Dynamic common-value contests*

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Abstract

In this paper, I study dynamic common-value contests. Agents arrive over time and expend efforts to compete for prizes that are allocated proportionally according to efforts exerted. This model can be applied to a number of examples, including rent-seeking, lobbying, advertising, and R&D competitions. I provide a full characterization of equilibria in dynamic common-value contests and use it to study their properties, including comparative statics, earlier-mover advantage, and large contests. I show that information about other players' efforts plays an important role in determining the total effort and that the total effort is strictly increasing with the information that becomes available.

JEL: C72, C73, D72, D82

Keywords: contests, sequential games, contest design, rent-seeking, R&D, advertising

1 Introduction

Many economic interactions, including rent-seeking, R&D competitions, advertising, and litigation, have a contest structure. The agents choose costly efforts to compete for prizes that are allocated proportionally according to the amount of effort exerted. For example, in rent-seeking contests firms spend resources lobbying to achieve market power. As rent-seeking efforts are considered socially wasteful, the important policy question is how to limit this wasteful spending. In research and development the probability of a scientific breakthrough may be proportional to research efforts, which are typically considered

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socially desirable. The literature has largely focused on the simultaneous case, where players choose efforts simultaneously. In many situations, however, the players make their choices over time and may have information about the choices of other players.

I study dynamic common-value contests, where players arrive over time and – either exogenously or by the choice of the contest designer – some players observe the choices of earlier players before making their decisions. I show that information about other players' behavior has a significant impact on equilibrium outcomes and more information makes the total effort unambiguously larger.

Static common-value contests have well-known equilibrium properties: the equilibrium is unique and in pure strategies. Very little is known about dynamic contests. Using backward induction requires finding best-response functions recursively and this approach is not tractable with contests. I introduce an alternative method in which, instead of finding best responses, I pool all the best-response relationships for all players before solving the resulting equation. Using this approach, instead of finding roots of polynomials and inserting them into the next problems recursively, I can solve the problem by finding roots for a polynomial just once.

I show that the equilibrium in all dynamic common-value contests is still unique and still in pure strategies; using this new approach, equilibria are also straightforward to compute.[1]Using the characterization result, I show that the total effort increases with the values of the prizes, decreases with the cost of effort, and increases with the number of players. Moreover, as the number of players becomes large, the total cost of effort converges to the total value of prizes (full dissipation).

The main result of the paper is that information about other players' efforts strictly increases the total effort of all players. This answers a contest design question: how much information should the contest designer disclose to the later players? When efforts are desirable (as in R&D competitions) the optimal contest would be one with full transparency, whereas when the efforts are undesirable (as in rent-seeking), the optimal contest would be one with hidden efforts.

The basic intuition behind the result is that earlier players exerting higher efforts discourages later players, but this discouragement effect is not strong enough—earlier players increase their efforts more than later players decrease theirs. If the efforts are high, then they are strategic substitutes, i.e., higher levels of effort by other players reduce the incentive to exert effort. Therefore, making efforts observable provides an incentive to earlier players to exert more effort to discourage the followers. How does the discouragement effect change the total effort? Players' payoffs are decreasing in the total

[1] The program that computes the equilibrium for any dynamic common-value contest is available at `http://toomas.hinnosaar.net/contests/`.

effort and increasing in their own efforts. In equilibrium, discouragement cannot be so strong that increasing individual efforts decreases the total effort because otherwise players would continue increasing their efforts. This means that in equilibrium the discouragement effect is not strong enough and information increases the total effort.

The contest design result is even stronger. In fact, contests that are more homogeneous lead to higher total effort. This answers another contest design question: if the information about previous efforts can only be revealed a limited number of times, then when is it optimal to disclose it? The answer is that to maximize the total effort, it is best to reveal the information at regular intervals. For example, if the contest designer wants to maximize the total effort invested in a contest with four players and can only reveal information once, it is optimal to reveal the information after the first two players have acted. This divides the four players into two equal groups, whereas disclosing the information earlier or later would make the groups less equal.

Next, I generalize the first-mover advantage result from Dixit (1987). Dixit showed that a player who can pre-commit chooses higher effort and ensures higher payoff than the followers. In this scenario, the leader has two advantages: he moves earlier than the other players and he does not have any direct competitors. With the new characterization, I am able to explore this idea further and compare players' payoffs, depending on their position and the number of competitors more generally. I show that there is always an *earlier-mover advantage*: players who move earlier choose higher efforts and ensure higher payoffs than later players.

The final part of the paper studies contests with large numbers of players. It answers two questions. The first is the computational question of how to compute equilibria in large contests. Although the characterization result holds for an arbitrary number of players, it is not computationally reliable with very large number of players. I show that, in this case, there is a simple approximation method. The second question concerns how quickly the total cost of effort converges to the total value of prizes. I show that the convergence is much faster in the case of sequential contests than simultaneous contests. This points to a conclusion that the information provided to players may be even more important than the number of players participating in the contest.

Literature: Contests allocate a limited number of prizes among participants who make costly efforts. There are three main models of contests, which differ in terms of the contest success function, i.e., the criteria for allocating prizes. First, the type of contests I am studying in this paper are called Tullock contests (or simply contests), and they allocate the prizes proportionally based on effort. In the static framework, they typically give unique equilibria, which is in pure strategies. This model is often used to study rent-seeking, R&D races, advertising, and elections. The literature was initiated by Tullock (1967, 1974) and motivated by rent-seeking (Krueger, 1974; Posner, 1975).[2] I extend this literature to arbitrary dynamic common-value contests and study their equilibrium properties. There have been relatively few attempts to study sequential contests in this framework. Linster (1993) proved that in the two-player case, sequential and simultaneous games give the same equilibrium efforts and Dixit (1987) showed that there is first-mover advantage with at least three players. Glazer and Hassin (2000) extended the analysis to three sequential players and provided some conjectures on the equilibria in contests with four or more sequential players.

The second class of contests includes all-pay auction and war of attrition, where the player with the highest effort always wins. These models are often used to study lobbying, military activities, and auctions. The equilibria in these auctions are typically in mixed strategies. The characterization of equilibria in static common-value (first-price) all-pay auction is due to Baye, Kovenock, and de Vries (1996) and in second-price all-pay auction (also called war of attrition) is due to Hendricks, Weiss, and Wilson (1988). Siegel (2009) provides a general payoff characterization for static all-pay contest. In a broader sense, Tullock contest and all-pay auction are two members of the same family: generalized Tullock contest that allocates prizes proportionally on efforts to the power $r > 0$. If this power $r = 1$, then we get the standard Tullock contest, whereas if $r \to \infty$, we get an all-pay auction.

The third class of contests is rank-order tournaments that allocate prizes according to the highest output rather than the highest effort. The output is a noisy measure of effort. Tournaments were introduced by Lazear and Rosen (1981) and Rosen (1986), and they are most often used to model principal-agent relationships and contract design in personnel economics and labor economics.

This paper also contributes to the theory of contest design. Previous papers on contest design have mostly focused on the all-pay auction specification with asymmetric information about the effort costs (abilities). Examples include Glazer and Hassin (1988); Che and Gale (2003); Moldovanu and Sela (2001, 2006). In a recent paper, Olszewski and Siegel (2016) studied a general class of contests in this setting with an infinite number of players. An exception is Taylor (1995) who studied tournament games and showed that free entry is not optimal, since it decreases the effort of all contestants. I am not aware of papers studying contest design of common-value (Tullock) contests. In this paper, I show that prizes and the cost of effort have very little impact on the outcomes of these contests. In contrast, the information provided to the players about the previous players' efforts plays a crucial role.

[2]See Tullock (2001) and Konrad (2009) for literature reviews.

The paper is organized as follows. Section 2 introduces the general model of dynamic common-value contests and provides a useful normalization result that allows me to focus on contests where the values of prizes and the cost of effort are normalized. Section 3 characterizes equilibria in dynamic common-value contests. Section 4 studies the comparative statics. Section 5 provides the main result—it shows that total effort in contests is strictly increasing in information and homogeneity. Section 6 proves earlier-mover advantage and section 7 studies contests with a large number of players. Finally, section 8 concludes and discusses potential avenues for future research.

2 Model

2.1 Set up

A set[3] of prizes $\mathcal{V}$ is allocated at some fixed deadline. There are n players who arrive sequentially over time.[4] Each player i chooses effort $x_i \geq 0$ at arrival at marginal cost c. Prizes are allocated randomly with probabilities proportional to efforts. Both the values of prizes and the marginal cost are common to all players. In particular, with efforts $\mathbf{x} = (x_1, \ldots, x_n)$, player i's expected payoff is

$$u_i(\mathbf{x}) = \sum_{v \in \mathcal{V}} v \frac{x_i}{\sum_{j=1}^n x_j} - cx_i = V \frac{x_i}{\sum_{j=1}^n x_j} - cx_i, \tag{1}$$

where $V = \sum_{v \in \mathcal{V}} v$ is the sum of the values of all prizes.

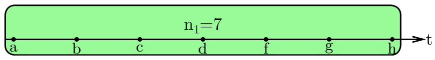
The players observe the efforts of previous players only at some specific points in time, which could be exogenous or chosen by a contest designer. Let there be $T-1$ points in time where the previous efforts are made public. These points of disclosure partition the n players into T groups, where all players inside the group have the same information, whereas players prior to a disclosure have less information than players after a disclosure.

It is convenient to denote the sizes of these groups in reverse order. In particular, let n_T be the number of players arriving before the first disclosure, n_{T-1} the number of players between the first and the second disclosure, and so on, until n_1, which is the number of players after the last time that efforts were disclosed. Notice that the number of players $n = \sum_{t=1}^T n_t$ and the exact arrival times of players do not affect the payoffs and therefore the equilibria.

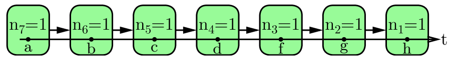
[3]It may be just one prize, which could be the value of becoming a monopolist (in rent-seeking) or the value of a scientific breakthrough (R&D). But the model also allows for multiple different prizes or even a continuum of prizes, as in a market share in advertising.

[4]To simplify the exposition, I assume that two players never arrive at the same time. As will become apparent in the later sections, it is straightforward to extend the analysis to allow simultaneous arrivals.

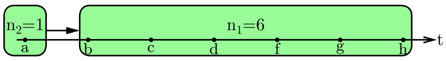
A general contest is a triple $(\mathcal{V}, c, \mathbf{n})$, where the information disclosure is characterized by a vector $\mathbf{n} = (n_T, \dots, n_1)$. For example, when $\mathbf{n} = (n)$ (see Figure 1(a)), the players do not receive any information about the efforts of other players and therefore play a simultaneous game. In the other extreme, when $\mathbf{n} = (1, 1, \dots, 1)$ (Figure 1(b)), all efforts are made public right after they are made, so that the game is fully sequential. When $\mathbf{n} = (1, n-1)$ (Figure 1(c)), the initial player is the first mover and all other players observe only this first player's effort, and when $\mathbf{n} = (n-1, 1)$ (Figure 1(d)) the last-mover observes the efforts of all other players, whereas all other players choose simultaneously without observing any efforts.



