An Impure Theory of Public Expenditure

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Abstract

Impure public goods, like environmentally-friendly and socially-responsible products, have garnered sustained interest. However, existing research on impure public goods focuses on individual consumer decision-making and Nash equilibrium outcomes; little has been done to analyze how policy can improve efficiency in such markets. We develop a cost sharing mechanism to implement a Pareto optimal allocation in impure public good markets. We prove the existence of the cost sharing solution and conditions for uniqueness. We also elucidate the efficiency and comparative static properties of the resulting equilibrium. Our analysis has many applications, ranging from renewable energy to international environmental protection.

Keywords: Cost sharing; impure public goods; private provision of public goods; green markets

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1 Introduction

The private provision of public goods is a critical topic in economics. While early work considered only pure public and pure private goods, impure public goods, which jointly produce public and private benefits, have increasingly entered the spotlight. The theory of impure public goods has grown in relevance, especially as a framework for analyzing environmentally-friendly and socially-responsible consumption.

Impure public goods are commonplace. In a simple trip to the grocery store, a shopper will find green goods like organic and sustainably harvested foods - products that jointly provide nutrition (private) and environmental benefits (public). While such products are marketed as a blessing for the environment, economic analysis suggests a need for caution. There are certainly cases where green goods can deliver on this promise, but the introduction of a new green good or the enhancement of an existing one need not result in greater environmental quality or welfare improvements; in fact, there are even scenarios in which green markets can cause immiseration (Kotchen, 2006). Thus, it is critical to explore corrective policies that can generate more efficient outcomes. While such policies have been examined extensively in markets with pure public goods, little work has been done in extending this work to green goods in particular or impure public goods in general.

This paper describes one such approach. We use a cost sharing mechanism to achieve a Pareto efficient allocation. Under this mechanism, each agent is assigned a cost share that functions effectively like a subsidy based on the marginal willingness to pay schedules of individuals throughout the economy. This approach follows the basic intuition of Pigovian subsidies, and it generalizes upon the Pigovian framework by allowing heterogeneous preferences across the population and by considering a public good rather than a pure externality. Moreover, it is budget balancing, so that there are zero net expenditures from the social planner. We find that this cost sharing approach will always have a solution that is Pareto optimal. We also consider some of the properties of the cost sharing equilibrium, including the change in welfare resulting from an improvement in technology. We find that in equilibrium, a technology improvement can cause Pareto improvement or welfare redistribution. However, immiseration is precluded, a notable departure from the Nash equilibrium case discussed by Kotchen (2006).

Our work contributes to several lines inquiry, including impure public goods, corresponding corrective policies, and cost sharing mechanisms. There is a large literature on impure public goods that extends the linear characteristics model of Lancaster (1966) and Gorman (1980) to goods that jointly provide private and public characteristics (Cornes & Sandler, 1984, 1994). Kotchen (2005) and Chan & Kotchen (2014) examine comparative statics in impure public good markets on the consumer level, while Kotchen (2006, 2009) considers Nash equilibria in markets with impure public goods and impure public bads, respectively. We extend existing work on impure public goods by considering corrective policies and optimality.

McMahon (2015) and Wichman (2016) both consider mechanisms for improving on the Nash equilibrium in a green good market. However, the policies studied by McMahon (2015) do not implement optimal solutions and focus instead on compensatory taxation and marginal improvements to public good provision. Meanwhile, Wichman (2016) introduces an incentive compatible mechanism for achieving an optimal allocation, a notable contribution given the information constraints that can arise in these markets. However, his solution relies on the assumption of quasilinear preferences, which diminishes the information burden. Moreover, he focuses on a special case of green goods where there are no conventional substitutes for providing the private or public characteristics. We consider a more general context where such substitutes exist, and our approach of cost sharing ensures optimality for a much broader range of preferences (i.e., most well-behaved preferences), although we abstract away from the question of preference revelation.

Our work is related to other research on cost sharing and matching schemes for public good provision (Lindahl, 1958; Foley, 1970; Kaneko, 1977a,b; Mas-Colell & Silvestre, 1989; Buchholz *et al.*, 2011). These face similar information burdens to ours. The solution concept we present is related but distinct, as the bundling of characteristics in the impure public good presents an additional constraint, a complication that is not addressed in prior work. As such, our model is uniquely positioned to address impure public goods and green markets. In addition to defining and analyzing the mechanism and resulting equilibrium, we also characterize the properties of equilibrium, the method of solution, and applications. Our analysis highlights the advantages and limitations of such a cost sharing approach, and it provides a roadmap for informing policy design.

2 The Model

We begin with the impure public good market model prevalent in recent literature, whereby consumers derive utility from characteristics of goods rather than goods *per se*. Following Kotchen (2006), consider a market with $i = 1, .., n$ consumers and three available goods: a pure private good c, a pure public good d, and a green good g.[1] Consumption of these goods provides private and public characteristics x and Y, respectively. Each consumer i has a limited income w_i that she allocates between the three goods in order to maximize her utility function $U_i(x_i, Y)$, where x_i is her consumption of the private characteristic and $Y = y_i + Y_{-i}$ is the aggregate level of public characteristic provided in the economy. We use y_i to represent i's contribution to the public good, while $Y_{-i} = \sum_{j \neq i} y_j$ captures spill-ins from others' contributions.

For example, we can think of consumers as valuing electricity consumption x_i and pollution abatement Y. Consumers can purchase conventional electricity c_i, which only provides x_i; carbon offsets d_i, which only provide y_i; and green electricity (e.g., from solar or wind generation) g_i, which simultaneously generates both x_i and y_i.

Define $mrs_i = \frac{\partial U/\partial Y}{\partial U/\partial x_i}$ as the rate at which an individual consumer is willing to substitute x_i for Y.

Assumption 1. *For all i, $U_i(x_i, Y)$ is such that:*

1. U_i is strictly increasing in both characteristics whenever both arguments are positive. That is, both characteristics are goods in the strict sense.

[1] We will use the terms *impure public good* and *green good* interchangeably.

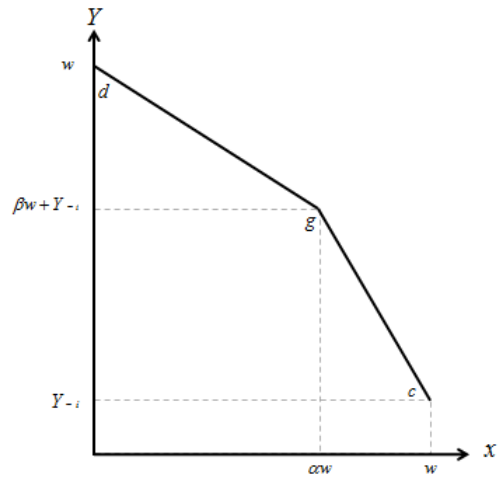


Figure 1: The individual budget frontier in characteristics space.

2. Both characteristics are indispensable, meaning that $lim_{x_i \longrightarrow 0} mrs_i = 0$ and $lim_{Y \longrightarrow 0} mrs_i = \infty$.

3. x_i and Y are strictly normal. This implies that $mrs_i(x_i, Y)$ is a continuous function strictly decreasing in Y and strictly increasing in x_i.

