_{i,0}, a_{i,-1}, \theta^i)_{i=1}^m, F, \delta, (\xi_t)_{t=0}^\infty \right).$$

Definition 1. Consider the economy \mathcal{E} . A sequence of prices and quantities

$(p_t, q_t, r_t, (c_{i,t}, k_{i,t+1}, a_{i,t})_{i=1}^m, K_t)_{t=0}^T$ is an equilibrium of the economy \mathcal{E} if the following conditions are satisfied:

(i) Price positivity: $p_t > 0, r_t > 0$ and $q_t \geq 0$ for $t \geq 0$.

(ii) Market clearing: at each $t \geq 0$,

$$\sum_{i \in I} (c_{i,t} + k_{i,t+1} - (1 - \delta)k_{i,t}) = e_t + F(K_t) + \xi_t \sum_{i \in I} a_{i,t-1} \quad (7)$$

$$K_t = \sum_{i \in I} k_{i,t}, \quad (8)$$

$$\sum_{i \in I} (a_{i,t} - a_{i,t-1}) \leq 0, \quad q_t \sum_{i \in I} (a_{i,t} - a_{i,t-1}) = 0 \quad (9)$$

where $e_t \equiv \sum_{i \in I} e_{i,t}$ the aggregate endowment.

(iii) Optimal consumption plans: for each i , $(c_{i,t}, k_{i,t+1}, a_{i,t})_{t=0}^T$ is a solution of the problem $(P_i(p, q, r))$.

(iv) Optimal production plan: for each $t \geq 0$, K_t is a solution of the problem $(P(p_t, r_t))$.

Comments. In this definition, we do not require that $q_t > 0$ for any t . The asset's market clearing condition (9) is in the spirit of [Arrow and Debreu \(1954\)](#), and $\sum_{i \in I} (a_{i,t} - a_{i,t-1}) = 0$ if price $q_t > 0$. As we will mention, in some cases when asset dividends are not positive, it is not easy to find out an equilibrium with $q_t > 0$. In condition (7), the term $\xi_t \sum_{i \in I} a_{i,t-1}$ will be ξ_t if $\sum_{i \in I} a_{i,t-1} = 1$. However, when nobody buys asset, we have $\sum_{i \in I} a_{i,t-1} = 0$.

Standard assumptions are required in our paper.

Assumption (H1): u_i is in C^1 , $u_i'(0) = +\infty$, and u_i is strictly increasing, concave, continuously differentiable.

Assumption (H2): $F(\cdot)$ is strictly increasing, concave, continuously differentiable, $F(0) = 0$.

Assumption (H3): At initial period 0, $k_{i,0}, a_{i,-1} \geq 0$, and $(k_{i,0}, a_{i,-1}) \neq (0, 0)$ for $i = 1, \dots, m$. Moreover, we assume that $\sum_{i=1}^m a_{i,-1} = 1$ and $K_0 := \sum_{i=1}^m k_{i,0} > 0$.

Definition 2. Given (ξ_t) , we say that a positive sequence of consumption and capital (C_t, K_t) is feasible if $C_t + K_{t+1} \leq e_t + F(K_t) + (1 - \delta)K_t + \xi_t$ for any t .

Let (D_t) be defined by

$$D_0 := e_0 + F(K_0) + (1 - \delta)K_0 + \max(0, \xi_0), \quad (10)$$

$$D_t := e_t + F(D_{t-1}) + (1 - \delta)D_{t-1} + \max(0, \xi_t) \quad \forall t \geq 0. \quad (11)$$

We see that D_t is exogenous and depends on the function F and $K_0, \delta, \xi_1, \dots, \xi_t$. Moreover, $C_t + K_{t+1} \leq D_t$ for every $t \geq 0$. This leads to the following result.

Lemma 1 (the boundedness of consumption and capital stocks). *Consider a feasible path (C_t, K_t) . We have*

1. *Capital and consumption are in a compact set for the product topology.*
2. *Moreover, they are uniformly bounded if $(e_t)_t$ and $(\xi_t)_t$ are uniformly bounded from above and there exists t_0 and there exists x_0 such that $F(x) + (1 - \delta)x + \sup_t(e_t + \xi_t) \leq x$ for every $x \geq x_0$.*

One can prove that conditions in point 2 are satisfied if $\sup_t(e_t + \xi_t) < \infty$ and $F'(\infty) < \delta$. The following assumption ensures that utility of each agent is finite.

Assumption (H4): For each agent i ,

$$\sum_{t=0}^{\infty} \beta_i^t u_i(D_t(F, \delta, K_0, \xi_0, \dots, \xi_t)) < \infty. \quad (12)$$

Price normalization. Since the utility function u_i is strictly increasing, at any equilibrium (if it exists), p_t must be strictly positive for any t . So, without loss of generality, we can normalize by setting $p_t = 1$ for any t . In this case, we also call $(q_t, r_t, (c_{i,t}, a_{i,t}, k_{i,t})_{i=1}^m, K_t)_t$ equilibrium.

2.1 Basis properties

We provide a necessary and sufficient condition to verify that a list of prices and allocations is an equilibrium.

Lemma 2. $(q_t, r_t, (c_{i,t}, a_{i,t}, k_{i,t})_{i=1}^m, K_t)_t$ is an equilibrium if and only if there exist sequences $(\sigma_{i,t}, \nu_{i,t})_{i,t}$ such that the following conditions are satisfied for any i and for any t .

(i) $c_{i,t} > 0, k_{i,t+1} \geq 0, a_{i,t} \geq 0, \sigma_{i,t} \geq 0, \nu_{i,t} \geq 0, K_t \geq 0, q_t \geq 0, r_t > 0$ for any t .

