Social Capital and Status Externality

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Abstract

This paper investigates how the presence of social capital affects status externality in a dynamic economy. It is assumed that the stock of social capital is accumulating jointly through jointly social interactions among individuals who are forward looking. In this setting, the presence of social capital mitigates the tendency of overconsumption over time, and hence makes the resulting allocation closer to the Pareto efficient one.

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JEL classification: O40; Q33

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1 Introduction

Social capital: those persistent and shared beliefs and values that help a group overcome the free rider problem in the pursuit of socially valuable activities. Guiso, Sapienza and Zingales (2011).

This concept is argued to pass Solow's (xxx) 4 tests for it to be a proper definition:

- To be distinct from others, in particular human capital;
- To be in principle measurable, even imperfectly;
- To have an own rate of return/payoff, in principle;
- To have a clear process of accumulation and decumulation/depreciation.

GSZ argue that this concept, termed by them 'civic capital', passes these tests.

We use this concept in relation to the status externality and the problem of free-riding that arises there.

GSZ: 'Since we consider as civic capital only values and beliefs that help a group overcome the free rider problem in the pursuit of socially valuable activities, by definition civic capital has a non-negative economic payoff. In other words, civic capital purposefully excludes from the definitions those values that favor cooperation in socially deviant activities, such as gangs.'

GSZ: 'As [...] in the Tabellini (2007b) model, investment in civic capital is the amount of resources that parents spend to teach more cooperative values to their children.'

Problem for us: we do not have an OLG model. But we can adopt/adapt the Varvarigos model?

Key aspect of social/civic capital: It's development by one individual depends on the amount others have or exhibit. GSZ: 'Second, these values and beliefs [NB: those underlying and supporting social/civic capital] do not represent civic capital if they are not shared by other members of the community.'

[It is this interaction that seems fundamental that is missed by the Varvarigos-Xin (2015) model of trust.]

2 The Model

There are n (≥ 2) individuals. The instantaneous utility of individual i depends positively on consumption, c_i, and social status c_i/C. The objective function of individual i is a discounted-sum of utilities over an infinite-time horizon:

$$u_i = \int_0^{\infty} \left[\log c_i + (1 - \theta_i(S)) \log\left(\frac{c_i}{\bar{c}}\right) \right] e^{-\rho t} dt, \quad \rho > 0, \qquad i = 1, 2, ..., n, \tag{1}$$

where the function $\log(c_i/C)$ captures the status externality and C and $\bar{c} = (\Sigma_{i=1}^{n} c_i)/n = C/n$, respectively, represent the aggregate amount of consumption and average consumption. The variable $\theta_i(S)$ represents the perception of social capital by individual i: $\theta_i(S) : R_+ \mapsto [0, 1)$; that is, the extent to which the status externality is internalized due to the spirit of social capital.

Individuals decide at each point in time how much they engage in social interaction as well as how much to consume. Their choices are constrained by a time (or budget) constraint. Consumption is constrained by income, which is earned from working at a wage rate w according to the following flow budget constraint for individual i at each point in time:

$$c_i = w(\bar{n} - a_i),$$

where a_i is the time allocated to accumulation activities for the stock of social capital and $\bar{n}$ is the total available time which is fixed through time. For notational simplicity, we assume that the relative price between private consumption and leisure (i.e., the wage rate), which is set equal to 1. As a result, we have

$$c_i = 1 - a_i, \tag{2}$$

Each individual maximizes u_i in (1) by selecting the sequences of c_i and a_i subject to (2) and the initial level of social capital, S_0 (> 0).

The aggregate stock of social capital will continuously change over time according to

$$\dot{S} = S^{\gamma}(\Sigma_{i=1}^{n} a_i) - \delta S, \tag{3}$$

where δ is a constant depreciation rate of social capital ($0 < \delta < 1$). For analytical simplicity, we also assume that $\gamma \in (0, 1)$.

The model described above is a differential game in which each player's strategies are its consumption and social interaction, which is measured by the time devoted to social activities, while the state variable of the game is the aggregate stock of social capital S. Following the existing studies, we focus on the Markov-perfect Nash (feedback Nash) equilibrium. That is, we assume that each agent's strategies, c_i and a_i, are functions of the current level of the aggregate capital, S, alone. This means that the value function of the i-th agent's optimization problem at time t can be written as

$$V_i\left(S_t\right) \equiv \max \int_t^{\infty} e^{-\rho(\tau-t)} u_i\left(c_\tau, a_\tau, S_t\right) d\tau.$$

This function satisfies the Hamilton-Jacobi-Bellman (HJB) equation such as

$$\rho V_i\left(S\right) = \max_{\{c_i, a_i\}} \left\{ \log c_i + \left(1-\theta_i\left(S\right)\right) \log\left(\frac{c_i}{C/n}\right) + V_i'\left(S\right)\left[S^{\gamma}\left(\Sigma_{j=1}^n\left(1-c_j\right)\right) - \delta S\right] \right\} \tag{4}$$

for all $t \geq 0$. In solving the maximization problem defined in the right-hand-side of (4) at each moment in time, the i-th agent takes the other players' strategies, $\left\{c_j, a_j\right\}_{j\neq i}$ $(j = 1, 2, .., n)$, as given. The first-order conditions for maximization are given by

$$\frac{1}{c_i} + \left(1-\theta_i\left(S\right)\right)\frac{1}{c_i/C}\frac{\Sigma_{j=1,\, j\neq i}^n c_j}{C^2} - V_i'\left(S\right)S^{\gamma} = 0. \tag{5}$$

To get a closed-form solution, we have to impose a special form on the function $\theta_i\left(S\right)$ such as

$$\theta_i\left(S\right) = \begin{cases} \theta_i S & \text{if } S < \bar{S}, \\ 1 & \text{if } \bar{S} \leq S, \end{cases} \tag{6}$$

where $\bar{S} = 1/\theta_i$. With this formulation, we can rewrite the above first order conditions as follows:

$$S^{\gamma} V_i'\left(S\right) = \frac{1}{c_i}\left[1 + \left(1-\theta_i S\right)\frac{\Sigma_{j=1,\, j\neq i}^n c_j}{C}\right]. \tag{7}$$

Equation (7) gives the Markov-perfect Nash solutions expressed as $\left\{c_i\left(S\right), a_i\left(S\right)\right\}$ for $i = 1.2., ..., n$. Substituting these optimal solutions back into the HJB equation (4) associated with agent i, together with (6), we obtain

$$\rho V_i\left(S\right) = \log c_i\left(S\right) + \left(1-\theta_i S\right)\log\left(\frac{c_i\left(S\right)}{C\left(S\right)/n}\right) + V_i'\left(S\right)\left[S^{\gamma}\Sigma_{j=1}^n\left(1-c_j\left(S\right)\right) - \delta S\right], \tag{8}$$

where $C(S) \equiv \Sigma_{j=1}^{n} c_j(S)$. By the use of the envelop theorem, we find that differentiating both sides of (4) with respect to S gives

$$\rho V_i'(S) = \frac{c_i'(S)}{c_i(S)} - \theta_i \log\left(\frac{nc_i(S)}{C(S)}\right) + (1-\theta_i S)\frac{C(S)}{c_i(S)}\frac{c_i'(S)C(S) - c_i(S)\Sigma_{j=1}^{n} c_j'(S)}{[C(S)]^2}$$
$$+V_i''(S)\left[S^\gamma \Sigma_{j=1}^{n}(1-c_j(S)) - \delta S\right] + V_i'(S)\left[\gamma S^{\gamma-1}\Sigma_{j=1}^{n}(1-c_j(S)) - S^\gamma \Sigma_{j=1}^{n} c_j'(S) - \delta\right],$$

$$\rho V_i'(S) = \frac{c_i'(S)}{c_i(S)} - \theta_i \log\left(\frac{nc_i(S)}{C(S)}\right) + (1-\theta_i S)\frac{c_i'(S)C(S) - c_i(S)\Sigma_{j=1}^{n} c_j'(S)}{c_i(S)C(S)}$$
$$+V_i''(S)\left[S^\gamma \Sigma_{j=1}^{n}(1-c_j(S)) - \delta S\right] + V_i'(S)\left[\gamma S^{\gamma-1}\Sigma_{j=1}^{n}(1-c_j(S)) - S^\gamma \Sigma_{j=1}^{n} c_j'(S) - \delta\right].$$

Assuming symmetry

$$\rho V'(S) = \frac{c'(S)}{c(S)} - \theta \log\left(\frac{nc(S)}{nc(S)}\right) + (1-\theta S)\frac{c'(S)nc(S) - c(S)nc'(S)}{c(S)nc(S)}$$
$$+V''(S)\left[S^\gamma n(1-c(S)) - \delta S\right] + V'(S)\left[\gamma S^{\gamma-1}n(1-c(S)) - S^\gamma nc'(S) - \delta\right],$$

Since $\log\left(\frac{nc(S)}{nc(S)}\right) = \log(1) = 0$ and $c'(S)nc(S) - c(S)nc'(S) = 0$,

$$\rho V'(S) = \frac{c'(S)}{c(S)} + V''(S)\left[S^\gamma n(1-c(S)) - \delta S\right]$$
$$+V'(S)\left[\gamma S^{\gamma-1}n(1-c(S)) - S^\gamma nc'(S) - \delta\right],$$

$$\rho V'(S) = \frac{c'(S)}{c(S)} - V'(S)S^\gamma c'(S) + V''(S)\left[S^\gamma n(1-c(S)) - \delta S\right]$$
$$+V'(S)\left[\gamma S^{\gamma-1}n(1-c(S)) - S^\gamma (n-1)c'(S) - \delta\right]. \tag{9}$$

On the other hand, under the assumption of symmetry, we can rewrite (7) as follows:

$$S^\gamma V'(S) = \frac{1}{c(S)}\left[1 + (1-\theta S)\frac{(n-1)c(S)}{nc(S)}\right],$$
$$\therefore S^\gamma V'(S) - \frac{1}{c(S)} = \frac{1}{c(S)}(1-\theta S)\frac{n-1}{n}.$$

Substituting this expression into (9) yields

$$\rho V'(S) = -\frac{c'(S)}{c(S)}(1-\theta S)\frac{n-1}{n} + V''(S)\left[S^\gamma n(1-c(S)) - \delta S\right]$$
$$+V'(S)\left[\gamma S^{\gamma-1}n(1-c(S)) - S^\gamma (n-1)c'(S) - \delta\right].$$

Rewriting the above HJB equation as follows:

$$\begin{aligned}
&\rho V'(S) - V'(S)\left[\gamma S^{\gamma-1} n(1-c(S)) - S^{\gamma}(n-1)c'(S) - \delta\right] \\
&= -\frac{c'(S)}{c(S)}(1-\theta S)\frac{n-1}{n} + V''(S)\left[S^{\gamma} n(1-c(S)) - \delta S\right],
\end{aligned}$$

$$\begin{aligned}
&\left[\rho + \delta - \gamma S^{\gamma-1} n(1-c(S)) + S^{\gamma}(n-1)c'(S)\right] V'(S) \\
&= -\frac{c'(S)}{c(S)}(1-\theta S)\frac{n-1}{n} + V''(S)\left[S^{\gamma} n(1-c(S)) - \delta S\right]. \qquad (10)
\end{aligned}$$

We once again differentiae (7) (i.e., $V_i'(S) = \frac{1}{S^{\gamma} c_i(S)}\left[1 + (1-\theta_i S)\frac{\Sigma_{j=1,\, j\neq i}^{n} c_j(S)}{C(S)}\right]$) with respect to S to get

$$\begin{aligned}
V_i''(S) = \frac{1}{[c_i(S)S^{\gamma}]^2}&\left[\left(-\theta_i \frac{\Sigma_{j=1,\, j\neq i}^{n} c_j(S)}{C(S)} + (1-\theta_i S)\cdot\right.\right. \\
&\left.\left(\frac{\Sigma_{j=1,\, j\neq i}^{n} c_j'(S)C(S) - \Sigma_{j=1,\, j\neq i}^{n} c_j(S)C'(S)}{(C(S))^2}\right)\right) c_i(S)S^{\gamma} \\
&\left. - \left(1 + (1-\theta_i S)\frac{\Sigma_{j=1,\, j\neq i}^{n} c_j(S)}{C(S)}\right)\left(c_i'(S)S^{\gamma} + c_i(S)\gamma S^{\gamma-1}\right)\right]
\end{aligned}$$

By symmetry

$$\begin{aligned}
V''(S) = \frac{1}{[c(S)S^{\gamma}]^2}&\left[\left(-\theta \frac{(n-1)c(S)}{nc(S)} + (1-\theta S)\cdot\right.\right. \\
&\left.\left(\frac{(n-1)c'(S)nc(S) - (n-1)c(S)nc'(S)}{(nc(S))^2}\right)\right) c(S)S^{\gamma} \\
&\left. - \left(1 + (1-\theta S)\frac{(n-1)c(S)}{nc(S)}\right)\left(c'(S)S^{\gamma} + c(S)\gamma S^{\gamma-1}\right)\right]
\end{aligned}$$

