Growth, longevity and endogenous health expenditures

Laurent Brembilla[*][†]
Janvier 2017

Abstract

Health expenditures are on the rise in developed countries. This paper analytically studies how they affect capital accumulation in a Diamond model in which individuals can spend resources to live longer in second period. We first derive the demand for health and show that the income share spent on health is an inverted U-shaped function of income. Second, we fully characterize the dynamic general equilibrium and determine the growth impacts of the health expenditures. Several cases can occur. Health expenditures can speed up or slow down economic growth. They can be a barrier to growth or they can be a necessity for growth to take place. A simple calibration of the model to OECD countries suggests that the latter case is the most likely one.

Keywords: Endogenous longevity, economic growth, overlapping generations and health expenditures
JEL classification: O41, I15, E13

1 Introduction

The share of health expenditures in GDP is on the rise in developed countries. In US, which is presently the country with the highest share of total health expenditures in GDP, the ratio has increased from 3,2%

[*]Université d'Evry Val-d'Essonne, Labex MME-DII. E-mail address: lbrembilla@hotmail.fr.

[†]This paper is a chapter of my Ph.D dissertation that I have prepared at the Université Paris-Dauphine. I am grateful to my supervisor Najat El Mekkaoui, Gregory Ponthiere and Hippolyte d'Albis for their helpful comments. I also thank seminar participants at Université d'Evry Val-d'Essonne.

in 1950 to 17,6% in 2015 (Chernew and Newhouse 2011). The same trend is observed in all other OECD countries (see Figure 1). Over the period 2000-2013, the income per capita of the OECD area grew at an average annual rate of 2.9%, while the total level of health expenditures per capita grew at an average annual rate of 4.75%.[1] Concerns have been raised according to which devoting so much resources to the health sector could endanger economic growth (see Kuhn and Prettner (2016)). This paper proposes a simple theoretical framework to assess such concerns. In a standard growth model augmented with endogenous health expenditures, we determine if individuals can voluntarily choose to spend a level of resources on health that harms or even impedes long-run economic growth.

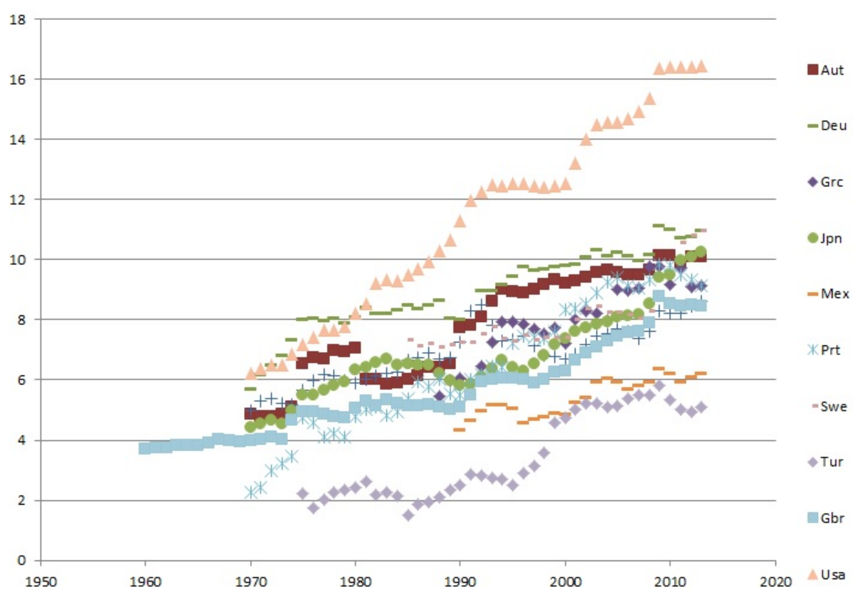

Figure 1: Ratio of total health expenditures to GDP in 10 OECD countries

More precisely, let us consider first a Diamond model with a AK technology. Because of constant returns to reproductible factors, there exists a mild condition on the parameters under which the economy perpetually grows. Second, let us add to this standard framework the possibility for young agents to make expenditures to live longer in second period. Does this economy perpetually grow under the mild condition of the AK model? In other words, can the possibility to spend resources to live longer impede long-run economic growth? If the economy perpetually

[1]These numbers and Figure 1 are obtained from the OECD database.

grows, then does it grow faster than the AK economy? Conversely, if the mild condition fails, can the economy grow? These questions are not trivial because health expenditures produce both positive and negative effects on economic growth. In a AK framework, the growth engine is the physical capital accumulation.[2] When agents increase their level of health expenditures, this creates two opposite effects on savings, hence on economic growth. On the one hand, this increases their longevity which positively impacts their propensity to save (Bloom et al. 2003; Chakraborty 2004). On the other hand, in a previous work, Brembilla (2016), we underline that health expenditures also decrease their disposable income which negatively impacts their savings. Particularly, if young agents were to spend all their resources on health, then there would be no capital stock for the next period and the economy would be trapped to a null income perpetually. Thus, when young agents spend a too large fraction of their income on health, they force a growing economy to stop its expansion. Otherwise said, a high level of health expenditures can be optimal for the current generation, while exerting strong negative *intergenerational externalities* on future generations who could be trapped to a constant income level. This paper focuses on this trade-off that health expenditures create on economic growth and omits other possible channels, such as the impact of health on productivity, to analyze analytically the occurrence of such an event. This enables to shed light on key parameters to take into account for the introduction of a health system to be a growth success.

Doing this, the paper connects two strands of the literature. The first one is a rich theoretical literature on the health-growth nexus, which has investigated the role of longevity in various dynamic general equilibrium settings. This literature first focused on the causal impact of longevity on economic growth by applying shocks on the longevity parameter in different growth models. For example, Boucekkine et al. (2002) and de la Croix and Licandro (1999) study the effect of a longevity increase on economic growth in a model with human capital investments. On the one hand, the human capital supply is stimulated through the Ben-Porath effect following a longevity increase. On the other hand, this creates more retirees and more people educated a long time before, leaving the total impact on economic growth ambiguous. This type of analysis has also been performed in models with a different growth engine. Prettner (2013) examines the consequences of an exogenous longevity shift

[2]It could be argued that human capital is also an important growth engine to take into account to study the health-growth nexus. However, as we consider longevity improvements in retirement period, the Ben-Porath mechanism does not operate (Cervellati and Sunde 2013).

in a R&D based growth model, while Chakraborty (2004) shows that the higher the longevity in retirement period, the higher the propensity to save and then economic growth when it is driven by physical capital accumulation. Second, this literature has studied the joint dynamics of income and longevity, when the latter is determined by health expenditures.[3] In an infuential paper, Chakraborty (2004) proposes a Diamond model in which the survival probability into second period depends on public health expenditures. Bhattacharya and Qiao (2007) propose an OLG model in which there are public and private health expenditures that affect the longevity. Kuhn and Prettner (2016) introduce a health sector into an endogenous growth model and examine the impact of its size on the growth rate and on the welfare of the individuals. However, in these papers, individuals do not control completely the level of their health expenditures. Indeed, the tax rate that finances the public health expenditures is exogenously fixed in these papers.[4] To rationalize the upward trend of the ratio of health expenditures to GDP and to analyze its consequences in terms of economic growth, we need a framework in which total health expenditures arise from the maximization of lifetime utility by agents. To achieve this, we abstract from the financing source of health and we follow Chakraborty (2004) and Bhattacharya and Qiao (2007) by letting second period longevity to depend on health expenditures, where, contrary to these two papers, the level of these expenditures is chosen by the agent by maximizing his lifetime utility under the budget constraints.

Thus, the paper also sheds light on the determinants of the demand for health.[5] This literature pioneered by Grossman (1972) has proposed various modelling strategies to incorporate health decisions into life-cycle models. In Grossman (1972) or Ehrlich and Chuma (1990), individuals live until their health capital, that depreciates each period and that can be increased through investment, falls under a threshold value. Dalgaard and Strulik (2014), criticize the possibility for individuals to increase their health stock and propose a framework based on research

[3]Some authors also study growth models in which longevity is determined by various externalities. Cipriani and Blackburn (2002) and Cervellati and Sunde (2005) both examine a model in which investments in education exert a positive externality on the longevity of individuals. In Mariani et al. (2010) and Raffin and Seegmuller (2014), pollution exerts a negative effect on the individuals'longevity. In Ponthiere (2011), good consumption influences the longevity.

[4]In Chakraborty and Das (2005), individuals fully control their longevity. Yet the authors focus on the transmission of inequalities in a small open economy framework that does not allow to study the growth consequences of these health expenditures.

[5]In line with this literature, we do not study supply side effects of the health sector.

in natural sciences, in which individuals make expenditures to slow the accumulation of deficits caused by aging. Here, we do not follow these approaches as they are not suitable for dynamic general equilibrium. We rather consider a framework in which health expenditures only allow to live longer to enjoy consumption utility.[6] In addition to its analytical convenience for the study of the dynamics of the economy, the formulation allows to establish new results on the income elasticity of health expenditures. Following Jones and Hall (2007), this elasticity is believed to be positively driven by the ratio of health elasticity to consumption elasticity (see also Acemoglu et al. 2013). The first contribution of the paper is to prove and explain why the ratio of these two elasticities is an imperfect picture of the income elasticity of health expenditures. The second one is to provide a complete characterization of the dynamics of the economy to assess the growth impacts of health expenditures.

The rest of the paper is as follows. Section 2 presents the model and characterizes the level of health expenditures. Section 3 studies the dynamics of the economy. Section 4 discusses alternative preferences and proposes a numerical illustration for OECD countries. Section 5 concludes.

2 The model

2.1 Outline

Individuals live for two periods. The young work while the old are retired. For a cohort-t individual, the length of the first and the second period are respectively 1 and p_t, with $p_t \leq 1$. There is a single good in the economy which is produced competitively. This good can be consumed or invested in physical capital or used to increase p_t. The size of each new cohort is constant and normalized to 1.

2.2 Firms

Here we introduce a AK technology for the production sector. There is a representative firm which uses labor and capital to produce the unique good of this economy. The production function F is homogeneous of degree 1 and satisfies Inada conditions:

$$Y_t = F(K_t, B_t L_t) \tag{1}$$

Where Y_t is output at time t, K_t the capital stock, L_t labor and B_t is the labor augmenting technological progress. There are positive

[6] See Azomahou et al. (2015) for a discussion on the inclusion of health in individuals'preferences.

externalities in the use of capital that linearly increase the productivity of workers: $B_t = K_t$. Factors are paid at their marginal productivity. At the equilibrium,

$$1 + r_t = F_1(1, 1) \tag{2}$$

$$w_t = K_t F_2(1, 1) \tag{3}$$

Where we have assumed that the capital fully depreciates at each period. It is convenient to define, as in Raffin and Seegmuller (2014), $A = F(1, 1)$ and $\alpha = \frac{F_1(1,1)}{F(1,1)}$ to write (2) and (3) as:

$$1 + r_t = A\alpha = 1 + r \tag{4}$$

$$w_t = K_t(1 - \alpha)A \tag{5}$$

2.3 Preferences

Individuals choose their consumption levels for both periods. They can also spend resources during their first period to increase their longevity p_t in second period. More precisely, each cohort-t member maximizes the following lifetime utility function:

$$U_t = u(c_{t,t}) + p(e_t)u(c_{t,t+1}) \tag{6}$$

With respect to $c_{t,t}$, $c_{t,t+1}$ and e_t subject to the budget constraints:

$$c_{t,t} + s_t + e_t = w_t \tag{7}$$

$$p(e_t)c_{t,t+1} = (1 + r)s_t \tag{8}$$

Where $c_{t,t}$ is the consumption level per unit of time in first period of a cohort-t member, $c_{t,t+1}$ the consumption level per unit of time in second period, e_t the level of health expenditures, s_t the savings. The function p specifies the relationship between the level of health expenditures and the longevity: $p_t = p(e_t)$. The function p is twice differentiable, increasing and strictly concave with:

$$p(0) = \underline{p} \geq 0, \lim_{e \to \infty} p(e) = \overline{p} \leq 1, p'(0) = \gamma \in (0, +\infty] \tag{9}$$

The set of survival functions p includes the ones that satisfy Inada conditions (this happens if $p(0) = 0$ and $\gamma = \infty$). Assuming that p is increasing and strictly concave is usual in the literature, however there is no consensus on the values of $p(0)$ (null or positive) and $p'(0)$ (finite or not). In Chakraborty (2004), $p(0)$ is null and $p'(0)$ is finite. In Boucekkine and Laffargue (2010), $p(0)$ is positive and $p'(0)$ is finite. Finally, in Chakraborty and Das (2005), $p(0)$ is null and $p'(0)$ is non-finite. We will consider both possibilities for $p(0)$ and $p'(0)$ in the analysis to determine how these assumptions change the results.

