A Shut Mouth Catches No Flies: Consideration of Issues and Voting

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Abstract

We study collective decision-making procedures involving the formation of an agenda of issues and the subsequent vote on the position for each issue on the agenda. Issues that are not on the agenda remain unsettled. We use a protocol-free equilibrium concept introduced by Dutta et al. (2004) and show for two prominent voting procedures that essentially any subset of issues may be excluded from the agenda in equilibrium. What is voted upon and what is not depends on the voters preferences in a subtle manner, suggesting a high degree of instability. We also discuss further conditions under which this "anything goes" result may be qualified. In particular, we study those cases where all issues will be put in the agenda.

1 Introduction

When preparing a constitution, legislators may decide to only contemplate a few basic issues, or else seek to regulate many aspects of political life. When getting ready to elect new members of a learned society, the present ones may propose

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many candidates to be in the ballot, or else refrain from doing so. These are instances of decision-making situations where the same agents who will have to vote on an agenda are given the chance to decide on what issues to concentrate.

In this paper we study how the number and nature of the issues contemplated by a voting body depend on the voting rules to be used, and on the expectations of voters regarding how others with whom they agree or do not agree will behave.

We consider a group of voters facing a set of potential issues. The alternatives they face are vectors that describe, for each issue, one of three possible results of their actions: either the issue is not put to vote, or it is, and in that case, one of two positions is adopted. All voters prefer, ceteris paribus, to have a vote on those issues where their preferred position will prevail, rather than leaving it undefined. Similarly, they prefer to avoid a vote on those issues where their worse position would be chosen. Our focus is placed on the process whereby some issues enter the agenda and others do not.

Our analysis of agenda formation is based on a protocol-free notion of equilibrium proposed by Dutta, Le Breton and Jackson (2004). Given a voting procedure, and a set of potential issues, we concentrate on the possible processes whereby an agenda can result from the proposals of different voters. Agendas are ordered sequences of issues. A full agenda is one that lists all possible issues. An agenda a' is a continuation agenda of agenda a if a is the list of first elements of a'. Notice that any agenda is always a continuation agenda of itself. Also, the only continuation of a full agenda is that same agenda. The primary objects on which we predicate our equilibrium notion are collections of continuation agendas. A collection is an equilibrium if it excludes continuations that fail to satisfy several mild conditions. An equilibrium collection thus consists of a set of equilibrium continuation agendas for each agenda a. The first condition is that any equilibrium continuation agenda at a is either a itself or is some equilibrium continuation at an agenda a' reached by adding one further issue to a. The remaining conditions depend on the voting procedure under consideration. The second condition is that a is an equilibrium continuation at a if and only if the vote on any agenda a', which is an equilibrium continuation when one further issue has been added to a, would result in a vector of positions on issues that is worse for all voters than what they get if the vote is restricted to a. The third condition is that essentially only rationalizable continuation agendas are equilibrium continuations, where a continuation agenda a' at a is defined to be rationalizable if there exists some voter who prefers a' over some other equilibrium continuation a'' at a. Equilibrium collections of sets of continuation agendas provide us with a representation of consistent expectations regarding what outcomes agents can expect from their own action and that of others, depending on the agendas for which a vote is taken. These expectations are consistent with each agent's behavior, but allow for a variety of possible timings in the actions of each one of them. In that sense, our equilibrium notion is not dependent of the specifics that would be introduced if we were to impose any single protocol on the actions of voters.

Although rather involved, an advantage of the proposed equilibrium notion is that it poses no existence problems. Moreover, the characterization of families of equilibrium continuation agendas closely follows the steps of a backward induction argument that is quite analogous to the pruning procedure suggested by Arieli and Aumann (2015) for the case of subgame perfect equilibria. Since any full agenda is its own continuation, we can start by asking whether an agenda a that contains all issues but one satisfies the equilibrium requirements. If it does, its continuation full agenda will be pruned. If it does not, then the expectation that a is its own equilibrium continuation is pruned. That leaves us with a family of potential equilibria regarding agendas where at most all issues but one are considered. Then we can continue a similar pruning process for agendas containing all but two issues, and proceed in a similar manner until we reach the case where the agenda is empty and no issue is put to vote. The family of continuation agendas that survives the pruning process is an equilibrium.

We prove that in many relevant cases, the exact number of issues that will be put to vote is highly dependent on the specification of the voting rule and the preferences of voters. Specifically, given some natural assumptions regarding preference domains we prove "anything goes" results of the following type: For a given voting procedure, give us an agenda of (almost) any length, and we'll show you a preference profile within the domain that has this agenda as its unique equilibrium. We prove such anything goes results for two prominent voting procedures, the amendment procedure and voting by quota. We consider these results to be relevant on several accounts. First of all, it is commonly observed that bills on the same subject can be lengthy or short, depending on time and country. Specifically, some constitutions only address a few aspects of principle, while others are much more specific. One could attribute these differences to exogenous factors: legislators may have deep beliefs regarding the correct level of detail that a constitution must go into. Our results show that one need not appeal to such exogenous and unexplained principles. Rather, the chances to prevail on certain issues and not on others are shown to drive legislators to support more or less detailed agendas. Also relevant is the remark that our "anything goes" results are valid for two prominent voting procedures, the amendment procedure and voting by quota, and even under significant domain restrictions. Different sized agendas can emerge under many circumstances. But it is also true that the specific sizes of equilibrium agendas hinge on rather volatile features of the voters' preferences, and therefore reflect a high level of potential instability. This remark fits well with the warnings of noted political scientists regarding the ever-present potential for instability and change in political situations (Riker, 1982, 1993; Cox and Shepsle, 2007). Our analysis also points at an additional form of manipulation that may be in the hands of a chair. It is the possibility of changing the agent's attitude on whether or not to discuss an issue by simply altering the order on which issues on the agenda will be presented for a vote. This type of manipulation could be an additional instrument in the hands of a chairperson, even of one who cannot directly determine what issues should be discussed.

In addition to the permissive results we have just emphasized, we also provide several sufficient conditions under which the only equilibrium agendas would be full, and all issues brought to the floor. This is important because previous work by Dutta et al. (2004) showed that, in a different context, only full agendas would result in equilibrium as long as the voting rule is efficient. By contrast, efficiency alone is by no means sufficient to precipitate this same result in our case. This is because our model distinguishes between two cases that are merged in their formulation. We treat separately the possibility that a position is not adopted on an issue because there is no vote, or because it was brought to the floor and defeated there.

The literature on agenda formation is rich, and the following overview is necessarily incomplete. Most previous work considers a specific protocol for the agenda formation and the subsequent voting stage. Austen-Smith (1987), Banks and Gasmi (1987), Baron and Ferejohn (1989), Miller, Grofman, and Feld (1990), Duggan (2006) and Penn (2008) all assume that voting takes place only after the agenda has been built. By contrast, Bernheim, Rangel, and Rayo (2006) analyse the case of "real-time" agenda setting, where any proposal is put to an immediate vote against the current default. In Eguia and Shepsle (2015) the bargaining protocol is endogenized and chosen by the members of a legislative assembly before the agenda is formed. While all these papers consider the case of complete information like we do, Godefroy and Perez-Richet (2013) use an incomplete information framework to study how the majority quota used to place alternatives on the agenda affects agents' behavior and hence the likelihood to change the status quo. There are only few papers that do not rely on a specific bargaining protocol. Among the notable exceptions is Dutta et al. (2004) who study equilibrium agendas in a model with farsighted agents. Their main result is that the set of equilibrium outcomes for Pareto efficient voting rules coincides with the outcomes when all full agendas are considered. Unlike in their paper, and as we shall discuss later, we show that in our model equilibrium agendas may not contain all issues even if the voting rule is Pareto efficient. Vartiainen (2014) also considers a protocol free agenda formation process, but this time, unlike in Dutta et al. (2004), in a real-time agenda setting.

There is also a related literature that focusses on strategic candidacy (Osborne and Slivinsky, 1996; Besley and Coate, 1997; Dutta et al., 2001, 2002). The main difference with models of agenda formation like ours is that in strategic candidacy problems, the agents who take the agenda formation decision, by choosing whether to run or not to, are different from those who will eventually vote.

Our main departure from previous work comes from the distinction between the issues that may be potentially discussed and the positions that society may eventually adopt regarding these issues. An agenda, in our setting, is a (possibly ordered) set of issues to be voted upon, and an alternative is a vector of positions on these issues. Previous models treat alternatives as the primitives of the problem, and then allow for agents to add an alternative to a previous agenda without changing the nature of those that were already in. In our model, expanding the agenda by including an additional issue completely changes the set of alternatives faced by agents, since the size of the vector of issues to be voted on is increased. That makes our formulation, and also our results, to be different than previous ones in the literature. In particular, we find that the size of the agenda may vary, and that it depends on characteristics of preferences that may be quite volatile. Hence, although our analysis allows for a notion of equilibrium agendas, it also indicates some of the reasons why societies may quickly abandon any equilibrium in favor of new objects of debate and conflict.

The outline of the paper is the following. In Section 2 we introduce our model and equilibrium concept. In Section 3 we present an example with a Pareto efficient voting rule where equilibrium agendas can be of arbitrary length. In Section 4 we study sufficient conditions for equilibrium agendas to be full agendas. Section 5 proves an anything goes result for two prominent voting procedures. Section 6 concludes. All proofs are in the appendix.

2 The Model

We consider a group of $n \geq 2$ agents facing a given set of issues $\mathcal{K} = \{1, \ldots, K\}$ with $K \geq 2$. The group may decide to keep silent on some issues, thus leaving its position on it undefined, while being ready to take a vote on others, in which case one of two positions will be voted upon and adopted for each one of those. We denote by "$-$" the decision to leave an issue out of the voting floor, and by 0 and 1 the two possible positions on issues that are voted upon. Social alternatives are then K-tuples indicating, for each issue, whether or not it was the object of a vote, and, if so, which stand was adopted on it. Accordingly, the set of alternatives then is given by $X = \{0, 1, -\}^{\mathcal{K}}$.

Our model applies in those cases where a voting body may bind itself to take positions on certain issues, and not on others. For example, the writers of a constitution may feel that they must take a stand regarding some fundamental aspects of political life, but leave others to be defined by practice or by legislation of a lower rank. Another example is the election of new members of a society, where agents may prefer not to nominate a candidate they like if they expect her to lose in the election.

Preferences

Every agent i has a strict preference ordering $\succ_i$ on X, which satisfies betweenness:

Definition 2.1 *A preference ordering $\succ$ on X satisfies **betweenness** if for all $k \in \mathcal{K}$, and for all $x \in X$, either*

$$(1, x_{\mathcal{K}\setminus\{k\}}) \succ (-, x_{\mathcal{K}\setminus\{k\}}) \succ (0, x_{\mathcal{K}\setminus\{k\}})$$

or

$$(0, x_{\mathcal{K}\setminus\{k\}}) \succ (-, x_{\mathcal{K}\setminus\{k\}}) \succ (1, x_{\mathcal{K}\setminus\{k\}}).$$ [1]

By $\mathcal{P}$ we denote the set of strict preference orderings that satisfy betweenness. Under betweenness, other things being equal, an agent strictly prefers the alternative that takes his preferred position on some issue k over leaving the position open, and he strictly prefers the latter to the alternative where the position is his worse. Observe that this assumption is compatible with the interpretation that agents perceive the resulting indeterminacy as creating a lottery between the competing positions, to be resolved in the future. Then, observe that our assumption will be satisfied whenever the agent's preference ordering can be represented by an expected utility function such that the utility of $(-, x_{\mathcal{K}\setminus\{k\}})$ is the expected utility of a lottery over the set $\{(0, x_{\mathcal{K}\setminus\{k\}}), (1, x_{\mathcal{K}\setminus\{k\}})\}$, where the agent assigns a positive probability to both outcomes, $(0, x_{\mathcal{K}\setminus\{k\}})$ and $(1, x_{\mathcal{K}\setminus\{k\}})$, that is independent of the corresponding probabilities for other open positions, if any.

For later use we introduce two separability properties of preference orderings.

[1] For $x \in \{0, 1, -\}^{\mathcal{K}}$ and $\mathcal{M} \subset \mathcal{K}$ we denote by $x_{\mathcal{M}}$ the projection of x onto $\{0, 1, -\}^{\mathcal{M}}$. Moreover, for $x, y \in \{0, 1, -\}^{\mathcal{K}}$ and $\mathcal{M} \subset \mathcal{K}$, $(x_{\mathcal{M}}, y_{\mathcal{K}\setminus\mathcal{M}})$ is the vector $z \in \{0, 1, -\}^{\mathcal{K}}$ with $z_k = x_k$ for all $k \in \mathcal{M}$ and $z_k = y_k$ for all $k \in \mathcal{K} \setminus \mathcal{M}$.