(ii) *First order conditions:*

$$\begin{aligned} \frac{1}{r_{t+1} + 1 - \delta} &= \frac{\beta_i u'_i(c_{i,t+1})}{u'_i(c_{i,t})} + \sigma_{i,t}, & \sigma_{i,t} k_{i,t+1} &= 0 \\ \frac{q_t}{q_{t+1} + \xi_{t+1}} &= \frac{\beta_i u'_i(c_{i,t+1})}{u'_i(c_{i,t})} + \nu_{i,t}, & \nu_{i,t} a_{i,t} &= 0. \end{aligned}$$

(iii) *Transversality conditions*

$$\lim_{t \rightarrow \infty} \beta_i^t u'_i(c_{i,t})(k_{i,t+1} + q_t a_{i,t}) = 0. \quad (13)$$

(iv) $\forall t, F(K_t) - r_t K_t = \max\{F(k) - r_t k : k \geq 0\}$.

(v) $c_{i,t} + k_{i,t+1} - (1 - \delta)k_{i,t} + q_t a_{i,t} = r_t k_{i,t} + (q_t + \xi_t)a_{i,t-1} + \theta_t^i \pi_t + e_{i,t}$
where $\pi_t = F(K_t) - r_t K_t$.

(vi) $K_t = \sum_{i \in I} k_{i,t}$

(vii) $\sum_{i \in I} (a_{i,t} - a_{i,t-1}) \leq 0$, and $\sum_{i \in I} (a_{i,t} - a_{i,t-1}) = 0$ if $q_t > 0$.

Transversality condition (13) which is not trivial can be proved by adapting the argument in the proof of Theorem 1 in [Kamihigashi \(2002\)](#). The proof of Lemma 2 is left to the reader.

Remark 1. Consider a finite T -period economy. If $\xi_t \leq 0$ for any $t \leq T$, there does not exist an equilibrium with $q_t > 0$ for any $t \leq T - 1$.

Let us denote, for each $t \geq 0$, $\gamma_{i,t+1}$ (respectively, $Q_{i,t}$) the agent i 's discount factor from date t to date $t + 1$ (respectively, from initial date to date t) as follows.

$$\gamma_{i,t+1} := \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}, \quad Q_{i,0} = 1, \quad Q_{i,t} := \gamma_{i,t} \cdots \gamma_{i1}. \quad (14)$$

We also define γ_{t+1} the discount factor of the economy from date t to $t + 1$ and Q_t the discount factor of the economy from date 0 to t

$$\gamma_{t+1} := \max_i \left\{ \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} \right\}, \quad Q_0 = 1, \quad Q_t := \gamma_1 \cdots \gamma_t. \quad (15)$$

According to point (iii) of Lemma 2, we have so-called non-arbitrage inequalities.

Lemma 3. At equilibrium, we have, for each t ,

$$q_t \geq \gamma_{t+1}(q_{t+1} + \xi_{t+1}) \text{ with equality if } \sum_i a_{i,t} > 0 \quad (16)$$

$$1 \geq \gamma_{t+1}(r_{t+1} + 1 - \delta) \text{ with equality if } K_{t+1} > 0 \quad (17)$$

Note that $Q_t k_{i,t+1} = (1 - \delta + r_{t+1})Q_{t+1} k_{i,t+1}$ for any t and for any i .

In the remainder of the paper, we will focus on equilibria where all prices are strictly positive, i.e. $q_t > 0$ for any t . In this case, we have $\sum_i a_{i,t} = 1$ for any t , and therefore

$$q_t = \gamma_{t+1}(q_{t+1} + \xi_{t+1}). \quad (18)$$

3 Negative dividend and production

The asset in our framework can be interpreted as an asset issued by the government. The government can choose negative dividends at some dates. However, such an action has no effect in the economy if the asset price is zero. This motivates us to introduce the following notion.

Definition 3. We say that the government can run negative dividends policy if there exists an equilibrium with $q_t > 0$ for any t .

The aim of this section is to find out conditions under which the government can run negative dividends policy.

3.1 Can we run negative dividends policy?

In this section, we will focus on the infinite-horizon model: $\bar{T} = \infty$.

First, we consider the case where asset dividend at only one date may be negative. We have the following result.

Proposition 1. *Assume that Assumptions (H1)-(H4) hold and $u_i(0) = 0$ for any i .*

Consider a date $s^ \geq 0$. Assume that $\xi_t \geq 0$ for any $t \neq s^*$, and there is an infinite sequence $(\xi_{t_n})_n$ such that $\xi_{t_n} > 0$ for any t . Then, there exists $\bar{\xi} > 0$ such that: for any $\xi_{s^*} \geq -\bar{\xi}$, there exists an equilibrium with $q_t > 0$ for any t .*

Proof. See Appendix 6.1. □

According to this result, the existence of equilibrium is ensured if asset dividend at some date is negative but not far from zero (in the sense that $B_{i,t} > 0$). In particular, we recover the existence result in [Le Van and Pham \(2016\)](#) for the case $\xi_t > 0$ for any t .

In what follows, we will consider more general cases where dividends at any date may be negative. We start by pointing out the behavior of asset price and value in the very long run.

Lemma 4. *Assume that $0 < \liminf_{t \rightarrow \infty} \xi_t \leq \limsup_{t \rightarrow \infty} \xi_t < +\infty$,² and conditions in point 2 of Lemma 1 hold. Then, for any equilibrium, we have $\lim_{t \rightarrow \infty} Q_t q_t = 0$ and $q_s = \sum_{t=s+1}^{\infty} Q_t \xi_t / Q_s$ for each $s \geq 0$. Consequently, $q_t > 0$ for any t high enough*

Proof. See Appendix 6.2. □

Lemma 4 provides a sufficient condition under which the present value $\sum_{t=1}^{\infty} Q_t \xi_t$ converges. Moreover, the equilibrium price at any date is equal to the present value of future dividends, which is equivalent to the fact that the discounted value of asset $\lim_{t \rightarrow \infty} Q_t q_t$ will converge to zero. Lemma 4 also gives a sufficient condition under which asset prices are positive in the very long run. Under assumptions in Lemma 4, aggregate consumption stocks are uniformly bounded from above.