As argued by Hall and Jones (2007), the shape of the function u is crucial for the health spending decision. We will pursue the analysis with the following functional form:

Assumption 1 $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, $\sigma < 1$.

Assumption 1 is a standard assumption in growth models incorporating health decisions. It can be found in Chakraborty and Das (2005) and Bhattacharya and Qiao (2007). It has the advantage to insure a positive flow utility and hence a positive marginal utility of longevity. We will also discuss and study numerically the utility function popularized by Hall and Jones (2007), $u(c) = \frac{c^{1-\sigma}}{1-\sigma} + b$, $\sigma > 1$, $b > 0$ in section 4.

2.4 Partial equilibrium results

In this subsection, we study the solution to the maximization problem of the consumer. We use (7) and (8) to eliminate the consumption levels in (6). Then the problem is reduced to maximize (6) with respect to s_t and e_t. The First-Order-Condition (FOC) on s_t yields:

$$s_t = \frac{p(e_t)}{p(e_t) + (1+r)^{\frac{\sigma-1}{\sigma}}}(w_t - e_t) \tag{10}$$

From (10), we can observe the trade-off that health expenditures create on savings: the propensity to save $\frac{p(e_t)}{p(e_t)+(1+r)^{\frac{\sigma-1}{\sigma}}}$ increases with e_t, while the disposable income $w_t - e_t$ decreases with e_t.

For an interior solution, the FOC on e_t yields:

$$c_{t,t}^{-\sigma} + p'(e_t)c_{t,t+1}^{1-\sigma} = p'(e_t)\frac{c_{t,t+1}^{1-\sigma}}{1-\sigma} \tag{11}$$

The Left-Hand-Side (LHS) of (11) is the marginal cost of health expenditures. It is composed of two terms. The first one is the loss of first period utility from foregone consumption. The second one is the loss of second period utility from diminishing per period resources due to longevity extension. The Right-Hand-Side (RHS), the marginal benefit of health expenditures, is the total second period utility gain due to longevity extension. At the optimal level of health expenditures, the marginal cost of health expenditures must equate its marginal benefit. It is useful to rewrite (11) as:

$$c_{t,t}^{-\sigma} = \sigma p'(e_t)\frac{c_{t,t+1}^{1-\sigma}}{1-\sigma} \tag{12}$$

Where the RHS of (12) is the net marginal benefit of longevity extension, which is positive. This means that despite the reduction of per period resources in retirement period, a longevity extension always increases welfare. Finally, (10) and (11) imply that an interior solution for the level of health expenditures must solve the following equation:

$$\frac{\sigma}{1-\sigma}\frac{p'(e_t)}{p(e_t)+(1+r)^{\frac{\sigma-1}{\sigma}}}(w_t - e_t) = 1 \tag{13}$$

The following proposition characterizes the solution of (13):

Proposition 1 *The optimal level of health expenditures is unique. Note it* $e(w_t)$*.*

(i) If $w_t \leq \frac{1-\sigma}{\sigma}\frac{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}$*, then* $e(w_t) = 0$*.*

(ii) If $w_t > \frac{1-\sigma}{\sigma}\frac{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}$*, then* $0 < e(w_t) < w_t$*.*

(iii) $w_t \longrightarrow e(w_t)$ *is increasing on* $[\frac{1-\sigma}{\sigma}\frac{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}, \infty)$.[7]

Proof. See Appendix A ■

When the initial marginal productivity of health expenditures γ is finite, Proposition 1 shows that individuals spend resources on health only if their income is above a certain threshold. As the marginal productivity of health expenditures in 0 is finite, the marginal utility of health expenditures in 0 is finite. On the contrary, the marginal utility of consumption is non-finite in 0. Thus, low-income individuals choose to spend their resources only on consumption.

When γ is non-finite, the corner solution for health expenditures disappears and individuals spend resources on health for all positive income levels.

Remark 2 *Consider the alternative assumption found in the literature according to which* p_t *is the probability to reach the second period. Assume as in Yaari (1965) that there is a perfect annuity market. If agents internalize the effect of health expenditures on the return of annuities, then the problem of the consumer is identical to the one studied in this*

[7] When $\gamma = \infty$, $[\frac{1-\sigma}{\sigma}\frac{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}, \infty)$ is simply read as $[0, \infty)$.

section. If agents do not internalize the effect of health expenditures on the return of annuities, then the optimal level of health expenditures, e_t^Y, would be the solution of the following equation:

$$\frac{p'(e_t^Y)}{p(e_t^Y) + (1+r)^{\frac{\sigma-1}{\sigma}}}(w_t - e_t^Y) = 1 \tag{14}$$

Thus, $e_t < e_t^Y$ if and only if $\sigma < \frac{1}{2}$. And all the propositions of the paper can be adapted to this alternative assumption on p_t given the similarity of the equations (13) and (14).

Proposition 1 also states that health is a normal good. We now sharpen this result by studying the income share spent on health, $x(w_t) := \frac{e(w_t)}{w_t}$. To get the exact shape of the function $x(.)$, we will need an additional assumption on the function p which is motivated by the following result:

Lemma 3 *$e \rightarrow \frac{(-p''(e))e}{p'(e)}$ is initially strictly smaller than 1 and ends strictly greater than 1.*

Proof. This follows from the fact that the function p is bounded ■

Then, for the rest of the paper we will assume that:

Assumption 2 *$e \rightarrow \frac{(-p''(e))e}{p'(e)}$ is increasing while it is smaller than* 1*.*

There are several reasons to believe that Assumption 2 is harmless. First, the survival functions used in the literature satisfy the condition that $e \rightarrow \frac{(-p''(e))e}{p'(e)}$ is increasing, which is a stronger condition than that of Assumption 2. Following an example given by Chakraborty (2004), Raffin and Seegmuller (2014) use a survival function of the form: $p(e) = \frac{\underline{p}+\overline{p}e}{1+e}$. This function is such that $e \rightarrow \frac{(-p''(e))e}{p'(e)}$ is increasing. This is also true for $p(e) = \frac{\underline{p}+\overline{p}e^{\beta}}{1+e^{\beta}}$ with $\beta \in (0, 1]$. Consider the following logistic function: $p(e) = \overline{p}\frac{\frac{\underline{p}}{\overline{p}-\underline{p}}}{e^{-ke}+\frac{\underline{p}}{\overline{p}-\underline{p}}}$ with $k > 0$. For this function to be an admissible survival function, $\frac{\underline{p}}{\overline{p}}$ must be greater than $\frac{1}{2}$. Then, it also satisfies the condition that $e \rightarrow \frac{(-p''(e))e}{p'(e)}$ is increasing. A simple way to build survival functions is to consider a probability density, f, on $[0, \infty)$ and to define $p(e) = \overline{p}\int_0^e f(a)da + \underline{p}$. Then, p is an admissible survival function if and only if f is decreasing. For usual decreasing density distributions (Gaussian, exponential, Weibull), the condition $e \rightarrow \frac{(-p''(e))e}{p'(e)}$ is increasing is also satisfied. We impose Assumption 2 which is weaker than imposing that $e \rightarrow \frac{(-p''(e))e}{p'(e)}$ is increasing because in our empirical illustration in section 4 we find that the function $p(e) = \frac{\underline{p}+\overline{p}\ln(1+\frac{e}{C})}{1+\ln(1+\frac{e}{C})}$ produces the best fit of the model to the data. In this case $e \to \frac{(-p''(e))e}{p'(e)}$ is not increasing, yet Assumption 2 is satisfied. It is possible to build survival functions that violate Assumption 2, however they will have unconventional shapes. Generalizing our results to such functions is possible, yet it would drastically complicate the exposition of the results. We are now able to prove the first main result of the paper.

Proposition 4 *(i)* $w_t \to x(w_t)$ *is inverted U-shaped on* $[\frac{1-\sigma}{\sigma}\frac{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}, \infty)$.
(ii) $\lim_{w_t \to \frac{1-\sigma}{\sigma}\frac{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}} x(w_t) = \lim_{w_t\to\infty} x(w_t) = 0.$

Proof. See Appendix B ■

Proposition 4 can be restated as follows:

Corollary 5 *The ratio of health expenditures to GDP can be written as a function of GDP,* $Y_t \to g(Y_t)$*, which is as follows:*

(i) $g(Y_t) = 0$ *for all* Y_t *in* $[0, \frac{1-\sigma}{\sigma}\frac{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}{(1-\alpha)\gamma}]$

(ii) $Y_t \to g(Y_t)$ *increases on* $[\frac{1-\sigma}{\sigma}\frac{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}{(1-\alpha)\gamma}, \frac{w^*}{1-\alpha}]$, *where* $w^* = \arg\max_{w\geq 0}(x(w))$.

(iii) $Y_t \to g(Y_t)$ *decreases on* $[\frac{w^*}{1-\alpha}, \infty)$.

Proof. Note that $w_t = (1-\alpha)Y_t$. Then $g(Y_t) = \frac{x((1-\alpha)Y_t)}{1-\alpha}$ and the result follows from Proposition 4 ■

Proposition 4 shows that health is a luxury good as individuals start spending resources on health. Thus, our simple framework is consistent with the joint increase of income and income share of health expenditures that OECD countries have known over the last forty years.[8] It also predicts that this income share should not keep rising perpetually with income. Before interpreting the result, notice that consumption in first and second periods are both normal goods so that the result is not driven by an undesirable feature of the preferences.[9] Note also that the result is not driven by the finiteness of γ. When γ is finite, the income share spent on health is equal to 0 for an income smaller than the threshold w^* and positive otherwise. Thus the income share spent on health must

[8]In section 4, we plot the cross-sectional relationship between income and income share of health expenditures for OECD countries for 2012. The curve is upward-sloping (see Figure 5).

[9]To see this, note that $c_{t,t} = \frac{w_t - e(w_t)}{p(e(w_t))+(1+r)^{\frac{\sigma-1}{\sigma}}}$. Using (13), $c_{t,t} = \frac{1-\sigma}{\sigma}\frac{1}{p'(e(w_t))}$. As $w_t \to e(w_t)$ is increasing, $c_{t,t}$ is increasing with w_t. The same proof applies for second period consumption.

increase with income for a range of income levels. However, when γ is non-finite, the corner solution for health expenditures vanishes, yet the income share spent on health is still initially increasing with income.

To understand, the result of Proposition 4, it is useful to rewrite equation (12) as follows:

$$\frac{\text{Health expenditures}}{\text{Income}} = g\left(\frac{2^{nd}\text{ period utility}}{1^{st}\text{ period marginal utility}}\right)\frac{1}{\text{Income}} \tag{15}$$

Where $g(x) = (p')^{-1}(\frac{1}{x}\frac{1}{\sigma})$ is an increasing function. This shows that the level of health expenditures increases with the second period utility level and decreases with the first period marginal utility. Indeed, the higher the second period utility level, the longer individuals want to live in second period, the higher their health expenditures. The health expenditures require a decrease of first period consumption, which implies a welfare loss equal to first period marginal utility. Thus, the higher the first period marginal utility, the higher the welfare costs of health expenditures, the lower the health expenditures. Therefore, to understand how the income share spent on health evolves with income, we need to understand how the second period utility and the first period marginal utility vary with income. For low income levels, the concavity of the utility function implies that the second period utility level increases a lot with income. The marginal utility is convex, which implies that for low income levels, it decreases a lot with income. Thus the ratio of second period utility to first period marginal utility increases by a large amount for low income levels. This implies that health expenditures increase by a large amount, superior to income. For large income levels, the concavity of the utility function implies that it does not increase by much with income. The convexity of the marginal utility also implies that the first period marginal utility does not increase by much with income for large income levels. Thus, the ratio of second period utility to first period marginal utility increases by a small amount for large income levels. This implies that health expenditures increase by a small amount, smaller than income.

One could argue that even a small increase of the ratio of second period utility to first period utility could be compatible with an increase of the income share spent on health. Indeed, as longevity is bounded, the marginal impact of health expenditures on longevity rapidly falls, which means that the function g can be very steep. Hence a small increase of the ratio of second period utility to first period utility can increase health expenditures by much. However, in our case, we show that this never suffices for health expenditures to increase faster than income for large income levels. The ratio of second period utility to first period utility behaves as $\frac{1}{w}$ for large income levels, and we show that $\frac{g(w)}{w}$ always tends to 0 with w.