Definition 2.2

(1) A preference ordering $\succ$ on $X = \{0,1,-\}^{\mathcal{K}}$ is ***separable*** *if for all $k \in \mathcal{K}$, $(x_k, x_{-k}) \succ (y_k, x_{-k})$ for some $x_{-k} \in \{0,1,-\}^{\mathcal{K}\setminus\{k\}}$ implies that $(x_k, x'_{-k}) \succ (y_k, x'_{-k})$ for all $x'_{-k} \in \{0,1,-\}^{\mathcal{K}\setminus\{k\}}$.*

(2) A preference ordering $\succ$ on X is ***additively separable*** *on X if there exist scalars $u_k(w) \in \mathbb{R}$ for $k \in \mathcal{K}$, and $w \in \{0,1,-\}$ such that for $x, y \in X$,*

$$x \succ y \iff \sum_{k=1}^{K} u_k(x_k) > \sum_{k=1}^{K} u_k(y_k).$$

If $\succ$ satisfies betweenness and additive separability, then for all issues $k \in \mathcal{K}$,

$$\max\{u_k(1), u_k(0)\} > u_k(-) > \min\{u_k(1), u_k(0)\}.$$

We consider a two-stage decision making process. In the first stage agents decide which issues to bring to the floor, and in what order they will be considered. The ordered list of these issues will be an agenda. After that, in the second stage, they vote on what position to take on each of the issues of the agenda. Since we allow for the case where the order in which issues are selected for vote plays a role in the second stage, agendas will not only specify what issues are put to vote, but also in what order, if that makes any difference. Notice that this order need not be the same as the one we have used to give name to the issues in our previous definition of the set $\mathcal{K}$.

Agendas

Let $m \in \{1, \ldots, K\}$. Then $a = (a_1, \ldots, a_m)$ with $a_l \in \mathcal{K}$ for $l = 1, \ldots, m$, and $a_l \neq a_{l'}$ for $l \neq l'$ is called an *agenda* of length m. The empty agenda $\varnothing$ where no issue is put to vote is defined to have length 0. By A^m we denote the set of agendas of length m, where $0 \leq m \leq K$, and by $A = \bigcup_{m=0}^{K} A^m$ we denote the set of all agendas.

Let $a \in A$. If an issue k is not on the agenda a, i.e. $k \notin a$, we call k a *free issue at a*.[2] For $a \in A^m$, where $0 \leq m \leq K-1$, and $k \in \mathcal{K}, k \notin a$, (a, k) denotes the agenda $a' \in A^{m+1}$ with $a'_l = a_l$ for $l = 1, \ldots, m$, and $a'_{m+1} = k$.

[2] Here and in what follows, for a given agenda $a = (a_1, \ldots, a_m) \in A$ we write, for short, $k \in a$ ($k \notin a$) whenever $k = a_l$ for some $l \in \{1, \ldots, m\}$ ($k \neq a_l$ for all $l = 1, \ldots, m$).

For a given agenda $a \in A$ we denote by $X(a)$ the set of *available alternatives* at a, i.e.

$$X(a) = \{x \in X \,|\, \text{for all } k \in \mathcal{K}, x_k \in \{0,1\} \text{ if and only if } k \in a\}.$$

Observe that $X(\varnothing) = \{(-, \ldots, -)\}$.

Voting

For the second stage we take as given a voting procedure that chooses a position for each issue on the agenda and leaves open the position for any free issue. Formally, a voting procedure on some domain of preference orderings $\mathcal{D} \subset \mathcal{P}$ is a mapping $V : A \times \mathcal{D}^n \to X$ with $V(a, P) \in X(a)$ for all $a \in A$ and $P \in \mathcal{D}^n$. Notice that a voting procedure, in our definition, associates a single outcome to each preference profile and each agenda a. Also, observe that the voting procedure need not be sensitive to the ordering of issues in a, and may only depend on the set of issues in the agenda.

One way to arrive at such voting procedures is to propose a game form, coupled with a solution that has unique equilibrium outcomes at each associated game. For example, we can select a sequential voting rule, defined by a binary tree and by a given majority rule to be used at each node, and associate to each profile of preferences the (unique) outcome under sophisticated voting induced by that profile (Farquharson, 1969), i.e. the Nash equilibrium outcome obtained by iterative elimination of weakly dominated strategies. Alternatively, we could assume that agents vote sincerely and obtain a different function for the same tree.[3]

Agenda Formation

In the first stage, starting from the empty agenda agents can unilaterally add issues to the agenda. This process stops when either a full agenda $a \in A^K$ is reached or no agent wants to add further issues. Instead of modeling the agenda

[3]Since it is well known that the same tree structure may lead to different outcomes depending on the order of vote on alternatives (see Barberà and Gerber, 2017, and references therein), this order must be determined when defining the voting rule. Notice that the order of vote on these alternatives may not be related to the order of issues in the agenda. See Section 3 for specific cases of application.

formation as a noncooperative game, where equilibrium agendas could potentially be very sensitive to the details of the game, we follow Dutta et al. (2004) and consider an equilibrium collection of sets of continuation agendas defined as follows.

For $a \in A^m$, where $m \in \{0, 1, \ldots, K\}$, let $A(a)$ be the set of *continuation agendas*, i.e.

$$A(a) = \{a' \in A \,|\, a'_k = a_k \text{ for all } k = 1, \ldots, m\}.^4$$

For given $P \in \mathcal{D}^n$, an *equilibrium collection of sets of continuation agendas* is a collection $(CE(a, P))_{a \in A}$, where $CE(a, P) \subset A(a)$ for each $a \in A$, that satisfies the following two conditions for all $a \in A$:

(E1) $CE(a, P)$ is a nonempty subset of $\bigcup_{k \notin a} CE((a, k), P) \cup \{a\}$.
(E2) $a \in CE(a, P)$ if and only if $V(a, P) \succ_i V(a', P)$ for all $a' \in \bigcup_{k \notin a} CE((a, k), P)$ and for all $i = 1, \ldots, n$.[5]

In what follows, for a given equilibrium collection of sets of continuation agendas $(CE(a, P))_{a \in A}$, we will refer to the continuation agendas in $CE(a, P)$ for $a \in A$ as *equilibrium continuations.*

Equilibrium collections of sets of continuation agendas express expectations about the agendas that will obtain starting from any given agenda a. Since issues are assumed to be added one after the other, expectations at agenda a have to be such that they either do not involve further additions of issues, or else are equilibrium continuations if one further issue is added to a (see condition (E1)). Moreover, no further additions of issues to an agenda a are expected if and only if no agent would be interested in adding any additional issue after having reached a, in view of what the expected continuations would be (see condition (E2)). Observe that this is a rather weak stopping requirement because an agent is assumed to stop adding issues to the agenda only if stopping is better than all equilibrium continuations reached when one further issue is added to the existing agenda. Nevertheless, as we will show, equilibrium continuations are not

[4]Observe that $a \in A(a)$ for all $a \in A$, i.e. any agenda is a continuation agenda for itself.

[5]Notice that $V(a', P) \neq V(a, P)$ for all $a' \in \bigcup_{k \notin a} CE((a, k), P)$ since any such agenda a' contains at least one issue $k \notin a$ which implies that $(V(a, P))_k = - \neq (V(a', P))_k$.

necessarily full agendas, even for restricted domains of preferences and very well behaved voting procedures.

In order to reduce the potential multiplicity of equilibrium collections of sets of continuation agendas we impose a third condition, which we call *consistency* (cf. Dutta et al., 2004). To this end, for $a \in A$ we define an agenda $a' = (a, k, \ldots) \in A$ to be *rationalizable* (relative to a) if $a' \in CE((a,k),P)$ and there exists an agent i and $a'' \in CE(a,P)$ with either $a'' = (a, l, \ldots)$ with $l \neq k$ or $a'' = a$ such that $V(a',P) \succ_i V(a'',P)$. Hence, the continuation agenda $(a,k,\ldots)$ is rationalizable relative to a if it is an equilibrium continuation at (a,k) and if some agent can gain from reaching it rather than sticking to some other equilibrium continuation at a. An equilibrium collection of sets of continuation agendas then is defined to be *consistent* if it satisfies the following condition:

(E3) If $a' \in \bigcup_{l \notin a} CE((a,l),P)$ is rationalizable, then $a' \in CE(a,P)$. Conversely, if $a' = (a,k,\ldots) \in CE(a,P)$ and either $a \in CE(a,P)$ or $a'' = (a,l,\ldots) \in CE(a,P)$ for some $l \neq k$, then a' is rationalizable.

Thus, consistency requires that an equilibrium collection of sets of continuation agendas contains all rationalizable continuation agendas. Moreover, it only contains rationalizable continuation agendas subject to the following two exceptions: The first is that the agenda a itself is an equilibrium continuation if all agents prefer to stop at a (condition (E2)). The second exception is when there is a unique equilibrium continuation $a' = (a,k,\ldots)$ at a which is then not required to be rationalizable. Observe, however, that the latter case only obtains if there is an agent who prefers continuing over stopping at a and if agents unanimously prefer a' to adding an issue different from k to agenda a.

We will be mainly interested in the agendas that would obtain at equilibrium when agenda formation starts from the point where all issues are still free. Hence, it is convenient to introduce the following terminology:

Definition 2.3 *Let $a^* \in A$ and $P \in \mathcal{P}^n$. Then a^* is a* ***(consistent) equilibrium agenda at P*** *if there exists a (consistent) equilibrium collection of sets of continuation agendas $(CE(a,P))_{a \in A}$ with $a^* \in CE(\varnothing,P)$.*

Before we consider an example with two issues we record the following result which is a straightforward implication of condition (E1): If an agenda a^* is an equilibrium continuation at some agenda a, then it is an equilibrium continuation at every agenda along the path from a to a^*.

Lemma 2.1 *Let $V : A \times \mathcal{D}^n \to X$ be a voting procedure and let $(CE(a,P))_{a \in A}$ be an equilibrium collection of sets of continuation agendas for some $P \in \mathcal{D}^n$. If $a = (a_1, \ldots, a_m) \in CE((a_1, \ldots, a_l), P)$ for some $l \leq m \leq K$, then*

$$a \in CE((a_1, \ldots, a_k), P) \text{ for all } k = l, \ldots, m.$$

In particular,

$$a \in CE(a, P).$$

3 An Example

We consider the election of new members to a society. There are two candidates, 1 and 2, i.e. the set of issues is $\mathcal{K} = \{1, 2\}$. In this case "$-$" means that the corresponding candidate is not nominated, "1" means that the candidate is nominated and elected and "0" means that the candidate is nominated and not elected. The set of alternatives then is

$$X = \{(-,-), (0,-), (1,-), (-,0), (-,1), (0,0), (0,1), (1,0), (1,1)\}.$$

Let there be three agents with preference orderings on the set of alternatives given in Table 1, where the alternatives in the table are listed in the order of decreasing preference. Note that all preference orderings are separable and satisfy betweenness.

Suppose that for any agenda a voting follows the amendment procedure (Farquharson, 1969; Miller, 1977, 1980) for some exogenously given ordering of the attainable alternatives in $X(a)$. That is, if $(x_1, x_2, \ldots, x_N)$ is the given ordering of the alternatives in $X(a)$, then the first vote is over x_1 and x_2, the second vote is over the winner of the first vote and x_3, and so on until all alternatives in $(x_1, x_2, \ldots, x_N)$ are exhausted. In every pairwise vote the winner is selected

$\succ_1$	$\succ_2$	$\succ_3$
$(0,1)$	$(1,0)$	$(1,1)$
$(0,-)$	$(-,0)$	$(-,1)$
$(-,1)$	$(1,-)$	$(0,1)$
$(-,-)$	$(-,-)$	$(1,-)$
$(0,0)$	$(0,0)$	$(-,-)$
$(-,0)$	$(0,-)$	$(0,-)$
$(1,1)$	$(1,1)$	$(1,0)$
$(1,-)$	$(-,1)$	$(-,0)$
$(1,0)$	$(0,1)$	$(0,0)$

Table 1: Preference orderings $\mathcal{K} = \{1, 2\}$.

according to simple majority voting. Notice that in our example the ordering of the attainable alternatives is relevant only if both issues are on the agenda.[6]

Agents are assumed to be sophisticated (Farquharson, 1969) and thus the voting outcome is given by the Nash equilibrium outcome obtained by iterative elimination of weakly dominated strategies. It is well known that the amendment procedure is efficient (Miller, 1977, 1980; Barberà and Gerber, 2017).

In order to solve for the equilibrium agendas we first determine the voting outcome for any agenda that contains at most one alternative. At the empty agenda the outcome is the unique attainable alternative $(-,-)$, i.e.

$$V(\varnothing, P) = (-,-).$$

At agenda $a = (1)$ the outcome is

$$V((1), P) = (1,-),$$

because a majority of agents prefer $(1,-)$ over $(0,-)$, and at agenda $a = (2)$ the outcome is

$$V((2), P) = (-,1),$$

[6]If there is only one issue on the agenda, e.g. issue 1, then there are only two attainable alternatives, $(1,-)$ and $(0,-)$, and the voting outcome is independent of the ordering of these alternatives.

since a majority of agents prefer $(-,1)$ over $(-,0)$. Finally, we determine the voting outcome at the full agendas, $(1,2)$ and $(2,1)$ with attainable sets

$$X(1,2) = X(2,1) = \{(0,0),(0,1),(1,0),(1,1)\}.$$

Figure 1 shows the dominance relation on $\{(0,0),(0,1),(1,0),(1,1)\}$ that results from pairwise simple majority voting.