Since $(n-1)c'(S)nc(S) - (n-1)c(S)nc'(S) = 0$,

$$\begin{aligned}
V''(S) = \frac{1}{[c(S)S^{\gamma}]^2}&\left[\left(-\theta\frac{n-1}{n}\right) c(S)S^{\gamma}\right. \\
&\left. - \left(1 + (1-\theta S)\frac{n-1}{n}\right)\left(c'(S)S^{\gamma} + c(S)\gamma S^{\gamma-1}\right)\right]
\end{aligned}$$

$$V''(S) = \frac{1}{c(S)S^{\gamma}}\left[\left(-\theta\frac{n-1}{n}\right) - \left(1 + (1-\theta S)\frac{n-1}{n}\right)\left(\frac{c'(S)}{c(S)} + \gamma S^{-1}\right)\right],$$

$$\therefore V''(S) = \frac{1}{c(S)S^{\gamma}}\frac{n-1}{n}\left[-\theta - \left(\frac{n}{n-1} + (1-\theta S)\right)\left(\frac{c'(S)}{c(S)} + \gamma S^{-1}\right)\right]. \qquad (11)$$

Substituting $V'(S) = \frac{1}{cS^{\gamma}}\left[1+(1-\theta S)\frac{n-1}{n}\right]$ and (11) into (10) yields

$$\left[\rho+\delta-\gamma S^{\gamma-1}n(1-c(S))+S^{\gamma}(n-1)c'(S)\right]\frac{1}{c(S)S^{\gamma}}\left[1+(1-\theta S)\frac{n-1}{n}\right]$$
$$=-\frac{c'(S)}{c(S)}(1-\theta S)\frac{n-1}{n}+$$
$$\frac{1}{c(S)S^{\gamma}}\frac{n-1}{n}\left[-\theta-\left(\frac{n}{n-1}+(1-\theta S)\right)\left(\frac{c'(S)}{c(S)}+\gamma S^{-1}\right)\right]\left[S^{\gamma}n(1-c(S))-\delta S\right].$$

Multiplying both sides by $c(S)S^{\gamma}$ yields

$$\left[\rho+\delta-\gamma S^{\gamma-1}n(1-c(S))+S^{\gamma}(n-1)c'(S)\right]\left[1+(1-\theta S)\frac{n-1}{n}\right]$$
$$=-c'(S)(1-\theta S)\frac{n-1}{n}S^{\gamma}+$$
$$\frac{n-1}{n}\left[-\theta-\left(\frac{n}{n-1}+(1-\theta S)\right)\left(\frac{c'(S)}{c(S)}+\gamma S^{-1}\right)\right]\left[S^{\gamma}n(1-c(S))-\delta S\right].$$

Multiplying both sides $n/(n-1)$ yields

$$\left[\rho+\delta-\gamma S^{\gamma-1}n(1-c(S))+S^{\gamma}(n-1)c'(S)\right]\left[\frac{n}{n-1}+(1-\theta S)\right]$$
$$=-c'(S)(1-\theta S)S^{\gamma}+$$
$$\left[-\theta-\left(\frac{n}{n-1}+(1-\theta S)\right)\left(\frac{c'(S)}{c(S)}+\gamma S^{-1}\right)\right]\left[S^{\gamma}n(1-c(S))-\delta S\right].$$

Moreover,

$$\left[\rho+\delta-\gamma S^{\gamma-1}n(1-c(S))\right]\left[\frac{n}{n-1}+(1-\theta S)\right]$$
$$+S^{\gamma}(n-1)c'(S)\left[\frac{n}{n-1}+(1-\theta S)\right]+c'(S)(1-\theta S)S^{\gamma}$$
$$=-\theta\left[S^{\gamma}n(1-c(S))-\delta S\right]$$
$$-\left[\frac{n}{n-1}+(1-\theta S)\right]\frac{c'(S)}{c(S)}\left[S^{\gamma}n(1-c(S))-\delta S\right]$$
$$-\left[\frac{n}{n-1}+(1-\theta S)\right]\gamma S^{-1}\left[S^{\gamma}n(1-c(S))-\delta S\right],$$

$$\left[\frac{n}{n-1}+(1-\theta S)\right]\frac{c'(S)}{c(S)}\left[S^{\gamma}n(1-c\left(S\right))-\delta S\right]$$
$$+S^{\gamma}(n-1)c'\left(S\right)\left[\frac{n}{n-1}+(1-\theta S)\right]+c'\left(S\right)\left(1-\theta S\right)S^{\gamma}$$
$$=-\theta\left[S^{\gamma}n(1-c\left(S\right))-\delta S\right]$$
$$-\left[\frac{n}{n-1}+(1-\theta S)\right]\gamma S^{-1}\left[S^{\gamma}n(1-c\left(S\right))-\delta S\right]$$
$$-\left[\rho+\delta-\gamma S^{\gamma-1}n(1-c\left(S\right))\right]\left[\frac{n}{n-1}+(1-\theta S)\right],$$

$$c'(S)\left[\left(\frac{n}{n-1}+(1-\theta S)\right)\frac{1}{c(S)}\left(S^{\gamma}n(1-c\left(S\right))-\delta S\right)\right.$$
$$\left.+S^{\gamma}(n-1)\left(\frac{n}{n-1}+(1-\theta S)\right)+\left(1-\theta S\right)S^{\gamma}\right]$$
$$=\left[-\theta-\frac{n}{n-1}\gamma S^{-1}-\left(1-\theta S\right)\gamma S^{-1}\right]\left[S^{\gamma}n(1-c\left(S\right))-\delta S\right]$$
$$-\left[\rho+\delta-\gamma S^{\gamma-1}n(1-c\left(S\right))\right]\left(\frac{n}{n-1}+(1-\theta S)\right),$$

$$c'(S)\left[\left(\frac{n}{n-1}+(1-\theta S)\right)\left(\frac{1}{c(S)}\left(S^{\gamma}n(1-c\left(S\right))-\delta S\right)+S^{\gamma}(n-1)\right)+\left(1-\theta S\right)S^{\gamma}\right]$$
$$=\left[-\theta-\frac{n}{n-1}\gamma S^{-1}-\left(1-\theta S\right)\gamma S^{-1}\right]\left[S^{\gamma}n(1-c\left(S\right))-\delta S\right]$$
$$-\left[\rho+\delta-\gamma S^{\gamma-1}n(1-c\left(S\right))\right]\left[\frac{n}{n-1}+(1-\theta S)\right],$$

$$c'(S)\left[\left(\frac{n}{n-1}+(1-\theta S)\right)\left(S^{\gamma}n\frac{1}{c(S)}-S^{\gamma}n-\frac{\delta S}{c(S)}+S^{\gamma}(n-1)\right)+\left(1-\theta S\right)S^{\gamma}\right]$$
$$=-\theta\left[S^{\gamma}n(1-c\left(S\right))-\delta S\right]-\left[\frac{n}{n-1}+(1-\theta S)\right]\gamma S^{-1}\left[S^{\gamma}n(1-c\left(S\right))-\delta S\right]$$
$$-\left[\rho+\delta-\gamma S^{\gamma-1}n(1-c\left(S\right)\right]\left[\frac{n}{n-1}+(1-\theta S)\right],$$

$$c'(S)\left[\left(\frac{n}{n-1}+(1-\theta S)\right)\left(\frac{S^{\gamma}n}{c(S)}-\frac{\delta S}{c(S)}-S^{\gamma}\right)+\left(1-\theta S\right)S^{\gamma}\right]$$
$$=-\left[\frac{n}{n-1}+(1-\theta S)\right]\left[\rho+\delta-\gamma S^{\gamma-1}n(1-c\left(S\right))+\gamma S^{-1}\left(S^{\gamma}n(1-c\left(S\right))-\delta S\right)\right]$$
$$-\theta\left[S^{\gamma}n(1-c\left(S\right))-\delta S\right],$$

$$c'(S)\left[\frac{n}{n-1}\left(\frac{S^{\gamma}n}{c(S)}-\frac{\delta S}{c(S)}-S^{\gamma}\right)+(1-\theta S)\left(\frac{S^{\gamma}n}{c(S)}-\frac{\delta S}{c(S)}-S^{\gamma}\right)+(1-\theta S)\,S^{\gamma}\right]$$
$$=\left[\frac{n}{n-1}+(1-\theta S)\right]\left[\rho+\delta-\gamma S^{-1}\delta S\right]-\theta\left[S^{\gamma}n(1-c\,(S))-\delta S\right],$$

$$c'(S)\left[\frac{n}{n-1}\left(\frac{S^{\gamma}n}{c(S)}-\frac{\delta S}{c(S)}-S^{\gamma}\right)+(1-\theta S)\left(\frac{S^{\gamma}n}{c(S)}-\frac{\delta S}{c(S)}\right)\right]$$
$$=-\left[\frac{n}{n-1}+(1-\theta S)\right]\left[\rho+\delta-\gamma\delta\right]-\theta\left[S^{\gamma}n(1-c\,(S))-\delta S\right],$$

$$c'(S)\left[\left(\frac{n}{n-1}+(1-\theta S)\right)\frac{1}{c(S)}\left(S^{\gamma}n-\delta S\right)-\frac{n}{n-1}S^{\gamma}\right]$$
$$=-\left[\frac{n}{n-1}+(1-\theta S)\right]\left[\rho+(1-\gamma)\delta\right]-\theta\left[S^{\gamma}n(1-c\,(S))-\delta S\right].$$

Denoting the following expression as

$$\frac{n}{n-1}+(1-\theta S)=\frac{n}{n-1}+\frac{n-1}{n-1}-\theta S=\frac{2n-1}{n-1}-\theta S=A-\theta S,$$

$$c'(S)\left[(A-\theta S)\frac{1}{c(S)}\left(S^{\gamma}n-\delta S\right)-\frac{n}{n-1}S^{\gamma}\right]$$
$$=-\left[A-\theta S\right]R-\theta\left[S^{\gamma}n(1-c\,(S))-\delta S\right].$$

where $R\equiv\rho+(1-\gamma)\delta$ and $A=(2n-1)/(n-1)$.

$$\begin{aligned}c'\,(S) &= \frac{-\,(A-\theta S)\,R-\theta\left[S^{\gamma}n(1-c\,(S))-\delta S\right]}{\dfrac{A-\theta S}{c(S)}\left(S^{\gamma}n-\delta S\right)-\dfrac{n}{n-1}S^{\gamma}},\\ &= -\frac{(A-\theta S)\,R+\theta\left[S^{\gamma}n(1-c\,(S))-\delta S\right]}{\dfrac{A-\theta S}{c(S)}\left(S^{\gamma}n-\delta S\right)-\dfrac{n}{n-1}S^{\gamma}}.\end{aligned}\tag{12}$$

Furthermore, since $c\,(S)=1-a\,(S)$, $c'\,(S)=-a'\,(S)$,

$$a'\,(S) = -c'\,(S)=\frac{(A-\theta S)\,R+\theta\left[S^{\gamma}n(1-c\,(S))-\delta S\right]}{\dfrac{A-\theta S}{c(S)}\left(S^{\gamma}n-\delta S\right)-\dfrac{n}{n-1}S^{\gamma}},$$
$$\therefore\;\; a'\,(S)=\frac{(A-\theta S)\,R+\theta\left[S^{\gamma}na\,(S)-\delta S\right]}{\dfrac{A-\theta S}{1-a\,(S)}\left(S^{\gamma}n-\delta S\right)-\dfrac{n}{n-1}S^{\gamma}}.\tag{13}$$

3 Dynamics and Steady State

We will draw the representatives of Markov strategies in a control and state space in order to characterize qualitative solutions to the nonlinear differential equation (13).

We will draw the representatives of Markov strategies in a control and state space in order to characterize qualitative solutions to the nonlinear differential equation (13). To do this, let us denote by C_2 the loci where $\phi'(S)$ goes to plus/minus infinity, and by C_3 the loci where $\phi'(S)$ equals zero in the (S, a) space:

$$\begin{aligned} C_1 &:= \{(S,a) : \dot{S} = S^{\gamma} n \phi(S) - \delta S = 0\}, \\ C_2 &:= \{(S,a) : \phi'(S) \to \pm\infty\}, \\ C_3 &:= \{(S,a) : \phi'(S) = 0\}. \end{aligned} \tag{14}$$

First, we identify the steady state locus where $\dot{S} = 0$ in (3), called C_1 in the following.

$$\begin{aligned} 0 &= S^{\gamma} na - \delta S, \\ S^{\gamma-1} na &= \delta, \\ \therefore a &= \frac{\delta}{n} S^{1-\gamma}. \end{aligned}$$

noting that $a = 0$ at $S = 0$. Moreover, since the slope of the steady state line C_1 is characterized by

$$\begin{aligned} \frac{da(S)}{dS} &= \frac{\delta}{n}(1-\gamma)S^{-\gamma} > 0, \\ \frac{d^2a(S)}{dS^2} &= -\frac{\delta}{n}(1-\gamma)\gamma S^{-\gamma-1} < 0. \end{aligned} \tag{15}$$

These facts together imply that the steady-state line C_1 is a upward-sloping, concave line in the (S, a) space. It starts from the origin and is monotonically increasing in S. Moreover, it immediately follows from (15) not only that the slope of the steady-state line C_1 at the origin becomes plus infinity and goes to zero as S becomes indefinitely larger, but also that it crosses the budget line $a = 1$ at point $\left((n/\delta)^{\frac{1}{1-\gamma}}, 1\right)$. Moreover, it follows from (3) that any strategy $\phi(S)$ above C_1 implies that S declines in time, while any strategy $\phi(S)$ below C_1 entails an increase of Z over time. Taken together, we can draw the graph of the steady-state line C_1 in Figure 1.