Some interesting consequences of Lemma 4 should be mentioned.

Corollary 1. *Assume that $0 < \liminf_{t \rightarrow \infty} \xi_t \leq \limsup_{t \rightarrow \infty} \xi_t < +\infty$, and conditions in point 2 of Lemma 1 hold. Let us consider a date $s \geq 0$ such that $\xi_s < 0$. Consider an equilibrium. If $q_{s-1} > 0$, then $\sum_{t=s+1}^{\infty} Q_t \xi_t > Q_s |\xi_s| > 0$.*

Corollary 1 indicates that when dividend at some date, say t , is negative, the asset price at date $t - 1$ is strictly positive only if the present value of dividends at date t is strictly higher than the absolute discounted value of asset at date t .

The following result shows the importance of the productivity.

Corollary 2. *Assume that $0 < \liminf_{t \rightarrow \infty} \xi_t \leq \limsup_{t \rightarrow \infty} \xi_t < +\infty$ and conditions in point 2 of Lemma 1 hold. Let us consider a date s where $\xi_s < 0$. If there is an equilibrium with $K_t > 0$ for any t , then we have $\sum_{t=s+1}^{\infty} (F'(0) + 1 - \delta)^{t-s} |\xi_t| \geq |\xi_s| > 0$.*

²Note that there may be some t such that $\xi_t < 0$.

Note that when conditions $0 < \liminf_{t \rightarrow \infty} \xi_t \leq \limsup_{t \rightarrow \infty} \xi_t < +\infty$ are violated, $\sum_{t \geq s+1}^{\infty} Q_t \xi_t$ may be lower than $Q_s |\xi_s|$. This property will be addressed in Section 4.2.

According to transversality condition (13) in Lemma 2, we have the following result showing the role of intertemporal marginal rates of substitutions $\gamma_{i,t+1} := \beta_i u'_i(c_{i,t+1})/u'_i(c_{i,t})$.

Proposition 2 (role of agents' heterogeneity). *Let $s \geq 0$. Assume that $\xi_t \leq 0$ for any $t \geq s$. Then,*

1. *there is no equilibrium with positive prices such that $\gamma_{i,t} = \gamma_t$ for any $t \geq s+1$ and for any i ,*
2. *and consequently, there is no equilibrium with positive prices such that $a_{i,t} > 0$ for any $t \geq s$ and for any i .*

Proof. See Appendix 6.3. □

According to Proposition 2, when all dividends are negative, there is no equilibrium with positive prices, in which the intertemporal marginal rates of substitutions are the same at any period (this happens if agents are identical). The intuition of is the following: when the intertemporal marginal rates of substitutions are the same, agents' investment behavior are similar. In such a case, nobody buys assets with negative dividends. So, asset prices are zero at any date.

Point 2 of Proposition 2 indicates the role of borrowing constraints: It implies that at each equilibrium with positive prices, there exists an agent i and an infinite sequence $(t_n)_{n \geq 1}$ such that $a_{i,t_n} = 0$ for any $n \geq 1$. This point is consistent with

We now look at the role of productivity. We prepare our presentation by an intermediate step.

Lemma 5. *If there exists an equilibrium with $q_t > 0 \forall t$, then ξ_t is bounded from below by an exogenous parameter: $\xi_t \geq -e_t - F(D_{t-1}) - (1 - \delta)D_{t-1}$ for any t , where the sequence $(D_t)_t$ is defined by (10) and (11).*

Proof. If an equilibrium exists, we have $0 \leq C_t + K_{t+1} \leq F(K_t) + (1 - \delta)K_t + \xi_t$. By definition of (D_t) , we see that $\xi_t \geq -F(D_{t-1}) - (1 - \delta)D_{t-1}$. □

According to Lemma 5, the existence of equilibrium with positive prices ($q_t > 0$ for any t) requires that asset dividends must be bounded from below by exogenous parameters. This leads to the following result.

Proposition 3 (role of productivity). *Assume that $e_t = 0$ for any t .*

1. *Assume that there exists d such that $\xi_t \leq -d < 0$ for any t . If $F'(0) < \delta$ and $F(0) = 0$, then there is no equilibrium with $q_t > 0$ for any t .*
2. *(collapse). Assume that $\xi_t \leq 0$ for any t , $F'(0) < \delta$ and $F(0) = 0$. If there exists an equilibrium with $q_t > 0$ for any t , then $\lim_{t \rightarrow \infty} \xi_t = 0$ and $\lim_{t \rightarrow \infty} K_t = 0$.*

Proof. See Appendix 6.4. □

The first point shows that when dividends are negative and bounded above from zero, there is no equilibrium with positive prices if the productivity is low. Point 2 of Proposition 3 indicates that when dividends are negative and productivity are low, an equilibrium

exists only if dividends tend to zero and in this case the economy will collapse (aggregate consumption stocks converge to zero).

Let us explain the economic intuition of our result. When asset prices are positive at any date, there are always some agents buy this asset. At any date, if one agent buys asset whose future dividends are negative, she will be able resell this asset but have to pay an amount at the same time. In the aggregate level, the economy has to finance an amount (corresponding to negative dividends) at any date, which is bounded away from zero ($-\xi_t > d > 0$). However, when productivity is very low ($F'(0) < \delta$), the production level decreases in time and tends to zero, the economy collapses. By consequence, there will be some date, the economy will not able to pay for negative dividends. Therefore, asset prices cannot be positive.

Propositions 1 and 3 suggest that negative dividend policies may be sustained only if (1) the production sector is strong enough (high productivity) and (2) dividends are not so low.

3.2 Should we run negative dividends policy?

In this section, we wonder under which conditions we should run negative dividend policies. It is reasonable to assume that the government chooses dividends in order to maximize the welfare of agents in the decentralized economy.

