Stochastic Shadow Pricing of Renewable Natural Resources

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Abstract

By means of stochastic optimal control, this paper aims at studying the shadow pricing of renewable natural resources in uncertainty. Two cases are considered, respectively centralized and decentralized control processes. The latter is in the form of a stochastic control of the state vector distributed between several agents. In both cases, the optimal control path that minimizes the cost function, which is a decreasing function of time, corresponds to the real option valuation as a cost-effective optional investment in the resource stock preservation in uncertainty. The results obtained from numerical simulations show coherence with those encountered in the literature on option pricing.

Keywords: bioeconomics, renewable natural resources, stochastic optimal control, shadow pricing, real options

JEL Classification: D81, Q21, Q57

1 Introduction

Managers of natural resource systems are accustomed to uncertainty (Nichols et al., 2011). For example, adaptive management, as a process of system monitoring, was developed to deal with uncertainty in recurrent decision making (Williams et al., 2002). The four types of uncertainty listed by Johnson et al. (1997), which can be found in adaptive resource management, are environmental variation – through the changes in the state variables –, partial controllability – in form of imprecision in the management actions –, partial observability – by means of a series of estimates within the natural systems – and structural uncertainty – as imperfect understanding of the managed systems.

Sectors of renewable natural resources, such as agriculture, aquaculture and forestry, are highly exposed to uncertainty, provided the combination of natural hazards and of levels of depletion that can jeopardize the ability of resources to renew themselves. Sustainable management of renewable natural resources then consists in maintaining sufficient stocks of resources, through pluriannual management plans, while satisfying various economic and social needs over a given horizon of time (De Lara and Doyen, 2008). Nevertheless, the adaptive management of natural resources, the growth dynamics of which depend on biotic and abiotic factors, is contingent on the future climate mitigation accomplishments, which are known to be unknown (FAO, 2013). Indeed, the other aspect of management in uncertainty deals with the problem of climate change, which exacerbates uncertainty (Joyce et al., 2006). Accordingly, the experts on climate change alert against the importance of uncertainty while measuring their confidence in key findings (IPCC, 2010), such that, in the face of climate-induced problems, adaptive management found itself to shape the existing management programs to climate change (Nichols et al., 2011).[1]

Renewable natural resource management is usually considered as a bioeconomic problem in dynamic optimization in which the the resource in question is not only subject to harvesting plans but also capable of regeneration (Conrad, 1986). Resource problems thus need to be placed in a dynamic perspective due to the dynamics of resource depletion and to the ecological process of regeneration (Bretschger and Smulders, 2007). Thereby,

[1]For example, in forestry, climate change is expected to alter the growth of trees in a way that is far from being fully understood (Millar et al., 2007). On one side, the increase of the CO2 atmospheric concentration may lead to the carbon fertilization effect, according to which the growth rate of tree species should increase (Knapp et al., 2001; Soulé and Knapp, 2006). On the other side, climate change may accentuate the risk of tree mortality (Dale et al., 2000; Allen et al., 2010; Lindner et al., 2010).

sustainability of renewable natural resources, achievable through adaptive management plans, requires a pricing regime, built on shadow prices (van Soest et al., 2006), obtained from dynamic optimization models (Le Bel, 2005; Williams and Johnson, 2013). The shadow pricing of the resource, or its sustainability price, is then the cost of conserving a sufficient level of stock for an additional unit of time; it corresponds to the willingness to undertake control of the resource level by accepting the cost to be incurred for that purpose (Lyon, 1999).

Nevertheless, environmental management based on optimal control simplified complex socio-ecological systems by pushing toward some theoretical equilibrium (Erickson, 2009). Indeed, the optimal control method (Feenstra et al., 2000), usually encountered in the economic literature, does not take into account the conjunction of different types of uncertainty over the level of stock (Hambusch, 2008), so that the deterministic optimal control yields levels of prices which do not reflect the lack of knowledge on the upcoming state distribution. However, when the sources of uncertainty can be described in terms of probability distributions, maximizing the expected objective return can be an effective mechanism for coping with uncertainty (Williams and Johnson, 2013). In order to take into account the uncertainty that drives the evolution of the system (Sethi and Thompson, 2000), stochastic optimal control, where the state of a system is represented by a controlled stochastic process, has been introduced. It has become the main approach for dealing with most forms of uncertainty (Williams, 1985; Williams, 1989). Furthermore, modeling option values through stochastic processes should help improving decision-making under uncertainty (Erickson, 2009).

Following the literature on stochastic control, we consider the renewable natural resource dynamics to take the form of a stochastic differential equation. In order to obtain the co-state distribution of shadow prices, the equation is then submitted to the optimal control. We find that the optimal shadow price distribution that minimizes the cost function corresponds to the real option valuation. Two cases are considered, respectively centralized and decentralized control processes. The latter is in the form of a stochastic control of the state vector distributed between several agents and corresponds, for example, to the cooperative conservation management plans (U.S. Department of the Interior, 2004). The numerical simulations show that, in both cases, the optimal path of the expected willingness to undertake the resource management, observed from the mean value of the co-state distribution, is achievable through higher pricing in the launch of the control processes and lower pricing at the terminal time-steps. In the first case, the latitude in pricing, measured via the level of standard deviation, is large throughout the time-line and narrow in the end. In the second case, the latitude is narrow at the cardinal points of the time-line and large in between.

After this starting section, the stochastic optimal control model, studied through the cases of centralized and decentralized controls, is presented in Section 2. Section 3 is devoted to illustrating simulation examples. Section 4 concludes.

2 Model

Stochastic differential equations, where the resource stock becomes an Itô variable, such that a random process influences the evolution of the resource stock, are now used in modeling the dynamics of natural resources (Conrad, 2010). For example, a stochastic environment, accentuated by climate change, makes the resource growth a random variable. Therefore, consider random state $X(t)$ to be the unknown level of renewable natural resources, subject to a management plan, at time t. Following Palmer and Milutinovic (2011), the continuous state dynamics $dX(t)$, belonging to domain $D \subset \mathbb{R}$, obeys the Itô stochastic differential equation in form of

$$dX(t) = b(X(t), u(t))dt + L(X(t), u(t))dW(t) \tag{1}$$

where $X(t) = \{x_1(t), ..., x_n(t)\}$ is an n-dimensional state vector subject to a time-evolving stochastic process and $W(t)$ is an m-dimensional Wiener process, illustrating the standard Brownian motion.

In detail, the first term on the right-hand side, that is, vector $b(X(t), u(t)) : D \times [0, T] \times U \subset \mathbb{R}^n$, corresponds to the diffusion process with local drift. It depends on the relative rates of growth and harvest and can be understood, as in the ordinary differential equations, as a net level of change in the resource stock dependent on the biological rate of growth and on the rate of exploitation. The control vector $u(t)$, which belongs to the set U of dimension k, or $u(t) : D \times [0, T] \times U \subset \mathbb{R}^k$, denotes the level of harvest of the resource. We thus define the first term to be the benefits at period t from having a resource stock of size $X(t)$ and harvest at rate $u(t)$.

The second term, that is, a positive-definite matrix $L(X(t), u(t)) : D \times [0, T] \times U \subset \mathbb{R}^{n,m}$ represents the local volatility, or standard deviation rate, through the diffusion tensor.[2] As a matter of fact, the standard deviation rate will cause the harvested stock to fluctuate, which, in turn, will cause harvest to fluctuate as well. The stochastic differential equation can be interpreted as showing that in a small time interval, the change in $X(t)$ or in the resource stock is an amount distributed with expectation $b(X(t), u(t))$ and variance $L(X(t), u(t))$. This change is considered to be independent from the past process behavior (Xepapadeas, 2011).