For simplicity, we normalize prices and units so that c, d, and g, each cost \$1 per unit and so that one unit of c_i provides 1 unit of private characteristic x_i (e.g., electricity) while one unit of d_i provides one unit of y_i (e.g., pollution abatement). One unit of g_i, meanwhile, provides α units of x_i and β units of y_i. To focus on the interesting case, we assume that the impure public good provides an improvement upon the pure private and pure public goods ($\alpha + \beta > 1$), but does not strictly dominate the pure goods in the sole provision of x_i or y_i, respectively ($\alpha, \beta < 1$). The initial budget constraint in terms of goods is $c_i + d_i + g_i = w_i$, which we can rewrite in terms of characteristics:

$$\Pi_X x_i + \Pi_Y y_i \quad = \quad w_i \tag{1}$$

$$\Updownarrow$$

$$\Pi_X x_i + \Pi_Y Y \quad = \quad w_i + \Pi_Y Y_{-i} \tag{2}$$

where (Π_X, Π_Y) are the implicit prices, given by: $\left(1, \frac{1-\alpha}{\beta}\right)$, when $y_i < \beta w_i$, which we refer to as Facet I of the consumer's budget constraint; and $\left(\frac{1-\beta}{\alpha}, 1\right)$, for $y_i > \beta w_i$, or Facet II. The implicit prices capture the tradeoffs in characteristics space, and they are the standard implicit prices derived in prior work on impure public goods (Kotchen, 2005, 2006; Chan & Kotchen, 2014). Figure 1 provides a graphical representation. Thus, in terms of characteristics, the individual effectively faces a different budget constraint depending on which facet her consumption bundle occurs.

In Nash equilibrium, consumers will take Y_{-i} as given and maximize their objective function:

$$\max_{c_i, d_i, g_i} U(x_i, y_i + Y_{-i}) \text{ s.t. } c_i + d_i + g_i = w_i; y_i = d_i + \beta g_i; \text{ and } x_i = c_i + \alpha g_i.$$

Rewriting the objective function in terms of characteristics and as a choice on aggregate Y yields

$$\max_{x_i, Y} U(x_i, Y) \text{ s.t. } \Pi_X x_i + \Pi_Y Y = w_i + \Pi_Y Y_{-i} \text{ and } Y \geq Y_{-i}.$$

Solving first-order conditions implies that each individual will consume such that $mrs_i = \frac{\Pi_Y}{\Pi_X}$. As discussed by Kotchen (2005, 2006), consumers will not simultaneously purchase c_i and d_i, as g_i offers a more cost-effective means for obtaining x_i and y_i.

According to Samuelson (1954), a social optimum must satisfy

$$\sum_{i=1}^{n} mrs_i = MRT, \tag{3}$$

where MRT is the marginal rate of transformation at the societal level. Here, MRT can be defined as

$$MRT(Y) = \begin{cases} \frac{1-\alpha}{\beta} & Y < \beta W \\ \frac{\alpha}{1-\beta} & Y > \beta W \end{cases}, \tag{4}$$

where the total wealth in the economy is $W = \sum_{i=1}^{n} w_i$. This MRT function assumes that societal provision of Y will be done in a cost-effective manner (i.e., no simultaneous purchases of c_i and d_i). For simplicity of notation, we will refer to the MRT on Facet I as $MRT_I = \frac{1-\alpha}{\beta}$ and on Facet II as $MRT_{II} = \frac{\alpha}{1-\beta}$.

Expression 3 indicates that the sum of individual marginal rates of substitution (effectively, willingness to pay for Y) sums to the overall marginal rate of transformation (the cost to society of providing Y). Note that underprovision of Y will occur in Nash equilibrium because the utility-maximizing consumer will equate $mrs_i = \frac{\Pi_Y}{\Pi_X}$, whereas a social optimum must satisfy $\sum_{i=1}^{n} mrs_i = MRT$.[2]

3 Cost sharing

Because the free market Nash equilibrium leads to a suboptimal allocation, corrective mechanisms are of great interest, as they can enhance welfare. We construct and analyze a cost sharing mechanism for attaining efficient allocations.

[2] Optimality could additionally be violated if some consumers purchase c_i and g_i while others purchase g_i and d_i, as provision of $(x_1, ..., x_n)$ and Y will not be cost-effective at the societal level.

3.1 Lindahl sharing in pure public good markets

To provide intuition for our mechanism, we begin by describing the Lindahl equilibrium in a pure public good market (Lindahl, 1958). For ease of exposition, consider for a moment a scenario where the impure public good g_i is unavailable, so the prices for x_i and y_i are both unity. Then the budget constraint for the consumer in terms of characteristics is $x_i + y_i = w_i$ or equivalently $x_i + Y = w_i + Y_{-i}$. Thus, the MRT is constant and equal to one.

Suppose that the social planner assigns a Lindahl share τ_i to each consumer. This share holds consumers accountable for providing a proportion of the aggregate public good such that $y_i = \tau_i Y$. The consumer's maximization problem is now

$$\underset{x_i,Y}{max} U_i(x_i, Y) \text{ s.t. } x_i + \tau_i Y = w_i,$$

or equivalently,

$$\underset{x_i,y_i}{max} U_i\left(x_i, \frac{y_i}{\tau_i}\right) \text{ s.t. } x_i + y_i = w_i. \tag{5}$$

While these two expressions are equivalent, they demonstrate different interpretations for implementing the cost sharing rule as a policy. In the first, the consumer is given a subsidized rate for purchasing Y, and she will purchase Y for the entire economy at that rate. In this respect, the consumer's marginal cost becomes the marginal rate of transformation scaled by τ_i, i.e., $\tau_i MRT$. Meanwhile, the subsidy is funded through taxes on (or direct transfers from) other consumers who pay $(1-\tau_i)Y$ to aid her purchase, and these other consumers do not directly purchase Y themselves.

In the second interpretation, the share τ_i represents that prescribed proportion of overall Y that i is asked to purchase. She directly provides $y_i = \tau_i Y$ units of the public characteristics, and all other consumers $j \neq i$ do likewise by purchasing $y_j = \tau_j Y$.

A third interpretation is also instructive:

$$\underset{x_i,y_i}{max} U_i(x_i, y_i + \sum_{j\neq i} y_j) \text{ s.t. } x_i + \tau_i y_i + \sum_{j\neq i} \frac{\tau_i}{1-\tau_j}(1-\tau_j)y_j = w_i \tag{6}$$

This form matches most closely with the likely policy scenario, in which taxes and subsidies are levied to encourage greater production of Y. In this scenario, i purchases y_i at a discounted rate τ_i, implying that her purchase is subsidized at the rate $(1-\tau_i)$. However, we require budget balancing for the social planner, so i's subsidy must be funded by taxes on other members of the economy; likewise, i must pay taxes to fund other individuals' purchases of y_j. Each unit of y_j receives a subsidy rate of $(1-\tau_j)$, and i funds a fraction of this $\frac{\tau_i}{1-\tau_j}$.[3] Therefore, i's budget constraint is composed of three terms: 1) her purchases of x_i; 2) her direct purchases of y_i, which are subsidized; and 3) the tax payments she is compelled to make in order to subsidize all other $j \neq i$.