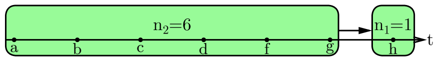
(a) $\mathbf{n} = (n)$, no information is disclosed, players make simultaneous decisions (Simultaneous contest).



(b) $\mathbf{n} = (1, \dots, 1)$, information is disclosed after each arrival (Sequential contest).



(c) $\mathbf{n} = (1, n-1)$, all players observe a's effort, but not the other players' efforts (First-mover).



(d) $\mathbf{n} = (n-1, 1)$, all players except h make their decisions without observing the efforts of other players, but h observes all efforts (Last-mover)

Figure 1: Examples of different information disclosure scenarios.

As all players between two disclosures have the same information, it is convenient to call the time intervals between disclosures *periods* (there are T periods), and use slightly different notation to address the players. As I am solving the game backwards, period 1 is the time interval after the last information $T-1$st disclosure, where the final n_1 players choose their efforts essentially simultaneously. Let me denote their efforts by $x_{i,1}$ for $i = 1, \dots, n_1$. Similarly, let period t be the period between information disclosures number $t-1$ and t. In this interval, n_t players simultaneously choose their efforts $x_{i,t}$, knowing all efforts from players in periods $T, \dots, t+1$ and anticipating the responses of all players from periods $t-1, \dots, 1$. There are n_T players who make their choices of $x_{i,t}$ before the first disclosure, in period T.

Finally, let $X_t = \sum_{s=t+1}^{T} \sum_{i=1}^{n_s} x_{i,s}$ denote the total effort prior to period t. This is the only relevant state variable available in the decisions of players in period t. For brevity of notation, $X_0 = X$ is the total effort at the end of the contest (i.e., 0 periods before the end) and $X_T = 0$ is the total effort in the beginning of the contest (T periods before the end). Moreover, by the argument above, it is clear that $X_t \in [0, 1]$ in equilibrium. The notation is illustrated in Table 1.

Players	First	Second	Third	Fourth	Fifth
Periods	Period 3	Period 2		Period 1	
Number of players	$n_3 = 1$	$n_2 = 2$		$n_1 = 2$	
Individual efforts	$x_{1,3}$	$x_{1,2}$	$x_{2,2}$	$x_{1,1}$	$x_{2,1}$
Total effort prior to period	$X_3 = 0$	$X_2 = X_3 + x_{1,3} = x_{1,3}$		$X_1 = X_2 + x_{1,2} + x_{2,2}$	

Table 1: Example of the notational change with $n = 5$ players such that the efforts are disclosed twice: between players 1 and 2 and then again between players 3 and 4.

2.2 Normalized contests

The first result (Lemma 1) shows that we can without loss focus on a normalized contest where the set of prizes is $\mathcal{V} = \{1\}$ and the marginal cost of effort is $c = 1$. The equilibria under all other parameter values are linear transformations of the equilibria in this normalized game $(\{1\}, 1, \mathbf{n})$, which I denote briefly by $\mathbf{n}$. The result comes from risk neutrality and the contest success function, which is proportional to efforts.

For a fixed $\mathbf{n}$, let $\mathbf{x}_{-i}$ denote the vector of efforts of other players *observable* to player i. For example, in the simultaneous contests $\mathbf{n} = (n)$, this means that $\mathbf{x}_{-i} = \varnothing$, since players do not see efforts of any other players, whereas in the sequential contests $\mathbf{n} = (1, 1, \ldots, 1)$, $\mathbf{x}_{-i} = (x_1, \ldots, x_{i-1})$, since players observe the efforts of all previous players. Let $\mathbf{x}^*$ denote an equilibrium strategy profile, where $x_i^*(\mathbf{x}_{-i})$ is the strategy of player i, as a function of the information available to him, i.e., where $\mathbf{x}_{-i}$. Finally, for $\alpha > 0$, let $\alpha\mathbf{x}^*$ denote a re-scaled strategy profile such that $x_i(\mathbf{x}_{-i}) = \alpha x_i^*(\alpha\mathbf{x}_{-i})$.

Lemma 1 (Linearity in $^V/_c$)**.** *Strategy profile* $\mathbf{x}^*$ *is an equilibrium in a contest* $(\mathcal{V}, c, \mathbf{n})$ *if and only if* $\frac{c}{V}\mathbf{x}^*$ *is an equilibrium in a normalized contest* $\mathbf{n}$.

Proof. Suppose $\mathbf{x}^*$ is an equilibrium in $(\mathcal{V}, c, \mathbf{n})$. This means that each player i, $x_i^*(\mathbf{x}_{-i})$ solves the maximization problem

$$\max_{x_i} V \frac{x_i}{X^*(x, \mathbf{x}_{-i}))} - cx_i = V \max_{x_i} \sum_{v \in \mathcal{V}} \frac{\frac{c}{V} x_i}{\frac{c}{V} X^*(x_{i,t}, \mathbf{x}_t))} - \frac{c}{V} x_{i,t}, \tag{2}$$

where $X^*(x_i, \mathbf{x}_{-i})$ is the total effort, given that player i chooses x_i, the profile of efforts prior to i is $\mathbf{x}_{-i}$, and all players except i are behaving according to their equilibrium strategies.

From the second representation, it is clear that if $\mathbf{x}^*$ is an equilibrium for contest $(\mathcal{V}, c, \mathbf{n})$, then rescaled strategy profile $\frac{c}{V}\mathbf{x}^i$ is an equilibrium for the normalized contest $\mathbf{n}$. $\square$

Using this result, I focus on normalized contests in the rest of the paper. For any other games $(\mathcal{V}, c, \mathbf{n})$, the equilibrium strategies are simply linear transformations with parameters ${}^{V}\!/_{c}$. Clearly, the total effort is also linear in ${}^{V}\!/_{c}$ and the payoffs are linear transformations with parameter V.

3 Characterization of equilibria

In this section, I characterize all subgame-perfect Nash equilibria in dynamic common-value contests. The standard backward-induction approach is not tractable with three or more periods (see Appendix A for examples that illustrate why). Instead of using backward induction directly, it is useful to characterize the responses of the following players via their *inverted best-response functions* and then pool all these functions. The advantage of this approach is tractability: instead of solving for the roots of polynomials recursively I aggregate the best-responses into one polynomial and then study its roots. I show that the equilibrium is unique, in pure strategies, easy to compute[5], and has some useful properties that I will explore further in the following sections.

Theorem 1 (Characterization theorem)**.** *Each contest* $\mathbf{n}$ *has a unique equilibrium. The total equilibrium effort* X^* *is the highest root of* $f_T(X) = 0$*, where* f_t *is defined recursively by*

$$f_0(X) = X \text{ and } f_t(X) = f_{t-1}(X) - n_t f'_{t-1}(X)X(1-X), \forall t = 1, \ldots, T. \tag{3}$$

The Individual equilibrium effort of a buyer from period t *is*

$$x^*_{i,t} = \frac{1}{n_t}[f_{t-1}(X^*) - f_t(X^*)] = f'_{t-1}(X^*)X^*(1-X^*). \tag{4}$$

Before proving the result, let me contrast the approach I'm taking with the standard backward induction. At period t, there are n_t players who observe the total effort prior to period t, denoted by X_t, and simultaneously choose their optimal efforts $x_{i,t}$. Therefore, the best-response functions would be $x^*_{i,t}(X_t)$. However, as this approach becomes quickly non-tractable, I am taking an alternative approach here. Instead of characterizing best responses as individual efforts $x^*_{i,t}(X_t)$, we can equivalently characterize the best responses of the preceding players as the total effort induced by X_t, i.e., function[6] $X(X_t)$. Finally, it is instead convenient to keep track their inverse functions $f_t(X)$, which I call *inverted best-response functions.* For a given total effort X, the inverted best-response $X_t = f_t(X)$

[5] The program is available at `http://toomas.hinnosaar.net/contests/`.

[6] That is,

$$X(X_t) = X_t + \sum_{i=1}^{n_t} x^*_{i,t}(X_t) + \sum_{i=1}^{n_{t-1}} x^*_{i,t-1}\left(X_t + \sum_{i=1}^{n_t} x^*_{i,t}(X_t)\right) + \ldots.$$

gives the total effort prior to period t that is consistent with the total effort X, given that the players in periods $t, \ldots, 1$ choose efforts optimally.

As we will see in the proof, optimality conditions are complex functions of X_t and therefore solving for best-response functions $x^*_{i,t}(X_t)$ or $X(X_t)$ recursively would not be tractable. On the other hand, the optimality conditions are linear in X_t, so finding the inverted best-response functions $f_t(X)$ is relatively straightforward. Appendix A illustrates this point with a few simple examples.

Proof. First, observe that there cannot be any equilibria where the total effort $X > 1$, because this means one or more players get strictly negative utilities and they could instead ensure zero utility by making no effort.

Fix any period t. There are $n_t \geq 1$ players who simultaneously choose their efforts $x_{i,t}$, knowing that the total effort by previous players is X_t, and if the total effort after period t is $X_{t-1} = X_t + \sum_{i=1}^{n_t} x_{i,t}$, then the optimal effort choices of all the following players lead to the total effort $X(X_{t-1})$.

Player i's maximization problem is

$$\max_{x_{i,t}} \frac{x_{i,t}}{X(X_t + \sum_{j=1}^{n_t} x_{j,t})} - x_{i,t} \quad \Rightarrow \quad \frac{1}{X(X_{t-1})} - \frac{x_{i,t}}{[X(X_{t-1})]^2} X'(X_{t-1}) = 1.$$

Adding up the conditions for all players in period t and observing that $f_{t-1}(X) = X_{t-1} = X_t + \sum_{i=1}^{n_t} x_{i,t}$ gives

$$\frac{n_t}{X(X_{t-1})} - \frac{X_{t-1} - X_t}{[X(X_{t-1})]^2} X'(X_{t-1}) = n_t.$$

As discussed above, $X(X_{t-1})$ is characterized by its inverse $f_{t-1}(X)$ and therefore $X'(X_t) = f'_{t-1}(X)^{-1}$, which gives[7]

$$\frac{n_t}{X} - \frac{f_{t-1}(X) - X_t}{X^2} \frac{1}{f'_{t-1}(X)} = n_t.$$

Therefore, the inverted best-response function for period t is

$$f_t(X) = f_{t-1}(X) - n_t f'_{t-1}(X) X (1 - X). \tag{5}$$

Note that after the final period, the total effort is X. If we define $f_0(X) = X$, then Equation (5) characterizes functions $f_1, \ldots, f_T$ recursively.

[7]This condition also illustrates why the standard approach fails and the inverted best-response approach is tractable: the condition is nonlinear in the choices of players in period t (i.e., $X_{t-1} - X_t$), so the best-response functions are complex expressions, especially in contests with many periods. However, the conditions are linear in the total effort prior to period t (i.e., X_t).

