Prudence and preference for flexibility gain*

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Abstract

Under the expected utility paradigm, prudence ($u''' > 0$) is usually associated with the amount of risk premium an individual requires in order to renounce to a certain current outcome in favour of an uncertain future outcome. A prudent individual requires a higher premium the lower her initial wealth. However, when the individual has to make a costly investment before obtaining the outcome, she may prefer to delay that investment. This translates into a preference for latter, not earlier outcome. Consequently, prudence cannot be associated with a risk premium. In this paper we show that, for an individual who prefers to delay the investment, prudence is actually associated with the economic benefit granted by that delay. Specifically, a lower expected unit cost of acquiring the good is associated with a greater benefit of the investment delay if and only if u''' is high, and, with a uniform distribution, $u''' > 0$. We also show that the preference for facing a lower expected unit cost and/or a wider support of the unit cost increases with u'''. We describe two applications of this result, namely, sequential learning in the delegation of a task and timing of investment decisions under multi-period uncertainty.

Keywords: Prudence; Risk aversion; Sequential screening; Real Options

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1 Introduction

Since Leland [9] and Sandmo [12] analyzed the precautionary savings behavior of individuals, it has become well known that risk aversion is insufficient to explain individual preferences over future outcomes when the latter are uncertain. The reason is that the notion of risk aversion refers to contemporary risk rather than to future risk. The authors show that an individual engages in precautionary savings today as a response to increased uncertainty about the future, if and only if the third derivative of her utility function is positive. The underlying concept is that of prudence, as defined by Kimball [13], which is related to, but distinct from, risk aversion. Broadly speaking, an individual is prudent if higher uncertainty about future outcomes induces her to pay a higher cost today (say, to reduce more her current consumption), provided that this allows her to decrease the risk associated with future outcomes.

Prudence has also been shown to be equivalent to aversion against downside risk (Menezes et al. [11]). It is useful to recall two interpretations of this result, which can be identified being based on the literature. First, a prudent individual requires a lower risk premium to accept facing an unknown future outcome, in place of the current one, if her initial wealth is raised (Hanson and Menezes [7]). Indeed, taking $\mathbb{E}[\widetilde{\varepsilon}] = 0$, the difference $u(x) - \mathbb{E}[u(x+\widetilde{\varepsilon})]$ decreases with x if and only if the third derivative of the utility function is positive. The reason is that the support of unknown consumption is shifted upwards if the initial wealth is higher. Second, as Menezes et al. [11] show themselves, when a prudent individual must choose between two lotteries over consumption levels, rather than between a certain outcome and an uncertain one, she will prefer a lottery in which some amount of risk is added to a better state of nature (like higher consumption level) to a lottery in which that same amount of risk is added to a worse state of nature. More recent papers, in which prudence is defined as a preference over lotteries, are Bigelow and Menezes [1] and Eeckhoudt and Schlesinger [6]. Also according to this interpretation, when the individual is called upon to choose between a certain outcome today and an unknown outcome tomorrow, prudence is viewed as a measure of the extent to which the risk premium varies with the initial wealth, as in Hanson and Menezes [7].

What about an investor who has a strict preference for a future outcome over the current one once she has decided to invest? In theory, the utility function $u(\cdot)$ of an investor is represented in the same manner as that of a consumer, except that the argument of the utility function is the amount of money spent/gained by the investor, rather than the quantity of a consumption good. This representation traces back to the Bernoulli and Cramer's conjecture that the investor derives a certain utility from money and obviously cares about that utility, rather than about money *per se*[1]. Hence, the investor will not be available to accept the unknown outcome unless she obtains a risk premium and, if she is

[1] See for instance Levy [10], page 25

prudent, the premium required will be greater the more uncertain that the future outcome is. Of course, the investor will prefer the future outcome to the current one, if and only if the net return thereof obtained exceeds the required risk premium, i.e. the downside risk is sufficiently low.

An essential characteristics of many investments is their irreversibility. This characteristics seems to be ignored in the representation of the investor's preferences previously described, as the investor may well face a higher downside risk from early consumption rather than from late consumption. As is well known from the literature on investment under uncertainty (Dixit and Pindyck [5]), the investor might have a strict preference for delaying an investment that is irreversible in nature and, hence, for delaying the outcome associated with that investment. Indeed, by doing so, she can acquire additional information, which will be useful to decide how much to invest. In such situations prudence cannot be related to the risk premium required by the individual, since the investor has a strict preference for the future outcome over the current one. In this paper, we show that, in such situations, prudence is actually related to the benefit drawn from the investment delay, which we will call the "flexibility gain," as in Dixit and Pindyck [5]. Intuitively, prudence is related to the flexibility gain because a delay in investment leads to a reduction in downside risk. Indeed, by not engaging in an irreversible investment today, the investor maintains the possibility of investing less tomorrow, if an unfavorable state of nature is realized.