The probability density function of the random state $X(t)$, $\rho(X(t)) : D \times [0, T] \subset \mathbb{R}^+$, evolves according to the Fokker-Planck equation of order two, which is the following parabolic equation (Sabelfeld and Simonov, 2013)

$$\begin{aligned} \frac{\partial \rho(X(t))}{\partial t} &= -\sum_{i,j=1}^{n} b_i(X,u)\frac{\partial \rho(X)}{\partial x_i} + \frac{1}{2}[LL^T]_{ij}(X,u)\frac{\partial^2 \rho(X)}{\partial x_i \partial x_j} \\ &= F(u)\rho(X) \end{aligned} \quad (2)$$

where $x_i \geq 0$ and $x_j \leq 0$ are ith and jth components of the state vector X.[3] By considering two distinguished elements of the state vector, we put emphasis on the fact that the state is uncertain, which could result either in increase or in fall of the resource level. Furthermore, whereas in option pricing, x_i and x_j are constrained to be nonnegative, we allow their levels to be both positive and negative. Indeed, the natural resource is considered to be renewable, or in capacity to regenerate, with respect to a threshold level of depletion or degradation. Should the threshold be exceeded, the resource is at risk of not being replenished. We normalize this threshold by setting it equal to zero. The second part of the equation can be written in form of a differential operator $F(u)$, as a function of the differentiation operator, which will serve to obtain the average property of dynamics by adjusting the rate of harvest.

Finally, under the Fokker-Planck equation, the total probability is conserved for $t \in$

[2] A diffusion process, as a continuous-time Markov process, is a solution to a stochastic differential equation. Brownian motion is one example of it.

[3] The Brownian motion in the dynamics of x_i is the same as that in the dynamics of x_j. Therefore, the probability density function of the random state $X(t)$ defines the probability of the paths of $x_i(t)$ and $x_j(t)$.

$[0, T]$, that is

$$\frac{\partial}{\partial t}\int_D \rho(X(t))dX = 0 \tag{3}$$

2.1 Centralized stochastic optimal control

Stochastic optimal control seeks to find the time-path of the control vector with minimum cost. Following Zvan et al. (2001), we solve the optimization program backwards in time from the terminal time-step to the initial one. We have

$$J(u) = \langle \phi(X(T)), \rho(X(T)) \rangle e^{-rT} + \int_0^T \langle f_0\left(X(t), u(t)\right), \rho(X(t)) \rangle e^{-r(T-t)} dt \tag{4}$$

where the first term is the expected final cost, discounted at rate r, in form of an inner product between a weighting function $\phi(X(T))$, reflecting the co-state distribution of shadow prices for sustaining the resource stock through the optimal control, and the probability density function $\rho(X(T))$ of the random state X at time T. The second term describes the expected costs, discounted at r, recorded during the stochastic process of the state trajectory under control $u(t)$. In this way, the present value at time t of X at time T is equal to the discounted expected future value (Allen and Ilic, 1999). The above equation is thus a two-dimensional convection-diffusion equation with diffusion matrix $[LL^T]_{ij}(X, u)$ and velocity vector $b_i(X, u)$, with exponential decay terms due to discounting (Petrosyan and Yeung, 2007).

The optimal control problem is formulated as the minimization of the cost function subject to the state dynamics, that is

$$\min_{u(t)} J(u) \quad \text{s.t.} \quad dX(t) \tag{5}$$

Thereby, the optimal control consists in finding the open-loop control sequence $u(t)$ that minimizes the cost function, which consists of a cost of the final state and of a cost that is integrated over finite time. We thus attempt to find an adaptive harvest policy that will stabilize the resource level by minimizing the cost functional, with respect to the resource stochastic laws, penalizing deviations from this level (Pereira et al. 2013).

A necessary condition for optimality is that, for $t \in [0, T]$, the optimal control value $u^\star(t)$ minimizes the Hamiltonian, which, according to the infinite dimensional minimum principle (Milutinovic, 2013), is defined as

$$H(\rho(X(t)), u(t), \pi(X(t))) = \langle \rho(X(t)), f_0(u(t)) + F'(u(t))\pi(X(t))\rangle \tag{6}$$

The Hamiltonian includes integral terms that depend on the solution of two systems of partial differential equations. The first system describes the state probability density function $\rho(X(t))$, while $F'(u(t))\pi(X(t))$ corresponds to the co-state distribution evolution, where $F'(u(t))$ is the adjoint operator of $F(u(t))$ satisfying the Green's identity

$$\langle F(u(t))\rho(X(t)), \pi(X(t))\rangle = \langle \rho(X(t)), F'(u(t))\pi(X(t))\rangle \tag{7}$$

The first-order optimality condition involves the partial derivative of $H(\rho, u, \pi)$, that is

$$\begin{aligned} \frac{\partial H(\rho, u, \pi)}{\partial u(t)} &= \frac{\partial J(u)}{\partial u(t)} \\ &= f_0(u(t)) + F'(u(t))\pi(X(t)) + \left\langle \rho(X(t)), \frac{\partial f_0(u(t))}{\partial u(t)} \right\rangle \end{aligned} \tag{8}$$

If $H(\rho, u, \pi)$ is convex in $u(t)$, it can be shown that the minimized Hamiltonian is convex in $X(t)$, a necessary and sufficient condition for the Hamiltonian to be minimized. Indeed, according to the Arrow-Mangasarian condition, if $H(\rho, u, \pi)$ is continuous in $u(t)$, has continuous first partial derivatives with respect to $u(t)$ and is convex in $u(t)$, then $H(\rho, u, \pi)$ is convex in $X(t)$, and $u(t)$ is a global minimizer of the functional (Framstad, 2006).

We do not know $\rho(X(t))$, so the inner product will be computed by evaluating the derivative of the co-state distribution. When computing the optimal control, the difficulty thus consists in evaluating the co-state distribution. With respect to the Feynman-Kac formula, the dynamics is defined by

$$
\begin{aligned}
-\frac{\partial \pi(X(t))}{\partial t} &= \frac{\partial H(\rho, u, \pi)}{\partial X(t)} \\
&= f_0(u) + \sum_{i,j=1}^{n} b_i(X,u)\frac{\partial \pi(X)}{\partial x_i} + \frac{1}{2}[LL^T]_{ij}(X,u)\frac{\partial^2 \pi(X)}{\partial x_i \partial x_j} \\
&= f_0(u) + F'(u)\pi(X)
\end{aligned} \tag{9}
$$

such that

$$
\pi(X(T)) = \phi(X(T)) \tag{10}
$$

where $\phi(X(T))$ is the terminal condition at the terminal time. By linearity of the inner product with respect to $\pi(X(t))$, given that $\rho(X(t))$ corresponds to the expected value of $\phi(X(T))$, and because the differential operator is defined when its coefficients are equal to zero (Albeverio and Kurasov, 2000), the co-state distribution from the optimal control amounts to

$$
\pi^{\star}(X(t)) = \phi(X(T))e^{-rT} + \int_0^T f_0\left(X(t), u^{\star}(t)\right) e^{-r(T-t)} dt \tag{11}
$$

By the transversality conditon, which is required for the integral to converge and for the optimization problem to be well-defined, we have

$$
\lim_{t \to T} f_0\left(X(t), u^{\star}(t)\right) e^{-r(T-t)} = 0 \tag{12}
$$

such that

$$
\lim_{t \to T} \pi^{\star}(X(t)) = \phi(X(T))e^{-rT} \tag{13}
$$

Given the nature of the underlying asset, which can be thought of as an investment in renewable natural resources, we are in presence of real option valuation (Brennan and

Schwartz, 1985; Cortazar et al., 1998). In that case, the value of the underlying asset is based upon an uncertain level of the resource stock. Indeed, a closed look enables us to make a parallel with the binomial options pricing model (Cox et al., 1979), where the expected value is calculated using the option values up and down – which illustrate two possible trends of evolution of the underlying asset value –, weighted by their respective probabilities and discounted, at the risk free rate, during the option lifetime.[4] In our framework, $\phi(X(T))$ encompasses the weighting function, with respect to the upward and downward trends of the state prone to stochastic processes, satisfying the total probability condition, while e^{-rT} portrays the discounting of the option value up to the terminal time.

In such an optional investment, costs relative to the resource development or preservation arise. They correspond to the exercise price of the option, where the expiration of the option occurs at the accomplishment of the management plan. At last, the variability in the resource stock corresponds to the standard variance in the value of the underlying asset. The option value thus depends on the likelihood to achieve sustainability of the natural resource stock through an optional cost-effective control.

