Efficiency, Equity, and Social Rationality under Uncertainty*

Kaname Miyagishima†

Department of Economics, Aoyama Gakuin University

This version: February 19, 2017

Abstract

In a simple model where agents' monetary payoffs are uncertain, this paper studies the aggregation of uncertainty preferences which are ordinal and interpersonally non-comparable. A maximin social welfare criterion is derived from axioms of efficiency, equity, and social rationality, as well as separability of unconcerned agents and independence of risk preferences in riskless situations. The criterion compares allocations by the values of the prospects composed of the statewise minimum payoffs evaluated by the certainty equivalences.

1 Introduction

Which social welfare criterion should be adopted to evaluate public policies under uncertainty? In this paper, we address this question by investigating the implications of equity, efficiency, and social rationality, which are central principles to the welfare economics of risk and uncertainty.[1]

*I am grateful to Marc Fleurbaey for invaluable discussions. The research grant from Aoyama Gakuin University is gratefully acknowledged.

†Email: Kaname1128@gmail.com

[1]For a comprehensive survey, see Mongin and Pivato (2015).

The path-breaking work is Harsanyi's (1955) aggregation theorem, which states that if individuals and the social planner are expected utility maximizers, the planner's utility function satisfying the standard ex ante Pareto principle[2] is represented by the weighted sum of individual expected utilities. The conditions of social expected utility and ex ante Pareto are, respectively, considered as requirements of social rationality and efficiency.

Harsanyi's result revealed serious tensions between equity (ex ante or ex post), efficiency, and social rationality. Among others, Diamond (1967) insists that the social expected utility condition is not desirable, because it conflicts with ex ante equity in the sense of inequality aversion to the distribution of individual expected utilities. Moreover, Grant (1995) shows that any ex ante egalitarian criterion cannot be compatible the "minimal" social rationality, *Statewise Dominance*. Fleurbaey and Voorhoeve (2013) argue that social welfare criteria satisfying the ex ante Pareto principle cannot help the ex post worst-off individual without violating *Statewise Dominance*.

A main purpose of this paper is to explore the implications of equity, efficiency, and social rationality. The exploration is important to construct a reasonable social criterion. In particular, we consider efficiency conditions weaker than the ex ante Pareto principle, since it is broadly admitted that judgment under uncertainty is difficult for individuals because of heuristics and biases, and that it is not compelling to fully respect ex ante preferences (Hammond, 1981). Another reason to weaken the ex ante Pareto condition is spurious unanimity pointed out by Mongin (1997). If agents' beliefs are different, unanimous agreement on uncertain allocations may be spurious because of the disagreement of expectations for future outcomes.[3] In this paper, agents may not have probabilistic beliefs (as explained below), but we consider Pareto conditions which avoid the problem of spurious unanimity.

We also require independence of risk preferences in riskless situations (Chambers and Echenique, 2012) and separability of unconcerned agents at sure allocations. It is meaningful to restrict the application of separability to riskless allocations in the following respects. First, an agent who has the same sure prospect in the allocations could be interpreted as a

[2]The ex ante Pareto principle claims that if all agents prefer one allocation of prospects to the other, then the planner also prefers the former.

[3]Recent contributions on aggregation of beliefs and tastes contain Mongin and Pivato (2015), Fleurbaey and Zuber (2015), and Hayashi and Lombardi (2016).

dead person. Then, it is not compelling to take well-being of the dead agent into account, and thus the separability requirement can be justified as a principle of independence of well-being of the dead (Blackorby et al., 2005; Fleurbaey, 2010; Fleurbaey and Zuber, 2013; and Fleurbaey et al., 2015). Second, if separability is fully applied to any unconcerned individuals under uncertainty, social evaluations cannot be sensitive to correlations of outcomes among agents (Fleurabey and Zuber, 2013). Moreover, the combination of full separability and quite weak conditions of equity and efficiency leads to the ex ante Pareto principle (Miyagishima, 2016, Lemma 1), and thus the worst-off individual cannot be helped under *Statewise Dominance* by the argument of Fleurbaey and Voorhoeve (2013).

This paper considers compatible five axioms in accordance with the above principles, which we refer to as *the basic axioms.* It is argued that the basic axioms have strong implications on inequality aversion. Though we just require a fairly weak equity condition, it is shown that the basic axioms imply *Strong Dominance Aversion*, which requires that if an agent's prospect strictly dominates the other's, this inequality should be reduced. In the main theorem, we characterize a maximin social welfare criterion by the axioms. The criterion assesses each allocation based on the prospect composed of the statewise minimum payoffs which is evaluated by the certainty equivalences. It is argued that this criterion is derived from a reasonable compromise between equity, efficiency, social rationality, and also separability of unconcerned individuals.