Since we are interested in the role of productivity, we allow for non-stationary production functions: the production function at date t is given by $F_t(K) = A_t F(K)$, where F is strictly increasing, strictly concave, $F(0) = 0$, $F'(0) = \infty$, $F(\infty) = \infty$, and $A_t \geq 0$.

For the reason of tractability, we assume that there is one representative household with instantaneous utility function u , the rate of time preference $\beta \in (0, 1)$, and endowments (e_t) . The agent's allocation is denoted by $(c_t, k_{t+1}, a_t)_{t \geq 0}$. In this case, according to definition of equilibrium, we have $a_t = 1$ and $r_t = F'_t(k_t)$, and therefore the welfare function is

$$W((\xi_t)_{t \geq 1}) := \max_{(c_t, k_{t+1})_{t \geq 0}} \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] \quad (19)$$

$$\text{subject to: } k_{t+1} \geq 0, \quad c_t + k_{t+1} \leq G_t(k_t) + e_t + \xi_t \quad (20)$$

where $G_t(k) = (1 - \delta)k + F_t(k)$.

Assume that the government's problem is to maximize the welfare function by choosing the sequence of dividend (ξ_t) subject to

$$\xi_t \geq -b_t \quad \forall t \geq 0, \quad \text{and} \quad \sum_{t=0}^{\infty} \xi_t \leq B, \quad (21)$$

where $B > 0$, $b_t > 0$ and $\xi_t > 0$ for any t .

Here, the government has an endowment: $B > 0$ units of consumption good. It have to distribute dividends across periods. It can choose negative dividend at each date but there is a lower bound b_t . To find the optimal choice (ξ_t) of the government, we will solve

the following problem.

$$(PW) : \max_{(c_t, k_{t+1}, \xi_t)_t} \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] \quad (22)$$

$$\text{subject to: } k_{t+1} \geq 0, \quad (23)$$

$$c_t + k_{t+1} \leq G_t(k_t) + e_t + \xi_t \quad (24)$$

$$\xi_t \geq -b_t \quad (25)$$

$$\sum_t \xi_t \leq B. \quad (26)$$

Here, we assume that (b_t) is not so high so that the set of choices of the problem (PW) is not empty.

The non-standard optimal growth problem (PW) is non-stationary. There is no closed-solution. Moreover, it is not easy to find out global property of the solution. Here, we can provide some qualitative analysis. The following result shows the role of productivity.

Proposition 4. *Let assumptions in this section be satisfied and $u(\infty) = \infty, u'(\infty) = 0$. Fix a date t and all parameters, except A_t . Then, there exists \bar{A}_t such that each solution of the problem (PW) satisfies: $\xi_t = -b_t < 0$ for any $A_t \geq \bar{A}_t$.*

Proof. See Appendix 6.5. □

The intuition of Proposition 4 is the following. Let us interpret date $(t - 1)$ as the present and date t as the future. When the productivity in the future is high enough, the government should issue an asset in the present, which will have negative dividend in the future. This action will provide investment for production sector, which will bring a high return in the future because the productivity in the future is high.

Proposition 5. *Let assumptions in this section be satisfied and $u(\infty) = \infty, u'(\infty) = 0$. Fix a date t . If $\delta = 1, A_t = 0, e_t = 0$, then $\xi_t > 0$.*

Under conditions in Proposition 5, $G_t(k_t) + e_t = 0$. In this case, the government should provide some resources for the economy because households need to consume and production sector needs investment. So, it chooses $\xi_t > 0$.

4 Asset valuation

By iterating (18) we have the following decomposition

$$q_0 = \left(\sum_{t=1}^T Q_t \xi_t \right) + Q_T q_T \quad \forall T. \quad (27)$$

The price q_0 , the value of 1 unit of asset at date 0, equals the sum of two terms: The first term $FV_0^T := \sum_{t=1}^T Q_t \xi_t$ is the sum of discounted values of dividends until date T , and the second term $Q_T q_T$, called *re-sold term*, is the discounted value of 1 unit of asset at date T .

We also have a similar decomposition for the equilibrium price at date t .

$$q_t = \left(\sum_{s=t+1}^T \frac{Q_s}{Q_t} \xi_s \right) + \frac{Q_T}{Q_t} q_T \quad \forall T. \quad (28)$$

Decomposition (28) is in terms of consumption good at date t . It can be rewritten in terms of consumption good at date 0 as follows

$$Q_s q_t = \left(\sum_{s=t+1}^T Q_s \xi_s \right) + Q_T q_T. \quad (29)$$

Consider the Lucas tree with the sequence of strictly positive dividends (ξ_t) . The standard literature of asset pricing in infinite-horizon models (Tirole, 1982; Kocherlakota, 1992; Santos and Woodford, 1997; Le Van and Pham, 2016) defines the fundamental value of this asset by $FV := \sum_{t=1}^{\infty} Q_t \xi_t$ and the bubble of this asset as the difference between the equilibrium price and the fundamental value $q_0 - FV$. This approach is suitable for assets with positive dividends because the series $\sum_{t=1}^T Q_t \xi_t$ always converges when $\xi_t \geq 0$ for any t . However when we consider assets whose dividends may be negative, there is a room for the divergence of the series $\sum_{t=1}^T Q_t \xi_t$. Hence, the standard approach cannot be applied.

For the reasons discussed above, we may introduce the following notions which generalizes the notion of asset bubbles.

Definition 4. 1. We say that the asset price at date t is high if

$$q_t > \limsup_{T \rightarrow \infty} \sum_{s=t+1}^T \frac{Q_s}{Q_t} \xi_s. \quad (30)$$

2. We say that the value of asset does not fluctuate in the long run if there exists the limit $\lim_{T \rightarrow \infty} Q_T q_T$.