As an illustration, take an individual who draws a utility of $u(y)-\theta y$ from the immediate investment in a capacity of size y, where θ is the expected cost of acquiring a capacity unit and the true cost is $\theta+\widetilde{\eta}$, where $\widetilde{\eta}\in\{-\eta,\eta\}$ with equal probabilities. Noticeably, the net utility is separable in the satisfaction derived from using y and the cost of θy that the purchase occasions. In this way the optimal choice of y is endogenous and depends on the technology of production (represented by θ and $\theta+\widetilde{\eta}$). Moreover, whereas we refer to y as to "capacity", y may well represent the decision variable in many other situations in which the decision is irreversible. The examples we have in mind include the delegation of a task with unknown cost of production in the contracting stage, and the regulation of a monopolist when the consumer surplus is known but the cost of production is unknown. We show that the individual prefers to delay her decision until after she will have observed the realization of $\theta+\widetilde{\eta}$, hence she has a flexibility gain. We find that a greater flexibility gain for the individual is associated with a lower value of θ if and only if u''' is positive. This is explained by the preference for less downside risk, as in Menezes et al. [11], with the caveat that here less downside risk reflects a greater flexibility gain rather than a lower risk premium.

In the development, we consider a somewhat more general utility function, namely $u(\pi(y))-(\theta+\widetilde{\eta})\,y$, where $\pi(y)$ is the profit obtained by using a capacity of y, and $\widetilde{\eta}$ is distributed on a continuous range of values. However, the link between flexibility gain and prudence follows the same principle as in the previous example.

Looking next at the preference over different distributions of optimal capacities, we consider cost functions of the form $\theta_i + \widetilde{\eta}_j$, where each distribution ij depends on the mean and the spread parameters. We show that the higher that u''' is, the more that distributions with lower θ_i and/or more dispersed $\widetilde{\eta}_j$ are preferable to other distributions. Hence, the shape of the marginal utility of capacity provides a measure of how much the individual is available to pay to be able to use a better technology, such as a technology with a lower expected cost of acquisition. Remarkably, this result depends finely on the fact that the distributions ij and $i'j'$ differ in terms of the support of the states of nature, i.e. some state of nature has a positive likelihood in some distribution and zero likelihood in some other distribution. In the literature, the most common way of expressing a preference for a certain distribution is to consider some stochastic ordering of distributions, among which first order stochastic dominance and mean preserving spread are most common. The support of states of nature does not necessarily differ across distributions. When the support is the same the distributions only differ in terms of the probabilities associated with the possible states, which are always positive. However, in that case, the magnitude of u''' does not necessarily have bite on the individual preference for some specific distribution.

As the final step of the analysis, we provide examples of applications to sequential learning in principal-agent models and to investment timing decisions when the uncertainty about the future lasts over an infinite number of periods.

The paper is first related to the studies, within the literature on Decision theory, in which prudence is defined as an averse attitude to downside risk (Menezes et al. [11], Bigelow and Menezes [1] and Eeckhoudt and Schlesinger [6]). We contribute to this literature by showing that, in addition to being associated with a type of preference over simple lotteries, prudence is also associated with a type of preference over different moments in time when an investment could take place.

The paper is further related to the literature on investment timing decisions under uncertainty (Dixit and Pindyck [5]). In this literature, the investor is usually assumed to be neutral to risk. An exception is the study of Henderson and Hobson [8]. Not surprisingly, they find that a risk averse investor invests less often under uncertainty than does a risk neutral investor. Hitherto no study has shown how an investment timing decision is affected by the fact that the individual is prudent. To make the point, we consider a setting in which uncertainty is resolved after one period, as is usual in Decision theory. In the application to investment timing decision under continuous uncertainty, we rely on a specific example to show that not only risk aversion but also prudence induces the investor to invest less often.

The outline of the paper is as follows. In Section 2 we analyze the link between prudence and flexibility gain to the individual in a two-period model. In Section 3 we describe applications in delegation, price discrimination and timing of investment decisions. Section 4 briefly concludes.

2 Prudence and flexibility gain

Consider an individual whose utility has the functional form $u(\pi(y)) - (\theta + \widetilde{\eta})\, y$, where y is referred to as a capacity of production, $\pi(y)$ as the profit obtained with a capacity of y, and $(\theta + \widetilde{\eta})\, y$ is the irreversible cost incurred to acquire y capacity units, where $\theta > 0$, $\mathbb{E}[\widetilde{\eta}] = 0$, $\widetilde{\eta} \in [-\eta, \eta]$ with cumulative distribution function $G(\widetilde{\eta})$, for some $\eta > 0$ and finite, and with frequency $g(\widetilde{\eta})$ positive everywhere. We take η to be finite in order to ensure that the support of cost values varies with θ and η, which is essential for our investigation. The function $u(\cdot)$ has derivatives $u' > 0$ and $u'' < 0$; moreover, u''' has a constant sign. Because the cost of acquisition is sunk once the investment is made, the capacity decision is irreversible. The choice of **a** utility function which is quasilinear in the cost of investment is made for expositional purposes, as it will become clear in a moment.