We adopt a simple economic model where agents' future monetary payoffs are uncertain.[4] It is not assumed that agents are expected utility maximizer or their preferences are interpersonally comparable, because these assumptions have been severely questioned. It is valuable to consider various preferences, since different individuals would follow different principles of decision-making under uncertainty and some of them may be probabilistically unsophisticated. We consider the aggregation of various ordinal preferences following the fair social ordering approach (Fleurbaey and Maniquet, 2011). Specifically, in this paper, certainty equivalence is derived as the measure to evaluate the prospect composed of the statewise minimum payoffs in each allocation.

The organization of this paper is as follows. In section 2, the model is presented. In

[4] An extension to the case of multi-dimensional outcomes is briefly discussed in the last section.

section 3, the basic five axioms are introduced. In section 4, the implications of the basic axioms are analyzed. In section 5, we show our main theorem which characterizes the social criterion by the basic axioms. In section 6, concluding remarks are given.

2 The Model

Let N be the set of agents such that $|N| \geq 2$. $S = \{s_1, ..., s_m\}$ is the finite set of states with $m \geq 2$. We denote $f_{is} \in \mathbb{R}_+$ the amount of money agent i receives under state $s \in S$. An act of agent i is denoted by $f_i = \left(f_{is}\right)_{s\in S} \in \mathbb{R}_+^S$, which is a vector of state-contingent monetary payoffs. Let $A = \mathbb{R}_+^S$ be the set of acts. $x = \left(x_s\right)_{s\in S} \in A$ is called a *constant act* if $x_s = x_{s'}$ for all $s, s' \in S$. Let $\bar{A}$ be the set of constant acts. We abuse notation in a standard way by denoting the value of money by x for each $x \in \bar{A}$. An allocation is denoted by $f_N = (f_i)_{i\in N}$. $X = A^N$ is the set of allocations. Let $\bar{X} = \bar{A}^N$ be the set of *constant allocations*, which are allocations composed of constant acts. Let us also denote $X^e = \{f_N \in X | f_i = f_j \text{ for all } i, j \in N\}$, which is the set of allocations where all agents have the equal acts.

R_i is agent i's preference relation over A, with strict part P_i and indifference part I_i. A binary relation is an quasi-ordering if it is reflexive and transitive. An ordering is a complete and transitive binary relation. We assume that R_i is an ordering satisfying convexity, continuity, and monotonicity in the sense that $f_i \geq f_i'$ implies $f_i R_i f_i'$ and $f_i \gg f_i'$ implies $f_i P_i f_i'$. Convexity is considered as a condition of risk aversion (e.g., Yaari, 1969) and uncertainty aversion (e.g., Gilboa and Schmeidler, 1989; Rigotti et al., 2008). Let $\mathcal{R}$ denote the set of preferences satisfying the above conditions.[5]

Given $f_j \in A$ and $R_j \in \mathcal{R}$, define $I(f_j, R_j) = \{h \in A | h I_j f_j\}$, $L(f_j, R_j) = \{h \in A | f_j R_j h\}$, $\mathring{L}(f_j, R_j) = \{h \in A | f_j P_j h\}$, $U(f_j, R_j) = \{h \in A | h R_j f_j\}$, and $\mathring{U}(f_j, R_j) = \{h \in A | h P_j f_j\}$. Let $\mathbf{1} = (1, \ldots, 1) \in \bar{A}$. Given $N' \subset N$, let us denote by $(f_{N'}, g_{N\backslash N'})$ an allocation such that each agent $i \in N'$ has f_i and each agent $j \in N\backslash N'$ has g_j.

For each $f \in A$ and each $s \in S$, let $f(s) \in \bar{A}$ be such that $f_{s'}(s) = f_{s''}(s) = f_s$ for all $s', s'' \in S$. $f_N(s) \in \bar{X}$ is similarly defined.

[5] Ertemel (2016), on the same domain, characterizes a maximin social ordering using the ex ante Pareto indifference principle.

A social quasi-ordering function (SQF) $\boldsymbol{R}$ is a mapping which, for every preference profile, determines a reflexive and transitive binary relation over the set of allocations. The domain is denoted by $\mathcal{D} = \mathcal{R}^N$. Given a preference profile $R_N \in \mathcal{D}$, $\boldsymbol{R}(R_N)$ is a *social quasi-ordering* over X. Also, let $\boldsymbol{P}(R_N)$ and $\boldsymbol{I}(R_N)$ be the strict and indifference parts of $\boldsymbol{R}(R_N)$, respectively.