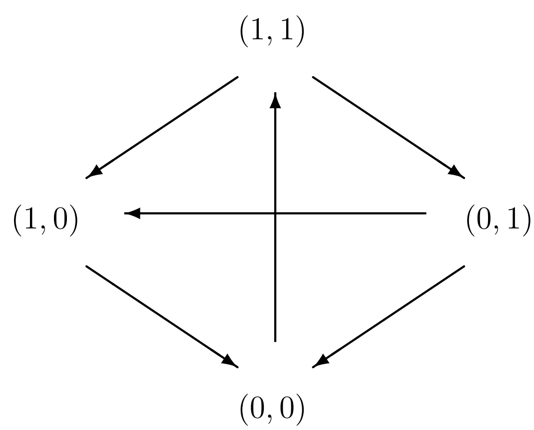

Figure 1: Dominance relation on $\{(0,0),(0,1),(1,0),(1,1)\}$ under pairwise simple majority voting. The arrows point to the alternatives that are beaten under simple majority voting.

It follows from the characterizations in Banks (1985, Theorem 3.1) and Barberà and Gerber (2017, Theorem 3.1) that any of the attainable alternatives except for $(1,0)$ is the outcome of sophisticated sequential voting under the amendment procedure for some ordering of the alternatives in $X(1,2) = X(2,1)$.[7]

First consider the case where the ordering of the alternatives in $X(1,2) = X(2,1)$ is such that

$$V((1,2),P) = V((2,1),P) = (0,0).$$

Observe that $(0,0)$ is Pareto dominated by $(-,-)$.

[7] This can also be verified directly: The ordering $((0,0),(0,1),(1,0),(1,1))$ yields outcome $(0,0)$, the ordering $((1,1),(1,0),(0,0),(0,1))$ yields outcome $(1,1)$ and the ordering $((0,1),(1,0),(1,1),(0,0))$ yields outcome $(0,1)$. Finally, no ordering gives outcome $(1,0)$.

We now solve for the equilibrium collection of sets of continuation agendas. To do that we proceed backwards starting from the full agendas $(1,2)$ and $(2,1)$. By (E1) it must be that

$$CE((1,2),P) = \{(1,2)\} \text{ and } CE((2,1),P) = \{(2,1)\}.$$

Now consider agenda (1). By condition (E1), $CE((1),P)$ is a nonempty subset of $\{(1),(1,2)\}$. By condition (E2), $(1) \in CE((1),P)$ is ruled out since agent 1 strictly prefers the equilibrium continuation $CE((1,2),P) = (1,2)$ over (1). Hence,

$$CE((1),P) = \{(1,2)\}.$$

Next consider agenda (2). By condition (E1), $CE((2),P)$ is a nonempty subset of $\{(2),(2,1)\}$. By condition (E2), $(2) \in CE((2),P)$ is ruled out since agent 2 strictly prefers the equilibrium continuation $CE((2,1),P) = (2,1)$ over (2). Hence,

$$CE((2),P) = \{(2,1)\}.$$

Finally, consider the empty agenda. By condition (E1), $CE(\varnothing,P)$ is a nonempty subset of $\{\varnothing\} \cup CE((1),P) \cup CE((2),P) = \{\varnothing,(1,2),(2,1)\}$. Since all agents strictly prefer the empty agenda over any full agenda, $\varnothing \in CE(\varnothing,P)$ by (E2). Suppose by way of contradiction that $(1,2) \in CE(\varnothing,P)$. Then, since $\varnothing \in CE(\varnothing,P)$, condition (E3) implies that $(1,2)$ is rationalizable relative to the empty agenda $\varnothing$. However, no agent prefers agenda $(1,2)$ over the empty agenda $\varnothing$ or agenda $(2,1)$. Hence, $(1,2)$ is not rationalizable which implies that $(1,2) \notin CE(\varnothing,P)$. Similarly, one proves that $(2,1) \notin CE(\varnothing,P)$. Therefore, we conclude that

$$CE(\varnothing,P) = \{\varnothing\}.$$

Thus, in this case the unique consistent equilibrium agenda is empty and no candidate is nominated and elected.

Now, we develop the same example but under the assumption that the exogenous order of vote under the amendment procedure is such that

$$V((1,2),P) = V((2,1),P) \in \{(0,1),(1,1)\}.$$

Observe that $(0, 1)$ is the best alternative for agent 1 and $(1, 1)$ is the best alternative for agent 3. Therefore, in this case only full agendas are equilibrium agendas because there is always one agent who is better off by adding an issue to the agenda that was a free issue before. Hence, for all agendas a, $CE(a, P)$ contains full agendas only. Thus, in this case any consistent equilibrium agenda is a full agenda, i.e. both candidates are nominated. However, depending on the order of vote under the amendment procedure, either both candidates or only candidate 2 is elected.

This example illustrates a number of notable points: (1) There are voting procedures and preference profiles for which equilibrium agendas are not full agendas. (2) The equilibrium collection of sets of continuation agendas can be very sensitive to the details of the voting rule, and in particular to the use of a fixed order of vote under sequential voting procedures. (3) As a consequence, if an agent can choose the order of vote under sequential procedures, he can not only influence the outcome for a given agenda, but also the set of issues that a society may choose to leave free.

4 Full Agendas

We start our general analysis by exploring cases where all equilibrium agendas are full agendas (cf. Dutta et al., 2004). One obvious case is when the voting procedure has the property that at any preference profile there is one agent for whom the outcome at any full agenda is this agent's most preferred alternative.[8] This agent will then keep adding issues until a full agenda is reached, i.e. any equilibrium agenda is a full agenda.

Rather than looking at specific voting procedures we may also ask under which conditions on individual preferences will there only be full agendas in equilibrium. To this end let $\mathcal{K} = \{1, \ldots, K\}$ be the set of issues and let $\mathcal{S}$ be the set of all preference orderings that satisfy separability and betweenness. Let $\succ_i \in \mathcal{S}$. Then

[8] Note that this does not imply that the voting procedure is dictatorial because the selected agent may change with the preference profile.

for all $k \in \mathcal{K}$ there exists a $w_k^i \in \{0,1\}$ such that

$$(-, x_{\mathcal{K}\setminus\{k\}}) \succ_i (w_k^i, x_{\mathcal{K}\setminus\{k\}}) \text{ for all } x \in \{0,1,-\}^K.$$

That is, w_k^i is the worst position on issue k for agent i. We then say that agent i is pessimistic about issue k if no position on that issue, i.e. "$-$", is almost as bad as getting the worst position, i.e. w_k^i, on issue k.

Definition 4.1 *Let $\succ_i \in \mathcal{S}$. Then i is **pessimistic** about issue k if for all $x \in \{0,1,-\}^K$, and for all $y \in \{0,1,-\}^K$,*

$$y \succ_i (w_k^i, x_{\mathcal{K}\setminus\{k\}})$$

implies that

$$y \succ_i (-, x_{\mathcal{K}\setminus\{k\}}).$$

The following theorem shows that if all agents are pessimistic about all issues and if the voting procedure is Pareto efficient, then all equilibrium agendas are full agendas.

Theorem 4.1 *Let $V : A \times \mathcal{S}^n \to X$ be a Pareto efficient voting procedure, i.e. $V(a,P)$ is Pareto efficient in $X(a)$ for all $a \in A$ and all $P \in \mathcal{S}^n$. If all agents are pessimistic about all issues, then any equilibrium agenda is a full agenda.*

The intuition for Theorem 4.1 is simple: If not deciding on an issue is almost as bad as getting the worst position on that issue, then nothing prevents an agent from adding further issues to an agenda. Hence only full agendas can be equilibrium agendas.

5 Anything Goes

We will now provide a comprehensive analysis of agenda formation under two prominent voting procedures, the amendment procedure, which was already defined in Section 3, and voting by quota, where there is a majority vote on the position for every issue on the agenda. Notice that these procedures have different properties: Voting by quota is strategy-proof on the domain of additively separable preferences, but it is not Pareto efficient. By contrast, the amendment procedure is Pareto efficient, but it is not strategy-proof, i.e. the equilibrium voting strategies will in general differ from sincere voting even on the restricted domain of additiely separable preferences.

For both these procedures we will show that apart from some minor qualifications, for any subset $\mathcal{F}$ of the set of issues $\mathcal{K} = \{1, \ldots, K\}$ there exists a set of agents $\{1, \ldots, n\}$, with n odd, and a profile of preference orderings $P = (\succ_1, \ldots, \succ_n) \in \mathcal{P}^n$, such that $\mathcal{F}$ is the set of free issues at any consistent equilibrium agenda at P. Thus, neither of the two procedures imposes any structure on the set of equilibrium agendas.

Amendment Procedure

Suppose voting is according to the amendment procedure for some ordering of the alternatives and simple majority is used throughout. It turns out that apart from some minor qualification, for any subset $\mathcal{F}$ of the set of issues $\mathcal{K} = \{1, \ldots, K\}$ there exists a set of agents $\{1, \ldots, n\}$, with n odd, and a profile of separable preference orderings $P = (\succ_1, \ldots, \succ_n) \in \mathcal{S}^n$, such that $\mathcal{F}$ is the set of free issues at any consistent equilibrium agenda at P. The precise result is as follows.

Theorem 5.1 *Let $n \geq 3$ be odd and let $\mathcal{F} \subset \mathcal{K} = \{1, \ldots, K\}$ be such that $\#\mathcal{F} > 1$ if $K = 2$.*[9] *Then there exists a profile of separable preferences $P \in \mathcal{S}^n$ and some ordering of the alternatives in $X(a)$ for all $a \in A$, such that $\mathcal{F}$ is the set of free issues at any consistent equilibrium agenda a^* at P if voting is according to the amendment procedure for the given orderings of the alternatives at any agenda a.*

Voting by Quota

We now consider another class of voting rules, namely voting by quota. Let $\bar{\mathcal{S}} \subset \mathcal{P}$ denote the set of strict preference orderings that satisfy additive separability

[9] By "#" we denote the number of elements in a set.

and betweenness and let $q \in \{1, \dots, n\}$. Then *voting by quota* q is the voting procedure $V : A \times \bar{\mathcal{S}}^n \to X$, such that for all $a \in A$, for all $k \in a$, and for all $P \in \bar{\mathcal{S}}^n$,

$$(V(a, P))_k = \begin{cases} 1, & \text{if } \#\{i \mid u_k^i(1) > u_k^i(0)\} \geq q \\ 0, & \text{otherwise} \end{cases}, \tag{5.1}$$

where $(u_k^i(\cdot))_{k \in \mathcal{K}}$ is the collection of scalars in the additively separable utility representation of agent i's preference ordering $\succ_i$.[10] Observe that (5.1) implies that $V(a, P)$ only depends on the issues in a but not on their specific ordering.

Notably, on the restricted domain of additively separable preferences, for any quota q and for (almost) any set $\mathcal{F}$, there exists a preference profile such that $\mathcal{F}$ is the set of free issues at some equilibrium agenda:

Theorem 5.2 *Let $V : A \times \bar{\mathcal{S}}^n \to X$ be voting by quota $q \in \{1, \dots, n\}$ and let $\mathcal{F} \subset \mathcal{K}$ be such that $\#\mathcal{F} > 1$ and $\#\mathcal{F} \neq 2$ if n is odd and $q = \frac{n+1}{2}$. Then there exists a preference profile $P \in \bar{\mathcal{S}}^n$ such that $\mathcal{F}$ is the set of free issues at any consistent equilibrium agenda a^* at P.*

We will now argue that the conditions in Theorem 5.2 are tight in the sense that any $\mathcal{F} \subset \mathcal{K}$ that does not satisfy the conditions in the theorem can never be a set of free issues at an equilibrium agenda. The following proposition deals with the case where $\#\mathcal{F} = 1$ and shows that equilibrium agendas never contain all but one issue.

Proposition 5.1 *Let $V : A \times \bar{\mathcal{S}}^n \to X$ be voting by quota q and let $P = (\succ_1, \dots, \succ_n) \in \bar{\mathcal{S}}^n$. If $(CE(a, P))_{a \in A}$ is an equilibrium collection of sets of continuation agendas and if $a^* \in CE(a, P)$ for some $a \in A$, then*

$$a^* \notin A^{K-1}.$$

In particular, no $a^* \in A^{K-1}$ is an equilibrium agenda at P.

We skip the proof of Proposition 5.1 since it is an immediate implication of Lemma 2.1 and the following result:

[10] Voting by quota is a special case of a larger class of voting procedures, called *voting by commmittees* (Barberà et al., 1991).

Lemma 5.1 *Let $V : A \times \bar{\mathcal{S}}^n \to X$ be voting by quota $q \in \{1, \ldots, n\}$ and let $P = (\succ_1, \ldots, \succ_n) \in \bar{\mathcal{S}}^n$. If $(CE(a,P))_{a \in A}$ is an equilibrium collection of sets of continuation agendas, then*

$$CE(a,P) \subset A^K \text{ for all } a \in A^{K-1}.$$

The intuition for Lemma 5.1 is that if there is only one free issue left, then there is always one agent who is in the winning coalition for that issue. This agent then is better off adding that issue to the agenda since further additions are impossible and hence nothing can deter the agent from her initial move.

The following proposition deals with the other exceptional case of simple majority voting with an odd number of agents, where there can never be only two free issues at an equilibrium agenda.

Proposition 5.2 *Let there be an odd number n of agents and let $V : A \times \bar{\mathcal{S}}^n \to X$ be voting by quota $q = \frac{n+1}{2}$. Let $P = (\succ_1, \ldots, \succ_n) \in \bar{\mathcal{S}}^n$. If $(CE(a,P))_{a \in A}$ is an equilibrium collection of sets of continuation agendas, then for all $a \in A$,*

$$CE(a,P) \subset A \setminus \left(A^{K-1} \cup A^{K-2}\right).$$

In particular, no $a^* \in A^{K-2}$ is an equilibrium agenda at P, i.e. the set of free issues at an equilibrium agenda never contains two issues only.