Real options can then be interpreted as the willingness, but not the obligation, to undertake, at a certain cost, the stabilization of the natural resource for an additional unit of time, the state of which is governed by a random process. Despite the fact that, in financial options, underlying stock values are also assumed to evolve according to a Brownian motion, real options differ from conventional financial options in that they are not traded as securities. Likewise, through optimal control, the option holders or resource managers can influence the value of the investment, which cannot be achieved with financial underlying assets.[5]

The stochastic optimal path of $\pi^\star(X(t))$, that minimizes the cost function, can now be obtained as

$$\begin{aligned} -dX(t)\frac{\partial^2\pi^\star(X(t))}{\partial t^2} &= dX(t)\left[f_0(u)+\sum_{i,j=1}^{n} b_i(X,u)\frac{\partial^2\pi^\star(X)}{\partial x_i^2}+\frac{1}{2}[LL^T]_{ij}(X,u)\frac{\partial^4\pi^\star(X)}{\partial x_i^2\partial x_j^2}\right] \\ &= dX(t)\left[f_0(u)+F'^2(u)\pi^\star(X)\right] \end{aligned} \quad (14)$$

[4]In detail, we have $C_{t-\Delta t,i} = e^{-r\Delta t}\left[pC_{t,i+1} + (1-p)C_{t,i-1}\right]$, with $C_{t-\Delta t,i}$ the binomial value, $C_{t,i+1}$ and $C_{t,i-1}$ the option values from the last two nodes, p and $(1-p)$ the weighting probabilities of moving up and down, and r the risk free rate.

[5]By considering several components of the state vector X, our framework can be compared to the real option valuation of multiple species preservation (Kassar and Lasserre, 2004).

where $F'^2(u)$ is the second-order differential operator, defined over the control vector, applied to $\pi^\star(X)$, that is, the measure of sensitivity of the option price distribution (Cont and Voltchkova, 2005) yielded by the control of the stochastic process. Thereby, we have a bounded subset of values that $\pi^\star(X)$ can take, with $\lim_{t\to T}\pi^\star(X(t)) = \phi(X(T))e^{-rT}$ being the price when the option is exercised.

In stationarity, where $-dX(t)\frac{\partial^2\pi^\star(X(t))}{\partial t^2} = 0$, the second-order optimality condition yields $-f_0(u(t)) = F'^2(u)\pi^\star(X)$. Therefore, the distribution of shadow prices, or that of real option values, equals the cost of control along the optimal time-path. This brings us to the following proposition.

Proposition 1 *Under centralized stochastic optimal control of the renewable natural resource, the optimal control path that minimizes the cost function, which is a decreasing function of time, corresponds to the real option valuation as a cost-effective optional investment in the resource stock preservation in uncertainty.*

2.2 Decentralized stochastic optimal control

Now consider the case where the stochastic optimal control is distributed between multiple agents, such that a collective of resource managers decide to jointly sustain the resource stock. Let $u_i(t)$ and $u_{-i}(t)$, where $i \neq -i$, be respectively the controls of agents i and $-i$ at time t. The control distribution is weighted by parameter $\alpha \in [0,1]$, such that $\alpha + (1-\alpha) = 1$. The aggregated cost function is represented as

$$\begin{aligned} J(u_{i,-i}) &= \langle\phi(X(T)),\rho(X(T))\rangle e^{-rT} + \int_0^T \langle f_0\left(X(t),\alpha u_i(t)\right),\rho(X(t))\rangle e^{-r(T-t)}dt \quad (15)\\ &+ \int_0^T \langle f_0\left(X(t),(1-\alpha)u_{-i}(t)\right),\rho(X(t))\rangle e^{-r(T-t)}dt \\ &= \langle\phi(X(T)),\rho(X(T))\rangle e^{-rT} \\ &+ \int_0^T \langle f_0\left(X(t),\alpha u_i(t)+(1-\alpha)u_{-i}(t)\right),\rho(X(t))\rangle e^{-r(T-t)}dt \end{aligned}$$

Once again, the necessary condition for optimality is that, for $t \in [0,T]$, the optimal control values $u_i^\star(t)$ and $u_{-i}^\star(t)$ minimize the Hamiltonian, which is now in form of

$$
\begin{aligned}
H(\rho(X(t)), u_{i,-i}(t), \pi_{i,-i}(X(t))) &= \langle \rho(X(t)), f_0(\alpha u_i(t)) + F'(\alpha u_i(t))\pi_i(X(t))\rangle \\
&+ \langle \rho(X(t)), f_0((1-\alpha)u_{-i}(t)) + F'((1-\alpha)u_{-i}(t))\pi_{-i}(X(t))\rangle \\
&= \langle \rho(X(t)), f_0(\alpha u_i(t) + (1-\alpha)u_{-i}(t))\rangle \\
&+ \langle \rho(X(t)), \alpha F'(u_i(t))\pi_i(X(t)) + (1-\alpha)F'(u_{-i}(t))\pi_{-i}(X(t))\rangle
\end{aligned}
\tag{16}
$$

The dynamics of the joint co-state distribution equals to

$$
\begin{aligned}
-\frac{\partial \pi_{i,-i}(X(t))}{\partial t} &= \frac{\partial H(\rho, u_{i,-i}, \pi_{i,-i})}{\partial X(t)} \\
&= f_0(\alpha u_i) + \sum_{i,j=1}^{n} b_i(X, u_i)\frac{\partial \pi_i(X)}{\partial x_i} + \frac{1}{2}[LL^T]_{ij}(X, u_i)\frac{\partial^2 \pi_i(X)}{\partial x_i \partial x_j} \\
&+ f_0((1-\alpha)u_{-i}) + \sum_{i,j=1}^{n} b_i(X, u_{-i})\frac{\partial \pi_{-i}(X)}{\partial x_i} + \frac{1}{2}[LL^T]_{ij}(X, u_{-i})\frac{\partial^2 \pi_{-i}(X)}{\partial x_i \partial x_j} \\
&= f_0(\alpha u_i + (1-\alpha)u_{-i}) + \alpha F'(u_i)\pi_i(X) + (1-\alpha)F'(u_{-i})\pi_{-i}(X)
\end{aligned}
\tag{17}
$$

From the foregoing, the joint co-state distribution issued from the decentralized optimal control amounts to

$$
\pi^\star_{i,-i}(X(t)) = \phi(X(T))e^{-rT} + \int_0^T f_0\left(X(t), \alpha u_i^\star(t) + (1-\alpha)u^\star_{-i}(t)\right) e^{-r(T-t)}dt \tag{18}
$$

Finally, the stochastic optimal path of $\pi^\star_{i,-i}(X(t))$ is in form of

$$
\begin{aligned}
-dX(t)\frac{\partial^2 \pi^\star_{i,-i}(X(t))}{\partial t^2} &= dX(t)\left[f_0(\alpha u_i) + f_0((1-\alpha)u_{-i})\right] \\
&+ dX(t)\left[\sum_{i,j=1}^{n} b_i(X, u_i)\frac{\partial^2 \pi_i^\star(X)}{\partial x_i^2} + \sum_{i,j=1}^{n} b_i(X, u_{-i})\frac{\partial^2 \pi^\star_{-i}(X)}{\partial x_i^2}\right] \\
&+ dX(t)\left[\frac{1}{2}[LL^T]_{ij}(X, u_i)\frac{\partial^4 \pi_i^\star(X)}{\partial x_i^2 \partial x_j^2} + \frac{1}{2}[LL^T]_{ij}(X, u_{-i})\frac{\partial^4 \pi^\star_{-i}(X)}{\partial x_i^2 \partial x_j^2}\right] \\
&= dX(t)\left[f_0(\alpha u_i + (1-\alpha)u_{-i})\right] \\
&+ dX(t)\left[\alpha F'^2(u_i)\pi_i^\star(X) + (1-\alpha)F'^2(u_{-i})\pi^\star_{-i}(X)\right]
\end{aligned}
\tag{19}
$$

We have a bounded subset of values that $\pi^{\star}_{i,-i}(X)$ can take. In stationarity, $-f_0(\alpha u_i(t)+(1-\alpha)u_{-i}(t)) = \alpha F'^2(u_i)\pi^{\star}_i(X)+(1-\alpha)F'^2(u_{-i})\pi^{\star}_{-i}(X)$. Thereby, along the optimal path, the joint distribution of shadow prices of i and $-i$, or their joint distribution of option prices, equals their joint cost of control. The following proposition ensues.