[3]The fraction $\frac{\tau_i}{1-\tau_j}$ has an intuitive interpretation. All agents besides j must help fund j's subsidy, so i's overall burden is her cost share τ_i, *conditional* on excluding j, which is why $1-\tau_j$ is in the denominator.

All three forms of the consumer problem are equivalent, and they offer different ways of interpreting cost sharing. Interestingly, because of the budget balancing nature of this solution, the social planner need not play a direct role in this scheme, as the individual agents in the economy can effectively cross-subsidize one another's purchases without government involvement. In any case, each consumer i will optimize such that $mrs_i = \tau_i MRT$.

If $(\tau_1, ..., \tau_n)$ are chosen appropriately, (i) the resultant bundles of (x_i, Y) will be utility maximizing for all i; (ii) each individual's budget constraint will be satisfied; and (iii) the payments from individuals will sum to the cost of providing Y. Formally, the Lindahl equilibrium is obtained by solving a set of $2n + 1$ equations:

$$\forall i : mrs_i(x_i^*, Y^*) = \tau_i^* MRT \tag{7}$$

$$\forall i : x_i^* + \tau_i^* Y^* = w_i \tag{8}$$

$$\sum_{i=1}^{n} \tau_i^* = 1 \tag{9}$$

Buchholz *et al.* (2008) have shown that, for a fixed income distribution, there exists a unique set of shares that will implement this equilibrium. Importantly, this Lindahl equilibrium is Pareto optimal; the Samuelson condition is clearly satisfied, and this can be shown by combining the conditions in Expression 7 and Expression 9. Moreover, this equilibrium is a Pareto improvement on the corresponding Nash equilibrium with the same income distribution (Walker, 1981). This is an important feature, as it is in individuals' best interest to participate in such a Lindahl policy if the alternative is a suboptimal Nash equilibrium. However, to implement this solution, the social planner needs full information regarding consumers' preferences; we discuss this challenge in subsequent sections.

3.2 Cost sharing in impure public good markets

Now let us return to our three-good model. There are several notable features that distinguish this setting from the simple Lindahl case and prior work on cost sharing. First, the availability of the green good presents consumers with two distinct consumption regimes with different tradeoffs between x_i and Y. Also, the necessity of obtaining characteristics via purchases of goods presents an additional constraint, which we will call the bundling constraint, that is not operative in other applications of cost sharing in Mas-Colell & Silvestre (1989) and Kaneko (1977a,b). Prior work on cost sharing assumes that x_i and Y can be provided directly (as opposed to indirectly via goods), and they can be divided and distributed in any way by the social planner.

In the impure public good context, the social planner is constrained by the technological correspondence between goods and characteristics. This complicates the social planner's problem, as the cost sharing system must respect the fact that the characteristics conferred by g_i cannot be unbundled. Therefore, an individual who purchases one unit of g_i necessarily consumes α units of x_i and provides β units of y_i; the characteristics from her purchase of g_i cannot be distributed to someone else. Thus, if the equilibrium solution is such that i consumes x_i^* on Facet II, it must also be the case that i purchases $g_i^* = \frac{x_i^*}{\alpha}$ in the goods market.

Still, the general intuition for cost sharing parallels the Lindahl scheme described above. Consider a mechanism that holds consumers accountable for a proportion τ_i of overall Y. Analogous to Expression 5 above, the consumer maximization problem becomes

$$\underset{x_i,y_i}{max} U_i\left(x_i, \frac{y_i}{\tau_i}\right) \text{ s.t. } \Pi_X x_i + \Pi_Y y_i = w_i,$$

or

$$\underset{x_i,y_i}{max} U_i(x_i, y_i + \sum_{j \neq i} y_j) \text{ s.t. } \Pi_X x_i + \tau_i \Pi_Y y_i + \sum_{j \neq i} \frac{\tau_i}{1-\tau_j}(1-\tau_j)\Pi_Y y_j = w_i, \tag{10}$$

like in Expression 6.[4]

For interior solutions, the equilibrium conditions will be similar to that of the standard Lindahl equilibrium, except with different expressions for MRT due to the presence of an impure public good.

Definition 2. An interior cost sharing equilibrium is an allocation $(x_1^*, ..., x_n^*, Y^*)$ and a set of shares $(\tau_1^*, \tau_2^*, \ldots, \tau_n^*)$ such that

$$\forall i : mrs_i(x_i^*, Y^*) = \tau_i^* MRT(Y^*) \tag{11}$$

$$\forall i : w_i = \Pi_X x_i^* + \Pi_Y y_i^*. \tag{12}$$

$$\sum_{i=1}^{n} \tau_i^* = 1$$

$$y_i^* = \tau_i^* Y^*$$

Definition 3. A kink cost sharing equilibrium is an allocation $(x_1^*, ..., x_n^*, Y^*)$ and a set of shares $(\tau_1^*, \tau_2^*, \ldots, \tau_n^*)$ such that $\sum_{i=1}^n \tau_i^* = 1$; $Y^* = \beta W$; and $\forall i : \tau_i^* MRT_I < mrs_i(x_i^*, Y^*) < \tau_i^* MRT_{II}$, where MRT_I and MRT_{II} are as defined in Expression 4, and Expression 12 holds as before.

Proposition 4. *For any distribution of total wealth W such that $\forall i, w_i > 0$, there exists a cost sharing equilibrium. This equilibrium is Pareto efficient.*

Proof. Proof is given in Appendix. □

[4] As before, Π_X and Π_Y vary by facet, and in the case of cost sharing, we define facets based on the social budget. That is, we have Facet I defined by $Y < \beta W$ and Facet II by $Y > \beta W$.

This proposition establishes the existence and optimality of the cost sharing equilibrium. The detailed proof in the Appendix further outlines the conditions for uniqueness.

It is straightforward to see that summing the n equations represented by Expression 11 and using $\sum_{i=1}^{n} \tau_i^* = 1$ yields $\sum_{i=1}^{n} mrs_i = MRT$, the Samuelson condition. Thus the interior cost sharing equilibrium is optimal. Similarly, for the kink solution, summing the n equations in Definition 3 yields a kink version of the Samuelson condition $MRT_I < \sum_{i=1}^{n} mrs_i(x_i^*, Y^*) < MRT_{II}$.

Solving for the cost sharing equilibrium yields a set of optimal cost shares that depend on exogenous parameters $(\alpha, \beta, \mathbf{w})$ and individual preferences: $\{\tau_i^*(\alpha, \beta, \mathbf{w})\}_{i=1}^{n}$. An example for calculating these cost shares is given in the Appendix. Critically, these $\{\tau_i^*(\cdot)\}_{i=1}^{n}$ have a useful policy application: they map directly into corrective price instruments, as we describe in the next section.

3.3 Implementation

We have considered how the cost sharing solution manifests in the implied market for *characteristics*. It is also worth considering what this method of pricing looks like specifically in markets for *goods*, as policy makers in real world contexts often target policies on goods rather than embodied characteristics.[5] The implied subsidy $(1 - \tau_i)$ for characteristic y_i described above will map into a direct subsidy for goods g_i and d_i.