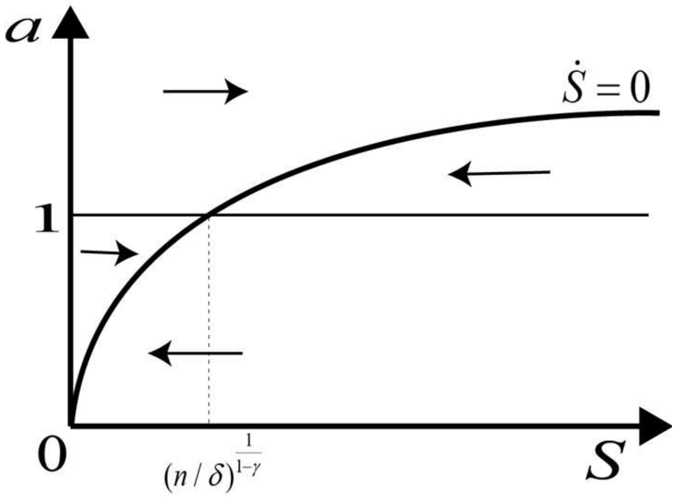


Figure 1: **Steady State Curve**

Let us consider C_2 the loci where $\phi'(S)$ goes to plus/minus infinity, and by C_3 the loci where $\phi'(S)$ equals zero in the (S, c) space. First, setting the denominator in (13) equal to zero, we obtain the locus of the curve C_2:

$$\frac{A - \theta S}{1 - a(S)} (S^\gamma n - \delta S) - \frac{n}{n-1} S^\gamma = 0,$$
$$\frac{A - \theta S}{1 - a(S)} (S^\gamma n - \delta S) = \frac{n}{n-1} S^\gamma,$$
$$(A - \theta S)(S^\gamma n - \delta S) = \frac{n}{n-1} [1 - a(S)] S^\gamma,$$
$$\frac{n}{n-1} [1 - a(S)] S^\gamma = (A - \theta S)(S^\gamma n - \delta S),$$
$$1 - a(S) = \frac{n-1}{n} (A - \theta S) \left(n - \delta S^{1-\gamma}\right),$$
$$\therefore a(S) = 1 - \frac{n-1}{n} (A - \theta S) \left(n - \delta S^{1-\gamma}\right),$$

Furthermore,

$$\begin{aligned}
a(S) &= 1 - \frac{n-1}{n}\left[A\left(n-\delta S^{1-\gamma}\right) - \theta S\left(n-\delta S^{1-\gamma}\right)\right], \\
&= 1 - \frac{n-1}{n}\left[An - A\delta S^{1-\gamma} - \theta S n + \theta\delta S^{2-\gamma}\right], \\
&= 1 - \frac{n-1}{n}\left[An - A\delta S^{1-\gamma} - \theta S n + \theta\delta S^{2-\gamma}\right], \\
&= 1 - A(n-1) - \frac{n-1}{n}\left[-A\delta S^{1-\gamma} - \theta S n + \theta\delta S^{2-\gamma}\right], \\
&= 1 - \frac{2n-1}{n-1}(n-1) + \frac{n-1}{n}\left[A\delta S^{1-\gamma} + \theta S n - \theta\delta S^{2-\gamma}\right], \\
&= 1 - (2n-1) + \frac{n-1}{n}\left[\frac{2n-1}{n-1}\delta S^{1-\gamma} + \theta S n - \theta\delta S^{2-\gamma}\right], \\
&= 2(-n+1) + \left[\frac{2n-1}{n}\delta S^{1-\gamma} + \frac{n-1}{n}\theta S n - \frac{n-1}{n}\theta\delta S^{2-\gamma}\right], \\
&= 2(1-n) + \frac{2n-1}{n}\delta S^{1-\gamma} + (n-1)\theta S - \frac{n-1}{n}\theta\delta S^{2-\gamma}, \\
\therefore a(S) &= 2(1-n) + \frac{2n-1}{n}\delta S^{1-\gamma} + (n-1)\theta S - \frac{n-1}{n}\theta\delta S^{2-\gamma}. \qquad (16)
\end{aligned}$$

which we call 'the non-invertibility locus' following Rowat (2007).

It follows from (16) that as $S \longrightarrow 0$, $a(S) \longrightarrow 2(1-n) < 0$, while as $S \longrightarrow \infty$, $a(S) \rightarrow -\infty$.

Substituting $(n/\delta)^{\frac{1}{1-\gamma}}$ into (16) yields

$$\begin{aligned}
a(S) &= 2(1-n) + \frac{2n-1}{n}\delta\left[(n/\delta)^{\frac{1}{1-\gamma}}\right]^{1-\gamma} + (n-1)\theta(n/\delta)^{\frac{1}{1-\gamma}} - \frac{n-1}{n}\theta\delta\left((n/\delta)^{\frac{1}{1-\gamma}}\right)^{2-\gamma}, \\
&= 2(1-n) + \frac{2n-1}{n}\delta(n/\delta) + (n-1)\theta(n/\delta)^{\frac{1}{1-\gamma}} - \frac{n-1}{n}\theta\delta\left((n/\delta)^{\frac{1}{1-\gamma}}\right)^{1-\gamma}(n/\delta)^{\frac{1}{1-\gamma}}, \\
&= 2(1-n) + (2n-1) + (n-1)\theta(n/\delta)^{\frac{1}{1-\gamma}} - \frac{n-1}{n}\theta\delta(n/\delta)(n/\delta)^{\frac{1}{1-\gamma}}, \\
&= 2(1-n) + (2n-1) + (n-1)\theta(n/\delta)^{\frac{1}{1-\gamma}} - (n-1)\theta(n/\delta)^{\frac{1}{1-\gamma}} \\
&= 2(1-n) + (2n-1) = 2 - 2n + 2n - 1 = 1,
\end{aligned}$$

which implies that the curve C_2 crosses the budget line $a = 1$ at point $((n/\delta)^{\frac{1}{1-\gamma}}, 1)$. Moreover, since there are two intersection points between the curve C_2 and the budget line $a = 1$, because

$$\begin{aligned}
a(S) &= 1 - \frac{n-1}{n}(A-\theta S)\left(n-\delta S^{1-\gamma}\right), \\
1 &= 1 - \frac{n-1}{n}(A-\theta S)\left(n-\delta S^{1-\gamma}\right), \\
0 &= (A-\theta S)\left(n-\delta S^{1-\gamma}\right),
\end{aligned}$$

which yields

$$\begin{aligned} S &= A/\theta = \frac{2n-1}{\theta(n-1)}, \\ S &= (n/\delta)^{\frac{1}{1-\gamma}}. \end{aligned}$$

Moreover, since the intersection point between the curve C_2 and the horizontal axis (i.e., $a = 0$) is given by

$$\begin{aligned} 0 &= 2(1-n) + \frac{2n-1}{n}\delta S^{1-\gamma} + (n-1)\theta S - \frac{n-1}{n}\theta\delta S^{2-\gamma}, \\ -2(1-n) &= \frac{2n-1}{n}\delta S^{1-\gamma} + (n-1)\theta S - \frac{n-1}{n}\theta\delta S^{2-\gamma}, \\ 2n &= \frac{2n-1}{n-1}\delta S^{1-\gamma} + n\theta S - \theta\delta S^{2-\gamma}, \end{aligned}$$

it is difficult to get an explicit solution in terms of S. Instead, substitute $S = \frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta(\rho+(2-\gamma)\delta)}$ into the curve C_2 yields

$$\begin{aligned} a &= 2(1-n) + \frac{2n-1}{n}\delta S^{1-\gamma} + (n-1)\theta S - \frac{n-1}{n}\theta\delta S^{2-\gamma} \\ &= 2(1-n) + \frac{2n-1}{n}\delta\left[\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta(\rho+(2-\gamma)\delta)}\right]S^{-\gamma} \\ &+ (n-1)\theta\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta(\rho+(2-\gamma)\delta)} - \frac{n-1}{n}\theta\delta\left[\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta(\rho+(2-\gamma)\delta)}\right]^2 S^{-\gamma} \\ &= 2(1-n) + \frac{2n-1}{n}\delta\left[\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta(\rho+(2-\gamma)\delta)}\right]S^{-\gamma} + \theta(2n-1)\frac{\rho+(1-\gamma)\delta}{\theta(\rho+(2-\gamma)\delta)} \\ &- \frac{2n-1}{n}\delta\left[\frac{1}{n-1}\frac{\rho+(1-\gamma)\delta}{\rho+(2-\gamma)\delta}\right]\left[\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta(\rho+(2-\gamma)\delta)}\right]S^{-\gamma} \\ &= 2(1-n) + \theta(2n-1)\frac{\rho+(1-\gamma)\delta}{\theta(\rho+(2-\gamma)\delta)} \\ &+ \frac{2n-1}{n}\delta\left[\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta(\rho+(2-\gamma)\delta)}\right]S^{-\gamma}\left[1 - \frac{1}{n-1}\frac{\rho+(1-\gamma)\delta}{\rho+(2-\gamma)\delta}\right] > 0 \end{aligned}$$

since

$$1 - \frac{1}{n-1}\frac{\rho+(1-\gamma)\delta}{\rho+(2-\gamma)\delta} = \frac{(n-1)(\rho+(2-\gamma)\delta) - (\rho+(1-\gamma)\delta)}{(n-1)(\rho+(2-\gamma)\delta)} > 0.$$

This implies that the curve C_2 is located above the horizontal axis at $S = \frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta(\rho+(2-\gamma)\delta)}$, as ilusarated in Figure 2.

On the other hand, the slope of the curve C_2 is given by

$$\begin{aligned}
\frac{da(S)}{dS} &= \frac{2n-1}{n}\delta\,(1-\gamma)\,S^{-\gamma}+(n-1)\,\theta-\frac{n-1}{n}\theta\delta\,(2-\gamma)\,S^{1-\gamma}, \\
&= (2n-1)\,\frac{\delta}{n}\,(1-\gamma)\,S^{-\gamma}+(n-1)\,\theta\left[1-\frac{\delta}{n}\,(2-\gamma)\,S^{1-\gamma}\right], \\
&= [(2n-1)\,(1-\gamma)-(n-1)\,\theta\,(2-\gamma)\,S]\,\frac{\delta}{n}S^{-\gamma}+(n-1)\,\theta,
\end{aligned}$$

which has the following slope at $(n/\delta)^{\frac{1}{1-\gamma}}$:

$$\begin{aligned}
&\frac{da(S)}{dS} = \left[(2n-1)\,(1-\gamma)-(n-1)\,\theta\,(2-\gamma)\,(n/\delta)^{\frac{1}{1-\gamma}}\right]\frac{\delta}{n}S^{-\gamma}+(n-1)\,\theta, \\
&= (n-1)\,\theta\left[\frac{2n-1}{(n-1)\,\theta}\,(1-\gamma)-(2-\gamma)\,(n/\delta)^{\frac{1}{1-\gamma}}\right]\frac{\delta}{n}S^{-\gamma}+(n-1)\,\theta \gtreqless 0, \\
&\iff (1-\gamma)\left[\frac{2n-1}{(n-1)\,\theta}-\frac{2-\gamma}{1-\gamma}\,(n/\delta)^{\frac{1}{1-\gamma}}\right]\frac{\delta}{n}\,(n/\delta)^{\frac{-\gamma}{1-\gamma}}+1 \gtreqless 0 \\
&\Longrightarrow \left[\frac{2n-1}{(n-1)\,\theta}-\frac{2-\gamma}{1-\gamma}\,(n/\delta)^{\frac{1}{1-\gamma}}\right](n/\delta)^{\frac{1-2\gamma}{1-\gamma}}+\frac{1}{1-\gamma}>0 \ \text{ iff } \ \frac{2n-1}{(n-1)\,\theta}>(n/\delta)^{\frac{1}{1-\gamma}}.
\end{aligned}$$

whose slope may be positive or ambiguous, while the curve C_2 has the following slope at $S=\dfrac{2n-1}{(n-1)\,\theta}$:

$$\begin{aligned}
\frac{da(S)}{dS} &= \left[(2n-1)\,(1-\gamma)-(n-1)\,\theta\,(2-\gamma)\,\frac{2n-1}{(n-1)\,\theta}\right]\frac{\delta}{n}\left[\frac{2n-1}{(n-1)\,\theta}\right]^{-\gamma}+(n-1)\,\theta, \\
&= [(2n-1)\,(1-\gamma)-(2-\gamma)\,(2n-1)]\,\frac{\delta}{n}\left[\frac{2n-1}{(n-1)\,\theta}\right]^{-\gamma}+(n-1)\,\theta, \\
&= (2n-1)\,[(1-\gamma)-(2-\gamma)]\left[\frac{2n-1}{(n-1)\,\theta}\right]^{-\gamma}+(n-1)\,\theta, \\
&= (2n-1)\,[-1]\left[\frac{(n-1)\,\theta}{2n-1}\right]^{\gamma}+(n-1)\,\theta, \\
&= -\,(2n-1)\left[\frac{(n-1)\,\theta}{2n-1}\right]^{\gamma}+(n-1)\,\theta,
\end{aligned}$$

which implies that

$$\begin{aligned}
\text{When } \gamma &= 1,\ \frac{da(S)}{dS}=-\,(2n-1)\left[\frac{(n-1)\,\theta}{2n-1}\right]+(n-1)\,\theta, \\
&= -\,(n-1)\,\theta+(n-1)\,\theta=0, \\
\text{When } \gamma &= 0,\ \frac{da(S)}{dS}=-\,(2n-1)+(n-1)\,\theta<0.
\end{aligned}$$

Moreover,

$$\frac{d^2a(S)}{dSd\gamma}<0 \ \text{ because } \ \frac{(n-1)\,\theta}{2n-1}<1.$$

Taken together, the slope of the curve C_2 at $S = \dfrac{2n-1}{(n-1)\,\theta}$ is negative (see Figure 2). This implies that the curve C_2 crosses the budget line $a = 1$ at the left point of interesection $\left(\dfrac{2n-1}{(n-1)\,\theta}, 1\right)$ from the below, whereas it crosses the budget line $a = 1$ at the right point of interesection $\left(\dfrac{2n-1}{(n-1)\,\theta}, 1\right)$ from the above. Moroever,

$$\frac{d^2a(S)}{dS^2} = -\frac{2n-1}{n}\delta\,(1-\gamma)\,\gamma S^{-\gamma-1} - \frac{n-1}{n}\theta\delta\,(2-\gamma)\,(1-\gamma)\,S^{-\gamma} < 0.$$

As $S \longrightarrow 0$, $da(S)/dS \longrightarrow \infty$, whereas as $S \longrightarrow \infty$, $da(S)/dS \longrightarrow -\infty$. This implies that its slope is positive when S is small, then its slope becomes negative when S is large.