3 Basic Axioms

In this section, we introduce basic five axioms. The first axiom is *Statewise Dominance*, which is often referred to as the *minimal criterion for rational decision.*

Statewise Dominance. For all $R_N \in \mathcal{D}$ and all $f_N, f'_N \in X$,

if $f_N(s)\boldsymbol{R}(R_N)f'_N(s)$ for all $s \in S$, then $f_N\boldsymbol{R}(R_N)f'_N$;

if $f_N(s)\boldsymbol{P}(R_N)f'_N(s)$ for all $s \in S$, then $f_N\boldsymbol{P}(R_N)f'_N$.

This axiom states that if every outcome of an allocation is weakly (resp. strictly) socially better than that of another allocation, the former allocation is socially weakly (resp. strictly) preferred to the latter. If the axiom is violated, the society may choose an allocation that results in a worse consequence.

Next, we introduce efficiency axioms. The first Pareto axiom takes into account ex post equality in the following form. In this paper, we say that f_N is more ex post equal than f'_N if $|f_i(s) - f_j(s)| \leq |f'_i(s) - f'_j(s)|$ for all $i, j \in N$ $(i \neq j)$ and $s \in S$, and the strict inequality holds for some $i, j \in N$ $(i \neq j)$ and $s \in S$.

Consensual Pareto for Ex-post Equality. For all $R_N \in \mathcal{D}$ such that $R_i = R_j$ for all $i, j \in N$, and all $f_N \in X$, $x_N \in \bar{X}$ such that f_N is more ex post equal than x_N, if $f_i P_i x_i$ for all $i \in N$, then $f_N\boldsymbol{P}(R_N)x_N$.

This axiom says that if all agents are willing to take risks (when f_N is uncertain) and the outcomes are more equal than those before the risk-taking, then such preferences for risk-taking should be socially supported. When f_N is also constant, the axiom is further compelling because all agents' monetary payoffs increase without any risk. This axiom is reasonable in terms of compatibility with ex post equality. Moreover, by the condition that all agents have the same preference, there is a consensus in the sense of Sprumont (2012) that everyone has a better prospect. This Pareto condition also avoids the problem of spurious unanimity (Mongin,1997), which is caused by different beliefs among agents. Spurious unanimity is problematic when agents may obtain very different outcomes. It is not a problem for *Consensual Pareto for Ex-post Equality* because individuals have more equal outcomes under the uncertain allocation than under the constant allocation.

The next efficiency condition is introduced by Fleurbaey and Zuber (2015).

Pareto for Equal or No Risk. For all $R_N \in \mathcal{D}$ and all $f_N, f'_N \in X^e \cup \bar{X}$, if $f_i P_i f'_i$ for all $i \in N$, then $f_N \boldsymbol{P}(R_N) f'_N$.

When comparing uncertain allocations where all agents have the equal acts and thus are under the egalitarian condition, it is compelling to judge that the unanimously preferred allocation should be socially more desirable (Fleurbaey, 2010). If allocations are constant, unanimous improvements are also socially desirable. *Pareto for Equal or No Risk* combines these ideas, but is still much weaker than the ex ante Pareto principle. This axiom also avoids spurious unanimity since all agents have the same act under the uncertain allocation.

Next, we introduce an equity condition.

Ex-post Transer among Equals. For all $R_N \in \mathcal{D}$ and all $x_N, x'_N \in \bar{X}$, if there exist j, k such that $R_j = R_k$, and $x_i = x'_i$ for all $i \neq j, k$, then for all $t > 0$,

$$\left[x_j = x'_j - t > x_k = x'_k + t\right] \Rightarrow x_N \boldsymbol{R}(R_N) x'_N.$$

This axiom requires that if there is an ex post inequality between two agents with the same preference, it should be socially accepted to reduce the inequality by transfers. The restriction to individuals with the same preference is meaningful in terms of equal treatment of equals.

The next invariance axiom was firstly introduced by Chambers and Echenique (2012).

Invariance to Risk Attitudes for Constant Acts (IRC). For all $R_N,\ R'_N \in \mathcal{D}$ and all $x_N, x'_N \in \bar{X}$, $x_N \boldsymbol{R}(R_N) x'_N$ if and only if $x_N \boldsymbol{R}(R'_N) x'_N$.

This axiom claims that social judgements over constant allocations should be invariant of risk preferences. The idea is that as long as riskless outcomes are compared, agents' risk preferences are irrelevant for the comparisons. This axiom would also be reasonable in terms of a strategic viewpoint that social decisions over constant allocations should be robust to agents' misreports of their risk preferences.