To which extent do the results depend on the utility per period specification? For low income levels, the convexity of the marginal utility and the concavity of the utility function apply for more general utility specifications. This is the case for example with the specification of Hall and Jones (2007): $u(c) = \frac{c^{1-\sigma}}{1-\sigma} + b$, $\sigma > 1, b > 0$. Hence, the result that health is a luxury good for low income levels should extend to more general utility specifications. For high income levels, our result relies on the asymptotic behavior of the ratio of utility to marginal utility. In our specification, this ratio behaves as $\frac{1}{w}$, and the income share spent on health behaves as $\frac{g(w)}{w}$ which always tends to 0 with w. In the specification of Hall and Jones (2007), the ratio behaves as $\frac{1}{w^{\sigma}}$, and the income share spent on health behaves as $\frac{g(w^{\sigma})}{w}$, which leaves the possibility for the income share spent on health to perpetually increase with income even though longevity is bounded.[10]

To complete the discussion, we compare our result with respect to the findings of Hall and Jones (2007). In their framework, individuals live for one period, whose length depends on the level of health expenditures. They choose their consumption level as well as their health spending. Hall and Jones show that the income share spent on health can be written as an increasing function of the ratio of consumption elasticity to health elasticity. Hence the income share spent on health is driven by the ratio of these two elasticities. For large income levels w, our discussion is indeed equivalent to a discussion on the ratio of the two elasticities. With $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, $\sigma < 1$, the ratio of the two elasticities behaves as $wp'(w)$, while with $u(c) = \frac{c^{1-\sigma}}{1-\sigma} + b$, $\sigma > 1, b > 0$, this ratio behaves as $w^{\sigma}p'(w)$. However, in our two-period framework, the ratio of the two elasticities does not completely governe the income share spent on health for all income levels. In Figures 2 and 3, we plot, for the survival function $p(e) = \frac{e^{0.5}}{1+e^{0.5}}$, the ratio of the two elasticities, which is decreasing in this case, and $w \to x(w)$ which is inverted U-shaped according to Proposition 4. Hence, the ratio of the two elasticities is an imperfect picture of the income share spent on health.

For the next section, we will maintain Assumption 1. Despite the existence of income levels for which health is a luxury good, there is no perpetual growth of the share of resources spent on health to 1. Otherwise, we could have concluded from the partial equilibrium analysis that the economy would not grow perpetually. Thus, in the next section,

[10]The result is stated formally in section 4.

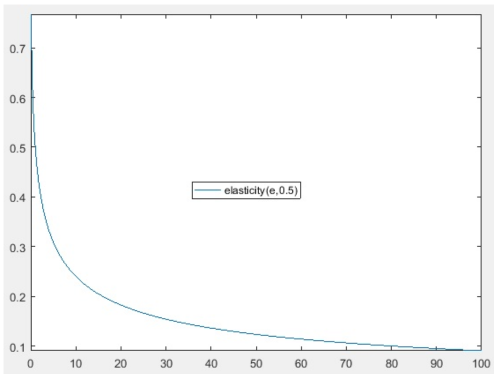

Figure 2: Health elasticity to consumption elasticity ratio (i.e. $e \to \frac{p'(e)e}{p(e)}(1-\sigma)$). $p(e) = \frac{e^{0.5}}{1+e^{0.5}}$, $\sigma = 0.5$.

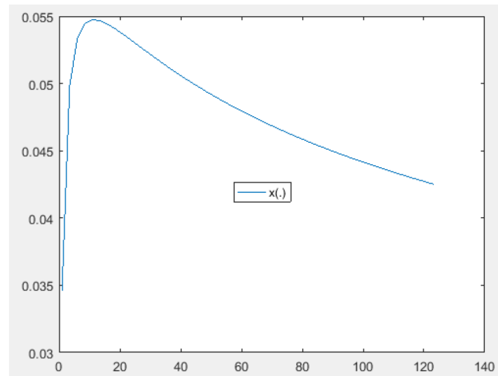

Figure 3: Income share spent on health as a function of income (i.e. $w \to x(w)$). $p(e) = \frac{e^{0.5}}{1+e^{0.5}}$, $\sigma = 0.5$, $R = 4.801$.

we study the dynamic general equilibrium to answer this question. In section 4, we rediscuss the alternative specification $u(c) = \frac{c^{1-\sigma}}{1-\sigma} + b$, $\sigma > 1, b > 0$.

3 The dynamic general equilibrium

In this section, we study the dynamics of the economy of section 2, which is obtained by imposing the capital market clearing condition:

$$K_{t+1} = s_t \tag{16}$$

Using (10) and (5), (16) is equivalent to:

$$w_{t+1} = \frac{p(e(w_t))}{p(e(w_t)) + (\alpha A)^{\frac{\sigma-1}{\sigma}}}(w_t - e(w_t))A(1-\alpha) \tag{17}$$

It is useful to study equation (17) separetely according to the value of $\underline{p}$.

3.1 Dynamics in the case $\underline{p} = 0$.

In this case, health expenditures are necessary for the economy not to collapse to a null income. Otherwise, without health expenditures, individuals do not live in second period, which implies that they do not save. As capital is an essential input, there is no production. There remains to determine how the economy behaves when individuals spend a positive amount of resources on health. The result is in the following proposition:

Proposition 6 *(i) If* $\frac{\overline{p}}{\overline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}A(1-\alpha) \leq 1$*, then for all* $w_0 \geq 0$*, the economy converges to a null income.*

(ii) If $\frac{\overline{p}}{\overline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}A(1-\alpha) > 1$*, then (17) has a unique positive steady state,* $w^* > \frac{1-\sigma}{\sigma}\frac{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}$*, which is unstable. Hence, if* $w_0 > w^*$*, then the economy perpetually grows, while if* $w_0 < w^*$*, then the economy converges to a null income.*

(iii) If the economy perpetually grows, hence if $\frac{\overline{p}}{\overline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}A(1-\alpha) > 1$ *and* $w_0 > w^*$*, then the growth rate monotonically increases along the trajectory.*

Proof. See Appendix C ■

Proposition 6 shows that the possibility to spend resources on health can have large benefits in terms of economic growth as it can allow the economy to perpetually grow instead of being trapped to a null income. For this to happen, the maximal longevity and the initial income must be high enough. These conditions yield a large enough initial longevity for the individuals to save a level of resources that allows the economy to grow. In this case, according to point (iii), health expenditures create a virtuous cycle: as income increases, health expenditures and then longevity increase, and the propensity to save increases more than the possible disposable income reduction due to greater health expenditures. According to Proposition 6, this happens when the maximal longevity $\overline{p}$ is large. This implies that health expenditures can increase longevity by a large amount and so that the benefits of health expenditures are large and larger than their costs. Thus, savings and the growth rate increase with income.

3.2 Dynamics in the case $\underline{p} > 0$.

In this case, health expenditures are no more necessary for this economy to grow. Consider the same economy as the one outlined in section 2 without the possibility for the individuals to spend resources on health. It is a standard AK economy, whose dynamics is governed by the following equation:

$$w_{t+1} = \frac{\underline{p}}{\underline{p} + (\alpha A)^{\frac{\sigma-1}{\sigma}}}A(1-\alpha)w_t \tag{18}$$

We easily get that the solution to (18) perpetually grows if and only if $\frac{\underline{p}}{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}A(1-\alpha) > 1$. We will write this condition as $A > \underline{A}$ where $\underline{A}$ satisfies $\frac{\underline{p}}{\underline{p}+(\alpha \underline{A})^{\frac{\sigma-1}{\sigma}}}\underline{A}(1-\alpha) = 1$. In the first part of this section, we will maintain this condition and we will determine if it is sufficient for the economy governed by (17) to perpetually grow. Alternatively, we can formulate the problem as follows. The economy perpetually grows if and only if $G(e(w_t), w_t) := \frac{p(e(w_t))}{p(e(w_t))+(\alpha A)^{\frac{\sigma-1}{\sigma}}}(w_t - e(w_t))A(1-\alpha) > w_t$ for all income levels. Under the condition $A > \underline{A}$, $e \to G(e, w_t)$ is decreasing or inverted U-shaped and the equation $G(e, w_t) = w_t$ has a unique root $\widehat{e}(w_t)$ such that if for some income levels $e(w_t) > \widehat{e}(w_t)$, then the economy does not perpetually grow. Otherwise said, if health expenditures are too large, then the disposable income reduction is greater than the increase of the propensity to save and savings become too low for economic growth to occur. The following proposition, which is the main result of the paper, gives necessary and sufficient conditions for this scenario to happen:

Proposition 7 *Note* $a := \lim_{e \to 0}(\frac{(-p''(e))e}{p'(e)}) < 1$. *The three following statements are equivalent:*

(i) $A > \underline{A}$ *and the solution to (17) does not perpetually grow.*

(ii) (17) has two steady states w_1^* *and* w_2^*, *with* $\frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma} < w_1^* < w_2^*$. w_1^* *is stable, while* w_2^* *is unstable.*

(iii) $\alpha > \frac{(\underline{p}\frac{\sigma}{1-\sigma}(1-a))^{\frac{\sigma}{\sigma-1}}}{\frac{1-a\sigma}{1-\sigma}+(\underline{p}\frac{\sigma}{1-\sigma}(1-a))^{\frac{\sigma}{\sigma-1}}}$ *and* $A \in (\underline{A}, A^*)$, *where* $A^* > \underline{A}$ *and* $w_0 \leq w_2^*$.

Proof. See Appendix D ■

Proposition 7 states that the mild condition that insures perpetual growth in the standard AK model ($A > \underline{A}$) is not sufficient to insure perpetual growth when individuals can spend resources to increase their longevity. This means that implementing a health system can have important repercussions on the long-term development of an economy. Assume that γ is finite and consider an economy that does not initially spend resources on health (hence $w_0 < \frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}$). Its trajectory is initially governed by the evolution equation of the AK model (18). Hence, under the condition $A > \underline{A}$, the economy grows at a positive rate. When income exceeds the threshold $\frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}$, individuals start spending resources on health. Under the conditions (iii), the economy still grows at a positive rate, yet the growth rate declines until income is trapped to the middle-income level w_1. Otherwise said, health expenditures create a strong negative intergenerational externality in this case, as they impede any possibility of growth for future generations.

According to conditions (iii), this scenario occurs in economies with a not too large technology level A and a high initial longevity $\underline{p}$. A high initial longevity implies that health expenditures cannot increase longevity by much since it is already large. This implies that the benefits of health expenditures, a greater propensity to save, are not important in this case. A influences the growth rate through three channels. First, the greater A, the greater the interest rate, which increases both savings and health expenditures because the inverse of the intertemporal elasticity of substitution (IES) is strictly smaller than 1. Second, an increase of A also acts as an increase of the disposable income (see the linear term in A in $G(e(w), w)$). The increase of savings and of the disposable income both increase the growth rate. The increase of health expenditures has the two opposite consequences on economic growth previously mentioned: it increases the propensity to save and it decreases the disposable income. In appendix D, we prove that the total effect of an increase of A on the growth rate is always positive. Hence a greater A can compensate negative growth effects of health expenditures. This means that once A is large enough (strictly greater than A^*), the economy perpetually grows.

To gain intuition, we reconsider the alternative formulation of the problem: are there income levels for which $G(e(w_t), w_t) < w_t$? We use (13) to rewrite this inequality as $\frac{1-\sigma}{\sigma}A(1-\alpha) < \frac{p'(e(w_t))w_t}{p(e(w_t))}$ and we note that it is sufficient to have $\frac{1-\sigma}{\sigma}A(1-\alpha) < \frac{p'(w_t)w_t}{p(w_t)}$ for this inequality to be true. Hence when the elasticity of the survival function is high (greater than $\frac{1-\sigma}{\sigma}A(1-\alpha)$), individuals choose to devote a large share of their resources to their health because health expenditures have a strong positive impact on their longevity and so on their welfare.

We now study the dynamics of the economy when it perpetually grows. Does this economy grow faster than the economy that does not spend resources on health? The following proposition gives necessary and sufficient conditions for the economy to grow perpetually under the condition $A > \underline{A}$ and examines if the growth rate is greater than the AK-growth rate:

Proposition 8 *Note $g(w_t)$ the growth rate of the economy governed by (17) and $g_{AK} = \frac{\underline{p}}{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}A(1-\alpha) - 1$. Note also $\widehat{A} = \frac{(\underline{p}(\frac{1-a\sigma}{1-\sigma}-1))^{\frac{\sigma}{\sigma-1}}}{\alpha}$. Assume $A > \underline{A}$.*

(i) The economy perpetually grows if and only if $\alpha < \frac{(\underline{p}\frac{\sigma}{1-\sigma}(1-a))^{\frac{\sigma}{\sigma-1}}}{\frac{1-a\sigma}{1-\sigma}+(\underline{p}\frac{\sigma}{1-\sigma}(1-a))^{\frac{\sigma}{\sigma-1}}}$ or $A > A^*$.