The equivalent subsidies for green goods and direct donations, respectively, are given by:[6]

$$s_{g_i} = \begin{cases} (1-\tau_i)(1-\alpha), & Y^* < \beta W \\ (1-\tau_i)\beta, & Y^* > \beta W \end{cases}, \tag{13}$$

and

$$s_{d_i} = 1 - \tau_i. \tag{14}$$

Without subsidies, the consumer's budget constraint $w_i = c_i + g_i + d_i$. When g_i and d_i are subsidized, this becomes

$$w_i = c_i + p_{g_i} g_i + p_{d_i} d_i + \sum_{j \neq i} \frac{\tau_i}{1-\tau_j} \left(s_{g_j} g_j + s_{d_j} d_j \right),$$

where $p_{g_i} = (1 - s_{g_i})$ and $p_{d_i} = (1 - s_{d_i})$ are the effective prices paid by i under the policy. Following the intuition of Expression 10, $\sum_{j \neq i} \frac{\tau_i}{1-\tau_j} \left(s_{g_j} g_j + s_{d_j} d_j \right)$ is i's tax liability to fund subsidies for others' purchases of g_j and d_j.

Note that the value of the subsidy for the green good differs in the two cases, so the policy maker must set subsidies according to the facet of the desired social outcome. As in the Lindahl case above, the social planner's budget is balanced, so she has no net revenues or expenditures. As such, it is conceptually possible for consumers to arrive at this solution in a decentralized fashion by agreeing among themselves to a schedule of transfers based on purchases of g and d.

[5] Examples include tax rebates, subsidies, expedited permits, and technical assistance for renewable energy projects and green buildings.

[6] The equivalency is shown in full in the appendix (6.3).

3.4 Properties of cost sharing equilibrium

Prior research on impure public goods has considered how green markets are affected by the introduction of new green goods or technology improvements to existing green goods, and it shows that such improvements can lead to a wide range of implications for public good provision and social welfare. That is, overall public good provision can increase or decrease as a result of improvements to a green good, which is rather counterintuitive. Furthermore, such changes can lead to Pareto improvements, but they may also lead to immiseration, whereby all consumers are worse off.

We consider these questions in the context of an optimal cost sharing policy, and our analysis has implications for both efficiency and equity. For example, does an improvement in solar technology necessarily improve welfare for all consumers? Intuition might suggest that this will be unambiguously beneficial. However, in spite of the presence of a corrective policy, we show that it is not always true that a Pareto improvement will result from an improvement in a green good.

First, we examine implications of a technology improvement for the public good provision (e.g., environmental quality) Y, and then we proceed to analyze welfare consequences.

Lemma. *For a given level of Y and τ_i, an increase in α or β will increase the amount of x_i that consumer i can afford. That is, a technology improvement expands the budget set.*

Proof. This follows from inspection of the consumer's budget constraint $w_i = \Pi_x x_i + \tau_i \Pi_Y Y$. □

Proposition 5. *In a cost sharing equilibrium on Facet I, a marginal increase in α or β will lead to higher level of equilibrium environmental quality $Y^{**} > Y^*$.*

Proof. When the original equilibrium is on Facet I, the technology improvement lowers the relative price for providing Y, taking cost shares as given. A technology improvement yields both an income effect and a substitution effect, and each one tends to increase demand for Y on Facet I. Since every consumer demands more Y for given shares, equilibrium provision of Y must increase. □

Proposition 6. *In a cost sharing equilibrium on Facet II, a marginal increase in α or β may result in either an increase or decrease in equilibrium environmental quality.*

Proof. When the original equilibrium is on Facet II, the technology improvement increases the relative price of Y, taking cost shares as given. Each consumer experiences a positive income effect, which increases demand for Y by normality. There is also a substitution effect, which tends to decrease demand for Y on Facet II. □

In summary, environmental quality may increase or decrease in response to a technology improvement. For solutions on Facet I, the effect is unambiguous, but on Facet II, the direction of this effect will depend upon whether x_i and Y are complements or substitutes. Although this parallels the findings of Kotchen (2006), it is perhaps somewhat more surprising in the present context given the implementation of a cost sharing policy. However, as we now proceed to show, the welfare consequences will differ in meaningful ways.

Proposition 7. *In a cost sharing equilibrium, a marginal increase in α or β results in Pareto improvements if $Y^{**} \geq Y^{*}$.*

Proof. Proof is given in Appendix. □

Proposition 8. *If $Y^{**} < Y^{*}$, Pareto improvement is not guaranteed, but immiseration is not possible.*

Proof. Proof is given in Appendix. □

While the full proofs are in the Appendix, we offer some intuition for these results here. We have established that in Facet II, equilibrium public good provision could decrease. Given heterogeneous preferences, it is possible for some people's cost shares to decrease while others experience higher shares. Those experiencing an increased burden might be worse off if the change in their burden outweighs the positive income effect from better technology. Thus Pareto improvement is not guaranteed. However, immiseration is not possible, because those who see their burdens decrease experience an increase in purchasing power both from the improved technology and from the smaller burden placed on them. Thus, a technology improvement can either cause Pareto improvement or a redistribution of welfare.

Together, Propositions 7 and 8 reveal key differences between the cost sharing equilibrium and the Nash equilibrium context characterized by Kotchen (2006). As described in Proposition 7, if public good provision rises as a result of improvements in a green technology, everyone will be made better off. This contrasts with the more ambiguous results in Nash equilibrium, where such improvements could yield Pareto improvements or redistribution of welfare (Kotchen, 2006). More broadly, a technology improvement can cause immiseration, welfare redistribution, or a Pareto improvement for the Nash equilibrium scenario. However, under cost sharing, a technology improvement can only lead to welfare redistribution or a Pareto improvement, while immiseration is precluded.

When considering distributional consequences, it is worth noting that our analysis assumes that the initial distribution of income will remain unchanged, leading to a particular cost sharing equilibrium. In principle, a policy maker can impose lump sum transfers to change the initial wealth distribution, leading to a different cost sharing equilibrium with a different allocation and level of public good provision. In the Lindahl context, any Pareto efficient allocation is attainable as a Lindahl equilibrium via such transfers (Sandler & Posnett, 1991). Altering the initial allotment of wealth will have a similar effect in our setting, allowing the social planner to attain different distributional outcomes via lump sum transfers.

However, such transfers may not be feasible in many policy contexts. Interestingly, even in a more restrictive setting that forbids transfers, impure public goods provide some latitude for changing distributional outcomes in cases with a kink solution. At the kink, there is a multiplicity of cost shares that can beget an optimal outcome. This provides the social planner some, albeit limited, power to manage for a desired distribution in equilibrium.

4 Discussion

One limitation of the cost sharing mechanism is that it does not satisfy the preference revelation criterion, a classic consideration in the management of public goods. To our knowledge, the only paper to investigate preference revelation in the impure public good context is Wichman (2016), who adapts the Clarke-Groves mechanism. However, this paper assumes quasilinear utility, which is restrictive. Another salient distinction is that the private characteristic is derived from warm glow, which is only attainable through purchases of the green good. We examine a more general model where there is a conventional good that provides an alternative means for obtaining the private characteristic. Because of these differences, the preference revealing mechanism of Wichman (2016) cannot be applied directly to the more general context we describe.