The last axiom is a separability condition.

Separability for Sure Prospects. For all $R_N \in \mathcal{D}$, all $x_N, x'_N \in \bar{X}$, if $x_i = x'_i$ for some $i \in N$, then for all $y_i \in \bar{A}$,

$$x_N \boldsymbol{R}(R_N) x'_N \Longleftrightarrow (x_{N\setminus\{i\}}, y_i) \boldsymbol{P}(R_N)(x'_{N\setminus\{i\}}, y_i).$$

This axiom requires that an agent should not affect the evaluation of *constant allocations* if the agent has the same act in the allocations. As mentioned in the introduction, it is important to restrict the application of separability to riskless situations.[6]

4 Implications of the Basic Axioms

In this section, we derive implications of our basic axioms. Those implications are not only interesting in their own light, but also useful to prove our main theorem.

The first lemma says that *Ex-post Transfer among Equals*, *Statewise Dominance*, and *IRC* together imply the following strong equity axiom.

Transfer. For all $R_N \in \mathcal{D}$ and all $f_N, f'_N \in X$, if there exist j, k such that $f_i = f'_i$ for all $i \neq j, k$, then for all $\Delta \in \mathbb{R}^S_{++}$,

$$\left[f_j = f'_j - \Delta \gg f_k = f'_k + \Delta\right] \Rightarrow f_N \boldsymbol{R}(R_N) f'_N.$$

This axiom states that for two agents, if one has more income in every state than the other, a transfer in each state to reduce the inequality should be acceptable.

Lemma 1. *Ex-post Transfer among Equals* and *Statewise Dominance* together imply *Transfer.*

[6] The reader may notice that *Separability for Sure Prospects* is redundant for our results if $|N| = 2$. This axiom is slightly weaker than the one employed by Fleurbaey and Zuber (2012) and Fleurbaey et al. (2015), but we can obtain the same results using the stronger separability condition.

Proof. Let $f_N, f'_N \in X$ be such that $f_i = f'_i$ for all $i \neq j, k$, and

$$f_j = f'_j - \Delta \gg f_k = f'_k + \Delta,\ \Delta \in \mathbb{R}^S_{++}.$$

Consider $f_N(s), f'_N(s) \in \bar{X}$ for each $s \in S$. Let R'_N be such that $R_i = R_j$ for all $i, j \in N$. By assumption and *Ex-post Transfer among Equals*, we have $f_N(s)\boldsymbol{R}(R'_N)f'_N(s)$ for all $s \in S$. It follows from *IRC* that $f_N(s)\boldsymbol{R}(R_N)f'_N(s)$ for all $s \in S$. Then, $f_N\boldsymbol{R}(R_N)f'_N$ follows from *Statewise Dominance.* □

Lemma 1 has an important normative implication. From the fundamental incompatibility of equity and efficiency shown by Fleurbaey and Trannoy (2003), we can see that there is no SQF satisfying both *Transfer* and the ex ante Pareto principle. Thus, we have to give up the ex ante Pareto if *IRC* and *Statewise Dominance* are required in addition to the quite weak equity condition, *Ex-post Transfer among Equals.* Intuitively, differences in preference become irrelevant for the equity axiom by *IRC*, and transfers among uncertain prospects become favorable for the society by *Statewise Dominance.*

Lemma 2. If $\boldsymbol{R}$ satisfies *Pareto for Equal or No Risk* and *Statewise Dominance*, then for all $R_N \in \mathcal{D}$ and all $f_N, f'_N \in X$,

$$f_N \gg f'_N \Longrightarrow f_N\boldsymbol{P}(R_N)f'_N.$$

Proof. Let $f_N, f'_N \in X$ be such that $f_N \gg f'_N$. Since $f_i(s) \gg f'_i(s)$ for all $i \in N$ and all $s \in S$, *Pareto for Equal or No Risk* implies $f_N(s)\boldsymbol{P}(R_N)f_N(s)$ for all $s \in S$. The desired conclusion follows from *Statewise Dominance.* □

The next lemma establishes infinite ex post inequality aversion, which is captured by the following axiom.