The individual optimality condition implies that the effort chosen by player i in period t, consistent with the total effort X, is

$$x_{i,t}(X) = \frac{1}{n_t}[X_{t-1} - X_t] = \frac{1}{n_t}[f_{t-1}(X) - f_t(X)]. \tag{6}$$

The equilibrium efforts have to satisfy two conditions. First, the total effort must be such that the implied total effort prior to the game is $X_T = f_T(X) = 0$. Second, all individual efforts must be non-negative, i.e., $x_{i,t}(X) \geq 0$ for each i and t. Proposition 7 in Appendix B shows that the highest root of $f_T(X) = 0$, is the only value that satisfies these conditions. As the total effort X determines all individual efforts, this completes the proof. □

Note that the proof relies on a technical result (Proposition 7), which is proven in the appendix, so let me give brief sketch of the proof here. It relies on the properties of functions $f_1, \ldots, f_T$. Namely, I show that f_t is a polynomial of degree $t+1$, with all roots in $[0,1]$, $f_t(0) = 0$, $f_t(1) = 1$, and the polynomials are interlaced, i.e., the t roots of $f_{t-1}(X) = 0$ are between the roots of $f_t(X) = 0$. This means that the highest root of f_T, denoted by X^*, is between $[0,1]$ and $f_t(X^*)$ is decreasing in t (see Figure 2 for illustration), which implies that X^* is indeed an equilibrium. Moreover, the only other candidates for equilibria are the other T roots of polynomial f_T, but I can exclude them, because at the second-highest root and all points below it, at $f_t(X) > f_{t-1}(X)$ for at least one t. This implies that at least one player chooses negative effort, and this is a contradiction.

4 Comparative statics

Using the characterization result from the previous section, we can now examine how changes in parameter values affect efforts and in particular how they impact the total effort. The first result is a straightforward corollary from the uniqueness of equilibria and normalization Lemma 1. It gives the natural conclusion that the total effort increases proportionally with the values of prizes and decreases proportionally with the marginal cost of effort.

Corollary 1 (Comparative statics of $\mathcal{V}$ and c)**.** *The total effort X^* as well as individual efforts $\mathbf{x}^*$ in contest $(\mathcal{V}, c, \mathbf{n})$ are proportional on $\frac{V}{c}$. The payoffs are proportional on $V = \sum_{v \in \mathcal{V}} v$.*

Proof. By Theorem 1, the equilibrium in normalized contest $(\{1\}, 1, \mathbf{n})$ is unique. Let the total effort in this equilibrium be X^1 and individual efforts $\mathbf{x}^1$. By Lemma 1, any

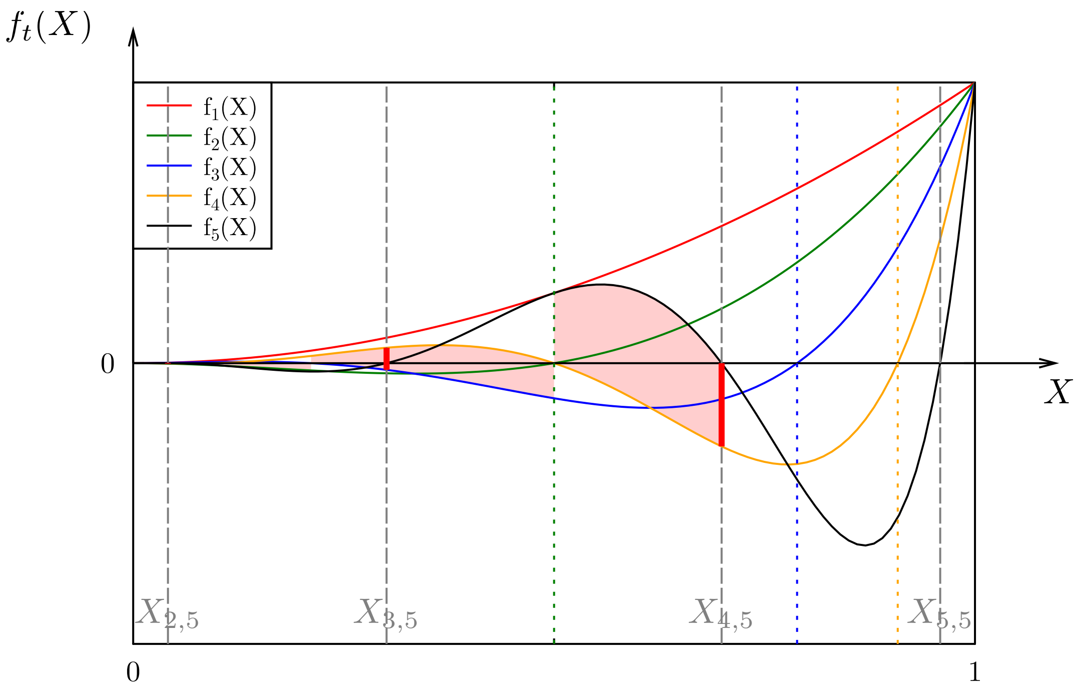


Figure 2: Illustration for $\mathbf{n} = (1, 1, 1, 1, 1)$. At $X^* = X_{5,5}$ all $f_t(X)$ is strictly decreasing in t, so that each $x_{i,t} > 0$. At any other root $X_{i,T}$ one of $f_{t-1}(X_{i,T}) < f_t(X_{i,T})$ for some t.

equilibrium in a general contest $(\mathcal{V}, c, \mathbf{n})$ must have a corresponding equilibrium in the normalized contest $(1, 1, \mathbf{n})$ with all efforts scaled by $\frac{c}{V}$. Since there is a unique equilibrium in $(1, 1, \mathbf{n})$, the equilibrium in $(\mathcal{V}, c, \mathbf{n})$ is also unique and has a property that $X^* = \frac{V}{c} X^1$ and $\mathbf{x}^* = \frac{V}{c} \mathbf{x}^1$. Therefore, the equilibrium efforts must be proportional on $\frac{V}{c}$. $\square$

The next Proposition 1 provides an interesting implication, as well as a useful intermediary result for the following analysis. Namely, the total equilibrium effort X^* is unchanged when we change the order of n_t's in $\mathbf{n}$. For example, a contest with $\mathbf{n} = (1, 2, 3)$ must therefore give the same total effort as $\hat{\mathbf{n}} = (3, 2, 1)$.[8]

Proposition 1 (Independence of permutations)**.** *Total equilibrium effort X^* of contest $\mathbf{n}$ is independent of permutations of $\mathbf{n}$.*

Proof. I show by induction that functions $f_t(X)$ are independent of permutations of respective $\mathbf{n}^t = (n_t, \ldots, n_1)$. Then, as $f_T(X)$ is independent of permutations of $\mathbf{n}^T = \mathbf{n}$, the highest root X^* is also independent of permutations of $\mathbf{n}$.

First, $f_0(X)$ does not depend on $\mathbf{n}$. Suppose now that $f_t(X)$ is independent of permutations of $\mathbf{n}^t$. By construction, $\mathbf{n}^t$ only enters to f_{t+1} through f_t and thus permutations of $\mathbf{n}^t$ do not change f_{t+1}. It suffices to verify that swapping n_t and n_{t+1} in $\mathbf{n}^{t+1}$ does not

[8]Note that the result does not say anything about games that are not permutations; for example, $\mathbf{n} = (1, 1, 4)$.

affect f_{t+1}. Define $\hat{\mathbf{n}}^{t+1} = (n_t, n_{t+1}, \mathbf{n}^{t-1})$. Then $f_{t-1}(X)$ is the same function for both $\mathbf{n}^{t+1}$ and $\hat{\mathbf{n}}^{t+1}$. To shorten the notation, let $\phi_{t-1}(X) = f'_{t-1}(X)X(1-X)$. Note that ϕ_{t-1} is also the same function for both sequences. Therefore

$$f_{t+1}(X) = f_{t-1}(X) - (n_t + n_{t+1})\phi_{t-1}(X) + n_{t+1}n_t\phi'_{t-1}(X)X(1-X)$$

Notice that since functions f_{t-1} and ϕ_{t-1} are the same for both sequences $\mathbf{n}^{t+1}$ and $\hat{\mathbf{n}}^{t+1}$, the expression for $f_{t+1}(X)$ is the same for both sequences. Therefore, indeed f_{t+1} is independent of permutations in $\mathbf{n}^{t+1}$. $\square$

We can now use this result to understand how increasing the number of players affects the total effort. The following Proposition 2 shows that X^* is strictly increasing in the number of players (in any period) and in the limit, when the total number of players becomes infinitely large, the total effort converges to 1 (or more generally, the total cost of effort converges to the total value of prizes).

Proposition 2 (Comparative statics of $\mathbf{n}$)**.** *The equilibrium X^* of a contest $(\mathcal{V}, c, \mathbf{n})$ is strictly increasing in each n_t and $\lim_{n\to\infty} X^* = \frac{V}{c}$.*

Proof. By Lemma 1, it suffices to prove the claims in the normalized contest $\mathbf{n}$.

First note that it suffices to show that X^* is increasing in n_T, because by Proposition 1 we can rearrange vector $\mathbf{n}$ without changing X^*. Consider $\mathbf{n}$ and $\hat{\mathbf{n}} = (n_T+1, n_{T-1}, \ldots, n_1)$ (i.e., a game with one more player in the initial period T). In both cases, $f_{T-1}(X)$ is the same, since the sequence $(n_{T-1}, \ldots, n_1)$ in unchanged. Then, X^* is the highest root of

$$f_T(X) = f_{T-1}(X) - n_T f'_{T-1}(X)X(1-X).$$

Now, in the case of $\hat{\mathbf{n}}$, we have that the new equilibrium $\hat{X}^*$ is the highest root of

$$\hat{f}_T(X) = f_{T-1}(X) - (n_T+1)f'_{T-1}(X)X(1-X).$$

By the proof of Theorem 1, both X^* and $\hat{X}^*$ are strictly higher than $X_{T-1,T-1}$, the highest root of $f_{T-1}(X)$. Moreover, in the interval $[X_{T-1,T-1}, 1]$, the function $f_{T-1}(X)$ is strictly increasing.

Now, we have $\hat{f}_T(X^*) - f_T(X^*) = -f'_{T-1}(X^*)X^*(1-X^*) < 0$. Therefore, $\hat{X}^* > X^*$.

Remark: as the characterization allows $n_T = 0$, the same argument also implies that increasing the number of periods with positive numbers of participants strictly increases the total effort X^*.

Remember that in the simultaneous case, i.e., $\mathbf{n} = (n)$, the equilibrium $X^* = \frac{n-1}{n}$. Therefore, in this case $\lim_{n\to\infty} X^* = 1$. Theorem 2 below will show that the total equilibrium effort for any other $\mathbf{n}$ is strictly higher than in the simultaneous case, which completes the proof. $\square$

In Section 7, I explore the limit further and compare large contests with different types of disclosure rules.

5 Contest design

In this section, I present the main result of this paper. I show that if we take any contest $\mathbf{n}$ and make it more informative, then the total equilibrium effort is strictly higher in the new contest. Contest $\hat{\mathbf{n}}$ is more informative than $\mathbf{n}$ if whenever a player observes the effort of another player in $\mathbf{n}$, he also observes the effort of this player in $\hat{\mathbf{n}}$, but the opposite is not true. In other words, if we start with $\mathbf{n}$, such that $n_t \geq 2$ at some period t, and create a modified game $\hat{\mathbf{n}}$ where the n_t players are split into two groups of one or more players, then in equilibrium the effort increases.