Proposition 2 *Under decentralized stochastic optimal control of the renewable natural resource, the optimal control path that minimizes the cost function, which is a decreasing function of time, corresponds to the joint real option valuation as a collective cost-effective optional investment in the resource stock preservation in uncertainty.*

3 Simulations

Based on the properties and conditions previously obtained, the aim of this section is to illustrate, through simplified simulations, the relationship between the cost of stochastic optimal control and of the willingness to control, in both centralized and decentralized settings, formulated in form of prices an agent is willing to pay to stabilize the resource stock for an additional unit of time.

The two-dimensional process under control, or the stock of the renewable natural resource, is $X(t) = \{x_1(t), x_2(t)\}$, with $x_1(t)$ and $x_2(t)$ the two possible levels of stock. The non-linear dynamics of the state vector, which corresponds to the resource growth in uncertainty and can evolve upward and downward, is formalized as a function of time, where $dx_1 = \sqrt{t} \times \text{rand}(\cdot)$ and $dx_2 = -\sqrt{t} \times \text{rand}(\cdot)$, with $t \in [0, 1]$ in increments of 0.01. The drift rate of the stochastic mean being zero, the expressions are summed up to a Wiener process.

The expected cost is represented by a discounted quadratic function $\mathbb{E}[f(t)] = f(t)e^{-r(T-t)}$, with $f(t) = f(0)(T-t)$ and $f(0) = \frac{1}{2}Qu(0)^2$, where $Q = 0.5$ and $r = 0.02$. The expected final cost is in form of a discounted quadratic function such that $\mathbb{E}[\phi(x_1(T), x_2(T))] = \phi(x_1(T), x_2(T))e^{-rT}$, with $\phi(x_1(T), x_2(T)) = \frac{1}{2}ax_1(T)^2 + \frac{1}{2}bx_2(T)^2$, where $a = 5$ and $b = 5$ give uniform weight to the final cost of control of any state component.

The control vector, which exemplifies the rate of harvest of the resource, is defined with reference to the minimum costs posted in initial time, terminal time and all those which occurred during the stochastic process of the state trajectory. Therefore, the optimal control is modeled as $u(t) = Q + \psi^2\left[a(T-t)^2 + b\right] + a(T-t)x_1(t) + \left[a(T-t)^2 + b\right]x_2(t)$, where $\psi = 0.5$ represents the boundary condition on the final cost. The state components $x_1(t)$ and $x_2(t)$ are subject to different constraints, for the cost of control is assumed to be higher in case of continuous shrinking of the resource stock level.

Finally, the state dynamics under the stochastic optimal control is defined as $dX(t) = [x_1(t) + x_2(t)]\, u(t)$, such that the stock dynamics under control ends up as a quadratic function of the state variable (Trentelman and Willems, 1997).

3.1 Centralized stochastic optimal control

Figs. 1 and 2 depict the mean dynamics, obtained from ten simulated trajectories, of resource levels x_1 and x_2 at $t \in [0, 1]$. The visible fluctuations of the states come from the stochastic processes previously defined.

In Fig. 1, we observe an increasing mean value of x_1, represented by the black curve, starting at 0.05 and reaching a final value of 0.57, which corresponds to a linearized increase of 0.39. The standard deviation, depicted by the grey curves, is in great rise along the time-line, starting at 0.03 and ending at 0.38, illustrating the increase in uncertainty over time, such that the level of stock of the renewable natural resource can either increase markedly or be placed in status quo.

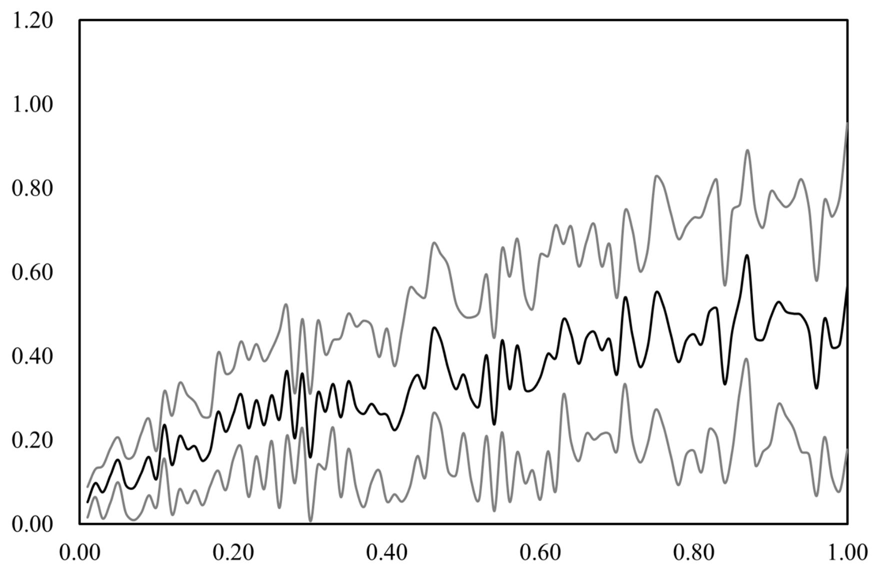

Figure 1: The distribution of $x_1(t)$ along the time-line.

In Fig. 2, the opposite is observed, where the mean value of x_2, also represented by the black curve, commences at -0.06 and closes at -0.70, with a linearized decrease of 0.37.

Like in the previous case, the standard deviation, illustrated by means of grey curves, also enlarges in time, ranging from 0.02 to 0.23, which depicts the possibility to observe an unchanged level of the renewable natural resource as well as its sharp fall.

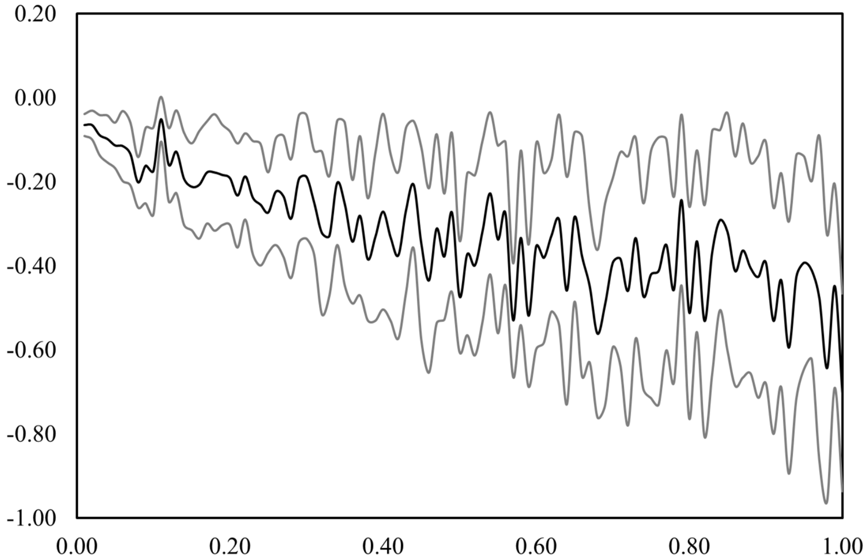

Figure 2: The distribution of $x_2(t)$ along the time-line.

Through the black curve, also in fluctuation due to the stochastic processes, Fig. 3 shows the mean value of the optimal control which starts at 1.20 and ends at 0.32. The linearized decrease is of 0.86. As for the level of standard deviation or uncertainty, depicted in grey curves, it is in overall stability, which implies that the variance of control values remains stable in time. The example also shows that the optimal cost of control is a decreasing function of time. Indeed, the decrease in the value of control reveals the decrease in cost.

As depicted in Fig. 4, the state dynamics under centralized stochastic optimal control is of cyclical nature, where the change in the level of the renewable natural resource continuously oscillates above and below zero. In detail, we observe a slight linearized fall of 0.01 or quasi-stability, with a mean value of -0.01 at $t = 0$ and of -0.06 at $t = 1$. The level of standard deviation is elevated and comparable to the one observed without optimal control, but is also stable in time. Therefore, engaging in controlled harvest of the renewable natural resource can stabilize its mean variation in the long-run, but does not necessarily reduce the uncertainty, which remains of significant level throughout the control process.