Strong Ex-post Inequality Aversion. For all $R_N \in \mathcal{D}$ and all $x_N, x'_N \in \bar{X}$, if there exist $j, k \in N$ such that $x_i = x'_i$ for all $i \neq j, k$, then

$$\left[x'_j > x_j > x_k > x'_k\right] \Rightarrow x_N\boldsymbol{P}(R_N)x'_N.$$

Lemma 3. The basic five axioms together imply *Strong Ex-post Inequality Aversion.*

Proof. Since x_N and x'_N are constant allocations, we can invoke *IRC* to arbitrarily modify the preferences. Then, suppose that all agents have R_0 defined as follows: For a probability distribution π over S and sufficiently small $\epsilon > 0$ (as explained below),

$$\begin{aligned}
&I(x'_j, R_0) = \Big\{ g \in A | \sum_{s \in S} \pi_s g_s = x'_j \Big\}, \\
&x_j I_0 (h^*_{s_1} - \epsilon, 0, \cdots, 0) \text{ where } h^*_{s_1} = \frac{x'_j}{\pi_{s_1}}, \\
&x_k I_0 (h^*_{s_1} - 2\epsilon, 0, \cdots, 0), \\
&I(x'_k, R_0) = \Big\{ g \in A | \sum_{s \in S} \pi_s g_s = x'_k \Big\}.
\end{aligned}$$

Moreover, $U(x_j, R_0)$ is the convex hull of $\Big\{ g \in A | \sum_{s \in S} \pi_s g_s \geq \pi_{s_1} \big(h^*_{s_1} - \epsilon \big) \Big\} \cup \{x_j\}$, and $U(x_k, R_0)$ is the convex hull of $\Big\{ g \in A | \sum_{s \in S} \pi_s g_s \geq \pi_{s_1} \big(h^*_{s_1} - 2\epsilon \big) \Big\} \cup \{x_k\}$.

First, we consider $y_N, y'_N \in \bar{X}$ such that $y_k = x_k$, $y'_k = x'_k$, $y_j = x_j$, and $y'_j = y_i = y'_i = x'_j$ for all $i \neq j, k$. We show $y_N \boldsymbol{P}(R_N) y'_N$. Let $g_k = (\epsilon_k + x'_k / \pi_{s_1}, 0, \cdots, 0)$ and $g_j = (\epsilon_j + x'_j / \pi_{s_1}, \epsilon'_j, \cdots, \epsilon'_j)$, where $\epsilon_k, \epsilon_j, \epsilon'_j > 0$ are determined so that these are sufficiently close to 0, $g_j \gg g_k$,

$$(\epsilon_j + x'_j / \pi_{s_1}) - (\epsilon_k + x'_k / \pi_{s_1}) < x'_j - x'_k,$$

and

$$g_k + \frac{n-1}{n}(g_j - g_k) \ll g_j - \frac{1}{n}(g_j - g_k) \in \mathring{L}(x_k, R_0).$$

g_j and g_k are well-defined by the construction of R_0 with sufficiently small ϵ. Define g_N such that agent k has g_k and other agents have g_j. Note that $g_i P_i y_i$ for all $i \in N$ and g_N is more ex post equal than y'_N, and hence we obtain $g_N \boldsymbol{P}(R_N) y'_N$ by Consensual Pareto for Ex-post Equality. Let us also define g'_N such that

$$\begin{aligned}
&g'_k = g_k + \frac{n-1}{n}(g_j - g_k), \\
&g'_i = g_j - \frac{1}{n}(g_j - g_k) \text{ for all } i \neq k.
\end{aligned}$$

Then, $g'_N \boldsymbol{R}(R_N) g_N$ follows from repeated applications of *Transfer* (Lemma 1) with $\Delta = \frac{1}{n}(g_j - g_k)$. By continuity of R_0, there exists $\hat{g} \in A$ such that $\hat{g} \gg g'_i$ for all i and $\hat{g} \in \mathring{L}(x_k, R_0)$. Let g''_N be such that all agents have $\hat{g}$. We obtain $g''_N \boldsymbol{P}(R_N) g'_N$ from Lemma 2. By *Pareto for Equal or No Risk*, $y_N \boldsymbol{P}(R_N) g''_N$. It follows from transitivity that $y_N \boldsymbol{P}(R_N) y'_N$.

Now, remember that $y_i = y'_i$ and $x_i = x'_i$ for all $i \neq j, k$. Thus, repeated applications of *Separability for Sure Prospects* imply $x_N \boldsymbol{P}(R_N) x'_N$. Applying *IRC* to adjust the preference profile, we have the desired result. $\square$

As a direct implication of Lemma 3, we obtain the following strong equity condition, which states that inequality of prospects between two agents should be reduced.