(ii) If $\alpha < \frac{(\underline{p}\frac{\sigma}{1-\sigma}(1-a))^{\frac{\sigma}{\sigma-1}}}{\frac{1-a\sigma}{1-\sigma}+(\underline{p}\frac{\sigma}{1-\sigma}(1-a))^{\frac{\sigma}{\sigma-1}}}$ and if $A \in [\underline{A}, \widehat{A}]$, then:

$$g(w_t)\begin{cases} = g_{AK} \text{ while } w_t \leq \frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma} \\ > g_{AK} \text{ and } \frac{dg(w_t)}{dt} > 0 \text{ for } w_t > \frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma} \end{cases}$$

(iii) If $\alpha < \frac{(\underline{p}\frac{\sigma}{1-\sigma}(1-a))^{\frac{\sigma}{\sigma-1}}}{\frac{1-a\sigma}{1-\sigma}+(\underline{p}\frac{\sigma}{1-\sigma}(1-a))^{\frac{\sigma}{\sigma-1}}}$ *and if* $A > \widehat{A}$ *or if* $A > A^*$ *and* $\alpha > \frac{(\underline{p}\frac{\sigma}{1-\sigma}(1-a))^{\frac{\sigma}{\sigma-1}}}{\frac{1-a\sigma}{1-\sigma}+(\underline{p}\frac{\sigma}{1-\sigma}(1-a))^{\frac{\sigma}{\sigma-1}}}$, *then there exists* $\widetilde{w} > \frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}$ *such that:*

$$g(w_t)\begin{cases} = g_{AK} \text{ while } w_t \leq \frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma} \\ < g_{AK} \text{ for } w_t \in (\frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}, \widetilde{w}) \\ > g_{AK} \text{ and } \frac{dg(w_t)}{dt} > 0 \text{ for } w_t > \widetilde{w} \end{cases}$$

Proof. See Appendix E ■

Proposition 8 states that there are two types of trajectories for an economy that perpetually grows. Along the first one, health expenditures create a virtuous cycle: as income grows, health expenditures increase and they have a positive impact on the growth rate. Then, at each period, the economy grows faster than the AK-economy. Along the second one, as individuals start spending resources on health, the growth rate is reduced compared to the one of the AK-economy. Yet, after an income threshold is reached, the growth rate increases and eventually exceeds the one of the AK-economy.

The reason why the economy finally grows faster than the AK-economy in all cases is that the income share spent on health ends decreasing and tends to 0 for large income levels. Thus, the negative effects of health expenditures, a reduced disposable income, vanish, while the positive effects of health expenditures, a higher propensity to save, increase because health expenditures keep rising.

There are two combinations of parameters under which the economy initially grows more slowly than the AK-economy. The first one requires that the initial longevity is high enough and the technology level is high enough to avoid the middle income trap. As previously explained, a high initial longevity implies that health expenditures cannot increase by much the longevity. Then, the benefits of health expenditures are low and are smaller than their costs. The second combination of parameters shows that a high initial longevity is sufficient but not necessary for the economy to initially grow more slowly than the AK-economy. Indeed, this happens also if the initial longevity is low and the technology level is large enough. As previously said, the growth rate of the economy increases with the technology level, however the growth rate of the AK-economy increases too. Recall that there are four channels through which A impacts the growth rate (interest rate, linear term, longevity, disposable income). The first two are common to both economies, while the last two are absent in the AK-economy. Hence the result is driven by an inverted U-shaped relationship between health expenditures and the growth rate. The increase of A initially increases more the propensity to save (which is concave in health expenditures) than it reduces the disposable income (which is linear in health expenditures). This implies that the increase of A initially increases more the growth rate of the economy than the one of the AK economy. Then the contrary happens.

Finally, we determine the dynamics of the economy under the condition $A < \underline{A}$. Can the economy perpetually grow under this condition? The following proposition provides the answer:

Proposition 9 *Assume $A < \underline{A}$.*

(i) If $A(1-\alpha)\frac{\overline{p}}{\overline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}} \leq 1$, then the economy converges to a null income.

(ii) If $A(1-\alpha)\frac{\overline{p}}{\overline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}} > 1$, then the dynamical system (17) has a unique unstable steady state, w^*. If $w_0 < w^*$, then the economy converges to a null income. If $w_0 > w^*$, then the economy perpetually grows and its growth rate increases along the trajectory.

Proof. See Appendix F ■

Proposition 9 shows that health expenditures can be necessary for economic growth to take place. Indeed, when the condition that insures perpetual growth in the AK model fails, it is possible for the economy to perpetually grow. For this to happen, the survival function must not take too small values, otherwise the propensity to save is too low for the economy to grow. Moreover, the initial income level must be high enough. When γ is finite, this poverty trap is due to the fact that individuals initially choose not to spend on health, which means that the economy behaves exactly as the AK economy, which cannot grow by assumption. Overall, my characterization of the dynamics of the economy shows that the introduction of health expenditures in the AK model yields completely different trajectories depending on the values of the parameters. Indeed, as suggested by Proposition 9, health expenditures can be necessary to perpetual growth, while as suggested by Proposition 7, health expenditures can annihilate the perpetual growth of the AK economy. This motivates the following section in which we provide a simple calibration of the model.

4 Discussion and numerical illustration

4.1 Alternative preferences

In the previous sections, utility per period is such that for any survival function, the income share spent on health falls to 0 for large income levels. As previously argued, this is a consequence of assuming an inverse of the IES strictly smaller than 1. There are both types of empirical evidence suggesting values above or below 1 for the IES. Thus, in this section, we briefly discuss our results when utility per period is $u(c) = \frac{c^{1-\sigma}}{1-\sigma} + b$, $\sigma > 1, b > 0$. In this case, the positive intercept b is required to avoid that agents unrealistically choose not to spend resources on health for any income level.

With the same notations as before, the FOC on health expenditures now writes:

$$c_{t,t}^{-\sigma} + p'(e_t)c_{t,t+1}^{1-\sigma} = p'(e_t)(\frac{c_{t,t+1}^{1-\sigma}}{1-\sigma} + b) \tag{19}$$

The LHS of (19) is the marginal cost of health expenditures: first period utility loss plus second period utility loss due to diminishing per period resources. The RHS of (19) is the marginal benefit of health expenditures, which is the total second period utility gain due to longevity extension. Note first that the marginal benefit of health expenditures is negative for low income levels. Hence, the marginal utility of longevity is negative for poor individuals, which implies that they do not spend resources on health. This means that contrary to the case $\sigma < 1$, the corner solution for health expenditures is unrelated to the finiteness of the initial marginal productivity of health expenditures γ. Rewrite (19) as:

$$c_{t,t}^{-\sigma} = p'(e_t)(\frac{\sigma c_{t,t+1}^{1-\sigma}}{1-\sigma} + b)$$

For low income levels, as income increases, the first period marginal utility decreases and the second period utility increases by a large amount due to the convexity of the marginal utility and the concavity of the utility per period function. This requires health expenditures to increase by a large amount, more than income, for the marginal cost and the marginal benefit to equate. For large income levels, the second period utility is equivalent to the intercept b, which does not depend on income. The first period marginal utility decreases by a small amount. Then, the adjustment of health expenditures to equal the marginal cost and the marginal benefit depends on how the maginal productivity of health expenditures behaves asymptotically. The result is in the following proposition:

Proposition 10 *The problem of the consumer has a unique solution. There exists $\underline{w} > 0$ such that $e(w) = 0$ for $w \leq \underline{w}$, while $w \to e(w)$ is increasing on $[\underline{w}, \infty)$.*

(i) $w \to x(w)$ is initially increasing on $[\underline{w}, \infty)$.

(ii) If $\lim_{w \to \infty} p'(w)w^{\sigma} = 0$, then $\lim_{w \to \infty} x(w) = 0$.

(iii) If $\lim\limits_{w\to\infty} p'(w)w^{\sigma} = l < \infty$, *then* $\lim\limits_{w\to\infty} x(w) = \frac{\frac{(bl)^{\frac{1}{\sigma}}}{(1+r)^{\frac{1-\sigma}{\sigma}}\overline{p}+1}}{1+\frac{(bl)^{\frac{1}{\sigma}}}{(1+r)^{\frac{1-\sigma}{\sigma}}\overline{p}+1}}$.

(iv) If $\lim\limits_{w\to\infty} p'(w)w^{\sigma} = \infty$, *then* $\lim\limits_{w\to\infty} x(w) = 1$.

Proof. See Appendix G ■

Point (i) of Proposition 10 confirms our intuition that health is initially a luxury good. The proposition also shows that the parametric specification of the survival function is crucial to determine the shape of the function $w \to x(w)$, which can perpetually increase with income towards 1 or falls to 0. This justifies to have worked with a general survival function. For example, with $p(e) = \frac{\underline{p}+\overline{p}e^{\epsilon}}{1+e^{\epsilon}}$, $\epsilon \in (0,1]$, case (ii) occurs if $\sigma < 1+\epsilon$, case (iii) occurs if $\sigma = 1+\epsilon$, while case (iv) occurs if $\sigma > 1+\epsilon$. With $p(e) = \frac{2}{\sqrt{\Pi}}\overline{p}\int_0^e e^{\frac{-u^2}{2}}du + \underline{p}$, only case (ii) occurs.

As an immediate corollary, we see that in case (iv) the economy does not perpetually grow under the mild condition that insures perpetual growth in the AK model. With this utility specification, we cannot provide a complete characterization of the dynamics of the economy as in section 3, so we calibrate the model to study the dynamical general equilibrium.

4.2 Numerical application

In this subsection, we propose a numerical calibration of the model for OECD countries, when $u(c) = \frac{c^{1-\sigma}}{1-\sigma} + b$, $\sigma > 1, b > 0$, to answer two questions: Will the income share spent on health continue to increase? How do health expenditures modify the trajectory of these economies?

We consider that a period is equal to 40 years and that individuals enter first period at the age of 25. r and α and σ are first set to standard values. The annual interest is 4%, which yields a value of 4.801 for R. α is set to 0.3. σ is set to 1.5, which yields a risk aversion coefficient in line with empirical estimates (Chetty 2006). The remaining unknowns are b and the survival function.

Usually in such models, the unknown parameters are calibrated from the equation that governes the dynamics of the economy (equation (17)) to match time series data. However, the two-period framework requires long time series to have a sufficient number of points to match. Here we rather notice that all the remaining unknowns enter the equation that determines the income share spent on health $x(w)$.[11] This equation links health expenditures to income, thus it can be estimated from cross-sectional data. We then choose our unknowns by minimizing the distance

[11]When $u(c) = \frac{c^{1-\sigma}}{1-\sigma} + b$, $\sigma > 1, b > 0$, this is the equation (33) in Appendix G.

between the income share spent on health generated by the model and the true ones. Regarding first the survival function, we have tried all the examples cited in the paper, yet the best fit of the model is obtained with the following specification:

$$p(e) = \frac{\underline{p} + \overline{p}\ln(1+\frac{e}{C})}{1+\ln(1+\frac{e}{C})} \tag{20}$$

Where C is a positive constant to be estimated. A bounded and concave function becomes rapidly flat. The parameter C allows not to have our health expenditures levels on the flat part of the curve, which could not yield a good fit of the cross-sectional variation of longevity. However, the higher the scaling parameter C, the lower the dispersion of the $\frac{e_i}{C}$, which also impedes a good fit of the cross-sectional variation of longevity. Thus, we need a survival function that does not become flat too fast in order not to have to use a too large scaling parameter. This explains why we obtain our best fit with the function (20). Thus, there are four parameters to be estimated: $(b, C, \underline{p}, \overline{p})$. Note $(w_i, x_i, p_i)_{i=1..33}$ our data, where w_i is wage in country i, x_i the income share of health expenditures and p_i the longevity.[12] Then, our parameters choice $(b^*, C^*, \underline{p}^*, \overline{p}^*)$ is:

$$\arg\min_{(b,C,\underline{p},\overline{p})}\left(\sum_{i=1}^{33}(x(w_i)-x_i)^2 + (p_m - p(w_m x(w_m))^2 + (p_M - p(w_M x(w_M))^2\right) \tag{21}$$

Where $m = \arg\min_{i=1..33}(p_i)$ and $M = \arg\max_{i=1..33}(p_i)$. We obtain $(b^*, C^*, \underline{p}^*, \overline{p}^*) =$

$(0.0206, 39991.876, 0.02, 0.65)$. Figure 5 plots $(w_i, x_i)_{i=1..33}$ and $(w_i, x(w_i))_{i=1..33}$. Figure 4 plots $(w_i x_i, p_i)_{i=1..33}$ and $(w_i x(w_i), p(w_i x(w_i)))_{i=1..33}$. The two graphs suggest that the model can replicate reasonably well the relationship between income and income share spent on health and health expenditures and longevity. There is one country, US, which is not well captured by the model: its income share on health is much larger than other countries, while its life expectancy is only in the middle of the distribution.