Fig. 5 shows the optimal distribution of the willingness to control, or the price a

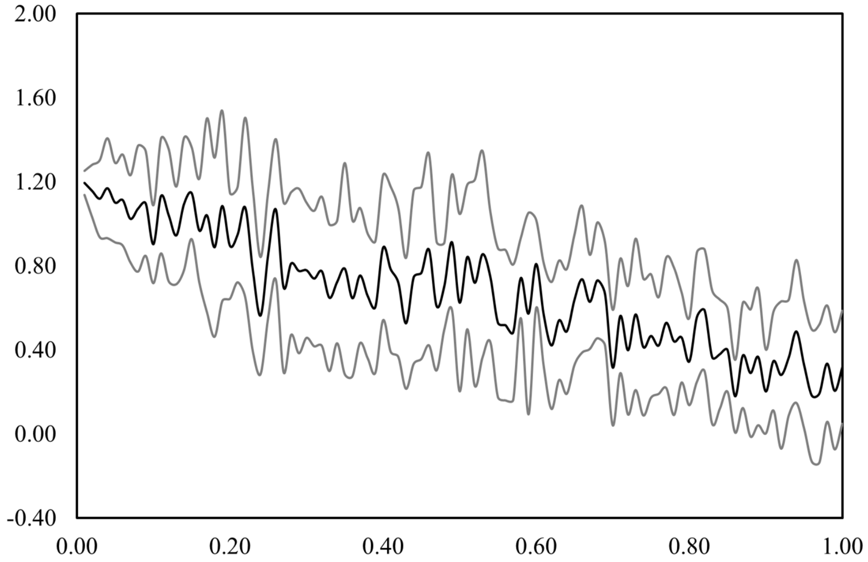

Figure 3: The centralized stochastic optimal control $u^{\star}(t)$ along the time-line.

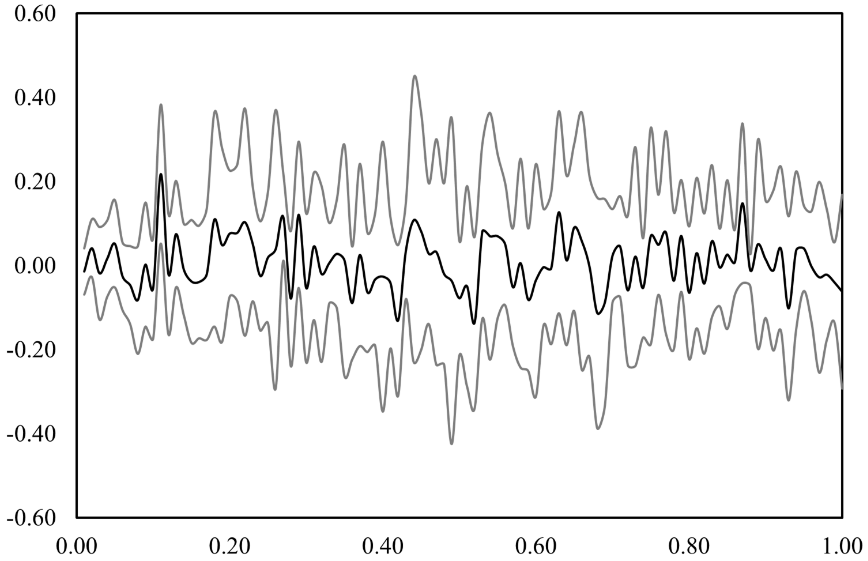

Figure 4: The state dynamics $dX(t)$ under centralized stochastic optimal control $u^{\star}(t)$ along the time-line.

resource manager is willing to pay to undertake the stabilization of the renewable natural resource in uncertainty. The mean value is of 0.37 at $t = 0$ and of 0.21 at $t = 1$, which represents a linearized fall of 0.18. The result is thus consistent with the decrease in control value, and thus in the cost, observed in Fig. 3. Greater variability in the values of $\pi^\star(X(t))$, as $t \to 1$, can be observed through the standard deviation, which starts at 0.10 and ends at 0.38. Such as predicted by the model, we observe a mean level of $\pi^\star(X(t))$ equal to that of the expected final cost at $t = 1$.

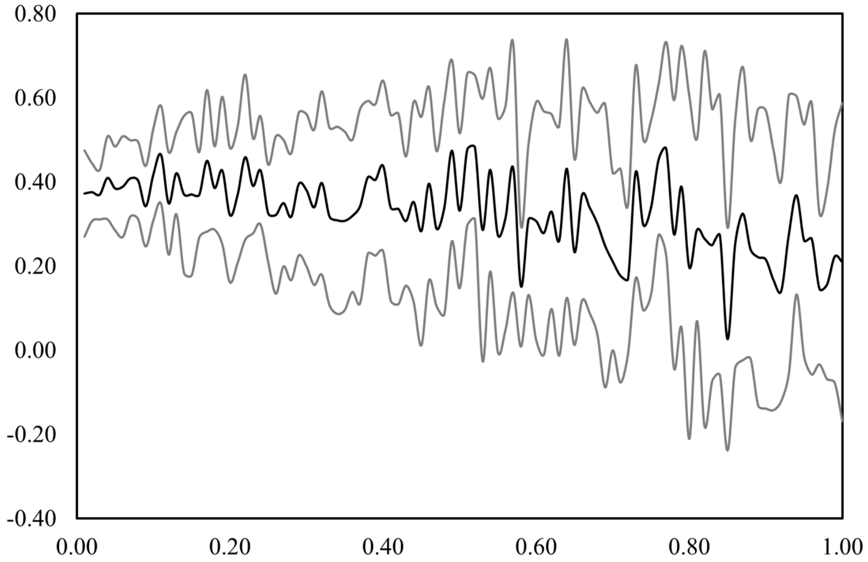

Figure 5: The optimal distribution of $\pi^\star(X(t))$ along the time-line.

As for Fig. 6, the stochastic optimal path that enables to achieve controllability, through management in uncertainty, shows a level of $\pi^\star(X(t))$ of 0.59 at $t = 0$ and of 0.05 at $t = 1$, with a linearized decrease of 0.61. The interesting result is related to the level of standard deviation, which starts at 0.16, reaches 0.30 at $t = 0.14$ and closes at 0.09, denoting a narrower range of price possibilities in the course of time. Therefore, despite the increasing variance of the distribution of the willingness to control along the time-line, the optimal path suggests a greater level of latitude in pricing in the beginning of the stochastic control and a tight one in the terminal steps of time, which in all cases heads toward 0. In addition, we observe that the control input through a channel with a higher variance is less expensive than that through a channel with a lower variance, which is in accordance with the result obtained by Kappen (2005). The optimal path reveals that the mean values from the distribution ought to be spread out in time, with higher pricing in early stages and lower pricing in the end.

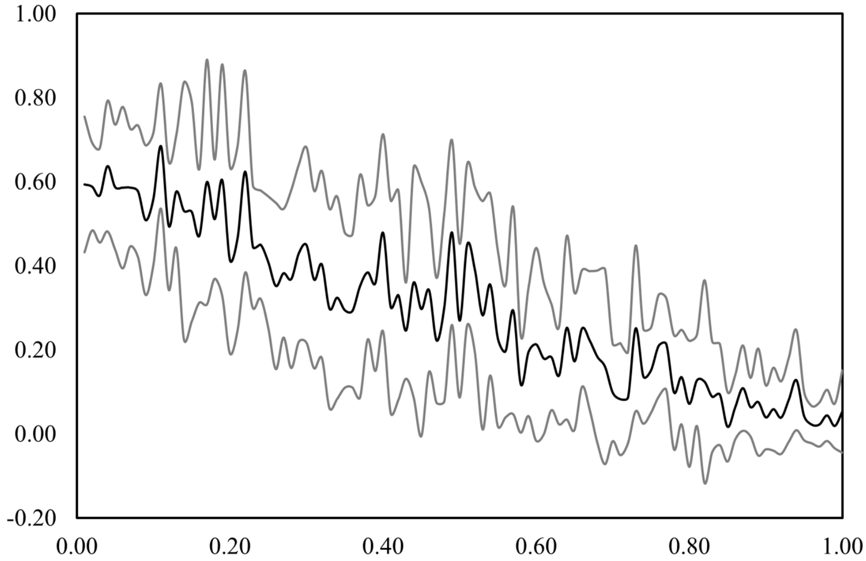

Figure 6: The stochastic optimal path of $\pi^\star(X(t))$ along the time-line.