It is easy to see that the price q_t is high if and only if $\liminf_{T \rightarrow \infty} Q_T q_T > 0$. This means that there exist $x > 0$ and t_0 such that the discounted value of one unit of asset from date t_0 is higher than x (i.e., $Q_t q_t > x$ for any $t \geq t_0$).

To understand better our concept, let us mention some of its particular cases.

- The first case is when $\xi_t > 0$ for any t . In this case, we have, for any t , $q_t > 0$ and $\sum_{s=t+1}^T Q_s \xi_s$ is positive and increasing in T . So, there exists the limit $\lim_{T \rightarrow \infty} \sum_{s=t+1}^T Q_s \xi_s$. Therefore, q_t is high if and only if $\lim_{T \rightarrow \infty} Q_T q_T > 0$. Thus, q_0 is high if and only if q_t is high for any t . We also observe that q_0 is high if and only if $q_0 > \sum_{t=1}^{\infty} Q_t \xi_t$. By the way, we recover the notion of *asset price bubble* in standard literature on rational bubbles (Tirole, 1982; Kocherlakota, 1992; Santos and Woodford, 1997).
- The second case is when $\xi_t = 0$ for any t , we recover the notion of *pure bubble* (Tirole, 1985; Hirano and Yanagawa, 2013). In this case, note that q_0 is high if and only if q_t is high for any t .
- The third case is when $\xi_t < 0$ for any t . In this case, we have $\sum_{t=1}^{\infty} Q_t \xi_t < 0$. In this case, q_0 is high iff $q_0 > 0$.³

It should be noticed that, in general case, the notion of high asset price and asset price bubble are different because the sum $\sum_{s=t+1}^T \frac{Q_s}{Q_t} \xi_s$ and $Q_T q_T$ may diverge (see infra).

³By consequence, we have $q_0 > \sum_{t=1}^{\infty} Q_t \xi_t$.

4.1 Asset value at infinity

A natural question concerns the behavior of the discounted value of 1 unit of the asset, i.e., $Q_t q_t$ in the long run. When dividends are positive, thanks to the decomposition (27), $Q_t q_t$ is bounded and decreasingly converges to some value (which is referred to the bubble of asset price bubble). However, if dividends may be positive, the story becomes more complicated.

Proposition 6 (Value and price of asset).

1. (Montrucchio, 2004; Le Van and Pham, 2014). Assume that $\xi_t > 0$ for any t . At any equilibrium, both $\sum_{s=t+1}^T Q_s \xi_s$ and $Q_T q_T$ converge, and

$$Q_s q_t = \left(\sum_{s=t+1}^{\infty} Q_s \xi_s \right) + \lim_{T \rightarrow \infty} Q_T q_T.$$

Moreover, we have (i) $(Q_T q_T)$ is decreasing in time, and (ii) $\lim_{T \rightarrow \infty} Q_T q_T > 0$ if and only if $\sum_{t=1}^{\infty} \frac{\xi_t}{q_t} < \infty$.

2. Assume that $\xi_t \leq 0$ for any t . At any equilibrium with $q_t > 0$ for any t , both $\sum_{s=t+1}^T Q_s \xi_s$ and $Q_T q_T$ converge, and

$$Q_t q_t = \left(\sum_{s=t+1}^{\infty} Q_s \xi_s \right) + \lim_{T \rightarrow \infty} Q_T q_T.$$

Moreover, we have

(i) $(Q_T q_T)_T$ is increasing in time,

(ii) $\lim_{T \rightarrow \infty} Q_T q_T < \infty$ if and only if $\lim_{T \rightarrow \infty} \prod_{t=1}^T \left(1 + \frac{\xi_t}{q_t}\right) > 0$, which is equivalent to $\sum_{t=1}^{\infty} \frac{-\xi_t}{q_t + \xi_t} < +\infty$, and this implies that $\sum_{t=1}^{\infty} \frac{-\xi_t}{q_t} < +\infty$.

Proof. See Appendix 6.6. □

In Proposition 6, we see that $Q_t q_t$ converges because either $\xi_t \geq 0 \forall t$ or $\xi_t \leq 0 \forall t$. However, in more general cases, $Q_t q_t$ may diverge. This issue will be addressed in the next section.

To understand the meaningful of Proposition 6's point 2, let us look at budget constraint:

$$c_{i,t} + k_{i,t+1} - (1 - \delta)k_{i,t} + q_t a_{i,t} \leq r_t k_{i,t} + (q_t + \xi_t) a_{i,t-1} + e_{i,t} + \theta_t^i \pi_t.$$

We see that 1 unit of asset bought at date $t - 1$ will give one unit of asset and ξ_t units of consumption good (i.e., $(q_t + \xi_t)$ units of consumption good) at date t . When $\xi_t < 0$, $\frac{-\xi_t}{q_t + \xi_t}$ can be interpreted as the interest-to-value ratio (proportion of interest to asset value) of asset at date t . Point 2.ii shows that asset value at infinity is finite if and only if the sum (over time) of interest-value ratios is finite. This also implies that interest rate (in terms of asset) $\frac{-\xi_t}{q_t}$ must converge to zero.

4.2 Example: Positive asset prices with negative dividends

In this section, we will work under the following setup.

Fundamentals of the economy. that there are 2 consumers H and F

$$u_i(c) = \ln(c), \beta_i = \beta \in (0, 1) \quad \forall i = \{H, F\}. \quad (31)$$

Their initial endowments are respectively $k_{H,0} = 0$, $a_{H,-1} = 0$, $k_{F,0} > 0$, $a_{F,-1} = 1$. Their endowments of the profits are given by

$$(e_{2t}^H, e_{2t}^F) = (e_t, 0), \quad (e_{2t+1}^H, e_{2t+1}^F) = (0, e_{t+1}).$$

Assume that the production functions are given by $F(K) = a_t K$, where $a_t \geq 0$ and $\beta(1 - \delta + a_t) \leq 1$ for any t . Note that $\pi_t = 0$ for any t .