My calibration parameters have two direct consequences. First, without health expenditures, the economy does not grow, hence the condition

[12]The data are obtained from the OECD database for the year 2012. The sample includes all OECD members except Luxembourg (33 countries). We use GDP per capita in current US dollars to compute the corresponding GDP per worker and then the wage. Total health expenditures are also in current US dollars. For the estimation, we multiply these quantities by 40 to get their value on the model period. When reporting the results, we use annual values. p_i is computed from life expectancy at birth, E_i: $E_i = 65 + 40p_i$.

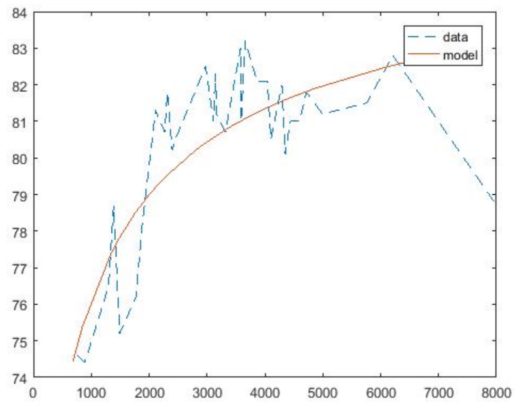

Figure 4: Health expenditures-Life expectancy

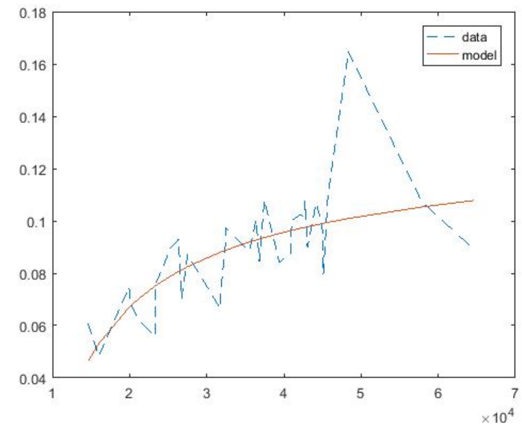

Figure 5: Income-Income share spent on health

that insures perpetual growth in the AK model fails. This means that there is a poverty trap. Second, the economy is in the case (iv) of Proposition 10, hence the income share spent on health tends to 1 for large income levels. This means that there is a positive stable steady state. We first compute the values of these steady states to assess the position of our sample with respect to these points. We find that the first steady-state income level is worth 12149. Two countries of the sample are below this level, which means that they are trapped. This suggests that health expenditures could be a barrier to convergence in income levels across countries. The second steady-state income level is greater than 1.10^9, which is well above the income levels of the sample. This means that the growing income share spent on health should not be an obstacle to economic growth in a not too far future for the countries which are not trapped. We now simulate the trajectory over 10 periods for the economy with the median income. Figure 6 reports the dynamics of the wage, while Figure 7 reports the dynamics of the income share spent on health. Over this period, the economy grows at an accelerating rate and its income share spent on health is increasing. After 10 periods, the income share spent on health reaches 0.33, which is more than the triple of its initial value. This does not prevent the economy from growing. This means that economic growth can take place despite a large amount of resources spent for health. The overall conclusion of this numerical analysis is that health expenditures are more necessary than detrimental to economic growth and that health expenditures endanger economic growth only if their level becomes much larger than their current level in OECD countries.

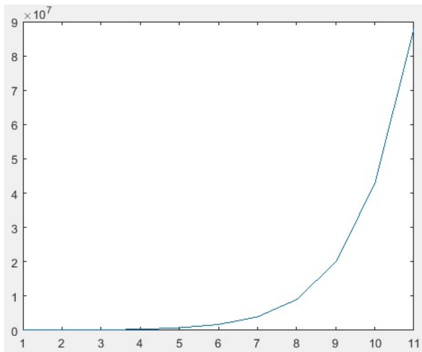

Figure 6: Wage dynamics over 10 periods

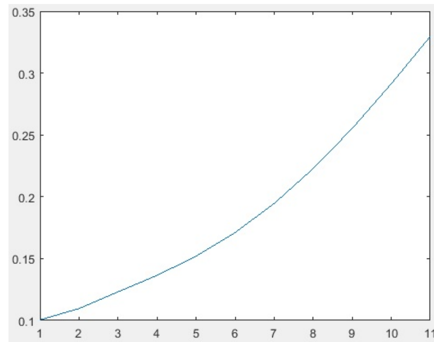

Figure 7: Income share spent on health dynamics over 10 periods

5 Conclusion

In a two-period OLG model with endogenous growth, we studied the consequences of allowing individuals to choose the level of health expenditures that increase their longevity in retirement period. We presented several results. With a CES utility function, with an IES strictly greater than 1, and a general survival function, we proved that the income share spent on health is an inverted U-shaped function of income. This implies that an increasing ratio of health elasticity to consumption elasticity is neither necessary nor sufficient for health to be a luxury good. Then, we gave a complete characterization of the dynamics of the economy. Under the condition that insures perpetual growth in the same AK economy except that health expenditures are constrained to be null, there are three types of trajectories. Along the first one, the economy perpetually grows and grows at each period faster than the AK economy. Along the second one, the economy perpetually grows, however its growth rate is initially reduced compared to the one of the AK economy, before growing faster than the AK economy. Finally, along the third trajectory, the economy is trapped to a middle income level and does not experience perpetual growth. This means that health expenditures create a strong negative intergenerational externality in this case by impeding any possibility of growth for future generations. We also found that when the condition that insures perpetual growth in the AK model fails, the economy can all the same experience perpetual growth. A simple calibration of the model to OECD countries suggests that this case might be the most likely one, hence that health expenditures are more necessary than detrimental to growth.

Acemoglu and Johnson (2007) conclude their study by noting that the decision to implement a health system in a country can be considered as orthogonal to its development policy given the weak impacts of longevity on economic growth they find. In this paper, we reach a different conclusion as we have shown theoretically that the implementation of a health system is not neutral for economic growth. Indeed, in the framework used, a standard Diamond model with health expenditures, there are economies in which the presence of these health expenditures produce drastic negative consequences by impeding any possibility of long-run economic growth. This should stimulate future empirical research on the health-growth nexus. Given its theoretical focus, the present analysis has omitted possibly important channels through which health expenditures can modify the growth path of an economy such as the impact of health on productivity or the introduction of a social security system. They could be included in future simulation studies.

References

[1] Acemoglu, D., Finkelstein, A. and Notowidigdo,M.J. (2013) Income and Health Spending: Evidence from Oil Price Shocks. The Review of Economics and Statistics, 95(4): 1079-1095.

[2] Acemoglu, D. and Johnson, S (2007) Disease and development: the effect of life expectancy on economic growth. Journal of Political Economy, 115(6): 925-985.

[3] Azomahou, T., Diene, B., Diene, M. and Soete, L. (2015) Optimal health investment and preference structure. Economic Theory 60(3): 521-565.

[4] Bhattacharya, J. and Qiao, X. (2007). Public and Private Expenditures on Health in a Growth Model. Journal of Economic Dynamics and Control, 31(8): 2519-2535.

[5] Blackburn, K and Cipriani, GP, (2002) A model of longevity, fertility and growth. Journal of Economic Dynamics and Control, 26(2): 187-204.

[6] Bloom, D.E., Canning, D. and Graham, B. (2003) Longevity and Life-cycle Savings. Scandinavian Journal of Economics, 105(3): 319-338.

[7] Boucekkine, R and de la Croix, D and Licandro, O (2002) Vintage human capital, demographic trends and endogenous growth, Journal of Economic Theory, 104(2): 340-375.

[8] Boucekkine, R. and Laffargue, J-P. (2010) On the distributional consequences of epidemics. Journal of Economic Dynamics and Control, 34(2): 231-245.

[9] Brembilla, L. (2016) Endogenous lifetime and economic growth: the role of the tax rate. Economic Theory Bulletin, 4(2): 247-263..

[10] Cervellati, M. and Sunde, U. (2005) Human Capital Formation, Life

Expectancy, and the Process of Development. American Economic Review, 95(5): 1653-1672.

[11] Cervellati, M. and Sunde, U. (2013) Life Expectancy, Schooling, and Lifetime Labor Supply: Theory and Evidence Revisited. Econometrica, 81(5): 2055-2086.

[12] Chakraborty, S (2004) Endogenous lifetime and economic growth. Journal of Economic Theory, 116(1):119-137.

[13] Chakraborty, S. and Das, M. (2005) Mortality, Human Capital and Persistent Inequality. Journal of Economic Growth, 10(2): 159-192.

[14] Chernew, N.E. and Newhouse, J.N. (2011) Health Care Spending growth. Handbook of health economics, Volume 2: 1-43.

[15] Chetty, R. (2006) A New Method of Estimating Risk Aversion. American Economic Review, 96(5): 1821-1834.

[16] Dalgaard, C-J and Strulik, H. (2014) Optimal Aging And Death: Understanding The Preston Curve. Journal of the European Economic Association 12(3): 672-701.

[17] de la Croix, D and Licandro, O (1999) Life expectancy and endogenous growth. Economic Letters, 65(2): 255-263.

[18] Ehrlich, I. and Chuma, H. (1990) A Model of the Demand for Longevity and the Value of Life Extension. Journal of Political Economy, 98(4):761-782.

[19] Grossman, M. (1972) On the Concept of Health Capital and the Demand for Health. Journal of Political Economy, 80(2): 223255.

[20] Hall, R.E. and Jones, C.I. (2007) The value of life and the rise in health spending. Quarterly Journal of Economics, 122 (1): 39-72.

[21] Kuhn, M. and Prettner, K. (2016) Growth and welfare effects of health care in knowledge-based economies. Journal of Health Economics, 46(C): 100-119.

[22] Mariani, F., Pérez-Barahona, A. and Raffin, N. (2010) Life expectancy and the environment. Journal of Economic Dynamics and Control, 34(4):798-815.

[23] Ponthiere, G. (2011) Existence and stability of overconsumption equilibria. Economic Modelling, 28 (1-2): 74-90.

[24] Prettner, K (2013) Population aging and endogenous economic growth. Journal of Population Economics, 26(2): 811-834.

[25] Raffin, N. and Seegmuller, T. (2014) Longevity, pollution and growth. Mathematical Social Sciences: 69(C): 22-33.

[26] Strittmater, A. and Sunde, U. (2013) Health and economic development-evidence from the introduction of public health care. Journal of Population Economics, 26(4): 1549-1584.