Informativeness here is a partial order on contests $\mathbf{n}$. For example, contest $(1, 2)$ is more informative than simultaneous contest (3) because the two followers in the second period now see what the leader does. On the other hand, I am not comparing contests $(1, 2)$ and $(2, 1)$ because, in both cases, there are players who have more information and players who have less information than in the other contest.[9] In particular, among all n-player contests, the simultaneous contest (n) is the least informative, and the fully sequential contest $\mathbf{n} = (1, 1, \ldots, 1)$ is the most informative.

Theorem 2 (Informativeness increases the total effort)**.** *Suppose X^* is the total equilibrium effort in a contest $\mathbf{n}$ with $n > 2$. Suppose $\hat{X}^*$ is the total equilibrium effort in another contest $\hat{\mathbf{n}} = (n_T, \ldots, n_{t+1}, \overline{n}_t, \underline{n}_t, n_{t-1}, \ldots, n_1)$, such that $\overline{n}_t + \underline{n}_t = n_t$, $\overline{n}_t > 0$, and $\underline{n}_t > 0$. Then $X^* < \hat{X}^*$.*

Theorem 2 is a special case of the following Proposition 3, which states that homogeneity increases the total effort. When we take any contest $\mathbf{n}$ with $n > 2$ and reallocate players in any two periods t and t' in a way that their sum $n_t + n_{t'}$ is unchanged by the product $n_t n_{t'}$ increases, then I say the new contest is more homogeneous than the initial one. Again, homogeneity is partial order on contests $\hat{n}$. Theorem 2 is then a corollary, as we can always pick period t with $n_t > 1$ and a period with $n_{t'} = 0$, so that the product is 0. Splitting n_t into two positive parts therefore makes the game more homogeneous and thus strictly increases the total effort.

[9] In fact, by Proposition 1, the total effort in contests $(1, 2)$ and $(2, 1)$ must be the same.

Proposition 3 (Homogeneity increases the total effort)**.** *Suppose X^* is total equilibrium effort in a contest* $\mathbf{n}$ *with* $n > 2$*. Suppose $\hat{X}^*$ is total equilibrium effort in another n-player contest* $\hat{\mathbf{n}}$*, such that for some t and t', $\hat{n}_t \hat{n}_{t'} > n_t n_{t'}$ and $\hat{n}_s = n_s$ for all $s \notin \{t, t'\}$. Then $\hat{X}^* > X^*$.*

Proof. By Proposition 1, we can always reorder periods so that $t = T$ and $t' = T - 1$, and therefore we need only compare $\mathbf{n}$ and $\hat{\mathbf{n}} = (\hat{n}_T, \hat{n}_{T-1}, n_{T-2}, \ldots, n_1)$. Then $f_{T-2}(X) = \hat{f}_{T-2}(X)$, as the subsequence $(n_{T-2}, \ldots, n_1)$ is identical.

It suffices to show that $\hat{f}_T(X^*) < 0 = f_T(X^*)$, because this implies that the highest root $\hat{X}^*$ of $\hat{f}_T(X)$, and since $\hat{f}_T(X)$ is strictly increasing in $[\hat{X}^*, 1]$, this means that $\hat{X}^*$ must be strictly above X^*. First consider the contest with $\mathbf{n}$. Then, denoting $\phi_{T-2}(X) = f'_{T-2}(X)X(1-X)$ for brevity, we get

$$\begin{aligned} f_{T-1}(X) &= f_{T-2}(X) - n_{T-1}\phi_{T-2}(X) \\ f_T(X) &= f_{T-2}(X) - [n_T + n_{T-1}]\phi_{T-2}(X) + n_T n_{T-1}\phi'_{T-2}(X)X(1-X). \end{aligned} \tag{7}$$

Analogously, for contest $\hat{\mathbf{n}}$, the corresponding function will be

$$\hat{f}_T(X) = \hat{f}_{T-2}(X) - [\hat{n}_T + \hat{n}_{T-1}]\hat{\phi}_{T-2}(X) + \hat{n}_T \hat{n}_{T-1}\hat{\phi}'_{T-2}(X)X(1-X). \tag{8}$$

Now, by assumptions $\hat{n}_T + \hat{n}_{T-1} = n_T + n_{T-1}$, and since $\hat{f}_{T-2}(X) = f_{T-2}(X)$, we also have $\hat{\phi}_{T-2}(X) = f'_{T-2}(X)X(1-X) = \phi_{T-2}(X)$. Combining Equations (7) and (8) gives

$$\hat{f}_T(X^*) - f_T(X^*) = [\hat{n}_T \hat{n}_{T-1} - n_T n_{T-1}]\, \phi'_{T-2}(X^*)X^*(1-X^*).$$

Note that $\hat{n}_T \hat{n}_{T-1} > n_T n_{T-1}$ by assumption and $X^*(1-X^*) > 0$ as $X^* \in (0,1)$; therefore, it remains to show that $\phi'_{T-2}(X^*) < 0$. This is shown in Appendix C. Namely, by Proposition 8, the highest root Z_t of polynomial $\phi'_{T-2}(X)$ is strictly below X^*, and by Lemma 3 $\phi'_{T-2}(X) < 0$ for all $X > Z_t$, which includes X^*. □

6 Earlier-mover advantage

In this section, I revisit Dixit's first-mover advantage result. He showed that in a contest with at least three players, when one player could pre-commit, it would be optimal to do so. The first mover chooses a strictly higher effort and achieves a strictly higher payoff than the followers. Using the tools developed above, I can explore this idea further. Namely, in the model studied by Dixit, the first-mover has two advantages compared to the followers. First, he moves earlier and his action may impact the followers. Second, he does not have any direct competitors in the same period.

I can now distinguish these two aspects. For example, what would happen if $n-1$ players chose simultaneously first and the remaining player chose after observing their efforts? Or more generally, in an arbitrary sequence of players, which players choose the highest efforts and which ones get the highest payoffs? The answer to all such questions turns out to be unambiguous. As Proposition 4 shows, it is always preferable to choose earlier.

Proposition 4 (Earlier mover advantage)**.** *At any contest* $(\mathcal{V}, c, \mathbf{n})$*, efforts and payoffs of earlier players are higher than for later players. If the later player has more information, the comparisons are strict.*

Proof. For any contest, total equilibrium effort X^* and individual efforts $x^*_{i,t}$ are determined by Theorem 1. For a fixed X^*, the individual payoffs are linear in individual efforts,

$$u^*_{i,t} = x^*_{i,t}\left[\frac{1}{X^*} - 1\right]. \tag{9}$$

Therefore, it suffices to show that efforts of earlier players are strictly higher. Moreover, it suffices to prove that $x^*_{i,t} > x^*_{j,t-1}$ for each $t = T, \dots, 2$ (and i, j). By Theorem 1, we have

$$\begin{aligned} x^*_{i,t} &= \frac{1}{n_t}\left[f_{t-1}(X^*) - f_t(X^*)\right] = \phi_{t-1}(X^*), \\ x_{j,t-1} &= \frac{1}{n_{t-1}}\left[f_{t-2}(X^*) - f_{t-1}(X^*)\right] = \phi_{t-2}(X^*), \end{aligned}$$

where $\phi_t(X) = f'_t(X)X(1-X)$ for brevity. In Appendix C, Lemma 2 shows that $\phi_t(X)$ can be written recursively. In particular,

$$x^*_{j,t-1} - x^*_{i,t} = \phi_{t-2}(X^*) - \phi_{t-1}(X^*) = n_{t-1}\phi'_{t-2}(X^*)X^*(1-X^*).$$

Finally, by Proposition 8 and Lemma 4, the highest root of $\phi'_{t-2}(X)$ is strictly below X^*, and by Lemma 3, $\phi'_{t-2}(X^*) < 0$. Therefore, indeed $x^*_{j,t-1} - x^*_{i,t} < 0$. □

7 Large contests

In this section, I study contests with a large number of players and focus on two questions. The first is computational: how to compute equilibria with large n. The equilibrium effort is the highest root of a polynomial $f_T(X)$ of degree $T+1$, with leading coefficient $T!\prod_{t=1}^{T} n_t$. Numeric analysis becomes difficult with large n and especially with large T

because it is difficult to find a root of a polynomial with very large coefficients. The second question is, how does the rate of convergence depend on the information provided to the players? Proposition 2 shows that as n increases, X^* converges to 1. I show that the rate of convergence is much larger with sequential contests (i.e., with more information) than with simultaneous contests.

7.1 Approximation result

Proposition 5 provides a simple approximation for equilibria with large n. For example, in the simultaneous contest $1 - X^* = \frac{1}{n}$ and $x^*_{i,1} = \frac{1}{n} - \frac{1}{n^2} \approx \frac{1}{n}$, whereas in the fully sequential contest $1 - X^* \approx \frac{1}{2^n}$ and $\mathbf{x}^* \approx \left(\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}\right)$.

Proposition 5 (Approximation)**.** *If n is large enough, then*

$$1 - X^* \approx \frac{1}{\prod_{t=1}^{T}(1+n_t)} \quad \text{and} \quad x^*_{i,t} \approx \frac{1}{\prod_{s=t}^{T}(1+n_s)}, \quad \forall t = 1, \ldots, T. \tag{10}$$

Proof. Remember that $\phi_{t-1}(X) = f'_{t-1}(X)X(1-X)$ is the individual effort of a player in period t that is consistent with total effort X. Lemma 2 (in Appendix C) shows that it can be recursively written as $\phi_0(X) = X(1-X)$ and $\phi_t(X) = \phi_{t-1}(X) - n_t\phi'_{t-1}(X)X(1-X)$.

I first show by induction that if $X^* \approx 1$, then $\phi_t(X^*) \approx X^*(1-X^*)\prod_{s=1}^{t}(1+n_s)$. Clearly, it holds for $\phi_0(X^*) = X^*(1-X^*)$. By Lemma 3, $\phi'_t(1) = -\prod_{s=1}^{t}(1+n_s)$. Since $\phi'_t(X)$ is a polynomial, it is continuous and therefore, for $X^* \approx 1$, we have $\phi'_t(X^*) \approx \phi'_t(1)$. Using this fact and assuming that the claim holds for $\phi_t(X^*)$, we get

$$\phi_{t+1}(X^*) = \phi_t(X^*) - n_{t+1}\phi'_t(X^*)X^*(1-X^*) \approx \prod_{s=1}^{t+1}(1+n_s)X^*(1-X^*).$$

Now, notice that $1 + \sum_{t=1}^{T} n_t \prod_{s=1}^{t-1}(1+n_s) = \prod_{t=1}^{T}(1+n_t)$, so that

$$X^* = \sum_{t=1}^{T}\sum_{i=1}^{n_t} x^*_{i,t} = \sum_{t=1}^{T} n_t\phi_{t-1}(X^*) \approx X^*(1-X^*)\left[\prod_{t=1}^{T}(1+n_t) - 1\right].$$

Therefore, we get an approximation for the total effort

$$1 - X^* \approx \frac{1}{\prod_{t=1}^{T}(1+n_t) - 1} \approx \frac{1}{\prod_{t=1}^{T}(1+n_t)}.$$

Using this approximation, we get an approximation for individual efforts

$$x^*_{i,t} = \phi_{t-1}(X^*) \approx X^*(1-X^*)\prod_{s=1}^{t-1}(1+n_s) \approx \frac{\prod_{s=1}^{t-1}(1+n_s)}{\prod_{s=1}^{T}(1+n_s)} = \frac{1}{\prod_{s=t}^{T}(1+n_s)}.$$

□

7.2 Rate of convergence

This approximation result allow us to study the rate of the convergence of X^* as n becomes large. Figure 3 illustrates that although with all contest types X^* converges to 1, the rate of convergence depends significantly on the type of contest—in the fully sequential contest the convergence seems to be much faster than in the simultaneous contest.