We offer a different angle on this problem by characterizing optimal policy for a broad set of potential preference structures following in the tradition of Lindahl (1958), Kaneko (1977b,a), Mas-Colell & Silvestre (1989), and others, leaving preference revelation to other means beyond the policy in question. However, it is worth noting that preference revelation should be possible via a Clarke-Groves mechanism if consumers have Bergstrom-Cornes preferences (Bergstrom & Cornes, 1983).

Because preference revelation is challenging in real-world applications, it is common to see Pigovian policies that impose a flat tax or subsidy for goods based on estimates of marginal damages or marginal benefits. As our analysis reveals, there are two shortcomings to this standard approach. For one, such policies are not budget-balancing, so funds must be raised to subsidize public goods, likely at a non-trivial cost to society (Bovenberg & Goulder, 1996). More importantly, Pigovian instruments impose a single set of prices for all consumers. This represents an inefficient policy for public good provision, except in very special cases such as situations where all agents have identical quasilinear preferences.

Our analysis thus brings to light important design principles for (impure) public good policies. In particular, there are clear ways to improve upon existing Pigovian instruments by incorporating variables that proxy for a consumer's strength of preference for the public good. For example, higher income individuals could be assigned larger cost shares, so that they receive relatively smaller subsidies and higher tax burdens. Such adjustments can enhance efficiency relative to standard Pigovian policies when preferences vary with income. While income is perhaps the most obvious and easy-to-observe preference shifter to consider, market behavior, survey data, choice experiments, or voting results can also offer insights into how preferences differ across demographics, social groups, or geographies, thus paving the way for better-tailored price instruments.

Overall, the mechanism we describe is especially useful for scenarios where there are relatively few agents and heterogeneous preferences over the public goods. For markets with an extremely large number of agents or with homogeneous preferences, the public good problem approaches a pure externality problem without strategic interactions. Moreover, in situations with many agents, administrative costs become a challenge, as our mechanism requires personalized prices for agents or groups based on individual utilities. For example, if the market of concern was organic food, supermarkets would have to give different discounts to each consumer, which would be difficult at best.

However, there are many situations where our model can be implemented to improve upon Nash equilibrium outcomes or even upon markets with typical Pigovian subsidies. Clean electricity offers one useful context for application, as administrative burdens may be less concerning. While the number of consumers in such a market may indeed be large, records for energy bills could be made available with relative ease. Consider a context where consumers can purchase conventional energy, green electricity, or make direct donations to environmental charities. Under our mechanism, each consumer would be assigned a share determining the subsidized prices she faces for green electricity and donations to environmental organizations; these shares generalize and improve upon classic Pigovian subsidies that are typically used to incentivize green energy. The share would also determine the new tax burden placed on each individual, offering a revenue-neutral and individually-rational means for increasing environmental quality. As discussed above, even if full preference revelation is challenging in such a context, the shares could at least be tailored based on income or preference proxies like property value, neighborhood, etc.

This model can also apply well to scenarios where groups, rather than individuals, are concerned. One example is international climate change mitigation. In such a case, countries would represent the consumers. We could suppose that countries value economic development x_i and environmental quality Y. A country i could spend its resources w_i on domestic improvements, which provide x_i; carbon sequestration, which augments Y; or it could invest in clean energy, which jointly provides x_i and Y.[7] We can imagine an international committee weighing available information, such as GDP and perhaps voting statistics, to assign cost shares to each country. Each country would then face international subsidies and taxes as outlined in Section 3.3. The international context is attractive for several reasons. For one, the smaller number of participants eases the administrative burden. Second, one could imagine that estimating countries' aggregate tradeoffs may be more reliable than estimating individuals' preferences for public goods.

5 Conclusion

This paper expands our understanding of impure public goods by describing how to apply a cost sharing framework to achieve a Pareto optimal allocation. While cost sharing and matching have been analyzed previously, prior work focuses on the simpler case of pure public goods rather than the more general case of impure public goods. Our approach implements the optimal allocation of *characteristics*, and we furthermore provide a clear set of policies on *goods* to achieve this. Our solution is attractive because it respects the bundling constraint presented by the impure public good as well as the individual budget constraints of all consumers. It is also budget-balancing for the social planner.

[7] At present, carbon sequestration technologies are still in development, and may not offer a cost-competitive means for improving Y. Even in such cases, our model is still applicable; the absence of c or d does not fundamentally change the analysis. In fact, our prior discussion remains important, as it demonstrates how sharing rules can be translated into prices for goods and characteristics.

In addition to characterizing a general solution procedure, we also consider the properties of the cost sharing equilibrium. We find that when a technology improvement causes equilibrium provision to decrease, it may lower the welfare for some, but never all, individuals. Additionally, the possibility of a kink solution in which only the green good is purchased allows for redistribution through manipulation of the cost shares, which can help a policymaker address fairness concerns in a situation where direct transfers are not allowed. Finally, we describe the implementation of cost sharing as a set of price policies, and make suggestions for possible applications in contexts such as green energy and international climate change mitigation.

6 Appendix

6.1 Proof of Proposition 4

Recall that under cost sharing, the individual maximization is

$$max_{x_i,Y} U_i(x_i, Y) \text{ subject to } \Pi_X x_i + \tau_i \Pi_Y Y = w_i.$$

where $(\Pi_X, \Pi_Y) = \begin{cases} \left(1, \frac{1-\alpha}{\beta}\right) & Y < \beta W \\ \left(\frac{1-\beta}{\alpha}, 1\right) & Y > \beta W \end{cases}.$

Each individual takes τ_i as given and solves the maximization problem. Using the first interpretation above, this yields a solution such that

$$\begin{aligned} \frac{\partial U}{\partial x_i} &= \Pi_X \\ \frac{\partial U}{\partial Y} &= \tau_i \Pi_Y \\ w_i &= \Pi_X x_i + \tau_i \Pi_Y Y. \end{aligned}$$

Combining the three optimality conditions gives

$$mrs_i \left(\frac{1}{\Pi_X} \left[w_i - \tau_i \Pi_Y Y \right], Y \right) \quad = \quad \tau_i MRT.$$

Define $\gamma(\tau_i, Y) \equiv mrs_i(\frac{1}{\Pi_X} \left[w_i - \tau_i \Pi_Y Y \right], Y)$. Note that $\gamma_{\tau_i} < 0$ and $\gamma_Y < 0$, where γ_{τ_i} and γ_Y represent the derivatives with respect to the first and second arguments, respectively.[8]

We can rewrite the combined optimality condition as

$$f(\tau_i, Y) = \gamma(\tau_i, Y) - \tau_i MRT = 0.$$

From this, we can use the implicit function theorem (except at the kink point $Y = \beta W$, where there is a discontinuity due to the change in implicit prices) to solve

$$\frac{\partial Y}{\partial \tau_i} = -\frac{\frac{\partial f}{\partial \tau_i}}{\frac{\partial f}{\partial Y}} \tag{15}$$

$$= -\frac{\gamma_{\tau_i} - MRT}{\gamma_Y} < 0. \tag{16}$$

This proves that demand Y is decreasing in τ_i for interior solutions.