We also need $\sum_{s=1}^{\infty} \beta^s \ln(e_s) < \infty$ to ensure that consumers' utilities are finite.

Computing equilibria. With the above setup, equilibria can be computed as follows. Allocations of the consumer H are given by

$$\begin{aligned} k_{H,2t} &= 0, \quad a_{H,2t-1} = 0, \quad k_{H,2t+1} = K_{2t+1}, \quad a_{H,2t} = 1 \\ c_{H,2t-1} &= (1 - \delta + r_{2t-1})K_{2t-1} + q_{2t-1} + \xi_{2t-1} \\ c_{H,2t} &= e_{2t} - K_{2t+1} - q_{2t} \end{aligned}$$

while allocations of the consumer F are

$$\begin{aligned} k_{F,2t} &= K_{2t}, \quad a_{F,2t} = 1, \quad k_{F,2t+1} = 0, \quad a_{F,2t} = 0 \\ c_{F,2t-1} &= e_{2t-1} - K_{2t} - q_{2t-1} \\ c_{F,2t} &= (1 - \delta + r_{2t})K_{2t} + q_{2t} + \xi_{2t}. \end{aligned}$$

Prices and the aggregate capital are given by the following system: for any t , $p_t = 1$, $r_t = a_t$, and

$$K_{t+1} + q_t = \frac{\beta}{1 + \beta} e_t \quad (32)$$

$$q_{t+1} + \xi_{t+1} = q_t(a_{t+1} + 1 - \delta) \quad (33)$$

$$q_t \geq 0, \quad K_t > 0. \quad (34)$$

By using Lemma 2, we can verify that this sequence of allocations and prices is an equilibrium. For short, we also call $(K_{t+1}, q_t)_{t \geq 0}$ equilibrium. It is easy to see that

$$Q_t = \frac{1}{(1 - \delta + a_1) \cdots (1 - \delta + a_t)}; \quad Q_t q_t = q_0 - \sum_{s=1}^t Q_s \xi_s \quad (35)$$

$$q_0 \in \left(0, \frac{\beta e_0}{1 + \beta}\right); \quad 0 \leq q_0 - \sum_{s=1}^t \frac{\xi_s}{(1 - \delta + a_1) \cdots (1 - \delta + a_s)} < \frac{\beta e_t}{1 + \beta} \quad \forall t \quad (36)$$

Example 1. Assume $a_t = \delta$ and $e_t = e$ for any t , then $\gamma_t = 1$ and $Q_t = 1$ for any t . For each q_0 such that

$$q_0 \in \left(0, \frac{\beta e}{1 + \beta}\right), \quad 0 < q_0 - \sum_{s=1}^t \xi_s < \frac{\beta e}{1 + \beta} \quad \forall t, \quad (37)$$

we determine

$$q_t := q_0 - \sum_{s=1}^t \xi_s, \quad K_{t+1} := \frac{\beta e}{1 + \beta} - q_t > 0. \quad (38)$$

It is easy to see that $(K_{t+1}, q_t)_{t \geq 0}$ is an equilibrium and $q_t > 0$ for any t .

In Example 1, we see that even when ξ_t is negative for any t , all assets prices are positive. Let us explain the intuition of this fact. A fluctuation on wealth (or endowments) creates a fluctuation on agents' income. In the odd periods $(2t + 1)$, agent H has no endowment. She wants to smooth consumption over time but she cannot transfer her wealth from future to this date because of borrowing constraint. By consequence, she needs to transfer her wealth from date $2t$ to date $2t + 1$, hence she accepts to buy financial asset at date $2t$ with positive prices even this asset brings negative dividends in the future. The same argument is applied from the even periods and agent F . Therefore, asset prices are positive at any date.

Remark 2. Let $s \geq 0$. Take $\xi_t = 0$ for any $t \geq s$ and $\xi_s < 0$. In this case $\sum_{t=s+1}^{\infty} Q_t \xi_t = 0 < -\xi_s < Q_s q_s$. This suggests that condition $\liminf_{t \rightarrow \infty} \xi_t > 0$ in Corollary 1 is essential in order to ensure that $\sum_{t=s+1}^{\infty} Q_t \xi_t > Q_s |\xi_s|$.

4.3 Fluctuations of asset price and (discounted) value

Given an equilibrium, the conventional view⁴ is that the discounted value of 1 unit of the asset (i.e., $(Q_t q_t)_t$) is bounded from above and converges. This property holds because the existing literature only considers the case where dividends are always positive.

In this section, we will investigate the behavior of $(Q_t q_t)$ to know whether it can diverge or converge when dividends are negative,

Example 2. Consider again the example in Section 4.2 but we only require $a_t = \delta$ for any t . It is easy to see that $(K_{t+1}, q_t)_{t \geq 0}$ determined by (32), (33), and (34), constitutes an equilibrium if

$$q_0 \in \left(0, \frac{\beta e_0}{1 + \beta}\right), \quad 0 \leq q_0 - \sum_{s=1}^t \xi_s < \frac{\beta e_t}{1 + \beta} \quad \forall t \quad (39)$$

Notice that under these conditions, we have $Q_t q_t = q_t = q_0 - \sum_{s=1}^t \xi_s$.

Let us point out some particular cases of Example 2.

1. *Asset price and value fluctuation.* When we choose (ξ_t) such that $(\sum_{s=1}^t \xi_s)_t$ diverges, then the sequence of asset prices (q_t) diverges and so does $(Q_t q_t)$

In particular, we can $q_0 \in \left(0, \frac{\beta e_0}{1 + \beta}\right)$ and (ξ_t) such that $q_0 - \sum_{s=1}^t \xi_s > 0$ for any t even and $q_0 - \sum_{s=1}^t \xi_s = 0$ for any t odd. Therefore, in general case, asset price q_t may be zero at infinitely many date and it may also be strictly positive at infinitely many date.