Result 1 *Under centralized stochastic optimal control of the renewable natural resource, the optimal path of the willingness to undertake the resource stock stabilization in uncertainty is achieved through higher pricing in the launch of the control process and lower pricing at the terminal time. The latitude in pricing is large throughout the time-line and narrow in the end.*

3.2 Decentralized stochastic optimal control

Figs. 7 and 8 depict the mean dynamics, obtained from ten simulated trajectories, of resource levels x_1 and x_2 at $t \in [0,1]$ in increments of 0.01. The stochastic control is distributed between agents 1 and 2, such that $u^\star_{1,2}(t) = \alpha u^\star_1(t) + (1-\alpha)u^\star_2(t)$, with respective weights of $\alpha = 0.3$ and $1-\alpha = 0.7$.

In Fig. 7, we observe an increasing mean value of x_1, represented by the black curve, starting at 0.05 and reaching a final value of 0.47, implying a linearized increase of 0.40. The standard deviation, depicted by the grey curves, is in great rise along the time-line, starting at 0.03 and ending at 0.35, which illustrates once again the increase in uncertainty of the stock of the renewable natural resource.

In Fig. 8, the mean value of x_2, also represented by the black curve, starts at -0.06 and closes at -0.50, with a linearized decrease of 0.42. The standard deviation, depicted in grey curves, enlarges in time, ranging from 0.02 to 0.16.

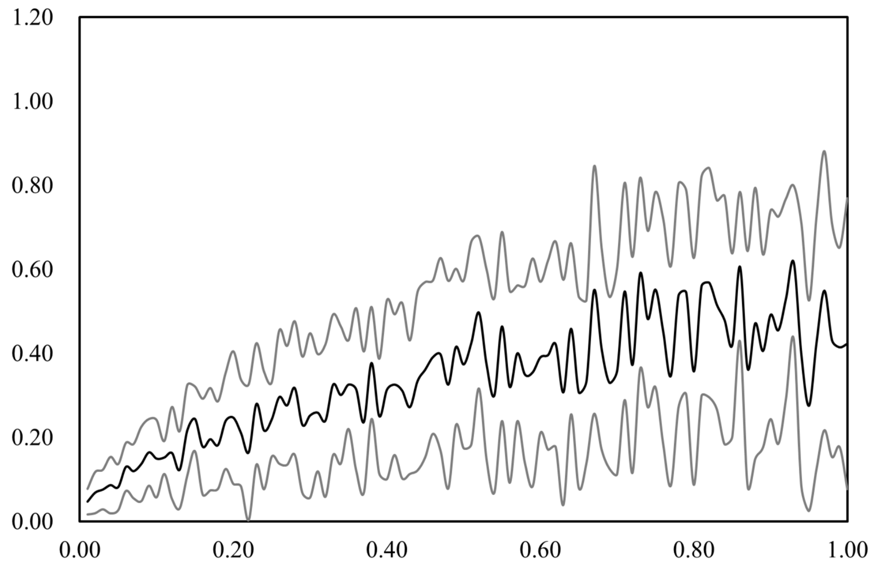

Figure 7: The distribution of $x_1(t)$ along the time-line.

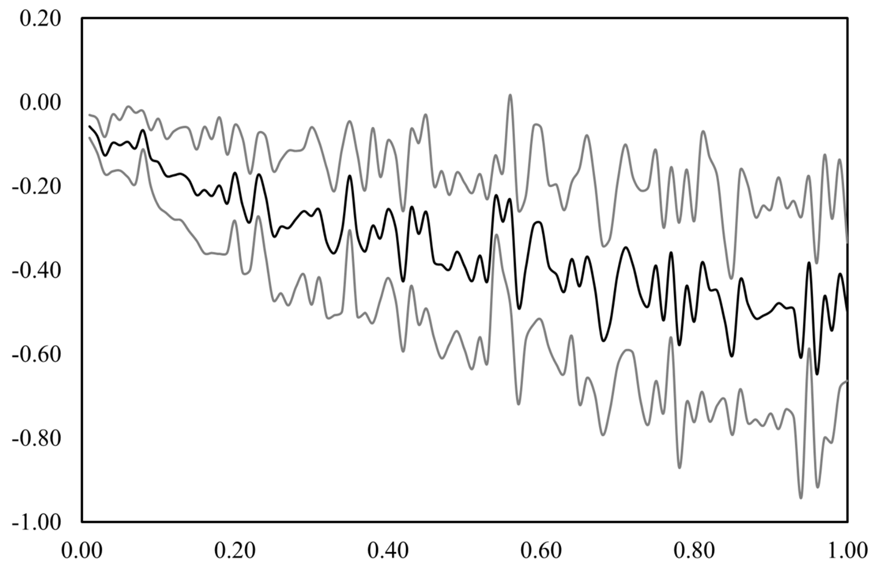

Figure 8: The distribution of $x_2(t)$ along the time-line.

The black curve in Fig. 9 shows the mean value of the decentralized optimal control, which commences at 1.01 and closes at 0.33. The linearized decrease amounts to 0.53. The standard deviation, illustrated by the grey curves, is once again in overall stability, signifying that the variance of control values is stable. The example shows that the optimal cost of distributed control is also a decreasing function of time.

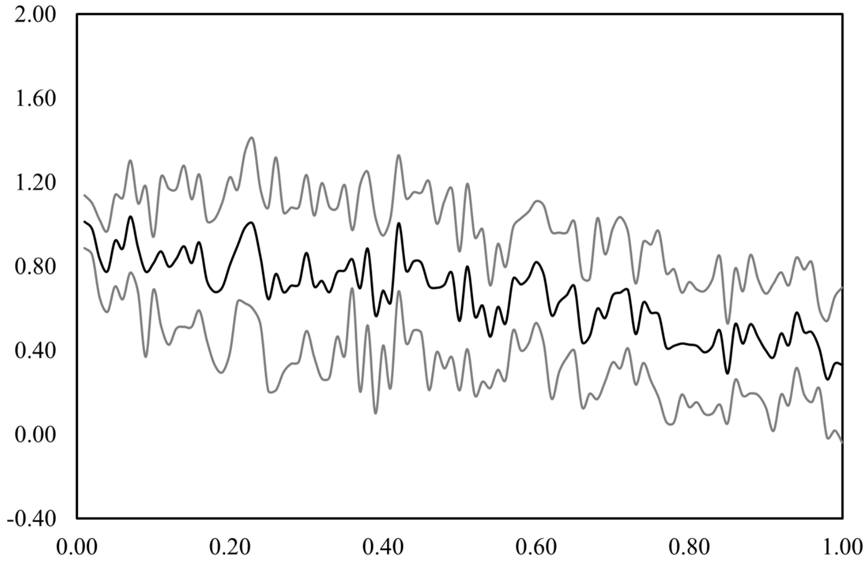

Figure 9: The decentralized stochastic optimal control $u^{\star}_{1,2}(t)$ along the time-line.

In Fig. 10, we observe that the state dynamics under decentralized stochastic optimal control is also of cyclical nature, where the change in the renewable natural resource level continuously oscillates around zero. In detail, we observe a slight linearized fall of 0.00^{-} or quasi-stability, with a mean value of -0.01 at $t = 0$ and of 0.00 at $t = 1$. Once again, the level of standard deviation is elevated and stable in time. Therefore, without reducing the uncertainty, engaging in decentralized optimal control stabilizes the variation of the renewable natural resource in the long-run.

Fig. 11 depicts the optimal distribution of the decentralized willingness to control in time. The mean value is of 0.28 at $t = 0$ and of 0.33 at $t = 1$, be it a linearized fall of 0.03. Unlike the scenario of centralized optimal control, the result shows that the fall in the distributed control value does not necessarily imply a proportional decrease in cost which, in turn, would imply a lower aggregated price of control. The standard deviation starts at 0.08 and ends at 0.28, which is comparable to the result in centralized control, that is, the increasing latitude in pricing as $t \to 1$. Once again, the mean level of $\pi^{\star}_{1,2}(X(t))$ equals that of the expected final cost at $t = 1$.