[27] Yaari, M. (1965) Uncertain lifetime, life insurance and the theory of the consumer. Review of Economic Studies, 32(2): 137-150

6 Appendix A

Assume first that $\underline{p} > 0$ and $\gamma < \infty$. The problem of the consumer is equivalent to maximize:

$$V(s_t, e_t) = \frac{(w_t - s_t - e_t)^{1-\sigma}}{1-\sigma} + p(e_t)^\sigma \frac{s_t^{1-\sigma}(1+r)^{1-\sigma}}{1-\sigma}$$

Subject to the constraints $s_t + e_t \leq w_t$ and $0 \leq e_t$ and $0 \leq s_t$. Note that these constraints are linear, which implies that any solution must satisfy the KKT conditions. The Lagrangian associated to this problem writes:

$$L(s_t, e_t, \chi_1, \chi_2, \chi_3) = V(s_t, e_t) + \chi_1(w_t - s_t - e_t) + \chi_2 e_t + \chi_3 s_t$$

Where χ_1 to χ_3 are the Lagrange multipliers. Note first that $\frac{\partial V}{\partial s}(0, e_t) = \infty$ for all $e_t \in [0, w_t)$. Thus, $s_t = 0$ is never optimal and $\chi_3 = 0$. Third, $\frac{\partial V}{\partial s}(w_t - e_t, e_t) = -\infty$ for all $e_t \in [0, w_t)$. Thus, χ_1 is equal to 0 and the KKT conditions for a point (s_t, e_t) to be an optimum can be written as:

$$(i)\ \frac{\partial V}{\partial s}(s_t, e_t) = -(w_t - e_t - s_t)^{-\sigma} + p(e_t)^\sigma s_t^{-\sigma}(1+r)^{1-\sigma} = 0$$

$$(ii)\ \frac{\partial V}{\partial e}(s_t, e_t) = -(w_t - e_t - s_t)^{-\sigma} + \sigma p'(e_t) p(e_t)^{\sigma-1} \frac{s_t^{1-\sigma}(1+r)^{1-\sigma}}{1-\sigma} = -\chi_2$$

$$(iii)\ \min(\chi_2, e_t) = 0$$

Consider now the possibility that $e_t = 0$. From (i), we get the optimal saving, $s_t = \frac{\underline{p}}{\underline{p} + (1+r)^{\frac{\sigma-1}{\sigma}}} w_t$. And χ_2 must be non-negative. Thus, (ii) writes $\frac{\partial V}{\partial e}(\frac{\underline{p}}{\underline{p} + (1+r)^{\frac{\sigma-1}{\sigma}}} w_t, 0) \leq 0$ which is equivalent to $w_t \leq \frac{1-\sigma}{\sigma} \frac{\underline{p} + (1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}$. Thus, for $w_t \leq \frac{1-\sigma}{\sigma} \frac{\underline{p} + (1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}$, $(\frac{\underline{p}}{\underline{p} + (1+r)^{\frac{\sigma-1}{\sigma}}} w_t, 0)$ satisfies the KKT conditions and is a possible solution. Consider now the case $e_t > 0$. Then, $\chi_2 = 0$ and the conditions (i) and (ii) imply the equations (10) and (13) of the text. Note that the left-hand-side (LHS) of (13) decreases and is worth 0 at $e_t = w_t$. Thus, (13) has a unique positive solution if and only if the LHS of (13) takes a value strictly greater than 1 at $e_t = 0$. This condition is equivalent to $w_t > \frac{1-\sigma}{\sigma} \frac{\underline{p} + (1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}$. Thus, for

$w_t > \frac{1-\sigma}{\sigma}\frac{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}$, there exists a unique pair (s_t, e_t) that satisfies the KKT conditions. Note finally that the problem of the consumer has always at least one solution because $(s,e) \to V(s,e)$ is continuous on the maximization domain, which is compact. Consequently, the unique pair satisfying the KKT conditions in the two cases $w_t \leq \frac{1-\sigma}{\sigma}\frac{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}$ and $w_t > \frac{1-\sigma}{\sigma}\frac{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}$ is the unique solution to the problem of the consumer. In the first case, the optimal level of health expenditures is equal to 0, while it is positive in the second case. The case $\underline{p} = 0$ follows by continuity. When $\gamma = \infty$, notice that the KKT conditions for a corner solution to exist cannot be satisfied. This completes the proof of Proposition 1.

7 Appendix B

Set $\underline{w} = \frac{1-\sigma}{\sigma}\frac{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}{\gamma}$, which is possibly equal to 0 when $\gamma = \infty$. We first need to compute $\lim\limits_{w\to\underline{w}} x(w)$. If $\gamma < \infty$, then $e(\underline{w}) = x(\underline{w})\underline{w} = 0$, then $\lim\limits_{w\to\underline{w}} x(w) = 0$. If $\gamma = \infty$, then $\underline{w} = 0$. We will need the following lemma:

Lemma 11 $\lim\limits_{w\to 0}(wp'(w)) = 0$

Proof. The first derivative of $w \to wp'(w)$ is $p'(w)(1 - \frac{(-p''(w))w}{p'(w)})$. The properties of the function p imply that $a := \lim\limits_{w\to 0}(\frac{(-p''(w))w}{p'(w)}) < 1$. This implies that $w \to wp'(w)$ increases in the neighborhood of 0. Consequently $\lim\limits_{w\to 0}(wp'(w))$ exists. It must be finite because $w \to wp'(w)$ is initially increasing. Assume $\lim\limits_{w\to 0}(wp'(w)) = l > 0$. Then there must exists $W > 0$ such that:

$$u < W \Rightarrow up'(u) > \frac{l}{2} \tag{22}$$

Consider $w < W$. Divide by u both terms of the previous inequality and integrate it from w to W to get:

$$\frac{l}{2}\ln(\frac{W}{w}) < p(W) - p(w)$$

$$\Longleftrightarrow p(w) + \frac{l}{2}\ln(W) < p(W) + \frac{l}{2}\ln(w)$$

This contradicts the fact that $\lim\limits_{w\to 0} p(w) > -\infty$. Then, it must be that $\lim\limits_{w\to 0}(wp'(w)) = 0$ ■

Then use (13):

$$\frac{\sigma}{1-\sigma}\frac{p'(e(w))e(w)}{p(e(w))+(1+r)^{\frac{\sigma-1}{\sigma}}}=\frac{x(w)}{1-x(w)}$$

Thus, $\lim_{w\to 0}(\frac{x(w)}{1-x(w)}) = \lim_{w\to 0}(\frac{\sigma}{1-\sigma}\frac{p'(e(w))e(w)}{p(e(w))+(1+r)^{\frac{\sigma-1}{\sigma}}}) = \frac{\sigma}{1-\sigma}\frac{\lim_{w\to 0}(wp'(w))}{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}} = 0$ according to Lemma 11. This implies that $\lim_{w\to 0}(x(w))$ exists and is equal to 0.

Apply now the implicit function theorem to (13) to get that:

$$x'(w_t)=\frac{1}{w_t}\frac{p'(x(w_t)w_t)(\sigma-x(w_t))+x(w_t)p''(x(w_t)w_t)\sigma w_t(1-x(w_t))}{p'(x(w_t)w_t)+(-p''(x(w_t)w_t))\sigma w_t(1-x(w_t))} \tag{23}$$

Thus, $x'(w_t)>0$ is equivalent to:

$$\frac{(-p''(x(w_t)w_t))x(w_t)w_t}{p'(x(w_t)w_t)}<\frac{\sigma-x(w_t)}{\sigma(1-x(w_t))} \tag{24}$$

Define $m(x,w_t)\equiv\frac{(-p''(xw_t))xw_t}{p'(xw_t)}$ and $g(x)\equiv\frac{\sigma-x}{\sigma(1-x)}$.

By Assumption 2, $x\to m(x,w_t)$ increases from a value strictly smaller than 1 while the function is smaller than 1. $x\to g(x)$ decreases from 1 to $-\infty$ on $[0,1]$. Thus, for any w_t there exists a unique root on $[0,1]$ to the equation $m(x,w_t)=g(x)$. Note it $\Sigma(w_t)$. $w_t\to\Sigma(w_t)$ decreases because $\frac{\partial m}{\partial w_t}(x,w_t)>0$ and $\lim_{w_t\to\infty}(\Sigma(w_t))=0$. Draw the curve $w_t\to\Sigma(w_t)$ on $[\underline{w},\infty)$. Note that $x'(w_t)>0$ if and ony if $x(w_t)\in\{y\geq 0, y<\Sigma(w_t)\}=\Delta$. Initially $x(w_t)\in\Delta$ because $\lim_{w_t\to\underline{w}}x(w_t)=0$. There necessarily exists w^* such that $x(w^*)=\Sigma(w^*)$ because Σ decreases towards 0. By definition of Σ, $x'(w^*)=0$, while $\Sigma'(w^*)<0$. Thus, $w_t\to x(w_t)$ enters Δ at w^* and $x(w_t)$ is trapped in Δ because wherever it hits the boundary of Δ, it has a greater slope than the boundary. Thus $w\to x(w)$ is inverted U-shaped.

There remains to compute the limit of $x(w_t)$ as w_t tends towards ∞. This limit exists as $w_t\to x(w_t)$ ends decreasing.

Lemma 12 $\lim_{w\to\infty}(wp'(w))=0$

Proof. We first show that $w\to wp'(w)$ ends decreasing. Its first derivative is $p'(w)(1-\frac{(-p''(w))w}{p'(w)})$ which is non-positive when w gets large as $\lim_{w\to\infty}\frac{(-p''(w))w}{p'(w)}>1$. Therefore, $\lim_{w\to\infty}(wp'(w))$ exists and it must be finite because $w\to wp'(w)$ ends decreasing. Note this limit l and assume that

l is positive. Then, there exists $M > 0$ such that $wp'(w) > \frac{l}{2}$ for all w greater than M. Integrate the previous inequality from M to $a > M$ to get:

$$p(a) - p(M) > \frac{l}{2} \ln(\frac{a}{M}) \tag{25}$$

As a tends towards ∞, the RHS of (25) tends towards ∞. This contradicts the fact that p is upper-bounded. Thus, $l = 0$ ∎

From (13):

$$\lim_{w \to \infty} (\frac{x(w)}{1 - x(w)}) = \lim_{w \to \infty} (\frac{\sigma}{1 - \sigma} \frac{p'(e(w))e(w)}{p(e(w)) + (1 + r)^{\frac{\sigma - 1}{\sigma}}}) = \frac{\sigma}{1 - \sigma} \frac{\lim_{w \to \infty} (wp'(w))}{\overline{p} + (1 + r)^{\frac{\sigma - 1}{\sigma}}} = 0 \tag{26}$$

This implies that $\lim_{w \to \infty} (x(w)) = 0$. This completes the proof of Proposition 4.

8 Appendix C

If $\frac{\overline{p}}{\overline{p} + (1+r)^{\frac{\sigma-1}{\sigma}}} A(1 - \alpha) < 1$, then $\frac{G(e(w),w)}{w} = \frac{p(e(w))}{p(e(w)) + (\alpha A)^{\frac{\sigma-1}{\sigma}}} \frac{(w - e(w))}{w} A(1 - \alpha) < \frac{\overline{p}}{\overline{p} + (1+r)^{\frac{\sigma-1}{\sigma}}} A(1 - \alpha) < 1$. Hence, the propagator of (17) is strictly below the 45° line for any income levels, which implies that the economy converges to 0.

If $\frac{\overline{p}}{\overline{p} + (1+r)^{\frac{\sigma-1}{\sigma}}} A(1 - \alpha) > 1$:

$\underline{p} = 0$ implies that $\frac{G(e(\frac{1-\sigma}{\sigma} \frac{(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}), \frac{1-\sigma}{\sigma} \frac{(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma})}{\frac{1-\sigma}{\sigma} \frac{(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}} = 0$ for all $\gamma \in (0, \infty]$.

Use (13) to write $\frac{G(e(w),w)}{w}$ as:

$$\frac{G(e(w), w)}{w} = \frac{1 - \sigma}{\sigma} \frac{p(e(w))}{wp'(e(w))} A(1 - \alpha) \tag{27}$$

Note first from (23) that:

$$e'(w) = \frac{\sigma p'(e(w))}{p'(e(w)) + (-p''(e(w)))\sigma(w - e(w))} \tag{28}$$

Then we can compute the derivative of $\frac{G(e(w),w)}{w}$ with respect to w. We find that:

$$\frac{d(\frac{G(e(w),w)}{w})}{dw} = \frac{(1 - \sigma)}{\sigma} \frac{A(1 - \alpha)p(e(w))}{w^2(p'(e(w)) + (-p''(e(w)))\sigma(w - e(w)))} [\frac{(1 - \sigma)A(1 - \alpha)}{\frac{G(e(w),w)}{w}} - 1 + \sigma \frac{(-p''(e(w))e(w)}{p'(e(w))}] \tag{29}$$

Thus, we get that:

$$\frac{d(\frac{G(e(w),w)}{w})}{dw} > 0 \Leftrightarrow \frac{A(1-\alpha)}{\frac{G(e(w),w)}{w}} > \frac{1-\sigma\frac{(-p''(e(w))e(w)}{p'(e(w))}}{1-\sigma} \tag{30}$$

At $w = \frac{1-\sigma}{\sigma}\frac{(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}$, this condition writes:

$$\infty > \frac{1-\sigma a}{1-\sigma}$$

As this condition holds, it must be that $w \to \frac{A(1-\alpha)}{\frac{G(e(w),w)}{w}}$ is initially decreasing. This function is also lower-bounded by 1. Consider now the RHS of the inequality (30). It is initially decreasing with w. It can increase with w, yet under Assumption 2 this can only happen when $\frac{(-p''(e(w))e(w)}{p'(e(w))}$ is greater than 1. When $\frac{(-p''(e(w))e(w)}{p'(e(w))}$ is greater than 1, the RHS of (30) is smaller than 1. Therefore, the LHS and the RHS of (30) can only be equal when both are decreasing with w. If $\frac{A(1-\alpha)}{\frac{G(e(w),w)}{w}}$ always stays above $\frac{1-\sigma\frac{(-p''(e(w))e(w)}{p'(e(w))}}{1-\sigma}$, then $w \to \frac{A(1-\alpha)}{\frac{G(e(w),w)}{w}}$ is always decreasing. If for some $y > \frac{1-\sigma}{\sigma}\frac{(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}$, $\frac{A(1-\alpha)}{\frac{G(e(y),y)}{y}}$ is equal to$\frac{(1-\sigma\frac{(-p''(e(y))e(y)}{p'(e(y))})}{1-\sigma}$, then $\frac{d(\frac{1}{\frac{G(e(w),w)}{w}})}{dw})_{w=y} = 0$, while the derivative of $\frac{(1-\sigma\frac{(-p''(e(w))e(w)}{p'(e(w))})}{1-\sigma}$ at $w = y$ is negative. Thus, $\frac{A(1-\alpha)}{\frac{G(e(w),w)}{w}} > \frac{(1-\sigma\frac{(-p''(e(w))e(w)}{p'(e(w))})}{1-\sigma}$ in the right neighborhood of y. This proves that $\frac{d\frac{G(e(w),w)}{w}}{dw} \geq 0$ for all $w > \frac{1-\sigma}{\sigma}\frac{(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}$.