Taken together, the nonlinear curve C_2 displays an inverse U-shape, which intersects the vertical axis at $(0, 2\,(1-n))$. The curve C_2 increases for smaller values of S and then decreases and crosses the horizontal axis at $S = (n/\delta)^{\frac{1}{1-\gamma}}$ and $\dfrac{2n-1}{\theta\,(n-1)}$.

The locus C_3 is obtained by setting the numerator in (12) equal to zero. Solving for a gives the following locus:

$$\begin{aligned}
AR - \theta\,(\rho + (2-\gamma)\delta)\,S + \theta S^{\gamma} na &= 0,\\
-AR + \theta\,(\rho + (2-\gamma)\delta)\,S &= \theta S^{\gamma} na,\\
\theta S^{\gamma} na &= -AR + \theta\,(\rho + (2-\gamma)\delta)\,S,\\
a &= -\frac{AR}{\theta S^{\gamma} n} + \frac{\theta\,(\rho + (2-\gamma)\delta)\,S}{\theta S^{\gamma} n},\\
\therefore a &= -\frac{2n-1}{n-1}\frac{\rho + (1-\gamma)\delta}{\theta n}S^{-\gamma} + \frac{\rho + (2-\gamma)\delta}{n}S^{1-\gamma}.
\end{aligned}$$

As $S \longrightarrow 0$, $a \longrightarrow -\infty$, while as $S \longrightarrow \infty$, $a \longrightarrow \infty$. In particular, it crosses the horizontal axis at point $\left(\dfrac{2n-1}{n-1}\dfrac{\rho + (1-\gamma)\delta}{\theta\,(\rho + (2-\gamma)\delta)}, 0\right)$, since

$$\begin{aligned}
0 &= -\frac{2n-1}{n-1}\frac{\rho + (1-\gamma)\delta}{\theta n}S^{-\gamma} + \frac{\rho + (2-\gamma)\delta}{n}S^{1-\gamma},\\
0 &= -\frac{2n-1}{n-1}\frac{\rho + (1-\gamma)\delta}{\theta n} + \frac{\rho + (2-\gamma)\delta}{n}S,\\
0 &= -\frac{2n-1}{n-1}\frac{\rho + (1-\gamma)\delta}{\theta} + [\rho + (2-\gamma)\delta]\,S,\\
\frac{2n-1}{n-1}\frac{\rho + (1-\gamma)\delta}{\theta} &= [\rho + (2-\gamma)\delta]\,S,\\
\therefore S &= \frac{\frac{2n-1}{n-1}\frac{\rho + (1-\gamma)\delta}{\theta}}{\rho + (2-\gamma)\delta} = \frac{2n-1}{n-1}\frac{\rho + (1-\gamma)\delta}{\theta\,(\rho + (2-\gamma)\delta)} > 0.
\end{aligned}$$

Moreover, the intersection between the curve C_3 and the budget line $a = 1$ is given by

$$
\begin{aligned}
1 &= -\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta n}S^{-\gamma} + \frac{\rho+(2-\gamma)\delta}{n}S^{1-\gamma}, \\
n &= -\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta}S^{-\gamma} + [\rho+(2-\gamma)\delta]\,S^{1-\gamma}, \\
nS^{\gamma} &- (\rho+(2-\gamma)\delta)\,S + \frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta} = 0,
\end{aligned}
$$

it is difficult to solve the above equation for S. Instead of it, we substitute $S = (n/\delta)^{\frac{1}{1-\gamma}}$ into S on the right-hand side of C_3, thus yielding

$$
\begin{aligned}
a &= -\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta n}S^{-\gamma} + \frac{\rho+(2-\gamma)\delta}{n}S^{1-\gamma}, \\
a((n/\delta)^{\frac{1}{1-\gamma}}) &= -\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta n}(n/\delta)^{\frac{-\gamma}{1-\gamma}} + \frac{\rho+(2-\gamma)\delta}{n}\left[(n/\delta)^{\frac{1}{1-\gamma}}\right]^{1-\gamma}, \\
&= -\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta n}(n/\delta)^{\frac{1-\gamma}{1-\gamma}}(n/\delta)^{\frac{-1}{1-\gamma}} + \frac{\rho+(2-\gamma)\delta}{n}(n/\delta), \\
&= -\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta n}(n/\delta)(n/\delta)^{\frac{-1}{1-\gamma}} + \frac{\rho+(2-\gamma)\delta}{\delta}, \\
&= -\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta\delta}(n/\delta)^{\frac{-1}{1-\gamma}} + \frac{\rho+(2-\gamma)\delta}{\delta}, \\
&= \frac{1}{\delta}\left[-\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta} + (\rho+(2-\gamma)\delta)\,(n/\delta)^{\frac{1}{1-\gamma}}\right](n/\delta)^{\frac{-1}{1-\gamma}} \\
&= \frac{1}{\delta}(\rho+(2-\gamma)\delta)\left[-\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta(\rho+(2-\gamma)\delta)} + (n/\delta)^{\frac{1}{1-\gamma}}\right](n/\delta)^{\frac{-1}{1-\gamma}} \gtreqless 0, \\
&\text{if } \frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta(\rho+(2-\gamma)\delta)} \lesseqgtr (n/\delta)^{\frac{1}{1-\gamma}},
\end{aligned}
$$

which implies that the curve C_3 may or may not be located above the horizontal axis at $S = (n/\delta)^{\frac{1}{1-\gamma}}$ depending on the relative size of $\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta(\rho+(2-\gamma)\delta)}$ and $(n/\delta)^{\frac{1}{1-\gamma}}$.

When $\gamma = 1$, it is clear that $a > 0$.

When $\gamma = 0$,

$$
\begin{aligned}
\frac{2n-1}{n-1}\frac{\rho+\delta}{\theta(\rho+2\delta)} &= (n/\delta), \\
(2n-1)\frac{\rho+\delta}{\theta(\rho+2\delta)} &= \frac{1}{\delta}n(n-1),
\end{aligned}
$$

whose solutions are $\frac{1}{4\theta\delta+2\theta\rho}\left(2\delta^2+2\theta\delta+\theta\rho+2\delta\rho-\sqrt{4\delta^4+4\theta^2\delta^2+\theta^2\rho^2+4\delta^2\rho^2+8\delta^3\rho+4\theta^2\delta\rho}\right)$

and $\frac{1}{4\theta\delta+2\theta\rho}\left(2\delta^2+2\theta\delta+\theta\rho+2\delta\rho+\sqrt{4\delta^4+4\theta^2\delta^2+\theta^2\rho^2+4\delta^2\rho^2+8\delta^3\rho+4\theta^2\delta\rho}\right)$.

$$\begin{aligned}
\text{If } \frac{1}{4\theta\delta+2\theta\rho}\left(2\delta^2+2\theta\delta+\theta\rho+2\delta\rho+\sqrt{4\delta^4+4\theta^2\delta^2+\theta^2\rho^2+4\delta^2\rho^2+8\delta^3\rho+4\theta^2\delta\rho}\right) &< n, \\
\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta(\rho+(2-\gamma)\delta)} &> (n/\delta)^{\frac{1}{1-\gamma}}, \\
\text{If } \frac{1}{4\theta\delta+2\theta\rho}\left(2\delta^2+2\theta\delta+\theta\rho+2\delta\rho+\sqrt{4\delta^4+4\theta^2\delta^2+\theta^2\rho^2+4\delta^2\rho^2+8\delta^3\rho+4\theta^2\delta\rho}\right) &> n, \\
\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta(\rho+(2-\gamma)\delta)} &< (n/\delta)^{\frac{1}{1-\gamma}}.
\end{aligned}$$

On the other hand,

$$\frac{da(S)}{dS}=\gamma\frac{AR}{\theta n}S^{-\gamma}+(1-\gamma)\frac{\rho+(2-\gamma)\delta}{n}S^{-\gamma}>0,$$

and

$$\frac{d^2a(S)}{dS^2}=-\gamma^2\frac{AR}{\theta n}S^{-\gamma-1}+(1-\gamma)(-\gamma)\frac{(\rho+(2-\gamma)\delta)}{n}S^{-\gamma-1}<0.$$

which shows that the nonlinear curve C_3 is a upward-sloping, concave line in the (S,c) space. The curve C_3 never intersects the vertical axis and thus goes to minus infinity as S approaches zero, while it goes to plus infinity as S becomes indefinitely large.

Since the intersection point between the curve C_2 and the curve C_3 is given by

$$\begin{aligned}
a(S) &= 2(1-n)+\frac{2n-1}{n}\delta S^{1-\gamma}+(n-1)\theta S-\frac{n-1}{n}\theta\delta S^{2-\gamma}, \\
a(S) &= -\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta n}S^{-\gamma}+\frac{\rho+(2-\gamma)\delta}{n}S^{1-\gamma}.
\end{aligned}$$

$$\begin{aligned}
& 2(1-n)+\left[\frac{2n-1}{n}\delta-\frac{\rho+(2-\gamma)\delta}{n}\right]S^{1-\gamma}+(n-1)\theta S-\frac{n-1}{n}\theta\delta S^{2-\gamma} \\
= & -\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta n}S^{-\gamma}
\end{aligned}$$

$$\begin{gathered}
2(1-n)n+[(2n-1)\delta-(\rho+(2-\gamma)\delta)]S^{1-\gamma}+n(n-1)\theta S-(n-1)\theta\delta S^{2-\gamma} \\
+\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta}S^{-\gamma}=0,
\end{gathered}$$

it is difficult to get an explicit solution for S.

On the other hand, the intersection point between the curve C_3 and the steady state line is given by

$$\begin{aligned}
a(S) &= \frac{\delta}{n}S^{1-\gamma}, \\
a(S) &= -\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta n}S^{-\gamma}+\frac{\rho+(2-\gamma)\delta}{n}S^{1-\gamma}.
\end{aligned}$$

$$\begin{aligned}
\frac{\delta}{n}S^{1-\gamma} &= -\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta n}S^{-\gamma}+\frac{\rho+(2-\gamma)\delta}{n}S^{1-\gamma}, \\
0 &= -\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta n}S^{-\gamma}+\frac{\rho+(1-\gamma)\delta}{n}S^{1-\gamma}, \\
\frac{2n-1}{n-1}\frac{\rho+(1-\gamma)\delta}{\theta n}S^{-\gamma} &= \frac{\rho+(1-\gamma)\delta}{n}S^{1-\gamma}, \\
&\therefore \quad \frac{2n-1}{n-1}=S,
\end{aligned}$$

which implies

$$a(S)=\frac{\delta}{n}\left(\frac{2n-1}{n-1}\right)^{1-\gamma}.$$

Taking natutal algarithm yields

$$\ln a(S)=\ln\frac{\delta}{n}+(1-\gamma)\left[\ln(2n-1)-\ln(n-1)\right]$$

When $\gamma=1$,

$$\ln a(S)=\ln\frac{\delta}{n}<0,$$

which implies $a(S)<1$, whereas when $\gamma=0$,

$$\begin{aligned}
\ln a(S) &= \ln\frac{\delta}{n}+\left[\ln(2n-1)-\ln(n-1)\right], \\
\ln a(S) &= \ln\frac{\delta}{n}-\ln(n-1)+\ln(2n-1), \\
\ln a(S) &= \ln\frac{\delta}{n(n-1)}+\ln(2n-1), \\
&= \ln\frac{\delta(2n-1)}{n(n-1)}>0 \text{ if } \frac{\delta(2n-1)}{n(n-1)}>1,
\end{aligned}$$

In order to focus on the maximum value for $\frac{\delta(2n-1)}{n(n-1)}$, setting $\delta=1$ yields

$$\frac{(2n-1)}{n(n-1)}\gtreqless 1,$$

Suppose that

$$\begin{aligned}
\frac{(2n-1)}{n(n-1)} &< 1, \\
2n-1 &< n^2-n, \\
n^2-3n+1 &> 0
\end{aligned}$$

$$n^2-3n+1=0$$

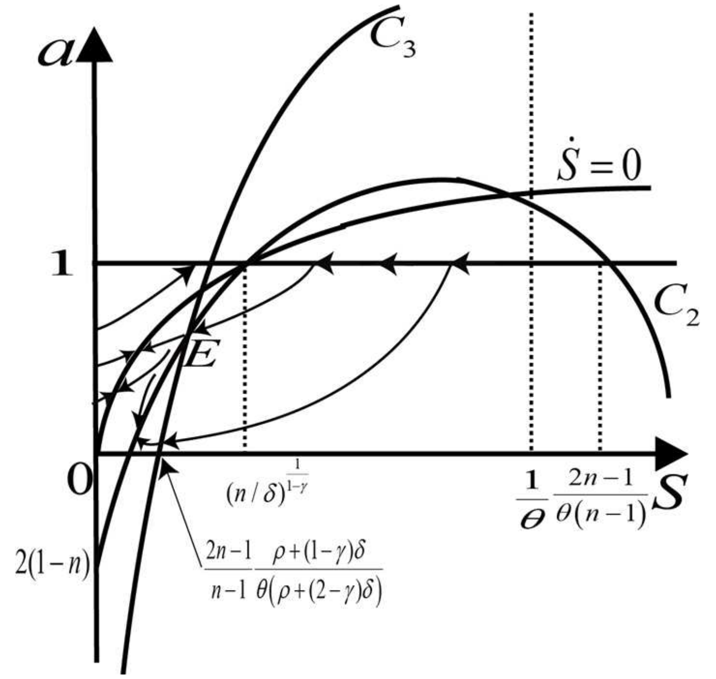


Figure 2:

whose solutions are

$$\begin{aligned} n &= \frac{3}{2}+\frac{1}{2}\sqrt{5}=2.618 \\ &= \frac{3}{2}-\frac{1}{2}\sqrt{5}=0.381\,97 \end{aligned}$$

which implies that if $n \geq 3$, $\dfrac{(2n-1)}{n\,(n-1)}<1$ and thus $\ln a(S)<0$ thus $a(S)<1$.