2. *Asset value converges to infinity.* When we take (e_t) such that $\lim_{t \rightarrow \infty} e_t = \infty$ and

$$\frac{\beta e_t}{1 + \beta} > \frac{\beta b_0}{1 + \beta} - \sum_{s=1}^t \xi_s, \quad \text{then: } Q_t q_t \text{ tends to infinity if and only if } \sum_{s=1}^{\infty} \xi_s = -\infty.$$

⁴See [Tirole \(1982, 1985\)](#), [Santos and Woodford \(1997\)](#), [Le Van and Pham \(2016\)](#) among others.

⁵For example, take $\xi_t = \xi < 0$ for any t and e_t such that $\frac{\beta e_t}{1 + \beta} > \frac{\beta e_0}{1 + \beta} + t\xi$.

5 Conclusion

The paper addresses the issue of prices and values of assets whose dividends may be negative. Because of the negativity of dividends, prices of assets are positive only when the productivity of the production sector is high.

It is hard to find robust behaviors of asset prices and values when dividends may be negative. For example, the discounted value of one unit of asset $Q_t q_t$ may fluctuate over time. It may also diverge or converge (even, converge to infinity). In the presence of financial market imperfection, there is no reason to expect that the price of asset equals the present value of future dividends.

6 Appendix: formal proofs

6.1 Proof of Proposition 1

Let $(B_{i,t})$ be defined by

$$B_{i,0} \equiv (1 - \delta)k_{i,0} + \xi_0 a_{i,-1}, \quad B_{i,t} \equiv (1 - \delta)B_{i,t-1} + \xi_t a_{i,t-1}.$$

We will show that: if $B_{i,t} > 0$ for any i, t then there exists an equilibrium with $q_t \geq 0$ for any t . This can be done by adapting the argument in [Le Van and Pham \(2016\)](#). $q_t > 0$ because there is an infinite sequence $(\xi_{t_n})_n$ such that $\xi_{t_n} > 0$ for any t

6.2 Proof of Lemma 4

We see that there exists $\xi > 0$ and $t_0 > s$ such that $\xi_t \geq \xi$ for any $t \geq t_0$. So, when T is high enough, the sequence $(FV_0^T)_T$ is increasing in T . Moreover, $FV_0^T \leq q_0$ for any T . By consequence, FV_0^T converges to $FV_0 := \sum_{t=1}^{\infty} Q_t \xi_t < \infty$ and hence $Q_t q_t$ converge. Since $\liminf_{t \rightarrow \infty} \xi_t > 0$, we get that $\sum_{t=1}^{\infty} Q_t < \infty$.

According to point 2 of Lemma 1, $e_t + F(K_t)$ is uniformly bounded from above. As a result, we obtain that $\lim_{T \rightarrow \infty} Q_T k_{i,T+1} = 0$ for any i , and so $\sum_{t=1}^{\infty} (e_t + F(K_t)Q_t) < \infty$ because e_t is also uniformly bounded from above.

For each agent i , we rewrite her/his budget constraint at date t as follows

$$Q_t c_{i,t} + Q_t k_{i,t+1} + Q_t q_t a_{i,t} = Q_t (r_t + 1 - \delta)k_{i,t} + Q_t (q_t + \xi_t) a_{i,t-1} + (e_{i,t} + \theta^i \pi_t) Q_t.$$

By summing the budget constraints from t equals 0 to t , and use (17), (18), we get that

$$\begin{aligned} & \left(\sum_{t=0}^T Q_t c_{i,t} \right) + Q_T k_{i,T+1} + Q_T q_T a_{i,T} \\ & = (r_0 + 1 - \delta)k_{i,0} + (q_0 + \xi_0) a_{i,-1} + \sum_{t=0}^T (e_{i,t} + \theta^i \pi_t) Q_t < +\infty \end{aligned}$$

where the last inequality is from the fact that $\sum_{t=1}^{\infty} (e_t + F(K_t)) Q_t < \infty$.

We have $Q_T q_T a_{i,T} + Q_T k_{i,T+1} \geq 0$, hence $\sum_{t=0}^{\infty} Q_t c_{i,t} < +\infty$, and then $(Q_T k_{i,T+1} + Q_T q_T a_{i,T})_T$ converges where T tends to infinity. Since $\lim_{T \rightarrow \infty} Q_T k_{i,T+1} = 0$, the sequence $(Q_T q_T a_{i,T})_T$ will converge.

If $\lim_{T \rightarrow +\infty} Q_T q_T > 0$, then $a_{i,T}$ converges for any i . By consequence, there exists i such that $\lim_{t \rightarrow +\infty} a_{i,t} > 0$. For such an agent, there exists T such that the $a_{i,t} > 0$ for any $t \geq T$. Thus, $\frac{Q_t}{Q_T} = \frac{Q_{i,t}}{Q_{i,T}}$ for any $t \geq T$. According to condition (13) in Lemma 2, we have

$$\lim_{t \rightarrow +\infty} Q_t q_t a_{i,t} = \frac{Q_T}{Q_{i,T}} \lim_{t \rightarrow +\infty} Q_{i,t} q_t a_{i,t} = 0$$

which is a contradiction. We conclude that $Q_T q_T$ converges to 0. Therefore, it is easy to see that $q_t = \sum_{s=t+1}^{\infty} Q_s \xi_s / Q_t > 0$ for any $t > t_0$.

6.3 Proof of Proposition 2

Point 1. Suppose that there is an equilibrium with positive prices such that $\gamma_{i,t} = \gamma_t$ for any $t \geq s+1$ and for any i . According to point (iii) of Lemma 2, we have $\lim_{t \rightarrow \infty} Q_t (q_t a_{i,t} + k_{i,t+1}) = 0$ for any i . This implies that $\lim_{t \rightarrow \infty} Q_t q_t = \lim_{t \rightarrow \infty} Q_t K_{t+1} = 0$. By combining this with (29), we obtain $Q_s q_s = \sum_{t=s+1}^{\infty} Q_t \xi_t \leq 0$ since $\xi_t \leq 0$ for any $t \geq s$. This is a contradiction because $q_t > 0$.