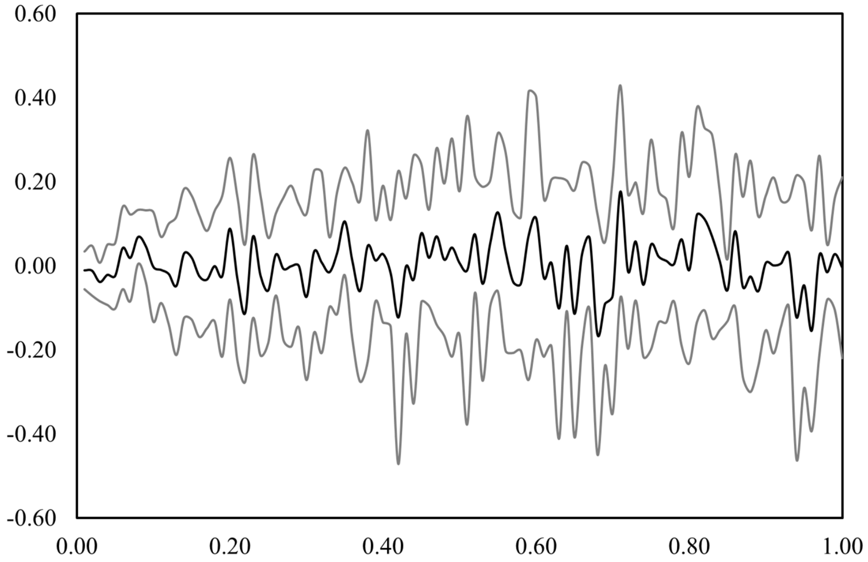

Figure 10: The state dynamics $dX(t)$ under decentralized stochastic optimal control $u^{\star}_{1,2}(t)$ along the time-line.

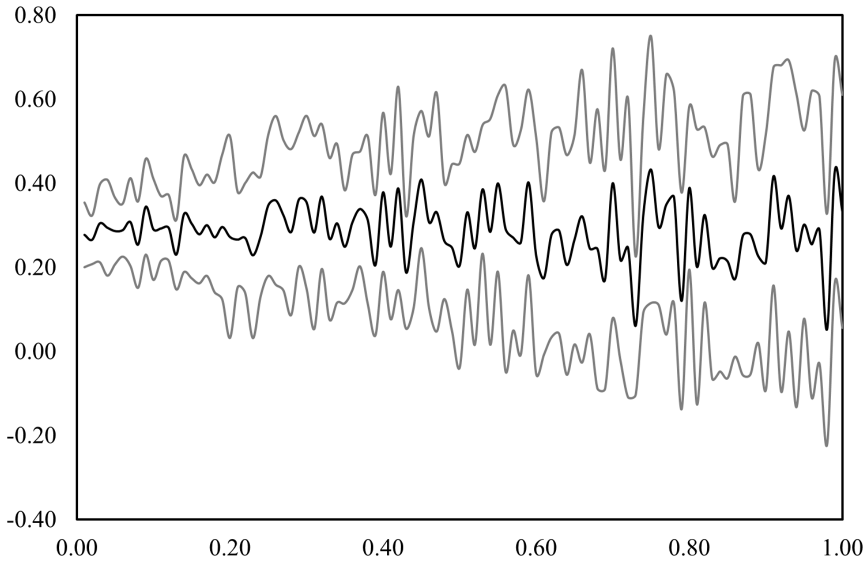

Figure 11: The optimal distribution of $\pi^{\star}_{1,2}(X(t))$ along the time-line.

Fig. 12 shows the stochastic optimal path that enables to achieve controllability in a decentralized or distributed manner. The level of $\pi^{\star}(X(t))$ is of 0.41 at $t = 0$ and of 0.07 at $t = 1$, with a linearized decrease of 0.38. This time, the standard deviation both starts and ends with 0.11, with a wider channel in the middle reaching the peak value of 0.28 at $t = 0.30$. In case of decentralized control, the optimal path reveals similar and narrower latitudes in pricing at the extreme values of the time-line and a wider variability in prices in between. The terminal values of prices once again head toward 0. Likewise, the optimal path suggests the spread of the distribution mean values in time, with higher pricing in early stages and lower pricing in the end.

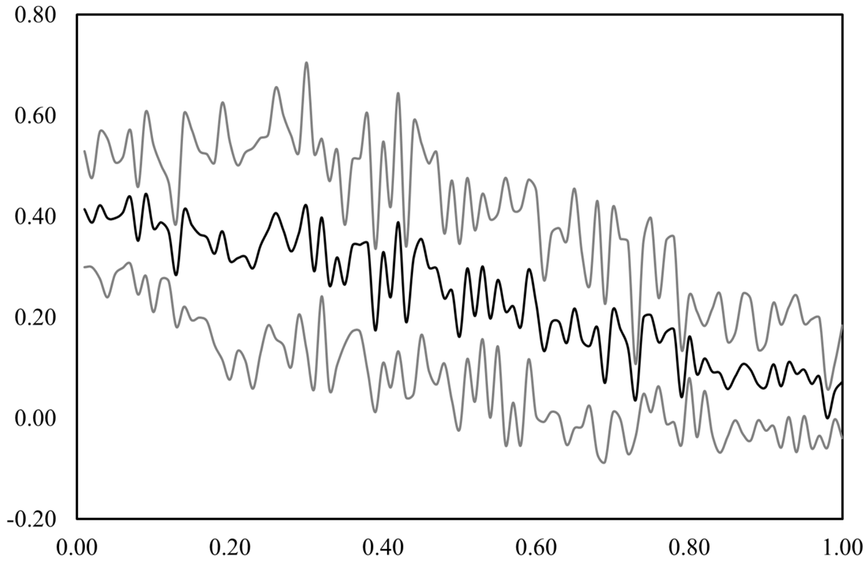

Figure 12: The stochastic optimal path of $\pi^{\star}_{1,2}(X(t))$ along the time-line.

Result 2 *Under decentralized stochastic optimal control of the renewable natural resource, the optimal path of the willingness to undertake the resource stock stabilization in uncertainty is achieved through higher pricing in the launch of the control process and lower pricing at the terminal time. The latitude in pricing is narrow at the cardinal points of the time-line, and large in between.*

4 Conclusion

Such as suggested by Erickson (2009), modeling option values through stochastic processes should help improving decision-making under uncertainty in the renewable natural resource management. By studying shadow pricing of renewable natural resources in a stochastic optimal control framework, we find that the control process takes on the allure of an optional investment, where the co-state distribution comes to resemble the real option value. We have therefore been able to give an interpretation to the distribution of shadow prices. In that respect, two key inputs in the option valuation, which are the value of the underlying asset and its variance, have been derived from the stochastic differential equation subject to optimal control.

Four of our results are consistent with those encountered in the literature on option pricing (Hull, 2011). The first relates to the trajectories of state dynamics, as the evolution of the underlying asset, and of co-state distribution, as the option value, which show similar patterns throughout the time-line. The second observation is about the increasing volatility of the mean value of the co-state distribution, that can be seen in option markets, which aligns with the increasing standard deviation with further expiration date. When it comes to the optimal path of the real option value, the third output applies to the decreasing mean value of the co-state distribution along the optimal path, which matches with the finding that, as the expiration approaches, the time value of the option decreases. With respect to the optimal path, we also find that the standard deviation decreases in time, which corresponds to the fact that, through the resource management control process, the resource stock is intended to be stabilized. In financial markets, this result suggests that, when the investors believe in a non-bearish market trend, a fall in implied volatility is noted.

The main difference between the centralized and decentralized control processes lies in the mean value of the co-state distribution, which is lower in the second case. This signifies that the collective action, in form of a distributed management of natural resources, is less onerous than that programmed under a centralized management. Nevertheless, the greater standard deviation observed in decentralized optimal control also implies higher uncertainty over the project completion. The greater latitude in pricing thus involves lower likelihood on the ability of agents to jointly sustain the renewable resource stock. To conclude on this point, our results ultimately show the presence of a trade-off between the ability to achieve stabilization of the resource in peril at a degree of uncertainty and the cost to bear in this endeavor.

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References

[1] Albeverio, S. and Kurasov, P. (2000). Singular Perturbations of Differential Operators. *London Mathematical Society: Lecture Notes Series*, 271: 111–142.

[2] Allen, E. and Ilic, M. (1999). Forward Contracts and Futures. *Price-Based Commitment Decisions in the Electricity Market*, Springer, Berlin.

[3] Allen, C., Macalady, A., Chenchouni, H., Bachelet, D., McDowell, N., Vennetier, M., Kitzberger, T., Rigling, A., Breshears, D., Hogg, E., Gonzalez, P., Fensham, R., Zhang, Z., Castro, J., Demidova, N., Lim, J.-H., Allard, G., Running, S., Semerci, A., Cobb, N. (2010). A Global Overview of Drought and Heat-Induced Tree Mortality Reveals Emerging Climate Change Risks for Forests. *Forest Ecology and Management*, 259: 660-684.