6 Conclusion

Agenda formation is an essential part of many decision-making processes. Before we take a decision we have to sort out those issues that we want to settle and those that shall remain unsettled. We have studied the resulting agenda formation process and have demonstrated that essentially any subset of issues may be excluded from an equilibrium agenda. This result holds even under the restrictive assumption that preferences are additively separable and that the voting rule is Pareto efficient or strategy-proof. We believe that these "anything goes" results are generic, i.e. except for peculiar rules like the voting procedure that always selects the best alternative for some voter at any full agenda (see Section 4), we do not expect to find another voting procedure where equilibrium agendas of a particular length do not obtain in general.

It is clear from our results that equilibrium agendas and outcomes are very sensitive to voters' perceptions about the preferences of others regarding not only the potential outcomes of a vote, but also their inclination to postpone the discussion of certain issues. Likewise, equilibrium agendas and outcomes are also sensitive to the choice of voting rules. In particular, when sequential voting rules are to be used, we have shown that agents who can control the order of vote can decisively influence what issues may reach the floor, and which ones will remain unsettled.

Appendix

Proof of Theorem 4.1: Let $V : A \times \mathcal{S}^n \to X$ be a Pareto efficient voting procedure and let all agents be pessimistic about all issues. Let $(CE(a,P))_{a \in A}$ be an equilibrium collection of sets of continuation agendas. We will prove by backwards induction that $CE(a,P) \subset A^K$ for all $a \in A$.

Obviously, the claim is true for any full agenda $a \in A^K$. Suppose the claim is true for all agendas $a \in A$ of length l, where $m+1 \leq l \leq K$ and $1 \leq m \leq K-1$. Let $a \in A^m$. By (E1), $CE(a,P)$ is a nonempty subset of $\bigcup_{k \notin a} CE((a,k),P) \cup \{a\}$. By our induction hypothesis $CE((a,k),P) \subset A^K$ for all $k \notin a$. Suppose by way of contradiction that $a \in CE(a,P)$. Then by (E2), for all agents i,

$$V(a,P) \succ_i V(a',P) \text{ for all } a' \in \bigcup_{k \notin a} CE((a,k),P).$$

Let $y = V(a',P)$ for some $a' \in \bigcup_{k \notin a} CE((a,k),P)$. Then a' is a full agenda, which implies that $y \in \{0,1\}^K$. Moreover, y is Pareto efficient in $\{0,1\}^K$. Let $x = V(a,P)$. Then $x_k = -$ for all $k \notin a$. Since every agent i is pessimistic about all issues $k \notin a$ it follows that

$$\big((w_k^i)_{k \notin a}, (x_k)_{k \in a}\big) \succ_i y \text{ for all } i.$$

Let $\bar{z} \in \{0,1\}^K$ be such that $z_k = x_k$ for all $k \in a$. Then, by definition of w_k^i for $k \notin a$ it follows that for all i either

$$\bar{z} = \big((w_k^i)_{k \notin a}, (x_k)_{k \in a}\big) \succ_i y$$

or

$$\bar{z} \succ_i \big((w_k^i)_{k \notin a}, (x_k)_{k \in a}\big) \succ_i y.$$

This contradicts our assumption that y is Pareto efficient. Hence, $a \notin CE(a,P)$ which implies that any agenda in $CE(a,P)$ is a full agenda. This proves the theorem.

□

Proof of Theorem 5.1:

We prove the claim for $n = 3$ agents and note that the extension to an arbitrary odd number of agents $n > 3$ is possible: For any preference profile $(\succ_1, \succ_2, \succ_3)$ for three agents we can define a preference profile for $n > 3$ agents, such that every agent's preference ordering is either $\succ_1, \succ_2$ or $\succ_3$ and such that the majority relation on X is preserved.[11]

From now on we assume that $n = 3$. The proof consists of four steps. Steps 1 and 2 deal with the case where there are at least two free issues. In Step 1 we show that for any $K \geq 2$ there exists a preference profile P such that all K issues are free at any consistent equilibrium agenda at P. in Step 2 we use a lexicographic extension of the preferences defined in Step 1 to prove that for $K \geq 3$ and $\#\mathcal{F} \geq 2$ there exists a preference profile P such that $\mathcal{F}$ is the set of free issues at any consistent equilibrium agenda. Steps 3 and 4 consider the case with one free issue. In Step 3 we prove that for $K = 3$ and $\#\mathcal{F} = 1$ there exists a preference profile P such $\mathcal{F}$ is the set of free issues at any consistent equilibrium agenda. Finally, in Step 4 we use a lexicographic extension of the preferences defined in Step 2 to extend the case with one free issue from $K = 3$ to an arbitrary number of issues $K \geq 4$.

Step 1: In the following we prove that for any set of issues $\mathcal{K} = \{1, \ldots, K\}$ with $K \geq 2$ there exists a preference profile $P = (\succ_1, \succ_2, \succ_3) \in \mathcal{S}^3$ such that the set of free issues at any consistent equilibrium agenda is $\mathcal{F} = \mathcal{K}$. Note that this means that $\varnothing$ is the unique consistent equilibrium agenda at P.

Let $\mathcal{K} = \mathcal{F} = \{1, 2\}$. Then Section 3 provides an example where $\mathcal{F}$ is the set of free issues at any consistent equilibrium agenda.

Let $\mathcal{K} = \mathcal{F} = \{1, 2, 3\}$ and let the agents' preference orderings be given by Table 2. Note that the agents' preferences are separable and satisfy betweenness.

[11] If $n = 5$ let the preference orderings $\succ'_i$ for agents $i = 1, \ldots, 5$, be given by $\succ'_i = \succ_1$ for $i = 1, 2$, $\succ'_i = \succ_2$ for $i = 3, 4$, and $\succ'_5 = \succ_3$. If $n \geq 7$ let $m \in \mathbb{N}$ and $k \in \{0, 1, 2\}$ be such that $n = 3m + k$, and let the preference orderings $\succ'_i$ for agents $i = 1, \ldots, n$, be given by $\succ'_i = \succ_1$ for $i = 1, \ldots, m$, $\succ'_i = \succ_2$ for $i = m + 1, \ldots, 2m$, and $\succ'_i = \succ_3$ for $i = 2m + 1, \ldots, n$. Let M be the simple majority relation at the preference profile $(\succ_1, \succ_2, \succ_3)$ and let M' be the simple majority relation at the preference profile $(\succ'_1, \ldots, \succ'_n)$. Then it is straightforward to show that for all $x, y \in X$, xMy if and only if $xM'y$.

It is immediate to see that there is a Condorcet winner for any agenda a which is the alternative that has position 1 for each issue on the agenda. Thus, for any agenda $a \in A$ the voting outcome under the amendment procedure satisfies

$$(V(a,P))_k = 1 \text{ for all } k \in a.$$

We now solve backwards for the equilibrium collection of sets of continuation agendas. By (E1) it follows that

$$CE(a,P) = \{a\}$$

for all full agendas a.

Now consider an agenda a of length 2. By condition (E1), $CE(a,P)$ is a nonempty subset of $\{a\} \cup CE((a,k),P) = \{a,(a,k)\}$, where $k \notin a$. By condition (E2), $a \in CE(a,P)$ is ruled out since there is always one agent who strictly prefers $(1,1,1)$, which is the voting outcome at agenda (a,k), over x with $x_k = -$ and $x_l = 1$ for all $l \neq k$, which is the voting outcome at agenda a. Hence,

$$CE(a,P) = \{(a,k)\}.$$

Next consider an agenda a of length 1, i.e. $a = (k)$ for some $k \in \{1,2,3\}$. Let $h,l \notin a, h \neq l$. By condition (E1), $CE((k),P)$ is a nonempty subset of $\{(k)\} \cup CE((k,h),P) \cup CE((k,l),P) = \{(k),(k,h,l),(k,l,h)\}$. By condition (E2), $(k) \in CE((k),P)$ is ruled out since there is always one agent who strictly prefers $(1,1,1)$, which is the voting outcome at agendas (k,h,l) or (k,l,h), over x with $x_k = 1$ and $x_l = x_h = -$, which is the voting outcome at agenda (k). Hence,

$$CE((k),P) \subset \{(k,h,l),(k,l,h)\}.$$

Finally, consider the empty agenda. By condition (E1), $CE(\varnothing,P)$ is a nonempty subset of $\{\varnothing\} \cup \bigcup_{k=1}^{3} CE((k),P)$. Since all agendas in $CE((k),P)$ for $k = 1,2,3$, are full agendas with voting outcome $(1,1,1)$ and all agents strictly prefer $(-,-,-)$ over $(1,1,1)$, all agents prefer the empty agenda over any full agenda. By (E2) this implies that $\varnothing \in CE(\varnothing,P)$.

It remains to prove that $\varnothing$ is the unique consistent equilibrium agenda. Suppose by way of contradiction that $a \in CE(\varnothing,P)$ for some $a \neq \varnothing$. Then a must

$\succ_1$	$\succ_2$	$\succ_3$
$(1,1,0)$	$(1,0,1)$	$(0,1,1)$
$(1,1,-)$	$(-,0,1)$	$(-,1,1)$
$(1,-,0)$	$(1,0,-)$	$(0,-,1)$
$(1,-,-)$	$(0,0,1)$	$(0,1,-)$
$(1,0,0)$	$(-,0,-)$	$(-,1,-)$
$(-,1,0)$	$(1,-,1)$	$(0,-,-)$
$(-,1,-)$	$(1,-,-)$	$(-,-,1)$
$(0,1,0)$	$(-,-,1)$	$(-,-,-)$
$(0,1,-)$	$(0,-,1)$	$(1,1,1)$
$(-,-,0)$	$(-,-,-)$	$(1,-,1)$
$(0,-,0)$	$(1,1,1)$	$(1,1,-)$
$(-,0,0)$	$(1,1,-)$	$(1,-,-)$
$(0,0,0)$	$(-,1,1)$	$(0,0,1)$
$(-,-,-)$	$(-,1,-)$	$(0,0,-)$
$(1,1,1)$	$(0,1,1)$	$(-,0,1)$
$(1,0,-)$	$(0,0,-)$	$(-,0,-)$
$(1,-,1)$	$(1,0,0)$	$(1,0,1)$
$(-,1,1)$	$(1,-,0)$	$(1,0,-)$
$(-,-,1)$	$(-,0,0)$	$(0,1,0)$
$(1,0,1)$	$(0,-,-)$	$(-,1,0)$
$(-,0,-)$	$(0,1,-)$	$(0,-,0)$
$(-,0,1)$	$(0,0,0)$	$(-,-,0)$
$(0,1,1)$	$(-,-,0)$	$(1,1,0)$
$(0,-,-)$	$(0,-,0)$	$(1,-,0)$
$(0,0,-)$	$(1,1,0)$	$(0,0,0)$
$(0,-,1)$	$(-,1,0)$	$(-,0,0)$
$(0,0,1)$	$(0,1,0)$	$(1,0,0)$

Table 2: Preference profile for $\mathcal{K} = \mathcal{F} = \{1,2,3\}$.

be a full agenda and since $\varnothing \in CE(\varnothing, P)$, condition (E3) implies that a is rationalizable relative to the empty agenda $\varnothing$. However, no agent prefers the voting outcome at a full agenda over the voting outcome at the empty agenda $\varnothing$ or any other full agenda a' which could be in $CE(\varnothing, P)$. Hence, a is not rationalizable which implies that $a \notin CE(\varnothing, P)$. Hence, we conclude that

$$CE(\varnothing, P) = \{\varnothing\}.$$

Thus, in this case the unique consistent equilibrium agenda is empty and the set of free issues is given by $\mathcal{K}$.

Let $\mathcal{K} = \mathcal{F} = \{1, 2, 3, 4\}$. We take the preference orderings $\succ_i$ for two issues in Table 1 (Section 3) and extend them in a lexicographic way to preference orderings $\succ_i'$ on $\{0, 1, -\}^{\mathcal{K}}$: For $i = 1, 2, 3$, let $\succ_i'$ be such that

$$(x_1, x_2, x_3, x_4) \succ_i' (y_1, y_2, y_3, y_4)$$

if and only if

$$(x_1, x_2) \succ_i (y_1, y_2)$$

or

$$(x_1, x_2) = (y_1, y_2) \text{ and } (x_3, x_4) \succ_i (y_3, y_4).$$

For illustration Table 3 gives the agents' preference orderings on $\{0, 1\}^{\mathcal{K}}$.