Strong Dominance Aversion. For all $R_N \in \mathcal{D}$ and all $f_N, f'_N \in X$, if there exist $j, k \in N$ such that $f_i = f'_i$ for all $i \neq j, k$,

$$\left[f'_j \gg f_j \gg f_k \gg f'_k\right] \Rightarrow f_N \boldsymbol{P}(R_N) f'_N.$$

The proof is straightforward. By assumption, $f'_j(s) > f_j(s) > f_k(s) > f'_k(s)$ for each $s \in S$. Then, it follows from Lemma 3 that $f_N(s) \boldsymbol{P}(R_N) f'_N(s)$. *Statewise Dominance* implies $f_N \boldsymbol{P}(R_N) f'_N$.

The following lemma is useful to prove our main theorem. For each $f_N \in X$, we denote $m(f_N) = (\min_{i \in N} f_{is})_{s \in S} \in A$, which is the prospect composed of the statewise minimum payoffs in f_N.

Lemma 4. Suppose that $\boldsymbol{R}$ satisfies the basic five axioms. Then, for all $R_N \in \mathcal{D}$ and all $f_N, f'_N \in X$, $m(f_N) \gg m(f'_N)$ implies $f_N \boldsymbol{P}(R_N) f'_N$.

Proof. Suppose $m(f_N) \gg m(f'_N)$. Consider $f_N(s), f'_N(s) \in \bar{X}$ for each $s \in S$. Note that by assumption,

$$\min_{i \in N} f_i(s) > \min_{i \in N} f'_i(s) \text{ for every } s \in S.$$

For our purpose, it is sufficient to show $f_N(s) \boldsymbol{P}(R_N) f'_N(s)$ for every $s \in S$, which implies $f_N \boldsymbol{P}(R_N) f'_N$ by *Stetewise Dominance*, as sought.

The rest of the proof is divided into two cases.

Case 1. Suppose $f'_j(s) = \min_{i \in N} f'_i(s)$ for all $j \in N$. Then,

$$f_j(s) \geq \min_{i \in N} f_i(s) > f'_j(s) \text{ for all } j \in N,$$

and hence $f_N(s) \boldsymbol{P}(R_N) f'_N(s)$ by *Pareto for Equal or No Risk*.

Case 2. Suppose $f'_j(s) > \min_{i\in N} f'_i(s)$ for some $j \in N$. Let $x_N \in \bar{X}$ be such that

$$x_j = \frac{\min_{i\in N} f_i(s) + \min_{i\in N} f'_i(s)}{2} \text{ for all } j \in N,$$

Then, it is straightforward to show that repeated applications of Lemma 3, *Pareto for Equal or No Risk*, and transitivity together imply $x_N \boldsymbol{P}(R_N) f'_N(s)$. $f_N(s)\boldsymbol{P}(R_N)x_N$ follows from *Pareto for Equal or No Risk.* We obtain $f_N(s)\boldsymbol{P}(R_N)f'_N(s)$ by transitivity. □

Though *Ex-post Transfer among Equals* is quite weak, if it is combined with other basic axioms, the social criterion should be sensitive to the statewise worst-off individuals. The statewise worst-offs are crucial for our main theorem in the next section.

5 The Social Criterion

In this section, we derive the social welfare criterion from the basic axioms. For convenience, we introduce a notation. Given $f_i \in A$ and $R_i \in \mathcal{R}$, let

$$C(f_i, R_i) = \inf\{c \in \mathbb{R}_+ | (c, \cdots, c) R_i f_i\},$$

which is the certainty equivalence of f_i with respect to R_i.

Then, we obtain the following result.

Theorem. Suppose that an SQF $\boldsymbol{R}$ satisfies the basic five axioms. Then, for all $R_N \in \mathcal{D}$ and all $f_N, f'_N \in X$,

$$\min_{i\in N} C\big(m(f_N), R_i\big) > \min_{i\in N} C\big(m(f'_N), R_i\big) \Longrightarrow f_N \boldsymbol{P}(R_N) f'_N.$$

The social criterion evaluates each allocation f_N by $m(f_N)$ based on the minimum value of certainty equivalences among individuals.

Proof of Theorem. Let f_N and f'_N be allocations satisfying the condition of the theorem. Without loss of generality, assume that $C(m(f'_N), R_1) = \min_{i\in N} C\big(m(f'_N), R_i\big)$. Consider $\boldsymbol{\epsilon} = (\epsilon, \cdots, \epsilon) \in \mathbb{R}^S_{++}$ and $g_N, g'_N \in X$ such that

$$g'_i = m(g'_N) = m(f'_N) + \boldsymbol{\epsilon} \text{ for all } i \in N,$$

$$g_i = m(g_N) = m(f_N) - \boldsymbol{\epsilon} \text{ for all } i \in N,$$

$$\min_{j\in N} C\big(m(g_N), R_j\big) > \min_{j\in N} C\big(m(g'_N), R_j\big) + 3\epsilon.$$

By Lemma 4, we have $f_N \boldsymbol{P}(R_N) g_N$ and $g'_N \boldsymbol{P}(R_N) f'_N$. In the following, we show $g_N \boldsymbol{P}(R_N) g'_N$. Then, by transitivity, we have the desired result.