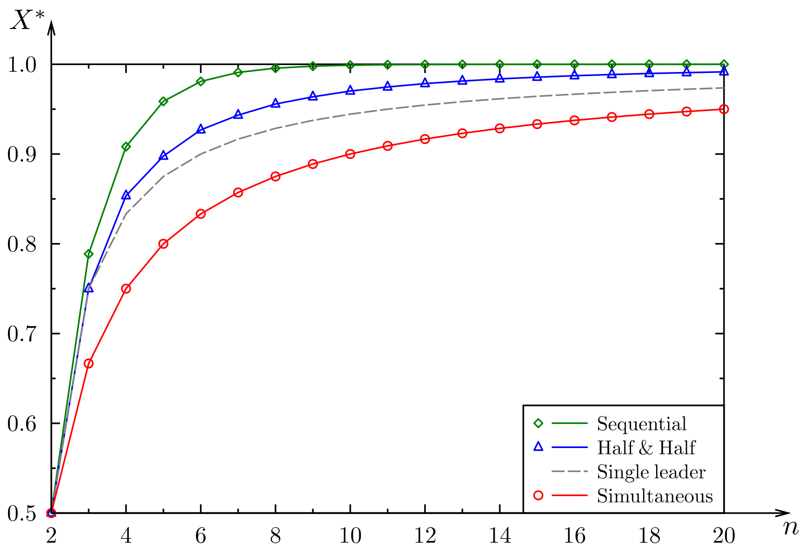


Figure 3: The total equilibrium effort in different contests: Sequential $\mathbf{n} = (1, \dots, 1)$, Half & Half $\mathbf{n} = (\lceil n/2 \rceil, \lfloor n/2 \rfloor)$, Single leader $\mathbf{n} = (1, n-1)$, and Simultaneous $\mathbf{n} = (n)$.

The following Proposition 6 formalizes this observation. The rate of convergence is always at least n^{-1} (simultaneous contest) and at most 2^{-n} (fully sequential contest). Moreover, when the number of announcements, T, is bounded, the growth rate is at most n^{-T}. Therefore, indeed, the convergence in the fully sequential case is much faster than in the simultaneous case. The main increase in the rate of convergence comes from the increase in the number of periods rather than the increase in the number of players in a particular period.

This result has two interesting implications. First, it highlights the importance of information in contest design. When maximizing the total effort, the information provided is more important than the exact number of players. For example, in a simultaneous contest with 20 players, the total effort is 0.95, whereas in fully sequential contests it suffices to have five players to achieve higher total effort (≈ 0.9587).

The result also sheds some light on the debate concerning rent dissipation. Namely, one concern with using linear Tullock contests for rent-seeking is rent under-dissipation, i.e., the fact that the total rent-seeking efforts are perhaps surprisingly low. As the result shows, this may be an implication of the fact that the literature has mainly focused on simultaneous rent-seeking contests, whereas in practice players might have more information.

Proposition 6 (Rate of convergence). *Let $\{\mathbf{n}^n\}_{n=2}^{\infty}$ be a sequence of contests, such that $\mathbf{n}^n = (n^n_{T_n}, \ldots, n^n_1)$ and $\sum_{t=1}^{T_n} n^n_t = n$ for each n. Let X^{n*} be the total equilibrium effort in contest $\mathbf{n}^n$.*

1. *Rate of convergence is at least n^{-1}. In particular, for $\mathbf{n}^n = (n)$ it is exactly n^{-1}.*
2. *Rate of convergence is at most 2^{-n}. In particular, for $\mathbf{n}^n = (1, 1, \ldots, 1)$ it is exactly 2^{-n}.*
3. *If $T_n \leq T$ for some T for all n, then the rate of convergence is at most n^{-T}. In particular, if all n_t are equal*[10]*, then it is exactly n^{-T}.*

Proof.

1. In the simultaneous contest $\mathbf{n}^n = (n)$ we have $1 - X^{n*} = n^{-1}$, and therefore the rate of convergence is n^{-1}. By Theorem 2, the total effort X^{n*} in all other contests is strictly higher; therefore, the rate of convergence must be at least as high.
2. In the fully sequential contest $\mathbf{n}^n = (1, 1, \ldots, 1)$, $n_t = 1$ for all t and $T_n = n$. Therefore, by Proposition 5, we have $1 - X^{n*} \approx 2^{-n}$, and therefore the rate of convergence is 2^{-n}. By Theorem 2, the total effort X^{n*} in all other contests is strictly lower; therefore the rate of convergence cannot be higher.
3. Consider first the case when $T_n = T$ (for n large enough) and each n_t is equal (up to an integer constraint). Then $n_t \approx n/T$. By Proposition 5, $1 - X^{n*} \approx n^{-T}T^{T}$, so that the rate of convergence is indeed n^{-T}. By Proposition 3, X^{n*} of any other (less homogeneous) sequence with $T_n \leq T$ is strictly lower; therefore the convergence rate is lower.

□

7.3 Connection to other forms of strategic interaction

It is straightforward to verify that the approximation formulas derived in Proposition 5 are in fact exact equilibrium quantities in a modified game, where the payoff of player

[10] Up to an integer constraint.

i in period t is $x_{i,t}(1-X)$ instead of $x_{i,t}(1-X)/X$, and the rest of the definitions are unchanged. Moreover, all the results derived in this paper continue to hold in the limit. Therefore, comparative statics, contest design, earlier-mover advantage, and convergence results extend to the modified game as well.[11]

The modified game with payoff function $x_{i,t}(1-X)$ corresponds to at least two well-known forms of strategic interaction. First, can represent the profit function of an oligopolistic firm, where $x_{i,t}$ is the quantity that the firm produces, X is the total quantity produced by all firms, $P(X) = 1 - X$ is the inverse demand function, and the production costs are zero.[12] Then $\mathbf{n} = (n)$ gives the Cournot oligopoly, $\mathbf{n} = (1, 1)$ gives the Stackelberg leadership model, and generally more informative $\mathbf{n}$ means the firms choose their quantities with more information about the other firms' quantities. The dead-weight loss is $(1-X)^2/2$, so that the social planner would be interested in maximizing the total equilibrium production X^*.[13]

Second, $x_{i,t}(1-X)$ is also a standard payoff function to model the tragedy of the commons, where $x_{i,t}$ is the individual use of resources and X is the total use of resources. It captures the idea that each player individually would prefer to use more, whereas higher depletion of resources is bad for all players. In this setting, players would jointly like to maximize $X(1-X)$, which would lead to the optimal resource use of $\frac{1}{2}$. Whenever $n > 2$, the total use of resource in equilibrium is $X^* > \frac{1}{2}$; therefore, a social planner is interested in minimizing X^*.

8 Discussion

In this paper, I characterize equilibria in dynamic common-value contests and study their properties. There is a unique equilibrium; it is in pure strategies and can be computed by finding the highest root of a recursively defined polynomial.

The main result of the paper answers a contest design question: how much information should be provided to the players to maximize or minimize the total effort? The answer is unambiguous: providing more information always increases the total effort. Moreover, information disclosures that are more evenly allocated increase the total effort. Therefore, the fully sequential contest maximizes the total effort and the simultaneous contest minimizes the total effort.

I also revisit the first-mover advantage result and show that the idea is much more

[11]Formal proofs are in Appendix D.

[12]Analogous to Lemma 1, this is generalizable to any linear demand function and constant marginal costs.

[13]The total profit is $P(X)X = X(1-X)$, which is decreasing when $X \geq \frac{1}{2}$, so the goal of a collusive arrangement would be to keep X^* as close to $\frac{1}{2}$ as possible. The consumer surplus is $X^2/2$.

general. The earlier players always exert greater effort and get higher payoffs, regardless of the number of players in any period.

Finally, I study large contests. I provide an approximation result to compute equilibria when the number of players is large and to study the rate of convergence. The rate of convergence of total cost of effort to total value of prizes is much faster with sequential contests than with simultaneous contests. This implies the information provided to the players may play an even more important role in determining the total effort than the number of players participating the contest.

The results are perhaps surprisingly strong, so it would be interesting to determine how robust they are to changes in the assumptions. The results in the paper already provide a partial answer to this. All the results provide comparisons with strict inequalities (with the one exception of Proposition 1, i.e., Independence of permutations). Therefore, these results are robust at least within some small range of parameter values. Moreover, the results also hold for an alternative specification for payoffs, which I interpreted as oligopoly or commons game.

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A Examples

In this appendix, I provide a few examples to show how to find the equilibria in some simple cases with the standard backward-induction approach, and why the approach fails in general dynamic contests. At the end of the appendix, I also show how to apply the characterization result (Theorem 1) in these cases. All the examples in this subsection are special cases of the general model.

A.1 Simultaneous contest

In the first example, $\mathbf{n} = (n)$, i.e,. players do not get any information about the efforts of other players.[14] This means they are essentially making their choices simultaneously. This will serve as a benchmark for later analysis. Each player solves the same maximization problem

$$\max_{x_i} \frac{x_i}{\sum_{j=1}^{n} x_j} - x_i \quad \Rightarrow \quad \frac{1}{X} - \frac{x_i}{X^2} - 1 = 0,$$

where $X = \sum_{i=1}^{n_1} x_j$. Adding up the first-order conditions gives a unique equilibrium, where the total effort is $X^* = \frac{n-1}{n}$ and the individual efforts are $x_i^* = \frac{n-1}{n^2}$.

A.2 Two-player sequential contest

To see how the analysis changes with sequential players, let us start with the simplest case: two-players, i.e., $\mathbf{n} = (1, 1)$. First note that if the effort by the first player $x_1 > 1$, then the payoff is negative. This means that in equilibrium $x_1 \leq 1$. The second player observes the effort choice of the first player and chooses his own effort x_2 such that

$$\max_{x_2} \frac{x_2}{x_1 + x_2} - x_2 \quad \Rightarrow \quad \frac{1}{x_1 + x_2} - \frac{x_2}{(x_1 + x_2)^2} - 1 = 0 \quad \Longleftrightarrow \quad x_2^*(x_1) = \sqrt{x_1} - x_1.$$

Now, the first player chooses x_1, knowing the best-response function $x_2^*(x_1)$, so that

$$\max_{x_1} \frac{x_1}{x_1 + x_2^*(x_1)} - x_1 = \max_{x_1} \sqrt{x_1} - x_1 \Rightarrow x_1^* = x_2^* = \frac{1}{4}, X^* = \frac{1}{2}.$$

Observe that in the two-player case, the equilibrium efforts are exactly the same, both in simultaneous and sequential contests.[15] To shed some light on the reason for this result, note that the reaction function $x_2^*(x_1) = \sqrt{x_1} - x_1$ is strictly increasing for all $x_1 < \frac{1}{4}$ and strictly decreasing for $x_1 > \frac{1}{4}$. See Figure 4 for graphical illustration.