[8]This follows from $\gamma_{\tau_i} = \frac{\partial mrs}{\partial x_i} \frac{dx_i}{d\tau_i} = \frac{\partial mrs}{\partial x_i} \underbrace{\left(-\frac{\Pi_Y}{\Pi_X} Y \right)}_{<0} < 0$ and $\gamma_Y = \frac{\partial mrs}{\partial x_i} \frac{dx_i}{dY} + \frac{\partial mrs}{\partial Y} \frac{\partial Y}{\partial Y} = \underbrace{\frac{\partial mrs}{\partial x_i}}_{>0} \underbrace{\left(-\tau_i \frac{\Pi_Y}{\Pi_X} \right)}_{<0} + \underbrace{\frac{\partial mrs}{\partial Y}}_{<0} < 0.$

Recognizing $\frac{\partial Y}{\partial \tau_i} < 0$, we can invert to obtain a share function $s_i(Y) = \tau_i = \frac{mrs_i}{MRT}$ defined for $Y \neq \beta W$ that describes the cost share that would yield a prescribed level of demand Y. $s_i(Y)$ is proportional to the consumer's inverse demand or the willingness to pay for Y at any given level of public good provision.

At the kink point, the behavior of $s_i(Y)$ is ambiguous. As Y increases from $Y < \beta W$ to $Y > \beta W$, two simultaneous changes occur: Π_X decreases and Π_Y increases. For the latter, both the income effect and the substitution effect decrease $s_i(Y)$. However, for the former, the substitution effect will lead to a drop in $s_i(Y)$, while the income effect works in the opposite direction.

Two cases are possible in the neighborhood of $Y = \beta W$: 1) $s_i(Y)$ decreases in Y for all i, and 2) $s_i(Y)$ increases in Y for some i, but $s_j(Y)$ decreases in Y for some $j \neq i$.[9]

Case 1. In the neighborhood of $Y = \beta W$, $s_i(Y)$ decreases in Y for all i. Define $S(Y) = \sum_{i=1}^{n} s_i(Y)$. Because share functions $s_i(Y)$ are decreasing in Y, so too is $S(Y)$. Therefore, there exists a unique Y^* satisfying the condition $S(Y^*) = 1$ for an interior solution, or $S(Y)$ crosses 1 at the point of discontinuity for a kink solution.[10] The intuition for this proof is illustrated in Figure 2. Using Y^*, optimal shares can be calculated as $\tau_i^* = s_i(Y^*)$ for all i (interior solution), or $s_i(\beta W - \varepsilon) > \tau_i^* > s_i(\beta W + \varepsilon)$, where $\varepsilon > 0$ is arbitrarily small (kink solution). Prescribing shares in this way will cause all parties to agree on the optimal level of Y^*.

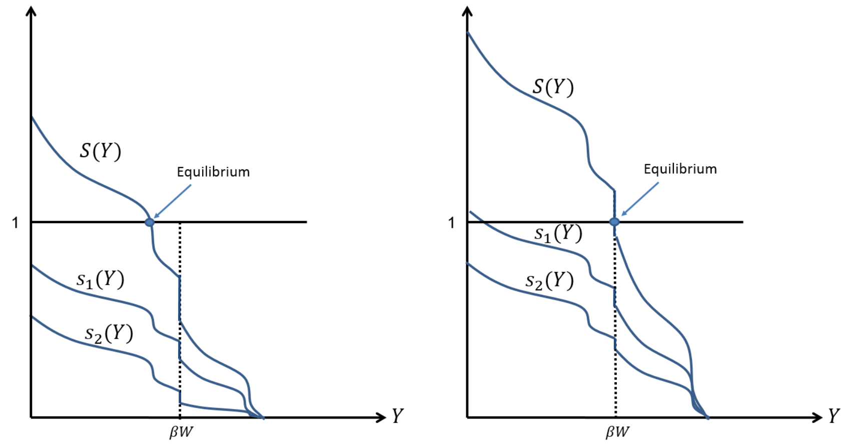
Case 2. In the neighborhood of $Y = \beta W$, $s_i(Y)$ increases in Y for some i, but $s_j(Y)$ decreases in Y for others j. Consider $S(Y) = \sum_{i=1}^{n} s_i(Y)$. There are two distinct subcases:

Case i. $S(Y)$ is non-increasing in Y. Then the proof follows Case 1 above, yielding a unique optimum.

Case ii. $S(Y)$ is non-increasing in Y, except at the kink point $Y = \beta W$ where there is a discrete increase in $S(Y)$. If there is a unique value of Y satisfying $S(Y) = 1$, then existence and uniqueness follow as in Case 1. However, it is possible that there are two values of Y for which $S(Y) = 1$. Denote these two values with subscripts a and b and let $Y_a < Y_b$. Then either will be implementable via cost sharing using shares $\{s_i(Y_a)\}_{i=1}^{n}$ or $\{s_i(Y_b)\}_{i=1}^{n}$, respectively, and the social planner can select her preferred equilibrium based on a social welfare function or otherwise. These possibilities are illustrated in Figure 3.

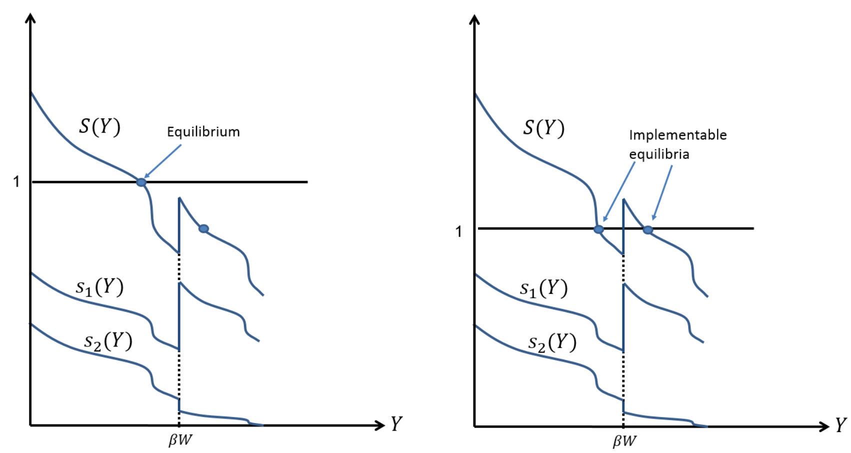
[9] A third configuration, where $s_i(Y)$ increases in Y for all i is not feasible, as this would violate the production constraint for the economy.

[10] As Y approaches infinity, $s_i(Y)$ approaches zero, and as τ_i approaches infinity, demand for Y approaches zero. Thus, from the intermediate value theorem, there exists a unique solution for $S(Y) = 1$ (interior solution) or $S(\beta W - \varepsilon) > 1 > S(\beta W + \varepsilon)$, where $\varepsilon > 0$ is arbitrarily small (kink solution).



(a) Interior solution. (b) Kink solution.