As a result, we can illustrate Figure 2. Moreover, the point of intersection between C_2 and C_3, labelled E, will be situated in the nonnegative quadrant of the (S,a) plane: $(S_E,a_E)=(,)$,which is called 'a singular point'. Note, however, that point E may be located below or above the resource constraint (2), since the value of a_E should be less than 1. as illustrated in Figure 2. Moreover, the point of intersection between C_2 and C_3, labelled E, will be situated in the nonnegative quadrant of the (S,a) plane:

$$(S_E,a_E)=(,)\,, \tag{17}$$

which is called 'a singular point'. Moreover, it follows from (3) that any strategy $\phi\,(S)$ above C_1 implies that S declines in time, while any strategy $\phi\,(S)$ below C_1 entails an increase of S over time.

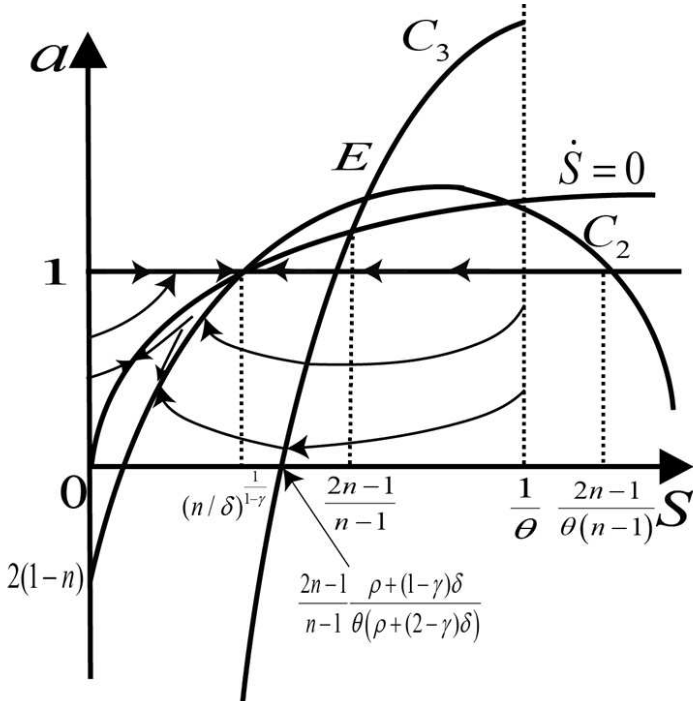


Figure 3:

Furthermore, by direct integration of (13) and manipulating we can obtain a general solution to the differential equation (13):

$$\phi(S) =, \tag{18}$$

where c_1 represents an arbitrary constant of integration and may take a positive, zero or negative value. When $c_1 = 0$, (18) simplifies to[1]

$$\phi_L(S) =, \tag{19}$$

It is seen form Figs 2 and 3 that the left branch of the linear strategy ϕ_L to the steady state line C_1 starts from the origin, and then reaches point S on C_1, while its right branch starts from any $S \geq \tilde{S}$, then reaching point S also. The right branch of ϕ_L goes through the singular point E.

[1] Long and Shimomura (1998) show that in the class of differential games if the integrand of the objective function is homogeneous of degree α and if constraints that are homogeneous of degree one, then the best replies to linear homogenous Markov strategies played the rivals are *linear homogenous*. The resulting linear strategy in our model is consistent with their characterization.

Collecting the arguments, we can illustrate an uncountable number of the curves corresponding to the (interior) solutions satisfying the HJB equation (4) in Figs. 1 and 2. These figures display representatives of those integral curves that are divided into five types of the families of strategies. Arrows on the *families* of integral curves ϕ_j, $j = 1, \ldots, 4$, and ϕ_L illustrate the evolution of S over time.

Furthermore, by direct integration of (13) and manipulating we can obtain a general solution to the differential equation (13):

$$\phi(S) =, \tag{20}$$

where c_1 represents an arbitrary constant of integration and may take a positive, zero or negative value. When $c_1 = 0$, (18) simplifies to[2]

$$\phi_L(S) =, \tag{21}$$

It is seen form Figs 1 and 2 that the left branch of the linear strategy ϕ_L to the steady state line C_1 starts from the origin, and then reaches point S on C_1, while its right branch starts from any $S \geq \tilde{S}$, then reaching point S also. The right branch of ϕ_L goes through the singular point E.

4 The Alternative Model

In this section, we assume that the utility weight of each player has the following form: $\theta_i(S) = e^{-S}$. The steady state locus locus C_1 is the same as before. The first-order conditions for maximization are modified as follows:

$$\begin{aligned} &\frac{1}{c_i} + (1 - e^{-S}) \frac{\Sigma_{j=1,\ j \neq i}^{n} c_j}{c_i C} - V_i'(S) S^{\gamma} = 0, \quad i = 1, 2., ..., n, \\ &\therefore S^{\gamma} V_i'(S) = \frac{1}{c_i}\left[1 + (1 - e^{-S}) \frac{\Sigma_{j=1,\ j \neq i}^{n} c_j}{C}\right], \quad i = 1.2., ..., n, \end{aligned} \tag{22}$$

[2] Long and Shimomura (1998) show that in the class of differential games if the integrand of the objective function is homogeneous of degree α and if constraints that are homogeneous of degree one, then the best replies to linear homogenous Markov strategies played the rivals are *linear homogenous*. The resulting linear strategy in our model is consistent with their characterization.

Equation (5) gives the Markov-perfect Nash solutions expressed as $\{c_i(S), a_i(S)\}$. Substituting these optimal solutions into the HJB equation (4), we obtain

$$\rho V_i(S) = \log c_i(S) + (1-e^{-S}) \log\left(\frac{c_i(S)}{C(S)/n}\right) + V_i'(S)\left[S^\gamma \Sigma_{j=1}^n (1-c_j(S)) - \delta S\right], \quad (23)$$

where $C(S) \equiv \Sigma_{j=1}^n c_j(S)$. By use of the envelop theorem, we find that differentiation of both sides of (23) with respect to S gives

$$\rho V_i'(S) = \frac{c_i'(S)}{c_i(S)} - e^{-S}\log\left(\frac{c_i(S)}{C(S)/n}\right) + (1-e^{-S})\frac{C(S)}{nc_i(S)} n \frac{c_i'(S)C(S) - c_i(S)\Sigma_{j=1}^n c_j'(S)}{[C(S)]^2}$$
$$+V_i''(S)\left[S^\gamma \Sigma_{j=1}^n (1-c_j(S)) - \delta S\right] + V_i'(S)\left[\gamma S^{\gamma-1}\Sigma_{j=1}^n (1-c_j(S)) - S^\gamma \Sigma_{j=1}^n c_j'(S) - \delta\right],$$

$$\rho V_i'(S) = \frac{c_i'(S)}{c_i(S)} - e^{-S}\log\left(\frac{c_i(S)}{C(S)/n}\right) + (1-e^{-S})\frac{c_i'(S)C(S) - c_i(S)\Sigma_{j=1}^n c_j'(S)}{c_i(S)C(S)}$$
$$+V_i''(S)\left[S^\gamma \Sigma_{j=1}^n (1-c_j(S)) - \delta S\right] + V_i'(S)\left[\gamma S^{\gamma-1}\Sigma_{j=1}^n (1-c_j(S)) - S^\gamma \Sigma_{j=1}^n c_j'(S) - \delta\right].$$

Assuming symmetry

$$\rho V'(S) = \frac{c_i'(S)}{c_i(S)} - e^{-S}\log\left(\frac{nc(S)}{nc(S)}\right) + (1-e^{-S})\frac{c'(S)nc(S) - c(S)nc'(S)}{c(S)nc(S)}$$
$$+V''(S)\left[S^\gamma n(1-c(S)) - \delta S\right] + V'(S)\left[\gamma S^{\gamma-1} n(1-c(S)) - S^\gamma nc'(S) - \delta\right],$$

Since $\log\left(\frac{nc(S)}{nc(S)}\right) = \log(1) = 0$,

$$\rho V_i'(S) = \frac{c_i'(S)}{c_i(S)} + V''(S)\left[S^\gamma n(1-c(S)) - \delta S\right] + V'(S)\left[\gamma S^{\gamma-1} n(1-c(S)) - S^\gamma nc'(S) - \delta\right].$$

Differentiating (22) (i.e., $V_i'(S) = \frac{1}{c_i S^\gamma}\left[1 + (1-e^{-S})\Sigma_{j=1,\, j\neq i}^n c_j\right]$) with respect to S yields

$$V_i''(S) = \frac{1}{[c_i(S)S^\gamma]^2}\left[\left(e^{-S}\Sigma_{j=1,\, j\neq i}^n c_j(S) + (1-e^{-S})\Sigma_{j=1,\, j\neq i}^n c_j'(S)\right)c_i(S)S^\gamma\right.$$
$$\left. - \left(1 + (1-e^{-S})\Sigma_{j=1,\, j\neq i}^n c_j\right)\left(c_i'(S)S^\gamma + c_i(S)\gamma S^{\gamma-1}\right)\right]$$

By symmetry

$$V''(S) = \frac{1}{[c(S)S^\gamma]^2}\left[\left(e^{-S}(n-1)c(S) + (1-e^{-S})(n-1)c'(S)\right)c(S)S^\gamma\right.$$
$$\left. - \left(1 + (1-e^{-S})(n-1)c(S)\right)\left(c'(S)S^\gamma + c(S)\gamma S^{\gamma-1}\right)\right]$$

$$V''(S) = \frac{(n-1)}{c(S)S^{\gamma}} \left[\left(e^{-S} c(S) + (1-e^{-S}) c'(S) \right) \right.$$
$$\left. - \left(\frac{1}{n-1} + (1-e^{-S}) c(S) \right) \left(\frac{c'(S)}{c(S)} + \gamma S^{-1} \right) \right]$$

$$V''(S) = \frac{n-1}{S^{\gamma}} \left[\left(e^{-S} + (1-e^{-S}) \frac{c'(S)}{c(S)} \right) \right.$$
$$\left. - \left(\frac{1}{c(S)(n-1)} + (1-e^{-S}) \right) \left(\frac{c'(S)}{c(S)} + \gamma S^{-1} \right) \right]$$

$$V''(S) = \frac{(n-1)}{S^{\gamma}} \left[e^{-S} + (1-e^{-S}) \frac{c'(S)}{c(S)} \right.$$
$$\left. - \frac{1}{c(S)(n-1)} \left(\frac{c'(S)}{c(S)} + \gamma S^{-1} \right) - (1-e^{-S}) \left(\frac{c'(S)}{c(S)} + \gamma S^{-1} \right) \right]$$
$$V''(S) = \frac{(n-1)}{S^{\gamma}} \left[e^{-S} - \frac{1}{c(S)(n-1)} \left(\frac{c'(S)}{c(S)} + \gamma S^{-1} \right) - (1-e^{-S}) \gamma S^{-1} \right].$$

Moreover, under symmetry the first order condition is given by

$$V'(S) = \frac{1}{cS^{\gamma}} \left[1 + \left(1 - e^{-S}\right) \frac{n-1}{n} \right].$$

Substituting these terms yields

$$\rho V'(S) = \left[\frac{1}{c(S)} - V'(S) S^{\gamma} \right] c'(S) + V''(S) \left[S^{\gamma} n (1 - c(S)) - \delta S \right]$$
$$+ V'(S) \left[\gamma S^{\gamma-1} n (1 - c(S)) - S^{\gamma}(n-1) c'(S) - \delta \right],$$

$$\rho V'(S) = -\left(1 - e^{-S}\right) \frac{n-1}{n} + V''(S) \left[S^{\gamma} n (1 - c(S)) - \delta S \right]$$
$$+ V'(S) \left[\gamma S^{\gamma-1} n (1 - c(S)) - S^{\gamma}(n-1) c'(S) - \delta \right],$$

$$\left[\rho + \delta - \gamma S^{\gamma-1} n (1 - c(S)) + S^{\gamma}(n-1) \right] V'(S) =$$
$$-\left(1 - e^{-S}\right) \frac{n-1}{n} + V''(S) \left[S^{\gamma} n (1 - c(S)) - \delta S \right].$$