Consider $x_N, y_N \in \bar{X}$ such that

$$x_1 = \min_{j \in N} C\big(m(g'_N), R_j\big) + \epsilon,\ x_i P_i g'_i,\ x_i > x_1 + 3\epsilon \text{ for all } i \neq 1,$$
$$y_1 = x_1 + 2\epsilon,\ y_i = x_1 + 3\epsilon \text{ for all } i \neq 1.$$

By *Pareto for Equal or No Risk*, we obtain $x_N \boldsymbol{P}(R_N) g'_N$. It follows from repeated applications of *Strong Ex-post Inequality Aversion* (Lemma 3) that $y_N \boldsymbol{P}(R_N) x_N$. Note that $m(g_N) = g_i P_i y_i$ for all $i \in N$. Hence, *Pareto for Equal or No Risk* implies $g_N \boldsymbol{P}(R_N) y_N$. By transitivity, we obtain $g_N \boldsymbol{P}(R_N) g'_N$ as sought. $\square$

Note that the characterization is partial. The reader may think that if the standard continuity is additionally required, we can obtain the full characterization of the following social criterion.

Definition. $\boldsymbol{R}_M$ is a social ordering function such that for all $R_N \in \mathcal{D}$ and all $f_N, f'_N \in X$,

$$x_N \boldsymbol{R}_M(R_N) x'_N \Longleftrightarrow \min_{i \in N} C\big(m(f_N), R_i\big) \geq \min_{i \in N} C\big(m(f'_N), R_i\big).$$

However, this criterion violates *Separability for Sure Prospects.* To fully characterize $\boldsymbol{R}_M$, we can use the standard continuity condition and the following weaker separability.

Well-off Separability for Sure Prospects. For all $R_N \in \mathcal{D}$, all $x_N, x'_N \in \bar{X}$, if

$$x_i = x'_i > \max\{\min_{i \in N} x_i, \min_{i \in N} x'_i\} \text{ for some } i \in N,$$

then for all $y_i > \max\{\min_{i \in N} x_i, \min_{i \in N} x'_i\}$,

$$x_N \boldsymbol{R}(R_N) x'_N \Longleftrightarrow (x_{N \setminus \{i\}}, y_i) \boldsymbol{P}(R_N)(x'_{N \setminus \{i\}}, y_i).$$

This axiom says that when evaluating constant allocations, an unconcerned individual cannot affect the evaluation as long as the agent has larger monetary payoffs than the worst-offs. In terms of egalitarian view, the information on the worst-offs is important and the social evaluation may well change if the situations of the worst-offs vary. This axiom captures the idea and restricts the separability principle to the case where the unconcerned agent is not the worst-off. It is straightforward to modify the analysis. One can restate Lemma 3 so that agent k is the worst-off in both x_N and x'_N and apply the above separability instead of *Separability for Sure Prospect*. The modification of the proof of Theorem is a little tedious but straightforward. Note that we consider an SQF satisfying the basic axioms. Completeness is obtained as a result of the characterization.

The basic axioms are satisfied by a leximin criterion similar to R_M. To introduce the leximin criterion, we introduce several notations. Given $f_N \in X$ and $s \in S$, $f_{(i)s}$ is the ith lowest payoff of f_N in s.[7] Let us denote $m_{(i)}(f_N) = (f_{(i)s})_{s\in S}$, which is the prospect composed of the ith lowest monetary payoffs in f_N.

Definition. $\boldsymbol{R}_{LM}$ is a social ordering function such that for all $R_N \in \mathcal{D}$ and all $f_N, f'_N \in X$,

$$f_N \boldsymbol{R}_{LM}(R_N) f'_N \Longleftrightarrow \Big(\min_{i\in N} C\big(m_{(j)}(f_N), R_i\big)\Big)_{j\in N} \geq_{lex} \Big(\min_{i\in N} C\big(m_{(j)}(f'_N), R_i\big)\Big)_{j\in N},$$

where $\geq_{lex}$ the standard lexicographic ordering. It remains for future research to fully characterize $\boldsymbol{R}_{LM}$.