Figure 2: These figures provides intuition for Case 1 in the proof of Proposition 4 using a hypothetical two-person economy. We can derive $s_i(Y)$ functions for each consumer, and the equilibrium will be where the sum of these functions is unity.



(a) Interior solution. (b) Multiple equilibria.

Figure 3: These figures provides intuition for Case 2 in the proof of Proposition 4 using a hypothetical two-person economy. We can derive $s_i(Y)$ functions for each consumer, and the equilibrium will be where the sum of these functions is unity.

6.2 Example: Deriving optimal cost shares and equilibrium

Consider an economy with with $\alpha = \beta = 0.6$. Suppose that there are two individuals with preferences given by $U_1(x_1, Y) = x_1^{0.2}Y^{0.8}$ and $U_2(x_2, Y) = x_2^{0.1}Y^{0.9}$, and let the initial wealth distribution be $w_1 = w_2 = 100$.

To solve for the cost sharing equilibrium, we must solve the system of equations described in Definition 2. In equilibrium, consumers must choose bundles on the same facet. However, we do not know *a priori* on which facet the solution will lie, so we begin by assuming $Y < \beta W = 120$. Now we must solve: $mrs_1 = \tau_1 MRT$ and $mrs_2 = \tau_2 MRT$. Differentiating each utility function by Y and x_i gives us the expression for each mrs_i, and since there are only two individuals, we have $\tau_1 + \tau_2 = 1$. We also find that $MRT_I = 2/3$ and $MRT_{II} = 3/2$. Finally, using Expressions 11 and 12, we get $\frac{4(100-\frac{2}{3}\tau_1 Y)}{Y} = \frac{2}{3}\tau_1$ and $\frac{9(100-\frac{2}{3}\tau_1 Y)}{Y} = \frac{2}{3}\tau_2$.

Solving these yields $Y^* = 255$, $\tau_1 = \frac{120}{255}$, and $\tau_2 = \frac{135}{255}$, which contradicts our initial assumption $Y^* < \beta W$. Repeating the process, this time assuming that $Y > \beta W = 120$, yields $\frac{4(150-\frac{3}{2}\tau_1 Y)}{Y} = \frac{3}{2}\tau_1$ and $\frac{9(150-\frac{3}{2}\tau_1 Y)}{Y} = \frac{3}{2}\tau_2$.

Thus, the solution is $Y^* = 170, \tau_1 = \frac{80}{170}, \tau_2 = \frac{90}{170}$, which is consistent with our assumption. In terms of private characteristics, we have $x_1 = 30$, and $x_2 = 15$.

6.3 Derivation of subsidies and subsidized prices

The consumer's budget constraint is $w_i = c_i + d_i + g_i$. In a cost sharing system where goods g_i and d_i are subsidized, this becomes

$$w_i = c_i + p_{g_i} g_i + p_{d_i} d_i + \sum_{j \neq i} \frac{\tau_i}{1-\tau_j} \left(s_{g_j} g_j + s_{d_j} d_j \right),$$

where $p_{g_i} = (1 - s_{g_i})$ and $p_{d_i} = (1 - s_{d_i})$ are the prices inclusive of subsidies. Following the intuition of Expression 10, $\sum_{j \neq i} \frac{\tau_i}{1-\tau_j} \left(s_{g_j} g_j + s_{d_j} d_j \right)$ is i's payment to fund subsidies for others' purchases. In what follows, we demonstrate that the subsidies on goods

$$s_{g_i} = \begin{cases} (1-\tau_i)(1-\alpha), & Y^* < \beta W \\ (1-\tau_i)\beta, & Y^* > \beta W \end{cases}, \tag{17}$$

and

$$s_{d_i} = 1 - \tau_i. \tag{18}$$

will yield the desired tradeoffs in terms of characteristics as in Equation 12.

First, consider Facet I, where $d_i = 0$ so that

$$w_i = c_i + p_{g_i} g_i + \sum_{j \neq i} \frac{\tau_i}{1-\tau_j} s_{g_j} g_j.$$

Given, $s_{g_i} = (1 - \tau_i)(1 - \alpha)$, we have $p_{g_i} = \alpha + \tau_i - \alpha\tau_i$. Substituting this into the budget constraint, we have

$$\begin{aligned} w_i &= c_i + [\alpha + \tau_i - \alpha\tau_i]\, g_i + \sum_{j\neq i} \frac{\tau_i}{1-\tau_j} \left[(1-\tau_j)(1-\alpha)\right] g_j \\ &= c_i + \alpha g_i + (1-\alpha)\tau_i g_i + \sum_{j\neq i} \tau_i (1-\alpha) g_j \end{aligned}$$

On Facet I, we have the following correspondence between goods and characteristics: $x_i = c_i + \alpha g_i$ and $y_i = \beta g_i$. Substituting these expressions into the equation above yields:

$$w_i = x_i + \tau_i \frac{1-\alpha}{\beta} y_i + \sum_{j\neq i} \tau_i \frac{1-\alpha}{\beta} y_j,$$

so the subsidies on goods achieve the desired budget in terms of characteristics.

$$w_i = c_i + p_{g_i} g_i + \sum_{j\neq i} \frac{\tau_i}{1-\tau_j} p_{g_j} g_j.$$

Now consider Facet II, where $c_i = 0$ so that

$$w_i = p_{g_i} g_i + p_{d_i} d_i + \sum_{j\neq i} \frac{\tau_i}{1-\tau_j} \left(s_{g_j} g_j + s_{d_j} d_j\right).$$

Given $s_{g_i} = (1-\tau_i)\beta$ and $s_{d_i} = 1-\tau_i$, we have $p_{g_i} = 1-(1-\tau_i)\beta$ and $p_{d_i} = \tau_i$. Substituting this into the budget constraint, we have

$$\begin{aligned} w_i &= p_{g_i} g_i + p_{d_i} d_i + \sum_{j\neq i} \frac{\tau_i}{1-\tau_j} \left(s_{g_j} g_j + s_{d_j} d_j\right) \\ &= \left[1-(1-\tau_i)\beta\right] g_i + \tau_i d_i + \sum_{j\neq i} \frac{\tau_i}{1-\tau_j} \left((1-\tau_j)\beta g_j + (1-\tau_j) d_j\right) \\ &= (1-\beta)\, g_i + \tau_i \left(\beta g_i + d_i\right) + \sum_{j\neq i} \tau_i \left(\beta g_j + d_j\right) \end{aligned}$$

On this facet, $x_i = \alpha g_i$ and $y_i = \beta g_i + d_i$. Substituting these into the equation above gives

$$w_i = \frac{1-\beta}{\alpha} x_i + \tau_i y_i + \sum_{j\neq i} \tau_i y_j,$$

so the subsidies achieve the desired budget in terms of characteristics.