Next we determine the voting outcome at all agendas. At the empty agenda $(-, -, -, -)$ is the unique attainable alternative which implies that

$$V(\varnothing, P) = (-, -, -, -).$$

At agenda (k) for $k \in \{1, 2, 3, 4\}$ there are only two attainable alternatives, x and y with $x_k = 1$, $y_k = 0$ and $x_l = y_l = -$ for $l \neq k$. If $k \in \{1, 3\}$, then agents 2 and 3 prefer position 1 over position 0 for issue k which implies that

$$V((1), P) = (1, -, -, -) \text{ and } V((3), P) = (-, -, 1, -).$$

If $k \in \{2, 4\}$, then agents 1 and 3 prefer position 1 over position 0 for issue k which implies that

$$V((2), P) = (-, 1, -, -) \text{ and } V((4), P) = (-, -, -, 1).$$

$\succ'_1$	$\succ'_2$	$\succ'_3$
$(0,1,0,1)$	$(1,0,1,0)$	$(1,1,1,1)$
$(0,1,0,0)$	$(1,0,0,0)$	$(1,1,0,1)$
$(0,1,1,1)$	$(1,0,1,1)$	$(1,1,1,0)$
$(0,1,1,0)$	$(1,0,0,1)$	$(1,1,0,0)$
$(0,0,0,1)$	$(0,0,1,0)$	$(0,1,1,1)$
$(0,0,0,0)$	$(0,0,0,0)$	$(0,1,0,1)$
$(0,0,1,1)$	$(0,0,1,1)$	$(0,1,1,0)$
$(0,0,1,0)$	$(0,0,0,1)$	$(0,1,0,0)$
$(1,1,0,1)$	$(1,1,1,0)$	$(1,0,1,1)$
$(1,1,0,0)$	$(1,1,0,0)$	$(1,0,0,1)$
$(1,1,1,1)$	$(1,1,1,1)$	$(1,0,1,0)$
$(1,1,1,0)$	$(1,1,0,1)$	$(1,0,0,0)$
$(1,0,0,1)$	$(0,1,1,0)$	$(0,0,1,1)$
$(1,0,0,0)$	$(0,1,0,0)$	$(0,0,0,1)$
$(1,0,1,1)$	$(0,1,1,1)$	$(0,0,1,0)$
$(1,0,1,0)$	$(0,1,0,1)$	$(0,0,0,0)$

Table 3: Lexicographic extension of the preference orderings in Table 1 to $\{0,1\}^{\mathcal{K}}$ for $\mathcal{K} = \{1,2,3,4\}$.

Next we determine the voting outcome at all agendas of length 2. The analysis of case $K = 2$ implies that there exists an ordering of the alternatives in $\{0,1\}^{\{1,2\}}$ such that

$$V((1,2),P) = V((2,1),P) = (0,0,-,-).$$

Similarly, there exists an ordering of the alternatives in $\{0,1\}^{\{3,4\}}$ such that

$$V((3,4),P) = V((3,4),P) = (-,-,0,0).$$

Consider agendas $(2,3)$ and $(3,2)$. Then $(-,1,1,-) \succ'_i (-,0,x_3,-)$ for all $x_3 \in \{0,1\}$ and $i = 1,3$, and $(-,1,1,-) \succ'_i (-,1,0,-)$ for $i = 2,3$. Hence, $(-,1,1,-)$ is the Condorcet winner in $X(2,3)$ which implies that

$$V((2,3),P) = V((3,2),P) = (-,1,1,-).$$

Similarly, we derive

$$\begin{aligned} V((1,3),P) &= V((3,1),P) = (1,-,1,-), \\ V((1,4),P) &= V((4,1),P) = (1,-,-,1), \\ V((2,4),P) &= V((4,2),P) = (-,1,-,1). \end{aligned}$$

Next consider all agendas of length 3. Let $a \in A^3$ with $4 \notin a$. We will now argue that there exists an ordering of the alternatives in $X(a)$ such that $(0,0,1,-)$ is the voting outcome under the amendment procedure. To see this note that by definition of the preference orderings $\succ'_i$, under simple majority voting $(0,0,1,-)$ is dominated by $(1,0,x_3,-)$ and $(0,1,x_3,-)$ for all $x_3 \in \{0,1\}$ and $(0,0,1,-)$ dominates all remaining alternatives in $X(a)$. Moreover, $(1,0,x_3,-)$ and $(0,1,x_3,-)$ are dominated by $(1,1,1,-)$ for all $x_3 \in \{0,1\}$. It follows from the characterizations in Banks (1985, Theorem 3.1) and Barberà and Gerber (2017, Theorem 3.1) that there exists an ordering of the alternatives in $X(a)$ such that $(0,0,1,-)$ is the voting outcome under the amendment procedure. We take this ordering and get

$$V(a,P) = (0,0,1,-) \text{ for all } \ a \in A^3 \text{ with } 4 \notin a.$$

In a similar way one shows that for all $a \in A^3$ with $4 \in a$ there exist orderings of the alternatives in $X(a)$ such that

$$V(a,P) = (-,1,0,0) \text{ for all } \ a \in A^3 \text{ with } 1 \notin a,$$

$$V(a,P) = (1,-,0,0) \text{ for all } \; a \in A^3 \text{ with } 2 \notin a,$$
$$V(a,P) = (0,0,-,1) \text{ for all } \; a \in A^3 \text{ with } 3 \notin a.$$

Finally, we determine the voting outcome at all full agendas $a \in A^4$. Note that by definition of the preference orderings $\succ_i'$, under simple majority voting $(0,0,0,0)$ is dominated by $(0,0,1,0), (0,0,0,1), (1,0,x_3,x_4)$ and $(0,1,x_3,x_4)$ for all $x_3, x_4 \in \{0,1\}$, while $(0,0,0,0)$ dominates all remaining alternatives in $X(a)$. Moreover, $(1,1,1,1)$ dominates $(0,0,1,0), (0,0,0,1), (1,0,x_3,x_4)$ and $(0,1,x_3,x_4)$ for all $x_3, x_4 \in \{0,1\}$. Again we use the characterizations in Banks (1985, Theorem 3.1) and Barberà and Gerber (2017, Theorem 3.1) to conclude that there exists an ordering of the alternatives in $X(a)$ such that $(0,0,0,0)$ is the voting outcome under the amendment procedure. We take this ordering and get

$$V(a,P) = (0,0,0,0) \text{ for all } \; a \in A^4.$$

We now solve backwards for the equilibrium collection of sets of continuation agendas. By (E1) it follows that

$$CE(a,P) = \{a\} \text{ for all } a \in A^4.$$

Now consider an agenda of length 3. Let $a \in A^3$ and let $1 \notin a$. By condition (E1), $CE(a,P)$ is a nonempty subset of $\{a\} \cup CE((a,1),P) = \{a,(a,1)\}$. By condition (E2), $a \in CE(a,P)$ is ruled out since agent 2 strictly prefers the voting outcome at agenda $(a,1)$, which is $(0,0,0,0)$ over the voting outcome at agenda a which is $(-,1,0,0)$. Hence,

$$CE(a,P) = \{(a,1)\}.$$

In the same way one proves that

$$CE(a,P) = \{(a,k)\} \text{ for all } k \notin a.$$

Next consider agendas of length 2. To begin with, let $a \in \{(1,2),(2,1)\}$. By condition (E1), $CE(a,P)$ is a nonempty subset of $\{a\} \cup CE((a,3),P) \cup CE((a,4),P) = \{a,(a,3,4),(a,4,3)\}$. By condition (E2), $a \in CE(a,P)$ since all agents prefer the outcome under (a), which is $(0,0,-,-)$, over the outcome under $(a, 3, 4)$ or $(a, 4, 3)$ which is $(0, 0, 0, 0)$. Moreover, given $a \in CE(a, P)$ none of the agendas $(a, 3, 4)$ or $(a, 4, 3)$ is rationalizable relative to a and hence (E3) implies that

$$CE((1, 2), P) = \{(1, 2)\} \text{ and } CE((2, 1), P) = \{(2, 1)\}.$$

In a similar way it follows that

$$CE((3, 4), P) = \{(3, 4)\} \text{ and } CE((4, 3), P) = \{(4, 3)\}.$$

Let $a \in \{(1, 3), (3, 1)\}$. By condition (E1), $CE(a, P)$ is a nonempty subset of $\{a\} \cup CE((a, 2), P) \cup CE((a, 4), P) = \{a, (a, 2, 4), (a, 4, 2)\}$. By (E2), $a \in CE(a, P)$ is ruled out since agent 1 strictly prefers the voting outcome at any full agenda, which is $(0, 0, 0, 0)$, over the voting outcome at agenda a, which is $(1, -, 1, -)$. Hence,

$$CE((1, 3), P), CE((3, 1), P) \subset A^4.$$

In a similar way it follows that

$$CE((1, 4), P), CE((4, 1), P) \subset A^4,$$
$$CE((2, 3), P), CE((3, 2), P) \subset A^4,$$
$$CE((2, 4), P), CE((4, 2), P) \subset A^4.$$

Next consider agendas of length 1. To begin with, let $a = (1)$. By condition (E1), $CE((1), P)$ is a nonempty subset of $\{(1)\} \cup \bigcup_{k=2}^{4} CE((1, k), P)$. By (E2), $(1) \in CE((1), P)$ is ruled out since agent 2 strictly prefers the voting outcome at agenda $(1, 2) \in CE((1, 2), P)$, which is $(0, 0, -, -)$, over the voting outcome at agenda (1), which is $(1, -, -, -)$. Moreover, any agenda in $CE((1, 3), P)$ or $CE((1, 4), P)$ is a full agenda with voting outcome $(0, 0, 0, 0)$. If any such agenda were in $CE((1), P)$, then $(1, 2)$ is rationalizable relative to (1) since all agents prefer the voting outcome under $(1, 2)$, which is $(0, 0, -, -)$, over $(0, 0, 0, 0)$. (E3) then requires that $(1, 2) \in CE((1), P)$ which in turn implies that no agenda in $CE((1, 3), P)$ or $CE((1, 4), P)$ is rationalizable. By (E3) we conclude that no agenda in $CE((1, 3), P)$ or $CE((1, 4), P)$ belongs to $CE((1), P)$. Hence,

$$CE((1), P) = \{(1, 2)\}.$$

In a similar way it follows that

$$CE((2), P) = \{(2,1)\},$$
$$CE((3), P) = \{(3,4)\},$$
$$CE((4), P) = \{(4,3)\}.$$

Finally, consider the empty agenda. By condition (E1), $CE(\varnothing, P)$ is a nonempty subset of $\{\varnothing\} \cup \bigcup_{k=1}^{4} CE((k), P) = \{\varnothing, (1,2), (2,1), (3,4), (4,3)\}$. Since all agents strictly prefer the voting outcome under the empty agenda, which is $(-,-,-,-)$, over the voting outcome under agendas $(1,2)$ or $(2,1)$, which is $(0,0,-,-)$, and the voting outcome under agendas $(3,4)$ or $(4,3)$ which is $(-,-,0,0)$, it follows that $\varnothing \in CE(\varnothing, P)$ by (E2).

It remains to prove that $\varnothing$ is the unique consistent equilibrium agenda. To this end note that (E3) implies that if any of the agendas $(1,2), (2,1), (3,4), (4,3)$ is in $CE(\varnothing, P)$, then it must be rationalizable. However, the voting outcome under agendas $(3,4)$ and $(4,3)$ is $(-,-,0,0)$ which is strictly worse for all agents than the voting outcome under the empty agenda, which is $(-,-,-,-)$, and the voting outcome under agendas $(1,2)$ and $(2,1)$, which is $(0,0,-,-)$. This implies that neither $(3,4)$ nor $(4,3)$ is rationalizable and by (E3) neither of these agendas is in $CE(\varnothing, P)$. But then neither $(1,2)$ nor $(2,1)$ are rationalizable relative to $\varnothing$ because all agents strictly prefer $(-,-,-,-)$ over the voting outcome under agendas $(1,2)$ and $(2,1)$, which is $(0,0,-,-)$. Hence, (E3) implies that

$$CE(\varnothing, P) = \{\varnothing\}.$$

Thus, the unique consistent equilibrium agenda is empty and the set of free issues is given by $\mathcal{K} = \{1,2,3,4\}$.

Let $\mathcal{K} = \mathcal{F} = \{1,2,3,4,5\}$. We then construct a preference profile using the preference orderings in Table 1 and Table 2. For $i = 1,2,3$, let $\succ_i^2$ be agent i's preference ordering in Table 1 and let $\succ_i^3$ be agent i's preference ordering in Table 2. We extend these preferences in a lexicographic way to preference orderings $\succ_i'$ on $\{0,1,-\}^{\mathcal{K}}$: For $i = 1,2,3$, let $\succ_i'$ be such that

$$(x_1, x_2, x_3, x_4, x_5) \succ_i' (y_1, y_2, y_3, y_4, y_5)$$

if and only if

$$(x_1, x_2) \succ_i^2 (y_1, y_2)$$

or

$$(x_1, x_2) = (y_1, y_2) \text{ and } (x_3, x_4, x_5) \succ_i^3 (y_3, y_4, y_5).$$

Then, similar to case $\mathcal{K} = \mathcal{F} = \{1, 2, 3, 4\}$ above we can show that the unique consistent equilibrium agenda is empty and hence the set of free issues is given by $\mathcal{K} = \{1, 2, 3, 4, 5\}$.

Let $\mathcal{K} = \mathcal{F} = \{1, \ldots, K\}$ with $K \geq 6$. Then either K is even or $K = 2m + 3$ for some $m > 1$. In both cases we can proceed as in the previous two cases with $K = 4$ and $K = 5$ to construct an example with three agents and separable preferences such that all issues are free at the unique consistent equilibrium agenda. We simply extend the agents' preference orderings for $K = 2$ and $K = 3$ in a lexicographic way to preference orderings for the given number of issues.