[4] Brennan, M. and Schwartz, E. (1985). Evaluating Natural Resource Investments. *The Journal of Business*, 58: 135–157.

[5] Bretschger, L. and Smulders, S. (2007). Sustainable Resource Use and Economic Dynamics. *Environmental and Resource Economics*, 36: 1–13.

[6] Conrad, J. (1986). Bioeconomics and the Management of Renewable Resources. *Biomathematics*, 17: 381–403.

[7] Conrad, J. (2010). A Markov Decision Model and Stochastic Dynamic Programming. *Resource Economics*, Cambridge University Press, Cambridge.

[8] Cont, R. and Voltchkova, E. (2005). Integro-Differential Equations for Option Prices in Exponential Lévy models. *Finance and Stochastics*, 9: 299–325.

[9] Cortazar, G., Schwartz, E. and Salinas, M. (1998). Evaluating Environmental Investments: A Real Options Approach. *Management Science*, 44: 1059–1070.

[10] Cox, J., Ross, S. and Rubinstein, M. (1979). Option Pricing: A Simplified Approach. *Journal of Financial Economics*, 7: 229–263.

[11] De Lara, M. and Doyen L. (2008). Sustainable Management of Natural Resources. *Mathematical Models and Methods*, Springer, Berlin.

[12] Dale, V., Joyce, L., McNulty, S., Neilson, R. (2000). The Interplay between Climate Change, Forests, and Disturbances. *Science of the Total Environment*, 262: 201-204.

[13] Erickson, J. (2009). Economics of Renewable Natural Resources. *Economics Interactions with Other Disciplines*, EOLSS Publications, Paris.

[14] FAO (2013). Climate Change Guidelines for Forest Managers. *FAO Forestry Paper No. 172*, Food and Agriculture Organization of the United Nations, Rome.

[15] Feenstra, T., Cesar, H. and Kort, P. (2000). Optimal Control Theory in Environmental Economics. *Handbook of Environmental and Resource Economics*, Edward Elgar, Cheltenham.

[16] Framstad, N. (2006). Arrow-Mangasarian Sufficient Conditions for Controlled Semimartingales. *Stochastic Analysis and Applications* , 24: 929-938.

[17] Hambusch, G. (2008). Essays on Financial Dynamic Optimization Under Uncertainty. *University of Wyoming Economics and Finance*, ProQuest, Ann Arbor.

[18] Hull, J. (2011). Options, Futures and Other Derivatives. *Eight Edition*, Prentice-Hall, Upper Saddle River.

[19] IPCC (2010). Guidance Note for Lead Authors of the IPCC Fifth Assessment Report on Consistent Treatment of Uncertainties. *IPCC Cross-Working Group Meeting*, Jasper Ridge.

[20] Johnson, F., Moore, C., Kendall, W., Dubosky, J., Caithamer, D., Kelley, J. and Williams, B. (1997). Uncertainty and the Management of Mallard Harvests. *Journal of Wildlife Management*, 61: 202–216.

[21] Joyce, L., Haynes, R., White, R., Barbour, R. (2006). Bringing Climate Change into Natural Resource Management. *Proceedings of a Workshop*, General Technical Report PNW-GTR-706, Portland.

[22] Kappen, H. (2005). Linear Theory for Control of Nonlinear Stochastic Systems. *Physical Review Letters*, 95: 1–4.

[23] Kassar, I. and Lasserre, P. (2004). Species Preservation and Biodiversity Value: A Real Options Approach. *Journal of Environmental Economics and Management*, 48: 857–879.

[24] Knapp, P., Soulé, P., Grissino-Mayer, H. (2001). Detecting Potential Regional Effects of Increased Atmospheric CO2 on Growth Rates of Western Juniper. *Global Change Biology*, 7: 903-917.

[25] Le Bel, P. (2005). Optimal Pricing of Biodiverse Natural Resources for Sustainable Economic Growth. *Journal of Development Alternatives and Area Studies*, 24: 5–38.

[26] Lindner, M., Maroschek, M., Netherer, S., Kremer, A., Barbati, A., Garcia-Gonzalo, J., Seidl, R., Delzon, S., Corona, P., Kolström, M., Lexer, M.J., Marchetti, M. (2010). Climate Change Impacts, Adaptive Capacity, and Vulnerability of European Forest Ecosystems. *Forest Ecology and Management*, 259: 698-709.

[27] Lyon, K. (1999). The Costate Variable in Natural Resource Optimal Control Problems. *Natural Resource Modeling*, 12: 413-426.

[28] Millar, C., Stephenson, N., Stephens, S. (2007). Climate change and the Forests of the Future: Managing in the Face of Uncertainty. *Ecological Applications*, 17: 2145-2151.

[29] Milutinovic, D. (2013). Utilizing Stochastic Processes for Computing Distributions of Large-Size Robot Population Optimal Centralized Control. *Distributed Autonomous Robotic Systems*, 83: 359-372.

[30] Nichols, J., Koneff, M., Heglund, P., Knutson, M., Seamans, M., Lyons, J., Morton, J., Jones, M., Boomer, G. and Williams, B. (2011). Climate Change, Uncertainty, and Natural Resource Management. *Journal of Widelife Management*, 75: 6–18.

[31] Palmer, A. and Milutinovic, D. (2011). A Hamiltonian Approach Using Partial Differential Equations for Open-Loop Stochastic Optimal Control. *Proceedings of the American Control Conference*, 29: 2056–2061.

[32] Pereira, F., Fontes, F., Ferreira, M., Pinho, M., Oliveira, V., Costa, E. and Silva, G. (2013). An Optimal Control Framework for Resources Management in Agriculture. *Conference Papers in Mathematics*, Hindawi, Cairo.

[33] Petrosyan, L and Yeung, D. (2007). Subgame-Consistent Cooperative Solutions in Randomly Furcating Stochastic Differential Games. *Mathematical and Computer Modelling*, 45: 1294–1307.

[34] Sabelfeld, K. and Simonov, N. (2013). Random Fields and Stochastic Lagrangian Models: Analysis and Applications in Turbulence and Porous Media. *Numerical and Computational Mathematics*, De Gruyter, Berlin.

[35] Sethi, S. and Thompson, G. (2000). Stochastic Optimal Control. *Optimal Control Theory: Applications to Management Science and Economics*, Springer, New York.

[36] Soulé, P., Knapp, P. (2006). Radial Growth Rate Increases in Naturally Occurring Ponderosa Pine Trees: A Late-20th Century CO2 Fertilization Effect? *New Phytologist*, 171: 379-390.

[37] U.S. Department of the Interior (2004). Cooperative Conservation: Success through Partnerships. *Communication in Conservation of Natural Resources*, U.S. Department of the Interior.

[38] van Soest, D., List, J. and Jeppesen, T. (2006). Shadow prices, Environmental Stringency, and International Competitiveness. *European Economic Review*, 50: 1151–1167.

[39] Trentelman, H. and Willems, J. (1997). Storage Functions for Dissipative Linear Systems are Quadratic State Functions. *Proceedings of the Conference on Decision and Control*, 36: 42–47.

[40] Williams, B. (1985). Optimal Management Strategies in Variable Environments: Stochastic Optimal Control Methods. *Journal of Environmental Management*, 21: 95–115.

[41] Williams, B. (1989). Review of Dynamic Optimization Methods in Renewable Natural Resource Management. *Natural Resource Modeling*, 3: 137–216.

[42] Williams, B., Nichols, J. and Conroy, M. (2002). Analysis and Management of Animal Populations. *Academic Press*, San Diego.

[43] Williams, B. and Johnson, F. (2013). Confronting Dynamics and Uncertainty in Optimal Decision Making for Conservation. *Environmental Research Letters*, 8: 1–16.

[44] Xepapadeas, A. (2011). Stochastic Analysis: Tools for Environmental and Resource Economics Modeling. *Research Tools in Natural Resource and Environmental Economics*, World Scientific, Singapore.

[45] Zvan, R., Forsyth, P. and Vetzal, K. (2001). A Finite Volume Approach for Contingent Claims Valuation. *IMA Journal of Numerical Analysis*, 21: 703–731.