6.3.1 Example: Implementation of subsidies

Suppose that in the market for green electricity, the technology parameters are given by $(\alpha, \beta) = (0.6, 0.5)$. First, the policymaker must solve for the optimal level of Y. If $Y^* > \beta W$, then $s_{g_i} = (1-\tau_i)\beta$. Otherwise, it is $s_{g_i} = (1-\tau_i)(1-\alpha)$. Either way, $s_{d_i} = 1-\tau_i$. Let us focus on a consumer

i with $\tau_i = 0.05$. If the social planner imposes subsidies on Facet II in accordance with Equations 13 and 14, consumer i need only pay $p_{g_i} = 1 - s_{g_i} = 1 - \beta + \tau_i = \$0.525$ for each unit of g_i. Although the true cost of producing g_i is \$1, the remainder is covered by a subsidy of \$0.475 per unit. Similarly, $p_{d_i} = 1 - s_{d_i} = \$0.05$ for each unit of d_i, with the remaining \$0.95 of the cost covered by subsidies. Meanwhile, i must also reciprocally subsidize others' purchases of g_i and d_i according to her cost share τ_i.

Suppose that consumer i buys 10 units each of g_i and d_i, and ten other consumers $j \neq i$ each have a cost share of $\tau_j = 0.095$ and purchase 10 units of d_j. In this case, i spends \$5.75 on her purchases, taking advantage of \$14.25 in subsidies. At the same time, she must pay $\sum_{j \neq i} \frac{\tau_i}{1-\tau_j} \left(s_{g_j} g_j + s_{d_j} d_j \right) = \sum_{j \neq i} \frac{0.05}{0.905} * 0.905 * 10 = \$5$ to others.

6.4 Proof of Propositions 7 and 8:

We consider a marginal increase in the technology parameters, so that the new equilibrium is characterized by the same facet as the previous equilibrium. Such an improvement will decrease MRT_I while increasing MRT_{II}. We prove our claim by considering two primary cases:

Case 1. Equilibrium is on Facet I. On Facet I, the technology improvement leads to $Y^{**} > Y^*$, as described in the proof of Proposition 5, implying that all consumers demand more environmental quality in equilibrium. According to individual maximization, $mrs_i(x_i, Y) = \tau_i MRT_I$ for all i in equilibrium. Suppose that some consumer i is not better off. For this to be the case, it must be that $\tau_i^{**} MRT_I^{**} \geq \tau_i^* MRT_I^*$, implying that her effective price for providing the public good has weakly increased. Then, according to individual maximization, it must also be true that $mrs(x_i^{**}, Y^{**}) \geq mrs(x_i^*, Y^*)$. However, by assumption, $Y^{**} > Y^*$ and $\tau_i MRT_I$ weakly increases. Since the consumer is by assumption not better off, we must have $x_i^{**} < x_i^*$. This contradicts $mrs_i(x_i^{**}, Y^{**}) > mrs_i(x_i^*, Y^*)$, as the inequality cannot hold for $x_i^{**} < x_i^*$ and $Y^{**} > Y^*$ given strict normality. Therefore, a Pareto improvement must occur.

Case 2. Equilibrium is on Facet II. Here, the technology improvement can yield higher or lower environmental quality in equilibrium, so we consider these as two subcases.

Case i. $Y^{**} \geq Y^*$. According to individual maximization, $mrs_i(x_i, Y) = \tau_i MRT_{II}$ for all i in equilibrium. Suppose that some consumer i is not better off. For this to be the case, it must be that $\tau_i^{**} MRT_{II}^{**} > \tau_i^* MRT_{II}^*$, implying that her effective price for providing the public good has increased.[11] Combining this condition with the individual maximization condition, we have

$$\tau_i^{**} MRT_{II}^{**} = mrs_i(x_i^{**}, Y^{**}) > mrs_i(x_i^*, Y^*) = \tau_i^* MRT_I^*,$$

[11] It is straightforward to see that i experiences a higher utility after the technology improvement if $\tau_i^{**} MRT_{II}^{**} < \tau_i^* MRT_{II}^*$.

Because $Y^{**} \geq Y^*$, we know $mrs_i(x_i^*, Y^*) \geq mrs_i(x_i^*, Y^{**})$ due to strict normality. Thus, $mrs_i(x_i^{**}, Y^{**}) > mrs_i(x_i^*, Y^{**})$ by transitivity, which in turn implies that $x_i^{**} > x_i^*$. Thus, a consumer who faces a higher effective cost $\tau_i^{**} MRT_{II}^{**} > \tau_i^* MRT_{II}^*$ will consume $x_i^{**} > x_i^*$ and $Y^{**} > Y^*$, leading to an increase in utility. Therefore, a Pareto improvement must occur.

Case ii. $Y^{**} < Y^*$. A Pareto improvement is possible, as shown in the example in the next section, but it is not guaranteed. Consider individual i. She may be worse off if her equilibrium cost share increases, as she will enjoy less Y and potentially less x_i. This possibility is also shown in the next section. However, immiseration is not possible. It follows from the Lemma that an individual can only be made worse off by a technology improvement if her cost share increases. By definition, the cost share cannot increase for everyone, as $\sum_{i=1}^n \tau_i = 1$. Therefore, immiseration is impossible.

Combining our results, we see that Pareto improvement results if $Y^{**} > Y^*$, and that immiseration is impossible.

6.5 Examples of welfare changes from technology improvements

We here provide an example of a Pareto improvement from an improvement in technology as well as an example in which welfare is redistributed.

Consider an economy with 2 individuals, each with $w_i = 100$, and utility given by $u_1(x_1, Y) = (x_i^{0.5} + Y^{0.5})^{1/0.5}$ and $u_2(x_2, Y) = (x_2^{0.9} + Y^{0.9})^{1/0.9}$. With technology parameters $(\alpha, \beta) = (0.6, 0.5)$, we find that in equilibrium, $\tau_1^* = 0.5402$, $\tau_2^* = 0.4598$, $x_1^* = 47$, $x_2^* = 58$, $Y^* = 112$, $u_1^* = 305$, and $u_2^* = 183$ (rounded to the nearest integer value). Consider a technology change so that $(\alpha', \beta') = (0.9, 0.5)$. Given existing shares τ_1^* and τ_2^*, type 2 would like to overprovide Y while type 1 would like to underprovide. Thus, individual 1 sees a fall in her cost share, while 2 sees an increase in her cost share. In the new equilibrium, $\tau_1^{**} = 0.509$, $\tau_2^{**} = 0.491$, $x_1^{**} = 86$, $x_2^{**} = 89$, $Y^{**} = 103$, $u_1^{**} = 376$, and $u_2^{**} = 207$. Consumer 2 experiences an increase in her cost share that is more than offset by the positive income effect from better technology, resulting in a Pareto improvement.

On the other hand, if we increase the number of type 1 individuals, the overproviding type 2 will experience a larger increase in her burden. Consider the same scenario as before but with 10 individuals of type 1. Now we get $\tau_1^* = 0.09401$, $\tau_2^* = 0.0599$, $x_1^* = 12$, $x_2^* = 51$ $Y^* = 956$, $u_1^* = 1184$, and $u_2^* = 1033$ for the initial equilibrium and $\tau_1^{**} = 0.09385$, $\tau_2^{**} = 0.0615$, $x_1^{**} = 26$, $x_2^{**} = 79$ $Y^{**} = 912$, $u_1^{**} = 1246$, and $u_2^{**} = 1025$ after the technology improvement. In this case, the type 2 consumer sees a larger increase in her share, which more than offsets the benefits from higher technology.

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