Finally, recall that $\lim_{w\to\infty}(x(w)) = 0$, which implies that $\lim_{w\to\infty}(\frac{G(e(w),w)}{w}) = \frac{\overline{p}}{\overline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}A(1-\alpha) > 1$. This proves that the propagator of (17) is equal to 0 while $w \leq \frac{1-\sigma}{\sigma}\frac{(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}$, then increases and crosses the 45° line exactly one time. This completes the proof of the point (ii).

Point (iii) follows from the fact that $w \to \frac{G(e(w),w)}{w}$ increases on $[\frac{1-\sigma}{\sigma}\frac{(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}, \infty)$.

For the rest of the paper it will be useful to get the variations of $w \to \frac{G(e(w),w)}{w}$ when $\underline{p} > 0$. Use first (30) at $w = \frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}$ to see that $w \to \frac{G(e(w),w)}{w}$ is initially increasing if and only if $\frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\underline{p}} > \frac{1-\sigma a}{\sigma}$. Use the same argument as in the case $\underline{p} = 0$ to get that $w \to \frac{G(e(w),w)}{w}$ is increasing in this case. If $\frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\underline{p}} < \frac{1-\sigma a}{\sigma}$, then $w \to \frac{G(e(w),w)}{w}$

is initially decreasing. Hence $w \to \frac{A(1-\alpha)}{\frac{G(e(w),w)}{w}}$ is initially increasing and is still lower-bounded by 1. $w \to \frac{(1-\sigma\frac{(-p''(e(w))e(w)}{p'(e(w))})}{1-\sigma}$ is decreasing as long as $\frac{(-p''(e(w))e(w)}{p'(e(w))}$ is smaller than 1. Moreover, $w \to \frac{(1-\sigma\frac{(-p''(e(w))e(w)}{p'(e(w))})}{1-\sigma}$ ends smaller than 1. Thus, there exists $y > \frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}$, such that $\frac{A(1-\alpha)}{\frac{G(e(y),y)}{y}} = \frac{(1-\sigma\frac{(-p''(e(y))e(y)}{p'(e(y))})}{1-\sigma}$. Then, $\frac{d(\frac{A(1-\alpha)}{\frac{G(e(w),w)}{w}})}{dw})_{w=y} = 0$, while the derivative of $\frac{(1-\sigma\frac{(-p''(e(w))e(w)}{p'(e(w))})}{1-\sigma}$ at $w = y$ is negative. Thus, $\frac{A(1-\alpha)}{\frac{G(e(w),w)}{w}} > \frac{(1-\sigma\frac{(-p''(e(w))e(w)}{p'(e(w))})}{1-\sigma}$ in the right neighborhood of y and $w \to \frac{G(e(w),w)}{w}$ remains increasing according to the previous argument. Thus, $w \to \frac{G(e(w),w)}{w}$ is U-shaped if $\frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\underline{p}} < \frac{1-\sigma a}{\sigma}$.

9 Appendix D

We will assume that γ is positive, yet by following the same steps, the proof can be adapted to the case $\gamma = \infty$.

It will be necessary here to write explicitly the dependence of $e(w_t)$ with respect to A, due to the dependence of the interest rate on A. Then define:

$$H(A,w) = \frac{p(e(A,w))}{p(e(A,w)) + (\alpha A)^{\frac{\sigma-1}{\sigma}}}\frac{(w-e(A,w))}{w}A(1-\alpha) \qquad (31)$$

Step 1: We prove that $\frac{\partial H}{\partial A}(A,w) > 0$ for any pair $(A,w) \in (0,\infty)^2$.

For $(A,w) \in (0,\infty)^2$ such that $w \leq \frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}$, $H(A,w) = \frac{\underline{p}}{\underline{p}+(\alpha A))^{\frac{\sigma-1}{\sigma}}}A(1-\alpha)$ and the result follows.

For $(A,w) \in (0,\infty)^2$ such that $w > \frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}$, apply the implicit function theorem to (13) to get that $\frac{\partial e}{\partial A}(A,w) > 0$.

Rewrite (31) as:

$$H(A,w) = \frac{1-\sigma}{\sigma}\frac{p(e(A,w))}{wp'(e(A,w))}A(1-\alpha) \qquad (32)$$

Thus, $\frac{\partial H}{\partial A} > 0$.

Step 2: We prove that for any $w > 0$, there exists a unique $Z(w) > 0$ such that: $A < Z(w) \iff H(A,w) < 1$.

This follows from the fact that $A \to H(A,w)$ increases (step 1) from 0 to ∞ on $[0,\infty)$.

Define now for each $w > \frac{1-\sigma}{\sigma}\frac{\underline{p}}{\gamma}$, $A(w)$ as $w = \frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha A(w))^{\frac{\sigma-1}{\sigma}}}{\gamma}$ and reciprocally for each $A > 0$, $w(A)$ as $w(A) = \frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}$.

Step 3: We prove that for $w \leq w(\underline{A})$, $Z(w) = \underline{A}$.

By definition, $H(\underline{A}, w(\underline{A})) = 1$. Thus, $Z(w(\underline{A})) = \underline{A}$.

If $w < w(\underline{A})$, then $A(w) > \underline{A}$. Thus $H(A(w), w) = \frac{\underline{p}}{\underline{p}+(\alpha A(w))^{\frac{\sigma-1}{\sigma}}}A(w)(1-\alpha)$ is strictly greater than 1.Thus, $Z(w) < A(w)$.

So, $H(Z(w), w) = \frac{\underline{p}}{\underline{p}+(\alpha Z(w)))^{\frac{\sigma-1}{\sigma}}}Z(w)(1-\alpha)$. As $H(Z(w), w) = 1$, it must be that $Z(w) = \underline{A}$.

This means that under the condition $A > \underline{A}$, the propagator of (17) is strictly above the 45° line for $w < w(\underline{A})$. Thus, any steady state of (17) is necessarily greater than $w(\underline{A})$ under the condition $A > \underline{A}$.

Step 4: We prove that $\max\limits_{w \geq w(\underline{A})}(Z(w))$ exists. And $\max\limits_{w \geq w(\underline{A})}(Z(w)) > \underline{A}$ $\Leftrightarrow \frac{1-a\sigma}{\sigma} > \frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha\underline{A})^{\frac{\sigma-1}{\sigma}}}{\underline{p}}$.

The continuity of $(A, w) \to H(A, w)$ implies the one of $w \to Z(w)$. Moreover, $Z(w(\underline{A})) = \underline{A}$. $Z(w)$ has also a limit in ∞, noted $Z(\infty)$, which satisfies:

$$\frac{\overline{p}}{\overline{p}+(\alpha Z(\infty))^{\frac{\sigma-1}{\sigma}}}Z(\infty)(1-\alpha) = 1$$

The fact that $\underline{p} < \overline{p}$ implies that $Z(\infty) < \underline{A}$. This proves that $w \to Z(w)$ has a maximum on $[w(\underline{A}), \infty)$.

Consider now the function $w \to H(\underline{A}, w)$ on $[w(\underline{A}), \infty)$. From the proof of Proposition 6. (Appendix C), this function is increasing (if $\frac{1-a\sigma}{\sigma} < \frac{w(\underline{A})p'(e(w(\underline{A})))}{p(e(w(\underline{A})))} = \frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha\underline{A})^{\frac{\sigma-1}{\sigma}}}{\underline{p}}$) or U-shaped (otherwise).

If $w \to H(\underline{A}, w)$ is increasing, then $H(\underline{A}, w) > 1$ for all $w \geq w(\underline{A})$. Thus, $Z(w) < \underline{A}$ for all $w \geq w(\underline{A})$. Hence, $\max\limits_{w \geq w(\underline{A})}(Z(w))$ is exactly $\underline{A}$ in this case.

If $w \to H(\underline{A}, w)$ is U-shaped, then there exists $z(\underline{A}) > w(\underline{A})$ such that $H(\underline{A}, w) < 1$ for all $w \in (w(\underline{A}), z(\underline{A}))$. Thus, $Z(w) > \underline{A}$ for all $w \in (w(\underline{A}), z(A))$. Hence, $\max\limits_{w \geq w(\underline{A})}(Z(w))$ is strictly greater than $\underline{A}$ in this case.

Thus, it must be that $\max\limits_{w \geq w(\underline{A})}(Z(w)) > \underline{A} \Leftrightarrow \frac{1-a\sigma}{\sigma} < \frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha\underline{A})^{\frac{\sigma-1}{\sigma}}}{\underline{p}}$.

Write $A^* = \max\limits_{w \geq w(\underline{A})}(Z(w))$.

Step 5: We prove that $\underline{A} < A^*$ and $A \in (\underline{A}, A^*) \Leftrightarrow$ (17) has exactly two positive steady states under the condition $A > \underline{A}$.

Assume first that (17) has exactly two positive steady states and $A > \underline{A}$. It means that $Z(w)$ takes values strictly greater than $\underline{A}$ on $(w(\underline{A}), \infty)$. Thus, $\underline{A} < A^*$. Note also that $A < A^*$ otherwise $H(A, w)$ would be strictly greater than 1 for all $w \geq 0$ which would contradict the fact that (17) has two steady states.

Assume now that $\underline{A} < A^*$ and $A \in (\underline{A}, A^*)$. Then, $w \to H(A, w)$ takes values strictly smaller than 1 on $(w(\underline{A}), \infty)$, $H(A, w) > 1$ if $w \leq w(\underline{A})$ and $\lim\limits_{w \to \infty}(H(A, w)) > 1$ because $A > \underline{A}$. Thus, (17) has at least two steady states. Denote $w_1(A)$ the smallest one, which is necessarily stable, and $w_2(A)$, the highest one, which is necessarily unstable.

From step 4, the condition $\underline{A} < \max\limits_{w \geq w(\underline{A})}(Z(w))$ is equivalent to $\frac{1-a\sigma}{\sigma} > \frac{1-\sigma}{\sigma} \frac{\underline{p}+(\alpha\underline{A})^{\frac{\sigma-1}{\sigma}}}{\underline{p}}$. Then, it must be that $\frac{1-a\sigma}{\sigma} > \frac{1-\sigma}{\sigma} \frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\underline{p}}$ and according to the proof of Proposition 6, it must be that $w \to H(A, w)$ is U-shaped. Consequently, $w_1(A)$ and $w_2(A)$ are the two only steady states of (17).

To complete the proof, we rearrange the condition $\frac{1-a\sigma}{\sigma} > \frac{1-\sigma}{\sigma} \frac{\underline{p}+(\alpha\underline{A})^{\frac{\sigma-1}{\sigma}}}{\underline{p}}$ to obtain condition (ii).

Step 6: We prove that $\frac{1-a\sigma}{\sigma} > \frac{1-\sigma}{\sigma} \frac{\underline{p}+(\alpha\underline{A})^{\frac{\sigma-1}{\sigma}}}{\underline{p}} \Leftrightarrow \alpha > \frac{(\underline{p}\frac{\sigma}{1-\sigma}(1-a))^{\frac{\sigma}{\sigma-1}}}{\frac{1-a\sigma}{1-\sigma}+(\underline{p}\frac{\sigma}{1-\sigma}(1-a))^{\frac{\sigma}{\sigma-1}}}$.