Step 2: Let $\mathcal{K} = \{1, \ldots, K\}$ with $K \geq 3$ and let $\mathcal{F} = \{1, \ldots, F\} \subset \mathcal{K}$ with $F \geq 2$. We will prove that there exists a preference profile $P = (\succ_1, \succ_2, \succ_3) \in \mathcal{S}^3$ such that the set of free issues at the unique consistent equilibrium agenda is $\mathcal{F}$. To this end take the preference orderings $\succ_i$ used for the case $\mathcal{K} = \mathcal{F}$ in Step 1 and extend them in a lexicographic way to preference orderings $\succ_i'$ on $\{0, 1, -\}^{\mathcal{K}}$: For $i = 1, 2, 3$, let $\succ_i'$ be such that

$$(x_1, \ldots, x_K) \succ_i' (y_1, \ldots, y_K)$$

if and only if one of the following two conditions is satisfied:

(i) There exists some l with $F + 1 \leq l \leq K$, such that $x_k = y_k$ for $k = F + 1, \ldots, l - 1$, and either

$$\begin{aligned} & x_l = 1 \text{ and } y_l \in \{-, 0\} \\ \text{or} \quad & x_l = - \text{ and } y_l = 0. \end{aligned}$$

(ii) $x_k = y_k$ for $k = F + 1, \ldots, K$, and

$$(x_1, \ldots, x_F) \succ_i (y_1, \ldots, y_F).$$

Hence, all agents first consider the positions on issues $F+1,\ldots,K$ (in that order) and all prefer position 1 over $-$ and $-$ over 0 on these issues. Only if two alternatives have the same positions on all issues $F+1,\ldots,K$, the positions on the remaining issues are relevant. In that case agent i's preference over the alternatives is determined by the preference $\succ_i$ over the positions on issues $1,\ldots,F$.

Then, by Pareto efficiency of the amendment procedure, $(V(a,P))_k = 1$ for all agendas a with $k \in a$ and $k \in \{F+1,\ldots,K\}$ independent of the ordering of the alternatives in $X(a)$ under the amendment procedure. Moreover, any consistent equilibrium agenda a must contain all issues in $\{F+1,\ldots,K\}$. Suppose this were not true, i.e. there exists a consistent equilibrium agenda a with $k \notin a$ for some $k \in \{F+1,\ldots,K\}$. Lemma 2.1 implies that $a \in CE(a,P)$. Hence, by (E2) it must be true that for all i and for all $a' \in CE((a,k),P))$,

$$V(a,P) \succ_i V(a',P).$$

However, by definition of $\succ_i$ this is impossible since $(V(a',P))_k = 1$ and $(V(a,P))_k = -$. Therefore, we conclude that any consistent equilibrium agenda contains all issues in $\{F+1,\ldots,K\}$. We will now prove that there are no additional issues on any consistent equilibrium agenda if the order of vote under the amendment procedure is chosen in an appropriate way.

Let $(CE(a,P))_{a\in A}$ be any consistent equilibrium collection of sets of continuation agendas and let a be any agenda that is a permutation of $(F+1,\ldots,K)$. Then, given the definition of agents' preferences, Step 1 implies that there exists an order of vote under the amendment procedure, such that

$$CE(a,P) = \{a\} \tag{6.2}$$

Moreover, all such agendas a yield the same voting outcome x with $x_k = 1$ for all $k = F+1,\ldots,K$, and $x_k = -$ for all $k = 1,\ldots,K$.

Let a' be any agenda that is either empty or only contains issues in $\{F+1,\ldots,K\}$. We will prove by backwards induction over the number of issues in a', that any agenda in $CE(a',P)$ is a permutation of $(F+1,\ldots,K)$. (6.2) implies that the claim is true if a' is a permutation of $(F+1,\ldots,K)$. Now suppose the claim is true for all agendas that contain at least $l+1$ issues in $\{F+1,\ldots,K\}$ and no issues in $\{1,\ldots,K\}$, where $0 \leq l < K-F$. Let $a' = (a_1,\ldots,a_l)$ be an agenda with $a_1, \ldots, a_l \in \{F+1, \ldots, K\}$. Since a' does not contain all issues in $\{F+1, \ldots, K\}$, (E1) implies that $CE(a', P)$ is a nonempty subset of $\bigcup_{k \notin a'} CE((a', k), P)$. By the induction hypothesis any agenda in $CE((a', k), P)$ is a permutation of $(F+1, \ldots, K)$ for all $k \notin a'$ with $k \in \{F+1, \ldots, K\}$. By definition of the agents' preferences and the proof in Step 1 it follows that there exists an order of vote under the amendment procedure, such that all agents have the same preferences over voting outcomes $V(a'', P)$ for all $a'' \in CE((a', k), P)$ and for all $k \notin a'$. Moreover, all agents prefer $V(a, P)$, where a is some permutation of $(F+1, \ldots, K)$, over $V(a'', P)$ for any agenda $a'' \in CE((a', k), P)$ for all $k \in \{1, \ldots, F\}, k \notin a'$. Hence, the voting outcomes $V(a'', P)$ for all $a'' \in CE((a', k), P)$ and for all $k \in \{1, \ldots, K\}, k \notin a'$, are Pareto ranked with $V(a'', P)$ being preferred over $V(a''', P)$ for all $a'' \in CE((a', k), P)$ with $k \notin a'$ and $k \in \{F+1, \ldots, K\}$, and for all $a''' \in CE((a', k'), P)$ for all $k' \notin a'$ and $k' \in \{1, \ldots, F\}$. Therefore, consistency (E3) implies that

$$CE(a', P) \subset \bigcup_{k \notin a', k \in \{F+1, \ldots, K\}} CE((a', k), P).$$

Hence, by the induction hypothesis any agenda in $CE(a', P)$ is a permutation of $(F+1, \ldots, K)$. This proves the claim.

We conclude that any agenda in $CE(\varnothing, P)$ is a permutation of $(F+1, \ldots, K)$, i.e. the set of free issues at any consistent equilibrium agenda is given by $\mathcal{F}$.

Step 3: Let $\mathcal{K} = \{1, 2, 3\}$ and $\#\mathcal{F} = 1$. W.l.o.g. let $\mathcal{F} = \{3\}$. We will prove that there exists a preference profile $P = (\succ_1, \succ_2, \succ_3) \in \mathcal{S}^3$ such that the set of free issues at the unique consistent equilibrium agenda is $\mathcal{F} = \{3\}$. Let preference orderings be given by Table 4. Note that the agents' preferences are separable and satisfy betweenness.

In order to solve for the equilibrium agendas we first determine the voting outcome at all agendas. At the empty agenda $(-, -, -)$ is the unique attainable alternative which implies that

$$V(\varnothing, P) = (-, -, -).$$

At agenda (1) there are only two attainable alternatives, $(1, -, -)$ and $(0, -, -)$.

$\succ_1$	$\succ_2$	$\succ_3$
$(0,1,1)$	$(1,0,1)$	$(1,1,0)$
$(0,-,1)$	$(-,0,1)$	$(1,-,0)$
$(0,0,1)$	$(0,0,1)$	$(1,1,-)$
$(0,1,-)$	$(1,0,-)$	$(1,-,-)$
$(0,-,-)$	$(-,0,-)$	$(-,1,0)$
$(0,0,-)$	$(1,-,1)$	$(0,1,0)$
$(-,1,1)$	$(0,0,-)$	$(-,-,0)$
$(1,1,1)$	$(1,-,-)$	$(-,1,-)$
$(-,1,-)$	$(1,1,1)$	$(0,1,-)$
$(1,1,-)$	$(-,-,1)$	$(0,-,0)$
$(-,-,1)$	$(-,1,1)$	$(1,0,0)$
$(-,0,1)$	$(0,-,1)$	$(-,-,-)$
$(1,-,1)$	$(0,1,1)$	$(1,0,-)$
$(-,-,-)$	$(-,-,-)$	$(0,-,-)$
$(-,0,-)$	$(1,1,-)$	$(-,0,0)$
$(1,0,1)$	$(0,-,-)$	$(0,0,0)$
$(1,-,-)$	$(1,0,0)$	$(-,0,-)$
$(1,0,-)$	$(1,-,0)$	$(0,0,-)$
$(0,1,0)$	$(-,0,0)$	$(1,1,1)$
$(-,1,0)$	$(-,-,0)$	$(1,-,1)$
$(0,-,0)$	$(-,1,-)$	$(-,1,1)$
$(0,0,0)$	$(0,0,0)$	$(-,-,1)$
$(-,-,0)$	$(0,-,0)$	$(0,1,1)$
$(-,0,0)$	$(0,1,-)$	$(0,-,1)$
$(1,1,0)$	$(1,1,0)$	$(1,0,1)$
$(1,-,0)$	$(-,1,0)$	$(-,0,1)$
$(1,0,0)$	$(0,1,0)$	$(0,0,1)$

Table 4: Preference orderings for $\mathcal{K} = \{1,2,3\}$ and $\mathcal{F} = \{3\}$.

Since agents 2 and 3 prefer $(1,-,-)$ over $(0,-,-)$ it follows that

$$V((1),P) = (1,-,-).$$

At agenda (2) there are only two attainable alternatives, $(-,1,-)$ and $(-,0,-)$. Since agents 1 and 3 prefer $(-,1,-)$ over $(-,0,-)$ it follows that

$$V((2),P) = (-,1,-).$$

At agenda (3) there are only two attainable alternatives, $(-,-,1)$ and $(-,-,0)$. Since agents 2 and 3 prefer $(-,-,1)$ over $(-,-,0)$ it follows that

$$V((3),P) = (-,-,1).$$

Next we consider all agendas of length 2. At agendas, $(1,2)$ and $(2,1)$ the attainable set is

$$X(1,2) = X(2,1) = \{(0,0,-),(0,1,-),(1,0,-),(1,1,-)\}.$$

Note that under simple majority voting $(0,0,-)$ is dominated by $(0,1,-)$ and $(1,0,-)$, where both of the latter agendas are dominated by $(1,1,-)$ which in turn is dominated by $(0,0,-)$. Hence, using the characterization results in Banks (1985, Theorem 3.1) and Barberà and Gerber (2017, Theorem 3.1) we conclude that $(0,0,-)$ is the outcome under the amendment procedure for some ordering of the alternatives.[12] If we take this ordering of vote under the amendment procedure we obtain

$$V((1,2),P) = V((2,1),P) = (0,0,-).$$

At agendas $(1,3)$ and $(3,1)$ the attainable set is

$$X(1,3) = X(3,1) = \{(0,-,0),(0,-,1),(1,-,0),(1,-,1)\}.$$

Since $(1,-,1)$ dominates any other attainable alternative in pairwise simple majority voting it is the unique outcome under the amendment procedure for any ordering of the alternatives. Hence, we have

$$V((1,3),P) = V((3,1),P) = (1,-,1).$$

[12] The reader may also verify directly that $(0,0,-)$ is the outcome if voting takes place in the ordering $((0,0,-),(0,1,-),(1,0,-),(1,1,-))$ or $((0,0,-),(1,0,-),(0,1,-),(1,1,-))$.

Similarly, at agendas $(2,3)$ and $(3,2)$ the attainable set is

$$X(2,3) = X(3,2) = \{(-,0,0),(-0,1),(-,1,0),(-,1,1)\}$$

and $(-,1,1)$ dominates any other attainable alternative in pairwise simple majority voting. It is therefore the unique outcome under the amendment procedure for any ordering of the alternatives and we get

$$V((2,3),P) = V((3,2),P) = (-,1,1).$$

Finally, consider all full agendas. The attainable set at any full agenda $a \in A^3$ is

$$X(a) = \{(1,1,1),(1,1,0),(1,0,1),(1,0,0),(0,1,1),(0,1,0),(0,0,1),(0,0,0)\}.$$

Note that $(0,0,1)$ is the unique alternative that dominates $(1,1,1)$ under pairwise simple-majority voting. Since $(0,0,1)$ is dominated by $(1,0,1)$ which in turn is dominated by $(1,1,1)$, the characterizations in Banks (1985, Theorem 3.1) and Barberà and Gerber (2017, Theorem 3.1) imply that there exists an ordering of the alternatives in $X(a)$ such that $(1,1,1)$ is the outcome under the amendment procedure. Hence, for this ordering

$$V(a,P) = (1,1,1) \text{ for all } a \in A^3.$$

We now solve backwards for the equilibrium collection of sets of continuation agendas. By (E1) it follows that

$$CE(a,P) = \{a\} \text{ for all } a \in A^3.$$

Now consider an agenda $a \in \{(1,2),(2,1)\}$. By condition (E1), $CE(a,P)$ is a nonempty subset of $\{a\} \cup CE((a,3),P) = \{a,(a,3)\}$. Since all agents prefer $(0,0,-)$ over $(1,1,1)$ condition (E2) implies that $a \in CE(a,P)$. Moreover, in this case $(a,3)$ is not rationalizable. Hence, by (E3) we get

$$CE((1,2),P) = \{(1,2)\} \text{ and } CE((2,1),P) = \{(2,1)\}.$$

Next consider an agenda $a \in \{(1,3),(3,1)\}$. By condition (E1), $CE(a,P)$ is a nonempty subset of $\{a\} \cup CE((a,2),P) = \{a,(a,2)\}$. By condition (E2), $a \in$