Since

$$V_i''(S) = \frac{(n-1)}{S^{\gamma}} \left[e^{-S} - \frac{1}{c(S)(n-1)} \left(\frac{c'(S)}{c(S)} + \gamma S^{-1} \right) \right.$$
$$\left. - (1 - e^{-S}) \gamma S^{-1} \right]$$

$$\left[\rho+\delta-\gamma S^{\gamma-1}n(1-c\left(S\right))+S^{\gamma}(n-1)\right]\cdot$$
$$\frac{1}{c\left(S\right)S^{\gamma}}\left[1+\left(1-e^{-S}\right)\frac{n-1}{n}\right]=$$
$$-\left(1-e^{-S}\right)\frac{n-1}{n}+\left[S^{\gamma}n(1-c\left(S\right))-\delta S\right]\frac{(n-1)}{S^{\gamma}}\cdot$$
$$\left[e^{-S}-\frac{1}{c(S)(n-1)}\left(\frac{c'(S)}{c(S)}+\gamma S^{-1}\right)-(1-e^{-S})\gamma S^{-1}\right]$$

Multiplying both sides by $S^{\gamma}c\left(S\right)$, we have

$$\left[\rho+\delta-\gamma S^{\gamma-1}n(1-c\left(S\right))+S^{\gamma}(n-1)\right]\left[1+\left(1-e^{-S}\right)\frac{n-1}{n}\right]$$
$$=-\left(1-e^{-S}\right)\frac{n-1}{n}S^{\gamma}c\left(S\right)+\left[S^{\gamma}n(1-c\left(S\right))-\delta S\right](n-1)\cdot$$
$$\left[c(S)e^{-S}-\frac{1}{(n-1)}\left(\frac{c'(S)}{c(S)}+\gamma S^{-1}\right)-c(S)(1-e^{-S})\gamma S^{-1}\right]$$

Multiplying both sides by $\dfrac{n}{n-1}$, we have

$$\left[\rho+\delta-\gamma S^{\gamma-1}n(1-c\left(S\right))+S^{\gamma}(n-1)\right]\left[\frac{n}{n-1}+\left(1-e^{-S}\right)\right]$$
$$=-\left(1-e^{-S}\right)S^{\gamma}c\left(S\right)+n\left[S^{\gamma}n(1-c\left(S\right))-\delta S\right]\cdot$$
$$\left[c(S)e^{-S}-\frac{1}{(n-1)}\left(\frac{c'(S)}{c(S)}+\gamma S^{-1}\right)-c(S)(1-e^{-S})\gamma S^{-1}\right]$$

$$\left[\rho+\delta-\gamma S^{\gamma-1}n(1-c\left(S\right))+S^{\gamma}(n-1)\right]\left[\frac{n}{n-1}+\left(1-e^{-S}\right)\right]=$$
$$-\left(1-e^{-S}\right)S^{\gamma}c\left(S\right)+n\left[S^{\gamma}n(1-c\left(S\right))-\delta S\right]\cdot$$
$$\left[c(S)\left(e^{-S}-(1-e^{-S})\gamma S^{-1}\right)-\frac{1}{(n-1)}\left(\frac{c'(S)}{c(S)}+\gamma S^{-1}\right)\right]$$

$$-n\left[S^{\gamma}n(1-c\left(S\right))-\delta S\right]\frac{1}{(n-1)}\left(\frac{c'(S)}{c(S)}+\gamma S^{-1}\right)$$
$$=-\left[\rho+\delta-\gamma S^{\gamma-1}n(1-c\left(S\right))+S^{\gamma}(n-1)\right]\left[\frac{n}{n-1}+\left(1-e^{-S}\right)\right]$$
$$-\left(1-e^{-S}\right)S^{\gamma}c\left(S\right)+n\left[S^{\gamma}n(1-c\left(S\right))-\delta S\right]c(S)\left[e^{-S}-(1-e^{-S})\gamma S^{-1}\right]$$

$$n\left[S^{\gamma}n(1-c\left(S\right))-\delta S\right]\frac{1}{(n-1)}\left(\frac{c'(S)}{c(S)}+\gamma S^{-1}\right)$$
$$=\left[\rho+\delta-\gamma S^{\gamma-1}n(1-c\left(S\right))+S^{\gamma}(n-1)\right]\left[\frac{n}{n-1}+\left(1-e^{-S}\right)\right]$$
$$+\left(1-e^{-S}\right)S^{\gamma}c\left(S\right)+n\left[S^{\gamma}n(1-c\left(S\right))-\delta S\right]c(S)\left[e^{-S}-(1-e^{-S})\gamma S^{-1}\right]$$

$$
\begin{aligned}
&n\left[S^{\gamma}n(1-c\left(S\right))-\delta S\right]\frac{1}{(n-1)}\frac{c'(S)}{c(S)}\\
&=-n\left[S^{\gamma}n(1-c\left(S\right))-\delta S\right]\frac{1}{(n-1)}\gamma S^{-1}\\
&\left[\rho+\delta-\gamma S^{\gamma-1}n(1-c\left(S\right))+S^{\gamma}(n-1)\right]\left[\frac{n}{n-1}+\left(1-e^{-S}\right)\right]\\
&+\left(1-e^{-S}\right)S^{\gamma}c\left(S\right)+n\left[S^{\gamma}n(1-c\left(S\right))-\delta S\right]c(S)\left[e^{-S}-(1-e^{-S})\gamma S^{-1}\right]
\end{aligned}
$$

Dividing both sides by $n\left[S^{\gamma}n(1-c\left(S\right))-\delta S\right]$

$$
\begin{aligned}
&\frac{1}{(n-1)}\frac{c'(S)}{c(S)}=-\frac{1}{(n-1)}\gamma S^{-1}\\
&\frac{\rho+\delta-\gamma S^{\gamma-1}n(1-c\left(S\right))+S^{\gamma}(n-1)}{n\left[S^{\gamma}n(1-c\left(S\right))-\delta S\right]}\left[\frac{n}{n-1}+\left(1-e^{-S}\right)\right]\\
&+\frac{\left(1-e^{-S}\right)S^{\gamma}c\left(S\right)}{n\left[S^{\gamma}n(1-c\left(S\right))-\delta S\right]}+c(S)\left[e^{-S}-(1-e^{-S})\gamma S^{-1}\right]
\end{aligned}
$$

Multiplying both sides by $(n-1)$

$$
\begin{aligned}
\frac{c'(S)}{c(S)}&=-\gamma S^{-1}\frac{\rho+\delta-\gamma S^{\gamma-1}n(1-c\left(S\right))+S^{\gamma}(n-1)}{n\left[S^{\gamma}n(1-c\left(S\right))-\delta S\right]}\left[n+(n-1)\left(1-e^{-S}\right)\right]\\
&+\frac{(n-1)\left(1-e^{-S}\right)S^{\gamma}c\left(S\right)}{n\left[S^{\gamma}n(1-c\left(S\right))-\delta S\right]}+(n-1)c(S)\left[e^{-S}-(1-e^{-S})\gamma S^{-1}\right]
\end{aligned}
$$

5 Competitive Statics

6 Concluding Remarks

The first message of this paper is that completely aggressive behavior is not necessarily a rational strategy for a contender in anarchic situations. Rather, every contender will individually and voluntarily choose “partial cooperation”, in which each contender devotes individual resources both to productive and appropriation activities at the same time, even though contenders act fully rational and are guided by their self-interest. The primary driving force is the **durability** of the common-pool stock in conjunction with the forward looking behavior of **patient** contenders. These intrinsically dynamic ingredients induce every contender to behave cooperatively to some extent, even without punishments and threats. In other words, unless the stock of common-accessible goods depreciates completely each period, contenders are completely myopic or the initial stock level is huge, contenders are always motivated to follow a cooperative behavior in producing that good.

The second major finding is that even if nonlinear Markov strategies are available, there is a unique MPE strategy in the class of continuos, globally defined strategies. This result is in sharp contrast with the results of Dockner and van Long (1993), and Rowat (2007) which provide multiplicity of equilibrium strategies and uncountable many long run equilibria including the better outcomes supported by the nonlinear MPE strategies. However, it remains an open question as to the extent to which the uniqueness result of our model may be model-specific or robust under different contest success, production or/and objective functions.

The model presented in this paper should be developed further in several directions. In particular, introducing asymmetric agents into the present model enables us to compare the results of the present model with those of the static conflict model played by asymmetric agents. Another interesting research agenda is to investigate non-Markovian equilibria supported by history-dependent strategies such as trigger ones in the present model, which would support multiple and more efficient, peaceful equilibria (see Benhabib and Radner, 1992).

Appendix A

Setting $\gamma = 1$ in (13) leads to

$$\begin{aligned} a'(S) &= \frac{(A-\theta S)\,R + \theta\,[Sna(S) - \delta S]}{\dfrac{A-\theta S}{1-a(S)}(Sn - \delta S) - \dfrac{n}{n-1}S}, \\ &= \frac{(A-\theta S)\,\dfrac{\rho}{S} + \theta\,[na(S) - \delta]}{\dfrac{A-\theta S}{1-a(S)}(n-\delta) - \dfrac{n}{n-1}}, \end{aligned} \tag{A.1}$$

where $R \equiv \rho$ and $A = (2n-1)/(n-1)$.

In this case, the steady state curve $\dot{S} = 0$ is reduced to

$$a = \frac{\delta}{n}S^{1-\gamma} = \frac{\delta}{n} > 0.$$

Turn to C_2. Setting the denominator in $(A.1)$ equal to zero yields the following straight line:

$$\frac{A-\theta S}{1-a\,(S)}\,(n-\delta)-\frac{n}{n-1}=0,$$
$$\frac{A-\theta S}{1-a}\,(n-\delta)=\frac{n}{n-1},$$
$$(A-\theta S)\,(n-\delta)=\frac{n}{n-1}\,(1-a)\,,$$
$$1-a=\frac{n-1}{n}\,(A-\theta S)\,(n-\delta)$$
$$\therefore a=1-\frac{n-1}{n}\,(A-\theta S)\,(n-\delta)\,,$$

with the following properties:

$$\begin{cases} a\neq 1, \\ a=1 \text{ only when } A-\theta S=0, \end{cases}$$

$$\frac{da}{dS}=\frac{n-1}{n}\theta\,(n-\delta)>0,$$

$$\begin{aligned}
\lim_{S\to 0} a &= 1-\frac{n-1}{n}A\,(n-\delta)=1-\frac{n-1}{n}\frac{2n-1}{n-1}\,(n-\delta) \\
&= 1-\frac{2n-1}{n}\,(n-\delta)=1-\frac{1}{n}\left(2n^2-2n\delta-n+\delta\right), \\
&= 1-\left(2n-2\delta-1+\frac{\delta}{n}\right), \\
&= 2\left[1-n+\left(1-\frac{1}{2n}\right)\delta\right]<2\left[1-n+\delta\right]<0, \text{ since } n\geq 2 \text{ and } \delta<1.
\end{aligned}$$

$$\lim_{S\to\infty} a=\lim_{S\to\infty}\left[1-\frac{n-1}{n}\,(A-\theta S)\,(n-\delta)\right]=\infty,$$

while when $a=0$,

$$0=1-\frac{n-1}{n}\,(A-\theta S)\,(n-\delta)\,,$$
$$1=\frac{n-1}{n}\,(A-\theta S)\,(n-\delta),$$
$$\frac{n}{(n-1)(n-\delta)}=A-\theta S,$$
$$\theta S=A-\frac{n}{(n-1)(n-\delta)},$$
$$\therefore S=\frac{2n-1}{\theta(n-1)}-\frac{n}{\theta(n-1)(n-\delta)},$$
$$S=\frac{1}{\theta(n-1)}\left[2n-1-\frac{n}{n-\delta}\right]>\frac{1}{\theta(n-1)}\left[2n-1-\frac{n}{n-1}\right]>0,\ \text{since } \delta<1.$$

Indeed,

$$\begin{aligned}
& \frac{1}{\theta(n-1)}\left[2n-1-\frac{n}{n-1}\right] \\
= & \frac{1}{\theta(n-1)}\left[\frac{(2n-1)(n-1)-n}{n-1}\right] \\
= & \frac{1}{\theta(n-1)}\left[\frac{2n^2-4n+1}{n-1}\right] \\
= & \frac{2}{\theta(n-1)}\left[\frac{n^2-2n+(1/2)}{n-1}\right] \\
= & \frac{2}{\theta(n-1)}\left[\frac{(n-1)^2-1+(1/2)}{n-1}\right] \\
= & \frac{2}{\theta(n-1)}\left[\frac{(n-1)^2-(1/2)}{n-1}\right] \quad > 0 \text{ since } n \geqq 2.
\end{aligned}$$

When $a = 1$,

$$\begin{aligned}
1 &= 1-\frac{n-1}{n}(A-\theta S)(n-\delta) > 0, \\
0 &= -\frac{n-1}{n}(A-\theta S)(n-\delta) \\
&\therefore \quad S = A/\theta = \frac{2n-1}{\theta(n-1)}.
\end{aligned}$$

When $a = \delta/n$,

$$\frac{\delta}{n} = 1-\frac{n-1}{n}(A-\theta S)(n-\delta),$$

$$\delta = n-(n-1)\left(\frac{2n-1}{n-1}-\theta S\right)(n-\delta),$$

$$\delta = n-(2n-1-(n-1)\,\theta S)(n-\delta),$$

$$\delta = n-(2n-1)(n-\delta)-(-(n-1)\,\theta S)(n-\delta),$$

$$\delta = n-(2n-1)(n-\delta)+(n-1)\,\theta S\,(n-\delta),$$

$$\delta-n+(2n-1)(n-\delta) = (n-1)\,\theta S\,(n-\delta),$$

$$(n-\delta)[-1+(2n-1)] = (n-1)\,\theta S\,(n-\delta),$$

$$2(n-\delta)(n-1) = (n-1)\,\theta S\,(n-\delta),$$

$$2 = \theta S, \text{ since } n \neq \delta,$$

$$\therefore S = \theta/2,$$

Taken together, the curve C_2 is the straight line starting from point $(0, 1 - \frac{2n-1}{n}(n-\delta))$ (note also that $1 - \frac{2n-1}{n}(n-\delta) < 0$) and its slope is given by $\frac{n-1}{n}\theta(n-\delta) > 0$. This is illustrated in Figure 3.