By definition of $\underline{A}$, $\frac{\underline{p}\underline{A}(1-\alpha)}{\underline{p}+(\alpha\underline{A})^{\frac{\sigma-1}{\sigma}}} = 1$.

Then, $\frac{1-a\sigma}{\sigma} > \frac{1-\sigma}{\sigma} \frac{\underline{p}+(\alpha\underline{A})^{\frac{\sigma-1}{\sigma}}}{\underline{p}} \Leftrightarrow \frac{\underline{p}}{\underline{p}+(\alpha\underline{A})^{\frac{\sigma-1}{\sigma}}} > \frac{1-\sigma}{1-a\sigma}$ where $\frac{\underline{p}}{\underline{p}+(\alpha\underline{A})^{\frac{\sigma-1}{\sigma}}} = \frac{1}{\underline{A}(1-\alpha)}$

Set $m(A) = \frac{\underline{p}}{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}$ and $n(A) = \frac{1}{A(1-\alpha)}$. m increases, while n decreases.

Then the previous statement is true if and only if m is strictly greater than n at the point at which n is worth $\frac{1-\sigma}{1-a\sigma}$. As this point is $\frac{1-a\sigma}{(1-\alpha)(1-\sigma)}$, this condition is $\frac{\underline{p}}{\underline{p}+(\alpha\frac{1-a\sigma}{(1-\alpha)(1-\sigma)})^{\frac{\sigma-1}{\sigma}}} > \frac{1}{\frac{1-a\sigma}{(1-\alpha)(1-\sigma)}(1-\alpha)}$.

This completes the proof of Proposition 7.

10 Appendix E

(i) is a corollary of Proposition 7.

Recall first from the proof of Proposition 6 that $w \to G(e(w), w)$ is increasing (if $\frac{1-a\sigma}{\sigma} < \frac{\frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}p'(e(\frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}))}{p(e(\frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\gamma}))} = \frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\underline{p}}$) or U-shaped (otherwise).

If $A > \underline{A}$ and $\alpha < \frac{(\underline{p}\frac{\sigma}{1-\sigma}(1-a))^{\frac{\sigma}{\sigma-1}}}{\frac{1-a\sigma}{1-\sigma}+(\underline{p}\frac{\sigma}{1-\sigma}(1-a))^{\frac{\sigma}{\sigma-1}}}$, then according to the proof of Proposition 7, $\frac{1-a\sigma}{\sigma} < \frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha\underline{A})^{\frac{\sigma-1}{\sigma}}}{\underline{p}}$. Thus for $A \in (\underline{A}, \widehat{A}), w \to G(e(w), w)$ is increasing, while the function is U-shaped for $A > \widehat{A}$.

If $A > A^*$ and $\alpha > \frac{(\underline{p}^{\frac{\sigma}{1-\sigma}}(1-a))^{\frac{\sigma}{\sigma-1}}}{\frac{1-a\sigma}{1-\sigma}+(\underline{p}^{\frac{\sigma}{1-\sigma}}(1-a))^{\frac{\sigma}{\sigma-1}}}$, then according to the proof of Proposition 7, $\frac{1-a\sigma}{\sigma} > \frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha\underline{A})^{\frac{\sigma-1}{\sigma}}}{\underline{p}}$ and then for all $A > A^* > \underline{A}$, $\frac{1-a\sigma}{\sigma} > \frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}}{\underline{p}}$ and $w \to G(e(w), w)$ is U-shaped.

This completes the proof of Proposition 8.

11 Appendix F

$w \to G(e(w), w)$ is an increasing or U-shaped function which is below 1 for $w \leq \frac{1-\sigma}{\sigma}\frac{\underline{p}+(\alpha A(w))^{\frac{\sigma-1}{\sigma}}}{\gamma}$. This means that the equation $G(e(w), w) = 1$ has a unique solution if and only if $\lim_{w\to\infty} G(e(w), w) = \frac{\overline{p}A(1-\alpha)}{\overline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}} > 1$. In this case, the unique solution to the fixed point equation is an unstable steady state. If $\frac{\overline{p}A(1-\alpha)}{\overline{p}+(\alpha A)^{\frac{\sigma-1}{\sigma}}} < 1$, then $G(e(w), w)$ is strictly smaller than 1 for all income levels, which implies that the economy converges to a null income.

12 Appendix G

We follow the proof of Proposition 3. Assume first that $\underline{p} > 0$ and $\gamma < \infty$. The agent maximizes:

$$V(s_t, e_t) = \frac{(w_t - e_t - s_t)^{1-\sigma}}{1-\sigma} + p(e_t)^{\sigma}\frac{s_t^{1-\sigma}(1+r)^{1-\sigma}}{1-\sigma} + bp(e_t)$$

Subject to the constraints $s_t + e_t \leq w_t$ and $0 \leq e_t$ and $0 \leq s_t$. The Inada condition of the utility function implies that the Lagrangian writes:

$$L(s_t, e_t, \chi_1) = V(s_t, e_t) + \chi_1 e_t$$

The KKT conditions for a point (s_t, e_t) to be an optimum can be written as:

$$(i)\ \frac{\partial V}{\partial s}(s_t, \tau_t) = 0$$

$$(ii)\ \frac{\partial V}{\partial e}(s_t, e_t) = -(w_t - e_t - s_t)^{-\sigma} + p'(e_t)[\frac{\sigma}{1-\sigma}(\frac{s_t(1+r)}{p(e_t)})^{1-\sigma} + b] = -\chi_2$$

$(iii)\min(\chi_2, e_t) = 0$

Consider now the possibility that $e_t = 0$. From (i), we get the optimal saving, $s_t = \frac{\underline{p}}{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}} w_t$. And χ_2 must be non-negative. Thus, (ii) writes $\frac{\partial V}{\partial e}(\frac{\underline{p}}{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}} w_t, 0) \leq 0$ which is equivalent to $\gamma(b - \frac{\sigma}{1-\sigma}\frac{w_t^{1-\sigma}(1+r)^{1-\sigma}}{(\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}})^{1-\sigma}}) \leq \frac{w_t^{-\sigma}(1+r)^{1-\sigma}}{(\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}})^{-\sigma}}$. The LHS of this inequality increases from $-\infty$ to b on $[0, \infty)$, while the RHS decreases from ∞ to 0 on $[0, \infty)$. Thus, there exists $\underline{w} \in (0, \infty)$ such that the previous inequality is satisfied if and only if $w_t < \underline{w}$. Thus, for $w_t < \underline{w}$, $(\frac{\underline{p}}{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}} w_t, 0)$ satisfies the KKT conditions and is a possible solution. Consider now the case $e_t > 0$. Then, $\chi_2 = 0$ and the conditions (i) and (ii) imply the equation (10) and:

$$\frac{1}{p'(e_t)} = (\frac{w_t - e_t}{p(e_t) + (1+r)^{\frac{\sigma-1}{\sigma}}})^{\sigma} R^{\sigma-1} b - \frac{\sigma}{\sigma-1}\frac{w_t - e_t}{p(e_t) + (1+r)^{\frac{\sigma-1}{\sigma}}} \quad (33)$$

Note the RHS of (33) as $h(\frac{w_t - e_t}{p(e_t)+(1+r)^{\frac{\sigma-1}{\sigma}}})$, where $h(x) = x^{\sigma}R^{\sigma-1}b - \frac{\sigma}{\sigma-1}x$. Note that h is increasing where it is non-negative and so $e_t \to h(\frac{w_t - e_t}{p(e_t)+(1+r)^{\frac{\sigma-1}{\sigma}}})$ is decreasing on $[0, w_t]$. Moreover, the RHS of (33) is equal to 0 at $e_t = w_t$. As the LHS of (33) is positive and increasing, (33) has a solution, which is also unique, if and only if $h(\frac{w_t}{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}) > \frac{1}{\gamma}$ which is equivalent to $\gamma(b - \frac{\sigma}{1-\sigma}\frac{w_t^{1-\sigma}(1+r)^{1-\sigma}}{(\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}})^{1-\sigma}}) > \frac{w_t^{-\sigma}(1+r)^{1-\sigma}}{(\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}})^{-\sigma}}$ and so to $w_t > \underline{w}$. Thus, for $w_t > \underline{w}$, there exists a unique pair (s_t, e_t) that satisfies the KKT conditions. Note finally that the problem of the consumer has always at least one solution because $(s, e) \to V(s, e)$ is continuous on the maximization domain, which is compact. Consequently, the unique pair satisfying the KKT conditions in the two cases $w_t \leq \underline{w}$ and $w_t > \underline{w}$ is the unique solution to the problem of the consumer. The case $\underline{p} = 0$ follows by continuity. When $\gamma = \infty$, the condition for $e_t = 0$ to satisfy the KKT conditions write now $s_t = \frac{\underline{p}}{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}} w_t$ and $\lim_{e_t \to 0} p'(e_t)(b - \frac{\sigma}{1-\sigma}\frac{w_t^{1-\sigma}(1+r)^{1-\sigma}}{(\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}})^{1-\sigma}}) \leq \frac{w_t^{-\sigma}(1+r)^{1-\sigma}}{(\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}})^{-\sigma}}$. The second inequality is true if and only if $w_t < (b\frac{1-\sigma}{\sigma})^{\frac{1}{1-\sigma}}(\frac{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}{1+r})^{1-\sigma}$. Thus, for $w_t < (b\frac{1-\sigma}{\sigma})^{\frac{1}{1-\sigma}}(\frac{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}{1+r})^{1-\sigma}$, $(\frac{\underline{p}}{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}} w_t, 0)$ satisfies the KKT conditions. Consider now an interior solution. It satisfies the equations (10) and (33). For (33) to have a solution, the necessary and sufficient condition is now: $\lim_{e_t \to 0}\frac{1}{p'(e_t)} < \lim_{e_t \to 0}(\frac{w_t - e_t}{p(e_t)+(1+r)^{\frac{\sigma-1}{\sigma}}})^{\sigma} R^{\sigma-1} b - \frac{\sigma}{\sigma-1}\frac{w_t - e_t}{p(e_t)+(1+r)^{\frac{\sigma-1}{\sigma}}}$, which is equivalent to $w_t > (b\frac{1-\sigma}{\sigma})^{\frac{1}{1-\sigma}}(\frac{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}{1+r})^{1-\sigma}$. Hence for $w_t > (b\frac{1-\sigma}{\sigma})^{\frac{1}{1-\sigma}}(\frac{\underline{p}+(1+r)^{\frac{\sigma-1}{\sigma}}}{1+r})^{1-\sigma}$, there is a unique pair (s_t, e_t) that satisfies the KKT conditions.

Note that $h(\frac{w_t-e_t}{p(e_t)+(1+r)^{\frac{\sigma-1}{\sigma}}})$ increases with w_t, which shows that $e(w_t)$ is increasing.

To see that $w \to x(w)$ is initially increasing on $[\underline{w}, \infty)$, note that $x(\underline{w}) = 0$, while $x(w) > 0$ on $[\underline{w}, \infty)$.

To get $\lim_{w_t \to \infty} x(w_t)$, rewrite first (33) at the optimum:

$$\frac{1}{p'(e(w_t))} = (\frac{w_t - e(w_t)}{p(e(w_t)) + (1+r)^{\frac{\sigma-1}{\sigma}}})^\sigma R^{\sigma-1} b - \frac{\sigma}{\sigma-1} \frac{w_t - e(w_t)}{p(e(w_t)) + (1+r)^{\frac{\sigma-1}{\sigma}}}$$

Note that $\lim_{w_t \to \infty} \frac{1}{p'(e(w_t))} = \infty$, so it must be that $\lim_{w_t \to \infty} \frac{w_t - e(w_t)}{p(e(w_t))+(1+r)^{\frac{\sigma-1}{\sigma}}} = \infty$. Thus, $h(\frac{w_t-e_t}{p(e_t)+(1+r)^{\frac{\sigma-1}{\sigma}}}) \underset{w_t\to\infty}{\sim} (\frac{w_t-e(w_t)}{p(e(w_t))+(1+r)^{\frac{\sigma-1}{\sigma}}})^\sigma R^{\sigma-1} b$. And then $(\frac{x(w_t)}{1-x(w_t)})^\sigma \underset{w_t\to\infty}{\sim} \frac{p'(e(w_t))e(w_t)^\sigma}{(\overline{p}+(1+r)^{\frac{\sigma-1}{\sigma}})^\sigma} R^{\sigma-1} b$. This gives the three possible cases of Proposition 10.