$CE(a,P)$ is ruled out since agent 1 strictly prefers the voting outcome at agenda $(a,2)$ over the voting outcome at agenda a. Hence,

$$CE((1,3),P) = \{(1,3,2)\} \text{ and } CE((3,1),P) = \{(3,1,2)\}.$$

Next consider an agenda $a \in \{(2,3),(3,2)\}$. By condition (E1), $CE(a,P)$ is a nonempty subset of $\{a\} \cup CE((a,1),P) = \{a,(a,1)\}$. By condition (E2), $a \in CE(a,P)$ is ruled out since agent 2 strictly prefers the voting outcome at agenda $(a,1)$ over the voting outcome at agenda a. Hence,

$$CE((2,3),P) = \{(2,3,1)\} \text{ and } CE((3,2),P) = \{(3,2,1)\}.$$

We then move to agendas of length 1. By condition (E1), $CE((1),P)$ is a nonempty subset of $\{(1)\} \cup CE((1,2),P) \cup CE((1,3),P) = \{(1),(1,2),(1,3,2)\}$. By condition (E2), $(1) \in CE((1),P)$ is ruled out since agent 1 strictly prefers the voting outcome at agenda $(1,2)$ over the voting outcome at agenda (1). Suppose by way of contradiction that $(1,3,2) \in CE((1),P)$. Then $(1,2)$ is rationalizable and (E3) implies that $(1,2) \in CE((1),P)$ and that $(1,3,2)$ must be rationalizable. However, the latter is not true since all agents prefer the voting outcome at agenda $(1,2)$ over the outcome at agenda $(1,3,2)$. Contradiction. Hence, $(1,3,2) \notin CE((1),P)$ and we conclude that

$$CE((1),P) = \{(1,2)\}.$$

By condition (E1), $CE((2),P)$ is a nonempty subset of $\{(2)\} \cup CE((2,1),P) \cup CE((2,3),P) = \{(2),(2,1),(2,3,1)\}$. By condition (E2), $(2) \in CE((2),P)$ is ruled out since agent 2 strictly prefers the voting outcome at the full agenda $(2,3,1)$ over the voting outcome at agenda (2). We conclude that

$$CE((2),P) \subset \{(2,1),(2,3,1)\}.$$

Suppose by way of contradiction that $(2,3,1) \in CE((2),P)$. Since all agents prefer the voting outcome $(0,0,-)$ at agenda $(2,1)$ over the voting outcome $(1,1,1,)$ at agenda $(2,3,1)$ it follows that $(2,1)$ is rationalizable. (E3) then implies that $(2,1) \in CE((2),P)$ and that $(2,3,1)$ is rationalizable. However, the latter is not true since no agent prefers $(1,1,1)$ over $(0,0,-)$. Contradiction. Therefore, we conclude that

$$CE((2),P) = \{(2,1)\}.$$

By condition (E1), $CE((3), P)$ is a nonempty subset of $\{(3)\} \cup CE((3,1), P) \cup CE((3,2), P) = \{(3), (3,1,2), (3,2,1)\}$. By condition (E2), $(3) \in CE((3), P)$ is ruled out since agent 1 strictly prefers the voting outcome at any full agenda over the voting outcome at agenda (3). We conclude that

$$CE((3), P) \subset \{(3,1,2), (3,2,1)\},$$

where all these agendas give the same outcome $(1,1,1)$.

Finally, consider the empty agenda $\varnothing$. By condition (E1), $CE(\varnothing, P)$ is a nonempty subset of $\{\varnothing\} \cup \bigcup_{k=1}^{3} CE((k), P)$, where $CE((1), P) = \{(1,2)\}$ and $CE((2), P) = \{(2,1)\}$ both give voting outcome $(0,0,-)$, and $CE((3), P)$ contains full agendas only, which give outcome $(1,1,1)$. Since agent 1 strictly prefers $(0,0,-)$ over $(-,-,-)$, (E2) implies that $\varnothing \notin CE(\varnothing, P)$. Suppose by way of contradiction that $CE(\varnothing, P)$ contains a full agenda. Since all agents strictly prefer $(0,0,-)$ over $(1,1,1)$, agenda $(1,2)$ is rationalizable and hence is an element of $CE(\varnothing, P)$ by (E3). Moreover, in this case no full agenda is rationalizable and hence $CE(\varnothing, P)$ must not contain any full agenda. From this contradiction we conclude that

$$CE(\varnothing, P) \subset \{(1,2), (2,1)\}.$$

Thus, in this case the set of free issues at any consistent equilibrium agenda is $\mathcal{F} = \{3\}$.

Step 4: Let $\mathcal{K} = \{1, \ldots, K\}$ with $K \geq 4$ and let $\#\mathcal{F} = 1$. W.l.o.g. let $\mathcal{F} = \{3\}$. We will prove that there exists a preference profile $P = (\succ_1, \succ_2, \succ_3) \in \mathcal{S}^3$ such that the set of free issues at any consistent equilibrium agenda is $\mathcal{F} = \{3\}$. To this end take the preference orderings $\succ_i$ in Table 4 and extend them in a lexicographic way to preference orderings $\succ_i'$ on $\{0,1,-\}^{\mathcal{K}}$: For $i = 1,2,3$, let $\succ_i'$ be such that

$$(x_1, \ldots, x_K) \succ_i' (y_1, \ldots, y_K)$$

if and only if one of the following two conditions is satisfied:

(i) There exists some l with $4 \leq l \leq K$, such that $x_k = y_k$ for $k = 4, \ldots, l-1$, and either

$$x_l = 1 \text{ and } y_l \in \{-, 0\}$$
$$\text{or} \quad x_l = - \text{ and } y_l = 0.$$

(ii) $x_k = y_k$ for $k = 4, \ldots, K$, and

$$(x_1, x_2, x_3) \succ_i (y_1, y_2, y_3).$$

Hence, all agents first consider the positions on issues $4, \ldots, K$ (in that order) and all prefer position 1 over $-$ over 0 on these issues. Only if two alternatives have the same positions on all issues $4, \ldots, K$, the positions on the remaining issues $1, 2, 3$, are relevant. In that case agent i's preference over the alternatives is determined by the preference $\succ_i$ over the positions on issues $1, 2, 3$.

Then, by Pareto efficiency of the amendment procedure, $(V(a, P))_k = 1$ for all agendas a with $k \in a$ and $k \in \{4, \ldots, K\}$ independent of the ordering of the alternatives in $X(a)$ under the amendment procedure. Then, analogously to Step 2, we can use the findings from Step 3 for $\mathcal{K} = \{1, 2, 3\}$ to prove that for any agenda $a \in A$ there exists an order of vote over the alternatives in $X(a)$ under the amendment procedure, such that the set of free issues at any consistent equilibrium agenda is $\mathcal{F} = \{3\}$.

□

Proof of Theorem 5.2: Let $V : A \times \bar{\mathcal{S}}^n \to X$ be voting by quota $q \in \{1, \ldots, n\}$ and let $\mathcal{F} = \emptyset$. Take any preference ordering $\succ \in \bar{\mathcal{S}}$ and let $P = (\succ_1, \ldots, \succ_n)$ be such that $\succ_i = \succ$ for all $i = 1, \ldots, n$. Then, for all $a \in A \setminus A^K$ and for all $a' = (a, k, \ldots)$ with $k \notin a$,

$$V(a', P) \succ_i V(a, P). \tag{6.3}$$

Let $(CE(a, P))_{a \in A}$ be any consistent equilibrium collection of sets of continuation agendas. (6.3) and (E2) then imply that $a \notin CE(a, P)$ for all $a \in A \setminus A^K$ and we conclude that

$$CE(\varnothing, P) \subset A^K$$

by Lemma 2.1. Hence, there are no free issues at any equilibrium agenda at P.

Now let $\emptyset \neq \mathcal{F} \subset \mathcal{K}$ be such that $\#\mathcal{F} \neq 1$ and $\#\mathcal{F} \neq 2$ if n is odd and $q = \frac{n+1}{2}$. Let $r = \#\mathcal{F}$ and w.l.o.g. let $\mathcal{F} = \{1, \ldots, r\}$.

We first consider the case where $r \leq n$. In this case, for all i, choose $(u_k^i(\cdot))_{k \in \mathcal{K}}$ such that

$$u_k^i(1) > u_k^i(-) > u_k^i(0) \text{ for all } k \notin \mathcal{F}. \tag{6.4}$$

If $q \geq \frac{n+1}{2}$, let $\{W_1 \ldots, W_r\}$ be a partition of the set of agents $\{1, \ldots, n\}$ into nonempty subsets W_h, $h = 1, \ldots, r$, such that $\#W_h < q$ for all $h = 1, \ldots, r$. Observe that such a partition exists for any n and any $q \geq \frac{n+1}{2}$ if $r \geq 3$. It also exists for $r = 2$ if either n is even or n is odd and $q > \frac{n+1}{2}$. Then define utility scalars $(u_h^i(\cdot))_{h=1,\ldots,r}$ as follows: For $h = 1, \ldots, r$, and for all $i \in W_h$ let

$$\begin{aligned} u_h^i(1) &> u_h^i(-) > u_h^i(0), &(6.5)\\ u_{h'}^i(0) &> u_{h'}^i(-) > u_{h'}^i(1) \text{ for all } h' \in \{1, \ldots, r\} \setminus \{h\}, &(6.6)\\ \sum_{h'=1}^{r} u_{h'}^i(-) &> \sum_{h'=1}^{r} u_{h'}^i(0). &(6.7) \end{aligned}$$

Observe that $(u_k^i(\cdot))_{k\in\mathcal{K}}$ can always be chosen such that conditions (6.4), (6.5), (6.6) and (6.7) are satisfied and such that the corresponding preference ordering is strict. With this specification less than q agents prefer position 1 over 0 for any $h \in \mathcal{F}$. This implies that for all $h \in \mathcal{F}$ and for all agendas $a \in A$ with $h \in a$,

$$(V(a, P))_h = 0. \tag{6.8}$$

If $q < \frac{n+1}{2}$, then it is straightforward to show that there are two cases: Either there exists a partition $\{W_1 \ldots, W_r\}$ of the set of agents $\{1, \ldots, n\}$ into nonempty subsets W_h, $h = 1, \ldots, r$, such that $\#W_h < q$ for all $h = 1, \ldots, r$, and we can use the same utility specification as in the case where $q \geq \frac{n+1}{2}$. Or there exists a partition $\{W_1 \ldots, W_r\}$ of $\{1, \ldots, n\}$ into nonempty subsets W_h, $h = 1, \ldots, r$, such that $\#W_h \geq q$ and $n - \#W_h \geq q$ for at least one $h \in \{1, \ldots, r\}$. In the latter case, choose the utility scalars $(u_h^i(\cdot))_{h=1,\ldots,r}$ as follows: For $h = 1, \ldots, r$, and for all $i \in W_h$ let

$$\begin{aligned} u_h^i(0) &> u_h^i(-) > u_h^i(1), &(6.9)\\ u_{h'}^i(1) &> u_{h'}^i(-) > u_{h'}^i(0) \text{ for all } h' \in \{1, \ldots, r\} \setminus \{h\}, &(6.10)\\ \sum_{h'=1}^{r} u_{h'}^i(-) &> \sum_{h'=1}^{r} u_{h'}^i(1). &(6.11) \end{aligned}$$

Again observe that $(u_k^i(\cdot))_{k\in\mathcal{K}}$ can always be chosen such that conditions (6.4), (6.9), (6.10) and (6.11) are satisfied and such that the corresponding preference ordering is strict. With this specification at least q agents prefer position 1 over 0 for any $h \in \mathcal{F}$. This implies that for all $h \in \mathcal{F}$ and for all agendas $a \in A$ with $h \in a$,

$$(V(a, P))_h = 1. \tag{6.12}$$

(6.5), (6.6) and (6.8) (resp. (6.9), (6.10)and (6.12)) imply that if an agenda contains at most $m \leq r-1$ issues from $\mathcal{F}$, then there exists at least one agent who gets his most preferred position on all m issues, and if an agenda contains all issues from $\mathcal{F}$, then every agent i gets his most preferred position on exactly $r-1$ issues in $\mathcal{F}$. Moreover, given (6.7) (resp. (6.11)) every agent prefers an agenda that contains all but the issues in $\mathcal{F}$ over any full agenda: For all agendas a that contain all but the issues in $\mathcal{F}$, and for all full agendas $a' \in A^K$,

$$V(a, P) \succ_i V(a', P) \text{ for all } i. \tag{6.13}$$

Let $(CE(a, P))_{a \in A}$ be any consistent equilibrium collection of sets of continuation agendas and let $a \in A \setminus A^K$ with $k \in a$ for some $k \in \mathcal{F}$. We will prove that $a \notin CE(a, P)$. By definition of the agents' utility functions there exists an agent i who gets his most preferred position on all issues $l \notin a$. To see this, note that all agents agree on the position for issues not in $\mathcal{F}$ and there are at most $r-1$ issues from $\mathcal{F}$ which are not on agenda a. Hence,

$$V(a', P) \succ_i V(a, P) \text{ for all } a' \in \bigcup_{l \notin a} CE((a, l), P).$$

By (E2) this implies that $a \notin CE(a, P)$. Lemma 2.1 then implies that

$$CE(a, P) \subset A^K \text{ for all } a \in A \text{ with } k \in a \text{ for some } k \in \mathcal{F}. \tag{6.14}$$

Moreover, if $a \in A$ contains all issues but those in $\mathcal{F}$, then $CE(a, P) = \{a\}$. To see this, observe that any $a' \in CE((a, k), P)$ with $k \notin a$ must be a full agenda by (6.14). By definition of the agents' utility function it follows that

$$V(a, P) \succ_i V(a', P) \text{ for all } i. \tag{6.15}$$

(E2) then implies that $a \in CE(a, P)$. Moreover, by (6.15) and the fact that any $a'' \in CE(a, l)$ for some $l \notin a$ is a full agenda by (6.14), we conclude that no

$a' \in CE((a,k),P)$ with $k \notin a$ is rationalizable relative to a. (E3) then implies that $CE(a,P) = \{a\}$.