When deciding how much to effort to exert, player 1 has to take into account how player 2 responds. With small $x_1 < \frac{1}{4}$, efforts are strategic complements and therefore increasing x_1 will induce an aggressive reaction, whereas with large $x_1 > \frac{1}{4}$, the efforts are strategic substitutes, so that increasing x_1 discourages effort by player 2. In particular, in simultaneous equilibrium $x_1^* = \frac{1}{4}$, which is the knife-edge case where efforts are neither complements nor substitutes. Therefore, small changes in x_1 have very little impact on x_2 and player 1 decides as if the decisions were independent.

This argument fails with $n > 2$, since at the equilibrium of the simultaneous game,

[14] This is the standard assumption in the literature and the equilibrium is known (Tullock, 2001).

[15] The equality of payoffs in simultaneous and sequential contests with two players was proved by Linster (1993).

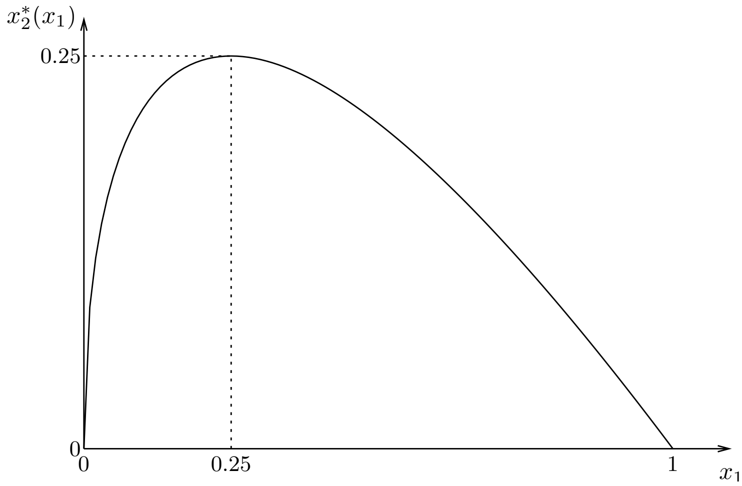

Figure 4: Best response of the last player in a two-player sequential contest.

the total quantity before the last player is strictly above $\frac{1}{4}$; therefore by choosing higher efforts, earlier players could discourage the last player, and thus would have an incentive to increase their efforts. I show in Section 5, for any fixed $n > 2$, the equilibrium has a property that the total effort is minimized when players choose simultaneously.

A.3 Three-player sequential contest

To illustrate how the analysis changes and why two-player case is not representative, consider also the three-player case with $\mathbf{n} = (1, 1, 1)$. I will first show how the standard approach of using backward induction is non-tractable, then describe the approach used in this paper in the general analysis.

Again, $x_1 \leq 1$ and $x_1 + x_2 \leq 1$ in equilibrium, because otherwise player 1 or player 2 would get negative payoffs, and they can ensure zero payoffs by making no effort. Consider again the decision of the last player 3, who observes x_1 and x_2 and chooses

$$\max_{x_3} \frac{x_3}{x_1 + x_2 + x_3} - x_3 \quad \Rightarrow \quad \frac{1}{X} - \frac{x_3}{X^2} = 1,$$

where $X = x_1 + x_2 + x_3$. We get the same reaction function as before,

$$x_3^*(x_1 + x_2) = \sqrt{x_1 + x_2} - (x_1 + x_2). \tag{11}$$

Now, player 2, who observes x_3, knows $x_3^*(\cdot)$, and maximizes

$$\max_{x_2} \frac{x_2}{x_1 + x_2 + \sqrt{x_1 + x_2} - (x_1 + x_2)} - x_2 = \max_{x_2} \frac{x_2}{\sqrt{x_1 + x_2}} - x_2.$$

Optimality condition

$$\frac{1}{\sqrt{x_3+x_2}} - \frac{x_2}{2(x_1+x_2)^{\frac{3}{2}}} = 1.$$

This gives us a relationship between x_2 and x_1. Taking into account that $x_1 \geq 0$ and $x_2 \geq 0$, it defines unique optimal x_2^* for each x_1. In particular,

$$x_2^*(x_1) = \frac{1}{12} - x_1 + \frac{\left(8\sqrt{27x_1^3(27x_1+1)} + 216x_1^2 + 36x_1 + 1\right)^{\frac{2}{3}} + 24x_1 + 1}{12\left(8\sqrt{27x_1^3(27x_1+1)} + 216x_1^2 + 36x_1 + 1\right)^{\frac{1}{3}}}. \tag{12}$$

Now, Player 1's problem is

$$\max_{x_1} \frac{x_1}{x_1 + x_2^*(x_1) + x_3^*(x_1 + x_2^*(x_1))} - x_1,$$

where $x_2^*(\cdot)$ and $x_1^*(\cdot)$ are defined by Equations (11) and (12). Although the problem is not complex, it is not very tractable. Moreover, it is obvious that the direct approach is not generalizable for arbitrary n.[16]

A.4 Examples revisited

I am now illustrating how the characterization result simplifies the task of finding equilibria in the examples above.

Simultaneous contest When $\mathbf{n} = (n)$ we have that $T = 1$ and

$$f_T(X) = f_1(X) = X - nX(1-X) = nX\left(X - \frac{n-1}{n}\right).$$

The polynomial $f_1(X)$ has two roots and the higher one is the total effort $X^* = \frac{n-1}{n}$. Individual efforts are $x_{i,1}^* = X^*(1-X^*) = \frac{n-1}{n^2}$.

Two-player sequential contest When $\mathbf{n} = (1,1)$, we get

$$f_1(X) = X - X(1-X) = X^2,$$
$$f_2(X) = X^2 - 2X^2(1-X) = 2X^2\left(X - \frac{1}{2}\right).$$

[16] Glazer and Hassin (2000) characterized the equilibrium in contests with three sequential players. I am not aware of any papers that have characterized equilibria for sequential contests with four or more periods.

Therefore, the total effort is the highest root $X^* = \frac{1}{2}$ and individual efforts $x^*_{1,2} = (X^*)^2 = \frac{1}{4}$ and $x^*_{1,1} = X^*(1 - X^*) = \frac{1}{4}$.

Three-player sequential contest When $\mathbf{n} = (1, 1, 1)$, we get the same $f_1(X)$ and $f_2(X)$, and

$$f_3(X) = 2X^3 - X^2 - \left(6X^2 - 2X\right) X(1 - X) = X^2 \left(6X^2 - 6X + 1\right).$$

The total effort is the highest root $X^* = \frac{1}{2} + \frac{1}{2\sqrt{3}} \approx 0.7887$ and individual efforts are $x^*_{1,3} = (X^*)^2 (2X^* - 1) \approx 0.3591$, $x^*_{1,2} = 2(X^*)^2(1 - X^*) \approx 0.2629$, and $x^*_{1,1} = X^*(1 - X^*) \approx 0.1667$.

B The final step of the proof of Theorem 1

In the proof of Theorem 1, I showed that the optimality constraints of all players can be combined into one expression $f_T(X) = 0$, where f_T is a recursively defined function and X is the total effort. Moreover, all players have to exert non-negative efforts, which is equivalent to $f_t(X)$ decreasing in t at equilibrium X. The following proposition shows that there is a unique value X^* that satisfies these conditions. This pins down the total effort in the equilibrium.

Proposition 7. *Suppose that f_t's are recursively defined as in Theorem 1. Then X^* is the highest root of $f_T(X) = 0$ if and only if*

$$0 = f_T(X^*) < f_{T-1}(X^*) < \cdots < f_1(X^*) < f_0(X^*) = X^*. \tag{13}$$

Proof. First, I show by induction that $f_t(1) = 1$ for all t. Clearly, $f_0(1) = 1$, so

$$f_t(1) = \underbrace{f_{t-1}(1)}_{=1} - \underbrace{n_t f'_{t-1}(1)}_{<\infty} \underbrace{1(1-1)}_{=0} = 1 \;\; \forall t = 1, \ldots, T.$$

Next, I show by induction that for all $t = 1, \ldots, T$, we can express $f_t(X)$ as

$$f_t(X) = \left[t! \prod_{s=1}^{t} n_s\right] (X - X_{0,t})(X - X_{1,t}) \ldots (X - X_{t,t}),$$

where $0 = X_{t,0} \leq X_{t,1} < X_{t,2} < \cdots < X_{t,t} < 1$. In other words, $f_t(X)$ is a $t+1$th order polynomial, which has $t+1$ real roots $X_{i,t}$ in the interval $[0, 1)$. In particular, one or two[17] lowest roots are 0 and the remaining roots are distinct and in the interval $(0, 1)$.

[17]If $n_s = 1$ for some $1 \leq s \leq t$, then there are two roots at 0; otherwise, there is one.

The representation holds for $f_0(X) = X$.[18] Suppose Appendix B is satisfied for f_t, then for all $j > 1$,

$$f'_t(X_{j,t}) = \left[t! \prod_{s=1}^{t} n_s\right] \prod_{i \neq j} (X_{j,t} - X_{i,t}) \neq 0,$$
$$f'_t(X_{j+1,t}) = \left[t! \prod_{s=1}^{t} n_s\right] \prod_{i \neq j+1} (X_{j+1,t} - X_{i,t}) \neq 0.$$

Moreover, $f'_t(X_{j,t})$ has j positive terms and $T - j$ negative terms in the product, whereas $f'_t(X_{j+1,t})$ has $j+1$ positive and $n - j - 1$ negative terms in the product, so they have different signs. Therefore, $f'_t(X_{j,t})$ must have alternating signs for all $j = 2, \ldots, T$.

Now, since $f_t(X)$ is a polynomial of order $t+1$, $f'_t(X)$ is a polynomial of order t and therefore continuously differentiable, so $f'_t(X)$ must have a root in $Y_{j+1,t} \in (X_{j,t}, X_{j+1,t})$ for each $j = 2, \ldots, T$. We also know that $Y_{1,t} = 0$ is a root of $f'_t(X)$. Therefore, we have found t roots $0 = Y_{1,t} < Y_{2,t} < \cdots < Y_{t,t}$. Because $f'_t(X)$ is a polynomial of degree t, it cannot have more roots. Therefore, we can write the derivative as

$$f'_t(X) = \left[t! \prod_{s=1}^{t} n_s\right] (X - Y_{1,t})(X - Y_{2,t}) \ldots (X - Y_{t,t}),$$

such that (1) $Y_{1,t} = 0$ and (2) $X_{j-1,t} < Y_{j,t} < X_{j,t}$ for all $k \in \{2, \ldots, t\}$.

By the argument above, the sign of $f'_t(X)X(1-X)$ at the points $X_{j,t}$ alternates between strictly positive and strictly negative for all $j > 2$. Moreover, since $f_t(1) = 1$, we must have that $f'_t(X_{t,t}) > 0$, $f'_t(X_{t-1,t}) < 0$, and so on.