Point 2 is a direct consequence of point 1.

6.4 Proof of Proposition 3

Proof of point (1). According to Lemma 5, we have $F(D_{t-1}) + D_{t-1} \geq \xi_t \geq d > 0$ for any t . So, D_t is bounded away from zero.

By definition, we have

$$\begin{aligned} D_t &= e_t + F(D_{t-1}) + (1 - \delta)D_{t-1} + \max(0, \xi_t) \\ &= F(D_{t-1}) + (1 - \delta)D_{t-1} < (F'(0) + 1 - \delta)D_{t-1}. \end{aligned}$$

Since $F'(0) < \delta$, we obtain that D_t converges to zero, a contradiction.

Proof of point (2). By definition, we have $D_t \geq C_t + K_{t+1}$, so both C_t and K_{t+1} converge to zero.

If ξ_t does not converge to zero, there exist $\xi > 0$ and an infinite sequence $(t_n)_n$ such that $\xi_{t_n} \leq -\xi$ for any n . Hence, $F(K_{t_n}) + (1 - \delta)K_{t_n} \geq -\xi_{t_n} \geq \xi > 0$. So, K_{t_n} is bounded away from zero, a contradiction.

6.5 Proof of Proposition 4

By Lemma 1 and Assumption H4, the problem (PW) has a solution. Let $(c_t, k_{t+}, \xi_t)_t$ be a solution of this problem. We have first order conditions

$$\lambda_t = \beta^t u'(c_t) \tag{40}$$

$$\lambda = \lambda_t + \mu_t, \quad \mu_t(\xi_t + b_t) = 0 \tag{41}$$

$$\lambda_t = \lambda_{t+1} G'_{t+1}(k_{t+1}) \tag{42}$$

for any t , where $\lambda_t, \mu_t, \lambda$ are non-negative multipliers associated to constraints (24), (25), (26) respectively.

We fix a date t .

Suppose that $\xi_t > -b_t$, then $\mu_t = 0$. We have

$$1 = \frac{\lambda}{\lambda} = \frac{\lambda_t}{\lambda_{t+1} + \mu_{t+1}} \leq \frac{\lambda_t}{\lambda_{t+1}} = G'_{t+1}(k_{t+1}). \quad (43)$$

We will claim that $G_t(K_t)$ tends to infinity when A_t tends to infinity. Indeed, suppose $G_t(K_t)$ is bounded, then the sequence $(c_t)_t$ is bounded and so is the welfare. However, it is easy to see the the welfare tends to infinity when A_t tends to infinity because $u(\infty) = F(\infty) = \infty$.

We now prove that K_{t+1} tends to infinity when A_t tends to infinity. Suppose that K_{t+1} is bounded, then $\lim_{A_t \rightarrow \infty} c_t = \infty$ (because $c_t + k_{t+1} = G_t(k_t) + e_t + \xi_t$). We see that (c_{t+1}) is bounded because K_{t+1} is bounded. Hence $u'(c_{t+1})$ and $G'_{t+1}(k_{t+1})$ are bounded away from zero

By FOCs, we have

$$1 = \frac{\beta u'(c_{t+1})}{u'(c_t)} G'_{t+1}(k_{t+1}).$$

Hence $u'(c_t)$ is also bounded away from zero. we have a contradiction because $\lim_{A_t \rightarrow \infty} c_t = \infty$ and $u'(\infty) = 0$.

Therefore, we have proved that K_{t+1} tends to infinity when A_t tends to infinity. This implies that $\lim_{A_t \rightarrow \infty} G'_{t+1}(k_{t+1}) = 1 - \delta < 1$, a contradiction to (43). Finally, we get $\xi_t = -b_t < 0$.

6.6 Proof of Proposition 6

According to (18), we have $Q_t q_t = Q_{t+1} q_{t+1} (1 + \frac{\xi_{t+1}}{q_{t+1}})$ for any t , so

$$\begin{aligned} q_0 &= (1 + \frac{\xi_1}{q_1}) q_1 Q_1 = (1 + \frac{\xi_1}{q_1}) (1 + \frac{\xi_2}{q_2}) q_2 Q_2 \\ &= \dots = (1 + \frac{\xi_1}{q_1}) \dots (1 + \frac{\xi_T}{q_T}) q_T Q_T. \end{aligned}$$

Point (1). $\lim_{T \rightarrow \infty} Q_T q_T > 0$ if and only if $\lim_{T \rightarrow \infty} (1 + \frac{\xi_1}{q_1}) \dots (1 + \frac{\xi_T}{q_T}) < \infty$, which is equivalent to $\sum_{t=1}^{\infty} \frac{\xi_t}{q_t} < \infty$.

Point (2). We see that $\lim_{T \rightarrow \infty} Q_T q_T < \infty$ if and only if $\lim_{T \rightarrow \infty} (1 + \frac{\xi_1}{q_1}) \dots (1 + \frac{\xi_T}{q_T}) > 0$.

Denote $d_t = -\xi_t \geq 0$. We observe that

$$1 + \frac{\xi_t}{q_t} = 1 - \frac{d_t}{q_t} = \frac{1}{1 + \frac{d_t}{q_t - d_t}}.$$

Therefore, $\lim_{T \rightarrow \infty} \prod_{t=1}^T (1 - \frac{d_t}{q_t}) > 0$ if and only if $\lim_{T \rightarrow \infty} \prod_{t=1}^T (1 + \frac{d_t}{q_t - d_t}) < \infty$ which is equivalent

to $\sum_{t=1}^{\infty} \frac{d_t}{q_t - d_t} < \infty$.

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