6 Concluding Remarks

In this paper, we analyzed the implications of equity, efficiency and social rationality under uncertainty. We obtained the social criterion which is sensitive to the statewise worst-off individuals. In the literature of welfare economics under risk and uncertainty, it is an important issue to construct a social welfare criterion satisfying separability and the three principles above (Fleurbaey, 2010; Fleurbaey and Zuber, 2013; Fleurbaey et al., 2015). Our result provided an answer to the problem.

To make our analysis simple, we considered the model where each agent's ex post well-being is measured in monetary terms and thus single dimensional. The analysis can be extended to the case where agents have preferences over multidimensional outcomes, following the approach developed by Fleurbaey and Zuber (2015). In that case, a criterion for interpersonal comparison is adopted to evaluate agents' well-beings ex post using the fair social ordering approach (Fleurbaey and Maniquet, 2011).

[7] Ties can be broken arbitrarily.

References

[1] Blackorby, C., Bossert, W., Donaldson, D. (2005). *Population Issues in Social Choice Theory, Welfare Economics and Ethics.* Econometric Society Monograph, Cambridge University Press.

[2] Chambers, C.P., Echenique, E. (2012). "When does aggregation reduce risk aversion?" *Games and Economics Behavior*, 76, 582-595.

[3] Diamond, P. (1967). "Cardinal welfare, individual ethics, and interpersonal comparisons of utility: Comment." *Journal of Political Economy*, 75, 765-766.

[4] Ertemel, S., (2016). "Welfare egalitarianism under uncertainty." Available at https://dl.dropboxusercontent.com/u/9401843/Welfare.pdf.

[5] Fleurbaey, M. (2010). "Assessing risky social situations." *Journal of Political Economy*, 118-4, 649-680.

[6] Fleurbaey, M., Gajdos, T., Zuber, S. (2015). "Social rationality, separability, and equity under uncertainty." *Mathematical Social Sciences*, 73, 13-22.

[7] Fleurbaey, M., Maniquet, F. (2011). *A Theory of Fairness and Social Welfare.* Econometric Society Monograph, Cambridge University Press.

[8] Fleurbaey, M., Voorhoeve, A. (2013). "Decide as you would with full information. An argument against ex ante Pareto." In N. Eyal, S.A. Hurst, O.F. Norheim, D. Wikler eds., *Inequalities in Health. Concepts, Measures, and Ethics*, Oxford University Press.

[9] Fleurbaey, M., Zuber, S. (2013). "Inequality aversion and separability in social risk evaluation." *Economic Theory*, 54, 675-692.

[10] Fleurbaey, M., Zuber, S. (2015). "Fair management of social risk." Unpublished manuscript.

[11] Gilboa, I., Schmeidler, D. (1989). "Maxmin expected utility with non-unique prior." *Journal of Mathematical Economics*, 18, 141-153.

[12]

[13] Grant, S. (1995). "Subjective probability without monotonicity: Or how Machina's mom may also be probabilistically sophisticated." *Econometrica*, 63-1, 159-189.

[14] Hammond, H. (1981). "Ex-ante and ex-post welfare optimality under uncertainty." *Economica*, 48, 235-50.

[15] Harsanyi, J. (1955). "Cardinal welfare, individualistic ethics and interpersonal comparisons of utility." *Journal of Political Economy*, 63, 309-321.

[16] Hayashi, T., Lombardi, M. (2016). "Fair social decision under uncertainty and responsibility for beliefs." Available at https://sites.google.com/site/tkshhysh/

[17] Miyagishima, K. (2016). "Fair criteria for social decisions under uncertainty." Unpublished manuscript.

[18] Mongin, P., (1997). "Spurious unanimity and the Pareto Principle." *Economics and Philosophy*, forthcoming.

[19] Mongin, P., Pivato, M. (2015). "Ranking multidimensional alternatives and uncertain prospects." *Journal of Economic Theory*, 157, 146-171.

[20] Mongin, P., Pivato, M. (2016). "Social preference and social welfare under risk and uncertainty." In M. Adler, M. Fleurbaey eds., *Oxford Handbook of Well-Being and Public Policy*, Oxford, University Press.

[21] Rigotti, L., Shannon, C., Strzalecki, T. (2008). "Subjective beliefs and ex-ante trade." *Econometrica 76-5*, 1167-1190.

[22] Sprumont, Y., (2012). "Resource egalitarianism with a dash of efficiency." *Journal of Economic Theory*, 147, 1602-1613.

[23] Yaari, M.E. (1969). "Some remarks on measures of risk aversion and on their uses." *Journal of Economic Theory*, 1-3, 315-329.