By definition of $f_{t+1}(X)$, we must have

$$f_{t+1}(X_{j,t}) = \underbrace{f_t(X_{j,t})}_{=0} - f'_t(X_{j,t}) \underbrace{n_{t+1} X_{j,t}(1 - X_{j,t})}_{>0},$$

which also has alternating signs at $X_{j,t}$ for all $j \geq 2$.

Now, since $f_{t+1}(1) = 1 > 0$ and $f_{t+1}(X_{t,t}) < 0$, there must be a root in $X_{t+1,t+1} \in (X_{t,t}, 1)$. Similarly, there must be a root $X_{j+1,t+1} \in (X_{j,t}, X_{j+1,t})$ for $j = 2, \ldots, t-1$. Finally, it is easy to see that f_{t+1} must have two roots at $0 = X_{0,t+1} = X_{1,t+1}$.[19] As we have identified $t+2$ roots and f_{t+1} is a polynomial of degree $t+2$, it cannot have any more roots.

Using the same steps, it is straightforward to verify that when $n_1 > 1$, f_{t+1} has $n+2$

[18] It also holds for $f_1(X) = X - n_1 X(1 - X) = n_1 X \left(X - \frac{n_1 - 1}{n_1}\right)$.

[19] Because $f_t(x)$ has two roots at 0 and thus $f'_t(X)$ has one root at 0, but it is multiplied by X in the definition of f_{t+1}.

roots in $[0, 1)$, the only difference being that the second lowest root $X_{1,t+1}$ is strictly positive, so that all $t+2$ roots are distinct.

Finally, since the multiplier of the term X^{t+1} was $t! \prod_{s=1}^{t} n_s$ in f_t, the term X^{t+2} comes from $n_{t+1} f_t'(X) X^2$ in f_{t+1} and therefore has a multiplier $(t+1)! \prod_{s=1}^{t+1} n_s$.

Therefore, we can write f_{t+1} as

$$f_{t+1}(X) = \left[(t+1)! \prod_{s=1}^{t+1} n_s\right] (X - X_{0,k+1})(X - X_{1,k+1}) \dots (X - X_{t+1,t+1})$$

such that $0 = X_{0,t+1} \leq X_{1,t+1}$, $X_{t+1,t+1} \in (X_{t,t}, 1)$, and for all $j \geq 2$, $X_{j,t+1} < 1$ and

$$X_{j-1,t} < X_{j,t+1} < X_{j,t}.$$

With this representation we can now conclude that $X_{T,T} > X_{t-1,t-1} > \dots > X_{1,1}$. Therefore, $f_{t-1}'(X_{T,T}) > 0$ for all t, so that $f_t(X_{T,T}) - f_{t-1}(X_{T,T}) = n_t f_{t-1}'(X_{T,T}) X_{T,T}(1 - X_{T,T}) > 0$, which proves that the highest root $X^* = X_{T,T}$ has the desired properties.

It remains to show $X_{T,T}$ is the unique point. For this, I argue that whenever $X \in [0, X_{T-1,T}]$, we must have $f_t(X) > f_{t-1}(X)$ for at least one t at X (monotonicity of $f_t(X)$ with respect to t). Let us start with $X_{T-1,T}$. By definition, $f_T(X_{T-1,T}) = 0$. However, we showed that $X_{T-2,T-1} < X_{T-1,T} < X_{T-1,T-1}$. In this interval $f_{T-1}(X) < 0$ and therefore $f_{T-1}(X_{T-1,T}) < 0 = f_T(X_{T-1,T})$, which violates the monotonicity condition for $t = T$. More generally, at any $X \in (X_{T-2,T-1} < X_{T-1,T}]$, we have $f_{T-1}(X) < 0 \leq f_T(X)$, so all of these points violate the monotonicity requirement.

For general proof, let us define the interval $[0, X_{T-1,T}]$ into sub-intervals $(X_{t-2,t-1}, X_{t-1,t}]$ for $t = 2, \dots, T$ (See Figure 2 in the text for an illustration). Take arbitrary $t \in \{2, \dots, T\}$. Now observe that $X \in (X_{t-2,t-1}, X_{t-1,t}] \subset X_{t-2,t-1}, X_{t-1,t-1})$ and therefore $f_{t-1}(X) < 0$, and also that $X \in (X_{t-2,t-1}, X_{t-1,t}] \subset (X_{t-2,t}, X_{t-1,t}]$, therefore $f_t(X) \geq 0$. This implies $f_{t-1}(X) < 0 \leq f_t(X)$ and therefore violates the monotonicity condition for t. □

C Properties of function ϕ_t

In the proofs of Propositions 3 and 4, I defined $\phi_t(X) = f_t'(X)X(1-X)$. By definition

$$f_{t+1}(X) = f_t(X) - n_{t+1} \underbrace{\phi_t(X)}_{=f_t'(X)X(1-X)} \iff \phi_t(X) = \frac{1}{n_{t+1}} \left[f_t(X) - f_{t+1}(X)\right]. \quad (14)$$

Thus $\phi_t(X^*)$ is the individual effort of a player from period $t+1$ that is consistent with the total effort X^*. I will first characterize some useful properties of $\phi_t(X)$ and its derivative.

Lemma 2 (Properties of ϕ_t). *$\phi_t(X)$ has the following properties:*

1. *$\phi_t(X)$ is a polynomial of degree $t+2$ and all its roots are in $[0,1]$.*
2. *$\phi_t'(X)$ is a polynomial of degree $t+1$ and all its roots are in $[0,1]$.*
3. *$\phi_t(0) = \phi_t(1) = 0$ for all t.*
4. *$\phi_t(X)$ and $\phi_t'(X)$ are independent of permutations of $\mathbf{n}^t = (n_t, \ldots, n_1)$.*
5. *$\phi_t(X)$ can be recursively expressed as $\phi_0(X) = X(1-X)$ and*

$$\phi_t(X) = \phi_{t-1}(X) - n_t \phi_{t-1}'(X) X(1-X). \tag{15}$$

Proof.

1. In the proof of Theorem 1, I showed that $f_t'(X)$ is a polynomial of degree t, with t real roots, and all roots in $[0,1]$. Therefore $\phi_t(X) = f_t'(X)X(1-X)$ is a polynomial of degree $t+2$ with all roots in $[0,1]$. The roots are the t roots of $f_t'(X)$ and two additional roots at 0 and 1, respectively.
2. Therefore, $\phi_t'(X)$ is a polynomial of degree $t+1$. By the Gauss-Lucas theorem, all of its roots are in the convex hull of the roots of $\phi_t(X)$, and therefore also in $[0,1]$.
3. As $f_t'(X)$ is a polynomial, it has finite values at 0 and 1.
4. This follows from the fact that $f_t(X)$ is independent of permutations.
5. By definition, $\phi_0(X) = f_0'(X)X(1-X) = X(1-X)$. Suppose that the recursive construction is correct for $\phi_t(X)$. Differentiating the recursive expression for $f_{t+1}(X)$ (see Equation (14) above) and multiplying it by $X(1-X)$ gives

$$\underbrace{f_{t+1}'(X)X(1-X)}_{=\phi_{t+1}(X)} = \underbrace{f_t'(X)X(1-X)}_{=\phi_t(X)} - n_{t+1}\phi_t'(X)X(1-X).$$

□

Next, let Z_t be the highest root of $\phi_t'(X)$. By Lemma 2, $Z_t \in [0,1]$. Remember that $\phi_0(X) = X(1-X)$, so that $\phi_0'(X) = 1-2X$ and thus has one root at $1/2$. Lemmas 3 and 4, presented below, show two properties of the function $\phi_t'(X)$. First, above its highest root $\phi_t'(X) < 0$ (thus $\phi_t(X)$ is strictly decreasing in $(Z_t, 1]$) and the highest roots are monotonically increasing in t. In particular, this means that in $(Z_t, 1]$ all functions $\phi_s'(X)$ for $s \leq t$ are strictly decreasing.

Lemma 3 (ϕ_t' near 1)**.** $\phi_t'(1) = -\prod_{s=1}^{t}(1+n_s) < 0$ *and* $\phi_t'(X) < 0$ *for all* $X > Z_t$*. Moreover,* $\phi_{t+1}(X) > \phi_t(X)$ *for all* $X \in (Z_t, 1)$*.*

Proof. I first confirm that $\phi_t'(1) = -\prod_{s=1}^{t}(1+n_s)$. Clearly, $\phi_0'(1) = 1 - 2 = -1$. Suppose that the claim holds for $\phi_t'(1)$. Then by Equation (15),

$$\phi_{t+1}'(X) = [1 - n_{t+1}(1-2X)]\phi_t'(X) - n_{t+1}\phi_t''(X)X(1-X).$$

Note that $\phi_t''(X)$ is a polynomial, therefore finite at $X = 1$. Thus indeed

$$\phi_{t+1}'(1) = (1+n_t)\phi_t'(1) = -\prod_{s=1}^{t+1}(1+n_s).$$

Next, by definition Z_t is the highest root of $\phi_t'(X)$, therefore $\phi_t'(X)$ has the same sign for all $X > Z_t$. As $\phi_t'(1) < 0$, the sign must be negative. Therefore at all $X \in (Z_t, 1)$

$$\phi_{t+1}(X) = \phi_t(X) - n_t \underbrace{\phi_t'(Z_t)}_{<0} \underbrace{Z_t(1-Z_t)}_{>0} > \phi_t(X).$$

$\square$

Lemma 4. *The highest roots of* $\phi_1'(X), \dots, \phi_T'(X)$ *are such that* $\frac{1}{2} < Z_1 < Z_2 < \dots Z_T < 1$*.*

Proof. As argued above, $Z_0 = \frac{1}{2}$. Now, suppose that the claim holds up to t, i.e., $\frac{1}{2} < Z_1 < \dots < Z_t < 1$. By Lemma 3, $\phi_t''(Z_t) < 0$ because $\phi_t'(X) < 0$ for all $X > Z_t$. Therefore,

$$\phi_{t+1}'(Z_t) = [1 - n_t(1-2Z_t)] \underbrace{\phi_t'(Z_t)}_{=0} - n_t \underbrace{\phi_t''(Z_t)}_{<0} \underbrace{Z_t(1-Z_t)}_{>0} > 0.$$

This implies that at Z_t, the function $\phi_{t+1}'(Z_t)$ is strictly positive. By Lemma 3, $\phi_{t+1}'(1) < 0$. Therefore, by continuity of $\phi_{t+1}'(X)$, there must be at least point in $(Z_t, 1)$, where $\phi_{t+1}'(X) = 0$. The highest such point must be the highest root Z_{t+1}. $\square$

The next Lemma 4 shows that Z_t is monotone in the vector $\mathbf{n} = (n_t, \dots, n_1)$, i.e., if we increase one or more n_t's, the highest root of $\phi_t'(X)$ moves upwards. The proof is analogous to Proposition 2 in terms of monotonicity, but not in terms of the limit.