The locus C_3 is obtained by setting the numerator in (A.1) equal to zero. Solving for a gives the following locus:

$$
\begin{aligned}
(A - \theta S)\frac{\rho}{S} + \theta\left[na(S) - \delta\right] &= 0, \\
\theta\left[na(S) - \delta\right] &= -\left(\frac{2n-1}{n-1}\frac{\rho}{S} - \theta\rho\right), \\
na(S) - \delta &= -\frac{1}{\theta}\left(\frac{2n-1}{n-1}\frac{\rho}{S} - \theta\rho\right), \\
na(S) &= \delta - \frac{1}{\theta}\frac{2n-1}{n-1}\frac{\rho}{S} + \rho, \\
a(S) &= -\frac{1}{\theta n}\frac{2n-1}{n-1}\frac{\rho}{S} + \frac{\rho+\delta}{n}, \\
\therefore a &= -\frac{2n-1}{n-1}\frac{\rho}{\theta n}S^{-1} + \frac{\rho+\delta}{n},
\end{aligned}
$$

with the following properties:

$$
\begin{aligned}
\lim_{S\to 0} a &= \lim_{S\to 0}\left[-\frac{2n-1}{n-1}\frac{\rho}{\theta n}S^{-1} + \frac{\rho+\delta}{n}\right] = -\infty, \\
\lim_{S\to\infty} a &= \lim_{S\to\infty}\left[-\frac{2n-1}{n-1}\frac{\rho}{\theta n}S^{-1} + \frac{\rho+\delta}{n}\right] = \frac{\rho+\delta}{n}, \\
\frac{da}{dS} &= \frac{2n-1}{n-1}\frac{\rho}{\theta n}S^{-2} > 0,
\end{aligned}
$$

and

$$
\frac{d^2a}{dS^2} = -\frac{2n-1}{n-1}2\frac{\rho}{\theta n}S^{-3} < 0.
$$

When $a = 0$,

$$
\begin{aligned}
0 &= -\frac{2n-1}{n-1}\frac{\rho}{\theta n}S^{-1} + \frac{\rho+\delta}{n}, \\
(2n-1)\frac{\rho}{\theta}S^{-1} &= (\rho+\delta)(n-1), \\
(2n-1)\frac{\rho}{\theta} &= (\rho+\delta)(n-1)S, \\
(\rho+\delta)(n-1)S &= (2n-1)\frac{\rho}{\theta}, \\
\therefore S = \frac{(2n-1)\frac{\rho}{\theta}}{(\rho+\delta)(n-1)} &= \frac{\rho(2n-1)}{\theta(\rho+\delta)(n-1)}.
\end{aligned}
$$

When $a = 1$,

$$\begin{aligned}
1 &= -\frac{2n-1}{n-1}\frac{\rho}{\theta n}S^{-1} + \frac{\rho+\delta}{n}, \\
n &= -\frac{2n-1}{n-1}\frac{\rho}{\theta}S^{-1} + \rho + \delta, \\
nS &= -\frac{2n-1}{n-1}\frac{\rho}{\theta} + (\rho+\delta)S, \\
\frac{2n-1}{n-1}\frac{\rho}{\theta} &= -nS + (\rho+\delta)S, \\
\frac{2n-1}{n-1}\frac{\rho}{\theta} &= \left[-n + (\rho+\delta)\right]S,
\end{aligned}$$

whose equality is impossible, because the left-hand side is positive, whereas the right-hand side is negative. This implies that the curve C_3 never crosses the line $a = 1$. The curve C_3 goes to minus infinity as S approaches 0, while it is monotonically increasing in S, and then approaches the horizontal line $\frac{\rho+\delta}{n}$.

Moreover, in order to compare the intersection point between the horizontal axis and the straight line C_2 with the intersection between the horizontal axis and the curve C_3 we take the difference between the values of S at these two intersections:

$$\begin{aligned}
&\frac{2n-1}{\theta(n-1)} - \frac{n}{\theta(n-1)(n-\delta)} - \frac{\rho\,(2n-1)}{\theta\,(\rho+\delta)\,(n-1)}, \\
&= \frac{2n-1}{\theta(n-1)}\left[1 - \frac{\rho}{\rho+\delta}\right] - \frac{n}{\theta(n-1)(n-\delta)}, \\
&= \frac{2n-1}{\theta(n-1)}\left[\frac{\rho+\delta-\rho}{\rho+\delta}\right] - \frac{n}{\theta(n-1)(n-\delta)}, \\
&= \frac{2n-1}{\theta(n-1)}\frac{\delta}{\rho+\delta} - \frac{n}{\theta(n-1)(n-\delta)}, \\
&= \frac{1}{\theta(n-1)}\left[(2n-1)\frac{\delta}{\rho+\delta} - \frac{n}{n-\delta}\right] \gtreqless 0.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
(2n-1)\frac{\delta}{\rho+\delta} - \frac{n}{n-\delta} &= 0, \\
(2n-1)\,(n-\delta)\,\delta &= n\,(\rho+\delta)\,, \\
(2n-1)\,(n-\delta)\,\delta - n\,(\rho+\delta) &= 0, \\
(2n-1)\,(n-\delta)\,\delta - n\,(\rho+\delta) &= 0, \\
2n^2\delta - 2n\delta^2 - 2n\delta - \rho n + \delta^2 &= 0,
\end{aligned}$$

whose solution is given by $\left\{\begin{array}{c} \frac{1}{2\delta}\left(\delta+\frac{1}{2}\rho+\delta^2-\frac{1}{2}\sqrt{4\delta^2+4\delta^4+\rho^2+4\delta\rho+4\delta^2\rho}\right), \\ \frac{1}{4\delta}\left(2\delta+\rho+2\delta^2+\sqrt{4\delta^2+4\delta^4+\rho^2+4\delta\rho+4\delta^2\rho}\right) \end{array}\right\}$

Since n should be positive, there is one solution and thus the positive solution is given by

$$
\begin{aligned}
n &= \frac{2\delta+\rho+2\delta^2+\sqrt{4\delta^2+4\delta^4+\rho^2+4\delta\rho+4\delta^2\rho}}{4\delta}, \\
&= \frac{1+(\rho/2\delta)+\delta+\sqrt{1+\delta^2+(\rho^2/4\delta^2)+(\rho/\delta)+\rho}}{2},
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\frac{2n-1}{\theta(n-1)}-\frac{n}{\theta(n-1)(n-\delta)}-\frac{\rho(2n-1)}{\theta(\rho+\delta)(n-1)}<0 \\
&\text{if } n<\frac{1+(\rho/2\delta)+\delta+\sqrt{1+\delta^2+(\rho^2/4\delta^2)+(\rho/\delta)+\rho}}{2}, \\
&\frac{2n-1}{\theta(n-1)}-\frac{n}{\theta(n-1)(n-\delta)}-\frac{\rho(2n-1)}{\theta(\rho+\delta)(n-1)}>0 \\
&\text{if } \; n>\frac{1+(\rho/2\delta)+\delta+\sqrt{1+\delta^2+(\rho^2/4\delta^2)+(\rho/\delta)+\rho}}{2}.
\end{aligned}
$$

In other words, the intersection point between the curve C_3 and the horizontal axis is located at the right of the intersection point between the straight line C_2 and the horizontal axis if

$$
\frac{1+(\rho/2\delta)+\delta+\sqrt{1+\delta^2+(\rho^2/4\delta^2)+(\rho/\delta)+\rho}}{2}<n,
$$

whereas the intersection point between the curve C_3 and the horizontal axis is located at the left of the intersection point between the straight line C_2 and the horizontal axis if

$$
\frac{1+(\rho/2\delta)+\delta+\sqrt{1+\delta^2+(\rho^2/4\delta^2)+(\rho/\delta)+\rho}}{2}>n.
$$

Note the following fact: when $\delta=0.1$and $\rho=0.03$

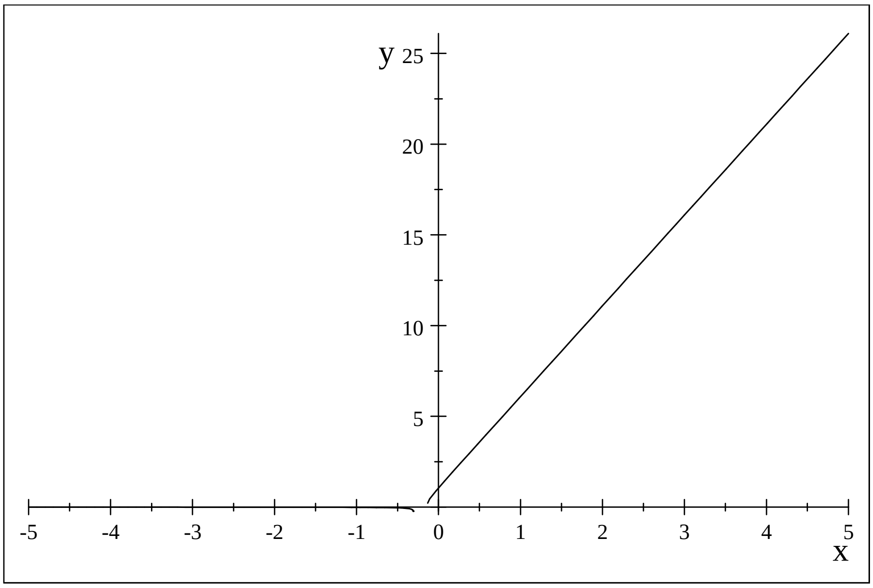
$$
\frac{0.2+0.03+2(0.1)^2+\sqrt{4(0.1)^2+4(0.1)^4+(0.03)^2+4(0.1)(0.03)+4(0.1)^2 0.03}}{0.4}
$$

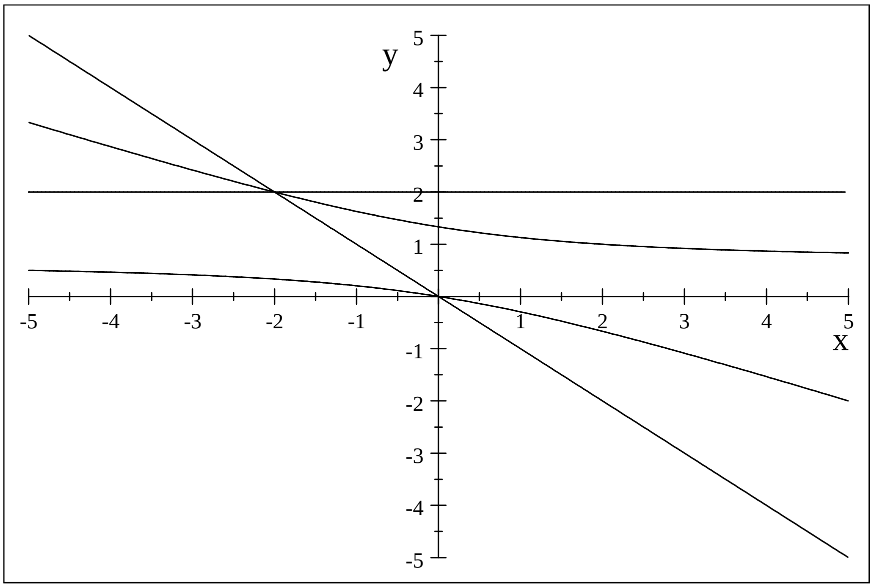
whose solution is given by 1.2086, which implies that when $n\geq 2$, it always holds that

$$
\frac{2n-1}{\theta(n-1)}-\frac{n}{\theta(n-1)(n-\delta)}-\frac{\rho(2n-1)}{\theta(\rho+\delta)(n-1)}<0\ .
$$

When $\delta=0.1$, the function of ρ is given by the following form:

$$
\frac{0.2+\rho+2(0.1)^2+\sqrt{4(0.1)^2+4(0.1)^4+\rho^2+4(0.1)\rho+4(0.1)^2\rho}}{0.4}<n.
$$





Since the maximum of δ is given by 1,

$$\frac{2\delta + 0.03 + 2\delta^2 + \sqrt{4\delta^2 + 4\delta^4 + (0.03)^2 + 0.12\delta + 0.12\delta^2}}{4\delta} < 2 \text{ when } \delta \in [0,1]$$