Let $0 \leq m \leq K - r$ and let $a \in A^m$ with $h \notin a$ for all $h \in \mathcal{F}$. We will now show inductively over m that $a' \in CE(a,P)$ implies that a' contains all issues but those in $\mathcal{F}$. We have shown above that this is true for $m = K - r$. Suppose that the claim has been proven for all $\overline{m}$ with $\overline{m} \leq m \leq K - r$, where $1 \leq \overline{m} \leq K - r$, and let $a \in A^{\overline{m}-1}$ with $h \notin a$ for all $h \in \mathcal{F}$. By (E1), if $a' \in CE(a,P)$, then either $a' = a$ or $a' \in \bigcup_{k \notin a} CE((a,k),P)$. Let $a' \in CE((a,k),P)$ for some $k \notin a$. If $k \in \mathcal{F}$, then $a' = (a,k,\ldots) \in A^K$ by (6.14). If $k \notin \mathcal{F}$, then by our induction hypothesis a' contains all issues but those in $\mathcal{F}$ which implies that

$$V(a',P) \succ_i V(a,P) \text{ for all } i,$$

since all agents agree on the position for all issues not in $\mathcal{F}$. (E2) then implies that $a \notin CE(a,P)$. Hence, $CE(a,P) \subset \bigcup_{k \notin a} CE((a,k),P)$ and any $a' \in CE(a,P)$ is either a full agenda or contains all issues but those in $\mathcal{F}$.

Suppose by way of contradiction that there exists $a' \in CE(a,P)$ with $a' \in CE((a,k),P)$ for some $k \in \mathcal{F}$. Then $a' = (a,k,\ldots) \in A^K$ by (6.14). Let $l \notin \mathcal{F}$ and $l \notin a$ and let $a'' \in CE((a,l),P)$. Then by the induction hypothesis $a'' = (a,l,\ldots)$ contains all issues but those in $\mathcal{F}$ which implies that

$$V(a'',P) \succ_i V(a',P) \text{ for all } i.$$

Hence, a'' is rationalizable relative to a and (E3) implies that $a'' \in CE(a,P)$. Moreover, if both $a' = (a,k,\ldots) \in CE(a,P)$ and $a'' = (a,l,\ldots) \in CE(a,P)$ with $k \neq l$, then (E3) implies that a' is rationalizable. Therefore, there exists an agent i and some $\hat{a} \in CE(a,P)$ with $\hat{a} = (a,h,\ldots)$ for some $h \neq k$ such that

$$V(a',P) \succ_i V(\hat{a},P). \tag{6.16}$$

However, by what we have shown above, any $\hat{a} \in CE(a,P)$ is either a full agenda or contains all issues but those in $\mathcal{F}$. Since a' is a full agenda, this contradicts (6.16). Hence, if $a' \in CE(a,P)$, then $a' \in CE((a,k),P)$ for some $k \notin \mathcal{F}$ and by the induction hypothesis we conclude that a' contains all issues but those in $\mathcal{F}$. This proves the claim for all m with $0 \leq m \leq K - r$. In particular, any consistent equilibrium agenda a^* at P has the property that $\mathcal{F}$ is the set of free issues at a^*.

It remains to consider the case, where $\mathcal{F} = \{1, \ldots, r\}$ with $r > n$. If $1 \leq q \leq n-1$, choose $(u_k^i(\cdot))_{k \in \mathcal{K}}$ as follows. For $i = 1, \ldots, n-1$, let

$$\begin{aligned} u_i^i(0) &> u_i^i(-) > u_i^i(1), && (6.17)\\ u_k^i(1) &> u_k^i(-) > u_k^i(0) \quad \text{for all } k \in \mathcal{K}, k \neq i, && (6.18)\\ \sum_{k=1}^{r} u_k^i(-) &> \sum_{k=1}^{r} u_k^i(1), && (6.19) \end{aligned}$$

and for $i = n$ let

$$\begin{aligned} u_k^n(0) &> u_k^n(-) > u_k^n(1) \quad \text{for all } k = n, \ldots, r, && (6.20)\\ u_k^n(1) &> u_k^n(-) > u_k^n(0) \quad \text{for all } k \in \mathcal{K}, k \notin \{n, \ldots, r\}, && (6.21)\\ \sum_{\substack{l=1\\ l\neq k}}^{r} u_l^n(-) &< \sum_{\substack{l=1\\ l\neq k}}^{r} u_l^n(1) \quad \text{for all } k = n, \ldots, r, && (6.22)\\ \sum_{k=1}^{r} u_k^n(-) &> \sum_{k=1}^{r} u_k^n(1), && (6.23) \end{aligned}$$

Observe that $(u_k^i(\cdot))_{k \in \mathcal{K}}$ can always be chosen such that conditions (6.17)-(6.23) are satisfied and such that the corresponding preference ordering is strict. Moreover, note that with this specification, for every issue k there are at least $n-1$ agents who prefer position 1 over 0, which implies that

$$(V(a, P))_k = 1 \quad \text{for all } a \in A \text{ and for all } k \in a.$$

If $q = n$, choose $(u_k^i(\cdot))_{k \in \mathcal{K}}$ as follows. For $i = 1, \ldots, n-1$, let

$$\begin{aligned} u_i^i(1) &> u_i^i(-) > u_i^i(0), && (6.24)\\ u_k^i(0) &> u_k^i(-) > u_k^i(1) \quad \text{for all } k \in \mathcal{K}, k \neq i, && (6.25)\\ \sum_{k=1}^{r} u_k^i(-) &> \sum_{k=1}^{r} u_k^i(0), && (6.26) \end{aligned}$$

and for $i = n$ let

$$\begin{aligned} u_k^n(1) &> u_k^n(-) > u_k^n(0) \quad \text{for all } k = n, \ldots, r, && (6.27)\\ u_k^n(0) &> u_k^n(-) > u_k^n(1) \quad \text{for all } k \in \mathcal{K}, k \notin \{n, \ldots, r\}, && (6.28) \end{aligned}$$

$$\sum_{\substack{l=1\\l\neq k}}^{r} u_l^n(-) \;<\; \sum_{\substack{l=1\\l\neq k}}^{r} u_l^n(0) \quad \text{for all } k = n,\dots,r, \tag{6.29}$$

$$\sum_{k=1}^{r} u_k^n(-) \;>\; \sum_{k=1}^{r} u_k^n(0), \tag{6.30}$$

Again observe that $(u_k^i(\cdot))_{k\in\mathcal{K}}$ can always be chosen such that conditions (6.24)-(6.30) are satisfied and such that the corresponding preference ordering is strict. Also, note that with this specification, for every issue k there is at most one agent who prefers position 1 over 0, which implies that

$$(V(a,P))_k = 0 \quad \text{for all } a \in A \text{ and for all } k \in a.$$

For all quotas q the preferences we have specified above have the following properties: Every agent $i \in \{1,\dots,n-1\}$ gets his most preferred position on all issues but issue i, and agent n gets his most preferred position on all issues but issues $n,\dots,r$. Moreover, all agents prefer to stop at an agenda that contains all but the issues in $\mathcal{F}$ rather than adding all issues in $\mathcal{F}$ to the given agenda. Finally, agent n prefers to add all remaining issues in $\mathcal{F}$ to any agenda that already contains some issue in $\{n,\dots,r\}$ and all issues not in $\mathcal{F}$. We will use these properties to prove that $\mathcal{F}$ is the set of free issues at any consistent equilibrium agenda.

Let $(CE(a,P))_{a\in A}$ be any consistent equilibrium collection of sets of continuation agendas and let $a \in A \setminus A^K$ with $k \in a$ for some $k \in \mathcal{F} = \{1,\dots,r\}$. We will prove that $CE(a,P) \subset A^K$. If $a \in A^K$ there is nothing to prove. Hence, let $a \in A^m$ for some $m < K$. The proof is by induction over m. Let $m = K-1$ and let $l \notin a$. If $l \notin \mathcal{F}$, then $V((a,l),P) \succ_i V(a,P)$ for all i and (E2) implies that $a \notin CE(a,P)$. It follows that $CE(a,P) = \{(a,l)\} \subset A^K$. If $l \in \mathcal{F}$, then there are $n-1$ agents who prefer $V((a,l),P)$ over $V(a,P)$ and again we conclude that $CE(a,P) = \{(a,l)\} \subset A^K$.

Now suppose that for all $\bar{m} \leq K-1$ it is true that $CE(a,P) \subset A^K$ for all $a \in A^m$ with $\bar{m} \leq m \leq K-1$ and $k \in a$ for some $k \in \mathcal{F}$. Let $a \in A^{\bar{m}-1}$. If $k \in \{1,\dots,n-1\}$, then agent k gets his most preferred position on all issues not in a, which implies that for all $l \notin a$,

$$V(a',P) \succ_k V(a,P) \text{ for all } a' \in CE((a,l),P).$$

Hence, (E2) implies that $a \notin CE(a,P)$ and

$$CE(a,P) \subset \bigcup_{l \notin a} CE((a,l),P).$$

From the induction hypothesis we conclude that $CE(a,P) \subset A^K$.

If $l \notin a$ for all $l \in \{1,\ldots,n-1\}$ and $k \in a$ for some $k \in \{n,\ldots,r\}$, let $a' \in CE((a,l),P)$ for some $l \in \{1,\ldots,n-1\}$. Then by the induction hypothesis it follows that $a' \in A^K$ and (6.22), respectively (6.29) imply that

$$V(a',P) \succ_n V(a,P).$$

Again, (E2) implies that $a \notin CE(a,P)$ and from the induction hypothesis we conclude that $CE(a,P) \subset A^K$.

Now let $a \in A$ contain all issues but those in $\mathcal{F}$. Then using the same argument as in the first part of the proof, where we considered the case $r \leq n$, (6.19), (6.23), (6.26), (6.30) imply that $CE(a,P) = \{a\}$. Moreover, as in the first part of the proof this implies that $\mathcal{F}$ is the set of free issues at any consistent equilibrium agenda. This proves the theorem.

□

Proof of Lemma 5.1: Let $V : A \times \mathcal{S}^n \to X$ be voting by quota $q \in \{1,\ldots,n\}$ and let $P = (\succ_1,\ldots,\succ_n) \in \mathcal{S}^n$. Let $(CE(a,P))_{a \in A}$ be an equilibrium collection of sets of continuation agendas. Then, for $a \in A^{K-1}$ and $k \notin a$ (E1) implies that

$$CE((a,k),P) = \{(a,k)\}.$$

Moreover, $(V(a,P))_k = -$ and $(V((a,k),P))_k \in \{0,1\}$. If $(V((a,k),P))_k = 1$ ($(V((a,k),P))_k = 0$) then there exists at least one agent i with $u^i_k(1) > u^i_k(0)$ ($u^i_k(0) > u^i_k(1)$). In either case the fact that $\max\{u^i_k(1), u^i_k(0)\} > u^i_k(-)$ implies that

$$V((a,k),P) \succ_i V(a,P)$$

for at least one agent i. Using (E2) we conclude that $a \notin CE(a,P)$ and hence $CE(a,P) = \{(a,k)\} \in A^K$ by (E1).

□

Proof of Proposition 5.2: Let n be odd and let $V : A \times \bar{\mathcal{S}}^n \to X$ be voting by quota $q = \frac{n+1}{2}$. Let $P = (\succ_1, \ldots, \succ_n) \in \bar{\mathcal{S}}^n$ and let $(CE(a,P))_{a \in A}$ be an equilibrium collection of sets of continuation agendas. Obviously, for any full agenda $a \in A^K$,

$$CE(a,P) = \{a\}.$$

Now consider any agenda $a \in A^{K-1}$ and let $k \notin a$. By condition (E1) and Lemma 5.1, we conclude that

$$CE(a,P) = \{(a,k)\}.$$

Now consider any agenda $a \in A^{K-2}$ and let k and l be the free issues at a. By condition (E1) and the previous reasoning, $CE(a,P)$ is a nonempty subset of $\{a\} \cup CE(a,k) \cup CE(a,l) = \{a, (a,k,l), (a,l,k)\}$. Observe that agendas (a,k,l) and (a,l,k) are outcome equivalent since the voting procedure does not depend on the ordering of the issues in the agenda. Let $x = V(a,P)$ and $y = V((a,k,l),P) = V((a,l,k),P)$. Then $y_m = x_m$ for all $m \in a$ and $y_k, y_l \in \{0,1\}$. By (E2) $a \in CE(a,P)$ if and only if for all agents i,

$$u_k^i(-) + u_l^i(-) > u_k^i(y_k) + u_l^i(y_l). \tag{6.31}$$

A necessary condition for (6.31) to hold for all agents is that no agent i is in the winning majority for both issues, k and l. However, if n is odd and $q = \frac{n+1}{2}$, there always exists at least one agent who belongs to the winning majority for both issues. This implies that (6.31) is violated for at least one agent i and hence,

$$CE(a,P) \subset A^K \text{ for all } a \in A^{K-2}.$$

Lemma 2.1 then implies that

$$CE(a,P) \subset A \setminus \left(A^{K-1} \cup A^{K-2}\right) \text{ for all } a \in A.$$

$\square$

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