Lemma 5. Z_t *is strictly increasing in* n*.*

Proof. By independence of permutations, it suffices to show the monotonicity in n_t. Consider two vectors $\mathbf{n} = (n_t, \dots, n_1)$ and $\hat{\mathbf{n}} = (n_t + 1, n_{t-1}, \dots, n_1)$. Then $\phi'_{t-1}(X)$ are unchanged. For $\mathbf{n}$, we get

$$\begin{aligned}\phi_t(X) &= \phi_{t-1}(X) - n_t\phi'_{t-1}(X)X(1-X),\\ \phi'_t(X) &= [1 - n_t(1-2X)]\phi'_{t-1}(X) - n_t\phi''_{t-1}(X)X(1-X).\end{aligned}$$

Analogously, for $\hat{n}$, we get

$$\hat{\phi}'_t(X) = [1-(n_t+1)(1-2X)]\phi'_{t-1}(X) - (n_t+1)\phi''_{t-1}(X)X(1-X).$$

Therefore at Z_t,

$$\hat{\phi}'_t(Z_t) - \underbrace{\phi'_t(Z_t)}_{=0} = \underbrace{-(1-2Z_t)}_{>0}\underbrace{\phi'_{t-1}(Z_t)}_{<0} - \underbrace{\phi''_{t-1}(Z_t)}_{<0}\underbrace{Z_t(1-Z_t)}_{>0} > 0,$$

where $\phi''_{t-1}(Z_t) < 0$ comes from the fact that, by the Gauss-Lucas theorem, the roots of $\phi''_{t-1}(X)$ must be in the convex hull of $\phi'_{t-1}(X)$, thus contained in $[0, Z_{t-1}]$, and we already argued that $\phi''_{t-1}(Z_{t-1}) < 0$; therefore, also $\phi''_{t-1}(Z_t)$ as $Z_t > Z_{t-1}$ by Lemma 4.

Therefore, $\hat{\phi}'_t(Z_t) > 0$ and since $\hat{\phi}'_t(1) < 0$, continuity of $\hat{\phi}'_t(X)$ implies that there is at least one root in $(Z_t, 1)$. Therefore its highest root is strictly above Z_t. □

The following Proposition 8 establishes a useful connection between the roots of $\phi'_t(X)$ and the original polynomials $f_t(X)$, i.e., Z_t and X^*, which helps to prove several important results in the paper. In particular, let us define

$$Z_t^\infty = \lim_{n_1\to\infty}\lim_{n_2\to\infty}\dots\lim_{n_t\to\infty} Z_t. \tag{16}$$

Then by monotonicity, Z_t^∞ is the upper bound for Z_t. Note that since each $Z_{t-1} < Z_t < 1$, we must have $\frac{1}{2} \le Z_1^\infty \le \dots \le Z_t^\infty \le 1$.

Let us also define X_t^1 as the highest root $X_{t,t}$ of $f_t(X)$ in the case when $\mathbf{n} = (1, 1, \dots, 1)$, i.e., a fully sequential game. Then by Proposition 2, X_T^1 is a lower bound for X^*, i.e., the total effort in equilibrium. Again $0 = X_1^1 < X_2^1 < \dots X_T^1 < 1$.

Proposition 8. *For each $T \ge 2$, $Z_{T-2}^\infty = X_T^1$. For any $\mathbf{n}$, we have $Z_{T-2} < Z_{T-2}^\infty = X_T^1 \le X_T$.*

Proof. Let $f_1, \dots, f_t$ be the sequence of polynomials we get from fully sequential games,

$\mathbf{n} = (1, \ldots, 1)$, and let us define another sequence of polynomials, $\underline{\phi}_t(X)$, as

$$\underline{\phi}_t(X) = \lim_{n_1 \to \infty} \ldots \lim_{n_t \to \infty} \frac{\phi_t(X)}{n_t \ldots n_1}. \tag{17}$$

For a given $\mathbf{n}$, dividing a polynomial by a constant does not affect its roots. Moreover, $\mathbf{n}$ only affects the coefficients of the polynomials[20] and therefore the limiting function $\underline{\phi}_t(X)$ has the same highest root as the limit Z_t^{∞}.

Then $\underline{\phi}_t$'s can be computed recursively as follows. The initial $\underline{\phi}_0(X) = \phi_0(X) = X(1-X)$ because it is independent of $\mathbf{n}$. Using the recursive definition of ϕ_{t+1}, it is straightforward to verify that

$$\underline{\phi}_{t+1}(X) = -\underline{\phi}'_t(X)X(1-X).$$

To complete the proof, I show by induction that $f_t(X) = -\underline{\phi}'_{t-2}(X)X^2$. Multiplying by $-X^2$ adds only two roots at 0 and thus does not affect the highest root. Therefore, $f_t(X)$ and $\underline{\phi}'_{t-2}(X)$ must have the same highest root, which would imply $X_t^1 = Z_{t-2}^{\infty}$.

Let us start with $t = 2$. Then $\underline{\phi}_{t-2}(X) = \phi_0(X) = X(1-X)$, so that $\underline{\phi}'_0(X) = 1 - 2X$. Therefore,

$$f_2(X) = 2X^2\left(X - \frac{1}{2}\right) = -X^2\underline{\phi}'_0(X).$$

Now, suppose that the claim holds up to t, i.e., $f_t(X) = -X^2\underline{\phi}'_{t-2}(X)$. Then

$$\begin{aligned} f_{t+1}(X) &= \underbrace{-X^2\underline{\phi}'_{t-2}(X)}_{=f_t(X)} - \underbrace{\left[-2X\underline{\phi}'_{t-2}(X) - X^2\underline{\phi}''_{t-2}(X)\right]}_{=f'_t(X)} X(1-X) \\ &= -X^2 \underbrace{\left[-(1-2X)\underline{\phi}'_{t-2}(X) - X(1-X)\underline{\phi}''_{t-2}(X)\right]}_{=\frac{d}{dX}\left[-\underline{\phi}'_{t-2}(X)X(1-X)\right]=\underline{\phi}'_{t-1}(X)} = -X^2\underline{\phi}'_{t-1}(X). \end{aligned}$$

Finally, the inequalities in the proposition follow from the monotonicity of Z_{T-2} and X_T. □

D Equilibrium for oligopoly and commons models

This appendix provides formal proof for the observation made in Section 7.3. Namely, it shows that the limiting equilibrium characterization in Proposition 5 for large n is also the unique equilibrium for any n in a modified model, which can be interpreted as oligopoly

[20] Because the roots of polynomials are continuous functions of coefficients.

model (with linear demand) or a simple model of commons.

Moreover, all the results derived in the paper continue to hold. That is, conclusions from Proposition 1, Proposition 2, Theorem 2, Proposition 3, Proposition 4, and Proposition 6 in the modified model are straightforward corollaries of Lemma 6.

I call a modified dynamic game an *oligopoly* or *commons* with parameter $\mathbf{n}$, when the payoff of an individual player is $x_{i,t}(1-X)$, where $x_{i,t}$ is the player's own effort and X is the total effort. All the other assumptions about the timing remain unchanged.

As the proofs show, the analysis in oligopoly and commons cases are considerably simpler. This comes from the fact that the optimality conditions are linear both in their own efforts and in the efforts of all other players. Therefore, all the best-response functions are linear and the usual backward induction approach is tractable with an arbitrary number of players and periods.

To shorten the notation, let $x_t = \sum_{i=1}^{n_t} x_{i,t}$ be the total effort in period t, $X_t = \sum_{s=t+1}^{T} x_s$ total effort of players arriving prior to period t, and let $X_t^+ = \sum_{s=1}^{t} x_s$ be the total effort of players arriving at period t or later. With slight abuse of notation, I use the same notation for best-response functions.

Lemma 6 (Equilibrium in oligopoly and commons games)**.** *For any oligopoly or commons game with parameter* $\mathbf{n}$*,*

$$1 - X^* = \frac{1}{\prod_{t=1}^{T}(1+n_t)} \quad \textit{and} \quad x_{i,t}^* = \frac{1}{\prod_{s=t}^{T}(1+n_s)}, \quad \forall t = 1, \dots, T. \tag{18}$$

Proof. I show by induction that the best-response functions $x_{i,t}(X_t)$ and the implied total proceeding efforts at period t following X_t are respectively

$$x_{i,t}(X_t) = \frac{1-X_t}{n_t+1} \quad \text{and} \quad X_t^+(X_t) = \left(1 - \frac{1}{\prod_{s=1}^{t}(n_s+1)}\right)(1-X_t). \tag{19}$$

Players in period $t=1$ observe X_1 and simultaneously choose $x_{i,1}$ to

$$\max_{x_{i,1}} x_{i,1}\,(1 - X_1 - x_1) \Rightarrow 1 - X_1 - x_1 - x_{i,1} = 0.$$

Combining the optimality conditions gives $x_1(X_1) = \frac{n_1}{n_1+1}(1-X_1)$; therefore, indeed $x_{i,1}(X_1) = \frac{1-X_1}{n_1+1}$ and $X_1^+(X_1) = x_1(X_1) = \left(1 - \frac{1}{n_1+1}\right)(1-X_1)$. Suppose that Equation (19) holds for $t-1$. Then in period t, players observe X_t, know responses of future players $X^+_{t-1}(X_t + x_t)$, and choose $x_{i,t}$ simultaneously, such that

$$\max_{x_{i,t}} x_{i,t} \underbrace{\left(1 - X_t - x_t - X^+_{t-1}(X_t + x_t)\right)}_{=\frac{1-X_t-x_t}{\prod_{s=1}^{t-1}(n_s+1)}} \Rightarrow \frac{1 - X_t - x_t - x_{i,t}}{\prod_{s=1}^{t-1}(n_s + 1)} = 0.$$

Combining the optimality conditions gives $x_t(X_t) = \frac{n_t}{n_t+1}(1 - X_t)$ and indeed $x_{i,t}(X_t) = \frac{1-X_t}{n_t+1}$ and

$$X^+_t(X_t) = x_t(X_t) + X^+_{t-1}(X_t + x_t(X_t)) = \left(1 - \frac{1}{\prod_{s=1}^{t}(n_s + 1)}\right)(1 - X_t)$$

As this argument works for any $t = 1, \ldots, T$, it gives us the equilibrium. Namely, at the initial period T, by definition there are no efforts yet, so $X_T = 0$ and the total effort on and after period T is equal to total equilibrium effort. Therefore,

$$1 - X^* = 1 - X^+_T(0) = \frac{1}{\prod_{t=1}^{T}(n_t + 1)}.$$

Finally, I show by induction that in equilibrium $x_{i,t} = 1 - X_{t-1} = \frac{1}{\prod_{s=t+1}^{T}(n_s+1)}$ for all $t = 1, \ldots, T$. Consider the initial period T first. Then as $X_T = 0$, by the results above, $x_{i,T} = x_{i,T}(0) = \frac{1}{n_T+1}$ and $x_T = \frac{n_T}{n_T+1}$. Therefore, $1 - X_{T-1} = \frac{1}{n_T+1}$. Suppose that the claim holds for $t + 1$. Then indeed,

$$x_{i,t} = \frac{1}{n_t + 1}(1 - X_t) = \frac{1}{\prod_{s=t}^{T}(n_s + 1)}, \quad 1 - X_{t-1} = \frac{1 - X_t}{n_t + 1} = \frac{1}{\prod_{s=t}^{T}(n_s + 1)}.$$

□