We investigate how the curves C_2 and C_3 intersect each other. To this end, we compute the intersection point:

$$
\begin{aligned}
a &= 1-\frac{n-1}{n}\left(A-\theta S\right)\left(n-\delta\right), \\
a &= -\frac{2n-1}{n-1}\frac{\rho}{\theta n}S^{-1}+\frac{\rho+\delta}{n}.
\end{aligned}
$$

$$
\begin{aligned}
1-\frac{n-1}{n}\left(A-\theta S\right)\left(n-\delta\right) &= -\frac{2n-1}{n-1}\frac{\rho}{\theta n}S^{-1}+\frac{\rho+\delta}{n}, \\
nS-(n-1)\left(AS-\theta S^2\right)(n-\delta) &= -\frac{2n-1}{n-1}\frac{\rho}{\theta}+S\left(\rho+\delta\right), \\
nS-(n-1)\,AS\,(n-\delta)+(n-1)\,\theta S^2\,(n-\delta)-S\,(\rho+\delta)+\frac{2n-1}{n-1}\frac{\rho}{\theta} &= 0, \\
(n-1)\,\theta S^2\,(n-\delta)+\left[n-(n-1)\,\frac{2n-1}{n-1}\,(n-\delta)-(\rho+\delta)\right]S+\frac{2n-1}{n-1}\frac{\rho}{\theta} &= 0, \\
(n-1)\,\theta\,(n-\delta)\,S^2+\left[n-(2n-1)\,(n-\delta)-(\rho+\delta)\right]S+\frac{2n-1}{n-1}\frac{\rho}{\theta} &= 0,
\end{aligned}
$$

When $S=1/\theta$

$$
\begin{aligned}
(n-1)\,(n-\delta)\,\frac{1}{\theta}+\left(n-(2n-1)\,(n-\delta)-(\rho+\delta)\right)\frac{1}{\theta}+\frac{2n-1}{n-1}\frac{\rho}{\theta} &= 0, \\
(n-1)\,(n-\delta)+\left(n-(2n-1)\,(n-\delta)-(\rho+\delta)\right)+\frac{2n-1}{n-1}\rho &= 0, \\
(n-1)^2\,(n-\delta)+\left(n-(2n-1)\,(n-\delta)-(\rho+\delta)\right)(n-1)+(2n-1)\,\rho &= 0, \\
f(n)=\delta-n-2n\delta+n\rho+n^2\delta+2n^2-n^3 &= 0,
\end{aligned}
$$

$$
\begin{aligned}
f(2) &= \delta-n-4\delta+2\rho+4\delta+6-8 \\
&= \delta-n+2\rho-2<0, \\
\frac{df}{dn} &= -1-2\delta+\rho+2n\delta+4n-3n^2 \\
&= -1-2\delta+\rho+n(2\delta+4-3n)<0 \text{ for } n\geq 2,
\end{aligned}
$$

which implies that $1-\frac{n-1}{n}\left(A-\theta S\right)\left(n-\delta\right)<-\frac{2n-1}{n-1}\frac{\rho}{\theta n}S^{-1}+\frac{\rho+\delta}{n}$ at $S=1/\theta$.

$$
\therefore PS^2+QS+R=0,
$$

where

$$\begin{aligned}
P &= (n-1)\,\theta\,(n-\delta) > 0, \\
Q &= n-(2n-1)\,(n-\delta)-(\rho+\delta)\,, \\
&= 2n-\delta+2n\delta-2n^2-(\rho+\delta)\,, \\
&= 2n\,(-n+1+\delta)-(\rho+2\delta) < 0,\ \text{since } n \geq 2 \text{ and } \delta < 1 \\
R &= \frac{2n-1}{n-1}\frac{\rho}{\theta} > 0.
\end{aligned}$$

The solution is given by

$$S = \frac{-Q \pm \sqrt{Q^2-4PR}}{2P}$$

The discriminant is given by

$$\begin{aligned}
Q^2-4PR &= (n-(2n-1)\,(n-\delta)-(\rho+\delta))^2-4\,(n-1)\,\theta\,(n-\delta)\,\frac{2n-1}{n-1}\frac{\rho}{\theta}, \\
&= (n-(2n-1)\,(n-\delta)-(\rho+\delta))^2-4\,(n-\delta)\,(2n-1)\,\rho, \\
&= ((n-\delta)-(2n-1)\,(n-\delta)-\rho)^2-4\,(n-\delta)\,(2n-1)\,\rho,, \\
&= [(n-\delta)\,(1-2n+1)-\rho]^2-4\,(n-\delta)\,(2n-1)\,\rho, \\
&= [2\,(n-\delta)\,(1-n)-\rho]^2-4\,(n-\delta)\,(2n-1)\,\rho, \\
&= [-2\,(n-\delta)\,(n-1)-\rho]^2-4\,(n-\delta)\,(2n-1)\,\rho, \\
&= [2\,(n-\delta)\,(n-1)+\rho]^2-4\,(n-\delta)\,(2n-1)\,\rho, \\
&= 4\,(n-1)^2\,(n-\delta)^2+4\,(n-\delta)\,(n-1)\,\rho+\rho^2-4\,(n-\delta)\,(2n-1)\,\rho, \\
&= 4\,(n-1)^2\,(n-\delta)^2-4\,(n-\delta)\,n\rho+\rho^2, \\
&= 4\,(n-\delta)\left[(n-1)^2\,(n-\delta)-n\rho\right]+\rho^2, \\
&= 4\,(n-\delta)\left[n-\delta+2n\delta-n\rho-n^2\delta-2n^2+n^3\right]+\rho^2, \\
&= 4\,(n-\delta)\left[n-\delta\left(n^2-2n+1\right)-n\rho+n^3\right]+\rho^2, \\
&= 4\,(n-\delta)\left[n-\delta\,(n-1)^2-n\rho+n^3\right]+\rho^2, \\
&= 4\,(n-\delta)\left[n(1-\rho)-\delta\,(n-1)^2+n^3\right]+\rho^2 > 0,
\end{aligned}$$

because $-\delta\,(n-1)^2+n^3 > 0$. Since the discriminate is always positive, this equation has two real roots. In other words, the lines C_2 and C_3 have two intersection points. Since $-Q > 0$, these two real roots of S are positive. Nevertheless, in order to more precisely identify how the curve C_3 intersect with the straight line C_2, we first identify the intersection between the steady state line and the line C_2. To do this, we substitute δ/n into the curve C_3 to get

$$\begin{aligned}
a &= -\frac{2n-1}{n-1}\frac{\rho}{\theta n}S^{-1}+\frac{\rho+\delta}{n}, \\
\frac{\delta}{n} &= -\frac{2n-1}{n-1}\frac{\rho}{\theta n}S^{-1}+\frac{\rho+\delta}{n}, \\
\delta S &= -\frac{2n-1}{n-1}\frac{\rho}{\theta}+(\rho+\delta)\,S, \\
\frac{2n-1}{n-1}\frac{\rho}{\theta} &= [(\rho+\delta)-\delta]\,S, \\
&\therefore \quad S=\frac{1}{\rho}\frac{2n-1}{n-1}\frac{\rho}{\theta}=\frac{2n-1}{n-1}\frac{1}{\theta}.
\end{aligned}$$

which implies that the curve C_3 at point $\left(\frac{2n-1}{n-1}\frac{1}{\theta},\frac{\delta}{n}\right)$ crosses the steady state line $\dot{S}=0$ as illustrated in Figure 3.

Taken together, it turns out that the nonlinear curve C_3 is a upward-sloping, concave line in the (S,c) space. Moreover, the curve C_3 does not intersect the vertical axis and diverges to minus infinity, while it approaches asymptotically the line $\frac{\rho+\delta}{n}$. Note that there are two possibilities as to the relative size depending on

$$\frac{2n-1}{\theta(n-1)}-\frac{n}{\theta(n-1)(n-\delta)}-\frac{\rho\,(2n-1)}{\theta\,(\rho+\delta)\,(n-1)}\gtreqless 0.$$

Appendix B

The comparative statics effects can be captured by the effects on point E.

$$\begin{aligned}
a &= 1-\frac{n-1}{n}\,(A-\theta S)\,(n-\delta)\,, \\
a &= -\frac{2n-1}{n-1}\frac{\rho}{\theta n}S^{-1}+\frac{\rho+\delta}{n}.
\end{aligned}$$

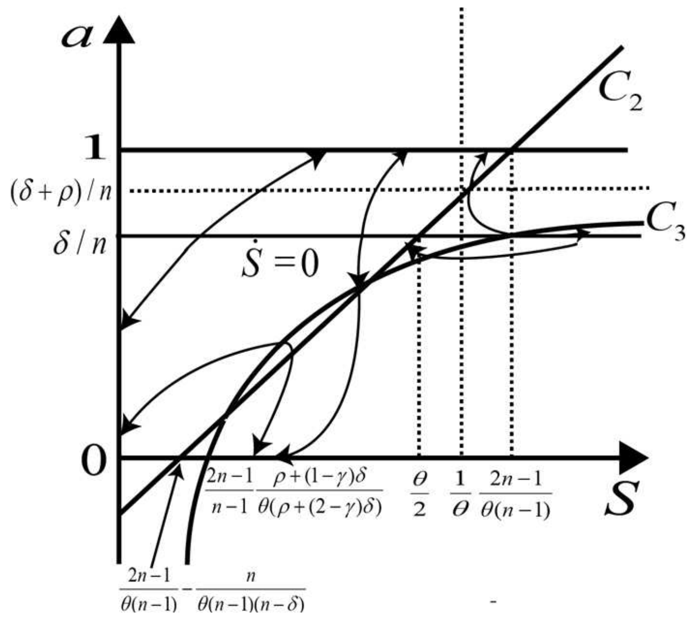


Figure 4:

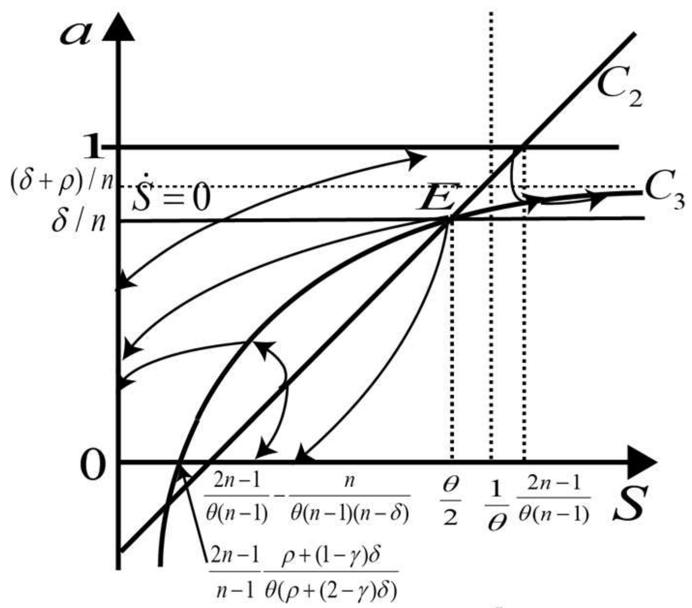


Figure 5:

$$
\begin{aligned}
1-\frac{n-1}{n}(A-\theta S)(n-\delta) &= -\frac{2n-1}{n-1}\frac{\rho}{\theta n}S^{-1}+\frac{\rho+\delta}{n}, \\
nS-(n-1)\left(AS-\theta S^{2}\right)(n-\delta) &= -\frac{2n-1}{n-1}\frac{\rho}{\theta}+S(\rho+\delta), \\
nS-(n-1)AS(n-\delta)+(n-1)\theta S^{2}(n-\delta)-S(\rho+\delta)+\frac{2n-1}{n-1}\frac{\rho}{\theta} &= 0, \\
(n-1)\theta S^{2}(n-\delta)+\left[n-(n-1)\frac{2n-1}{n-1}(n-\delta)-(\rho+\delta)\right]S+\frac{2n-1}{n-1}\frac{\rho}{\theta} &= 0, \\
F(S;n,\theta,\delta)=(n-1)\theta(n-\delta)S^{2}+[n-(2n-1)(n-\delta)-(\rho+\delta)]S+\frac{2n-1}{n-1}\frac{\rho}{\theta} &= 0,
\end{aligned}
$$

$$\frac{dS}{dn}=-\frac{\partial F/\partial n}{\partial F/\partial S}=-\frac{\partial F/\partial n}{\partial F/\partial S}$$

where

$$\frac{\partial F}{\partial n}=(n-1)\theta(n-\delta)S^{2}+[n-(2n-1)(n-\delta)-(\rho+\delta)]S+\frac{2n-1}{n-1}\frac{\rho}{\theta}$$

$$\frac{dS}{d\theta}=-\frac{\partial F/\partial\theta}{\partial F/\partial S}=-\frac{\partial F/\partial n}{\partial F/\partial S}$$

where

$$\frac{\partial F}{\partial\theta}=(n-1)(n-\delta)S^{2}-\frac{\rho}{\theta^{2}}$$

$$\frac{dS}{d\delta}=-\frac{\partial F/\partial\delta}{\partial F/\partial S}=-\frac{\partial F/\partial n}{\partial F/\partial S}$$

where

$$\frac{\partial F}{\partial S}=(n-1)\theta(n-\delta)2S+[n-(2n-1)(n-\delta)-(\rho+\delta)]$$

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