There are two decisions that the individual will have to make. First, whether to invest today or delay that decision. The advantage of delaying the investment decision resides in that, whereas in the current period only the expected value of $\theta + \widetilde{\eta}$ is known, the information about $\widetilde{\eta}$ will become available in the next period. Second, the individual will decide how many capacity units to acquire. The number of units is pinned down by the first order condition $u'(\pi(y))\, \pi'(y) = x$, where either $x = \theta$ or $x = \theta + \widetilde{\eta}$, depending on whether the investment is made immediately or it is delayed. It is optimal to delay the investment decision until the next period, when $\widetilde{\eta}$ will be known, if and only if $w(\theta, \eta) > 0$, where

$$w(\theta, \eta) = \mathbb{E}\left[u(\pi(y(\theta + \widetilde{\eta}))) - (\theta + \widetilde{\eta})\, y(\theta + \widetilde{\eta})\right] - (u(\pi(y(\theta))) - \theta y(\theta))$$

When $w(\theta, \eta) > 0$, the investor obtains a flexibility gain from delaying the investment from the first period, when $\widetilde{\eta}$ is unknown, to the next period, when its value will be realized.

Before analyzing the flexibility gain, we recall the notion of prudence. We know that an individual is prudent if and only if $u'''(y) > 0$, for any y exogenously given. Applying Jensen inequality, this is equivalent to $\mathbb{E}[u'(y + \widetilde{\varepsilon})] \geq u'(y)$, for some random $\widetilde{\varepsilon}$ such that $\mathbb{E}[\widetilde{\varepsilon}] = 0$. Using this equivalence, prudence is usually associated with the risk premium which the individual requires to be available to accept the unknown outcome. Indeed, supposing that the future capacity is $y + \widetilde{\varepsilon}$ and that it is *exogenously given* to the individual at no cost, she is prudent when $v'(y) < 0$, where

$$v(y) = \mathbb{E}[u(y + \widetilde{\varepsilon})] - u(y)$$

and $-v(y)$ is her risk premium (see, for instance, Eeckhoudt and Schlesinger [6]). A prudent individual asks for a lower risk premium $-v(y)$ as y is raised. If the support of $\widetilde{\varepsilon}$ is finite, then we can say that the reason why she asks for a lower risk premium is that the support of the unknown values $y + \widetilde{\varepsilon}$ shifts upwards and the downside risk is thus reduced. If the individual has to pay for that capacity and her utility is quasi-linear in the cost occasioned by that purchase, then the risk premium is $-v_1(y)$, where

$$v_1(y) = \mathbb{E}\left[u\left(y+\widetilde{\varepsilon}\right)\right] - u(y) - \left\{\mathbb{E}\left[\left(\theta+\widetilde{\varepsilon}\right)\left(y+\widetilde{\varepsilon}\right)\right] - \theta y\right\}.$$

Because $\mathbb{E}\left[\left(\theta+\widetilde{\varepsilon}\right)\left(y+\widetilde{\varepsilon}\right)\right] - \theta y = \mathbb{E}\left[\widetilde{\varepsilon}^2\right]$, the risk premium is positive and even higher than in the previous case, if $\widetilde{\varepsilon}$ has a positive variance.

Let us now turn to consider our setting, where $y(\theta)$ is *endogenous.* Using the following result, it is easy to deduce that the decision maker is very likely to obtain a flexibility gain if she delays her investment decision.

Lemma 1 *The flexibility gain is expressed as follows:*

$$w(\theta,\eta) = \int_{\theta-\eta}^{\theta} \left[y(x)\,g(x) - y(x+\eta)\,g(x+\eta)\right] dx \tag{1}$$

Proof. Considering that the optimal capacity is such that $u'(\pi(y(x)))\,\pi'(y(x)) = x$, $\forall x$, rewrite

$$\begin{aligned} w(\theta,\eta) &= \left[\mathbb{E}\left[u\left(\pi\left(y\left(\theta+\widetilde{\eta}\right)\right)\right)\right] - \left(\theta+\widetilde{\eta}\right)y\right] - \left[u\left(\pi\left(y(\theta)\right)\right) - \theta y\right] \\ &= \mathbb{E}\int_{\theta}^{\theta+\widetilde{\eta}} \left[u'(\pi(y(x)))\,\pi'(y(x)) - xy'(x) - y(x)\right] dx \\ &= -\mathbb{E}\int_{\theta}^{\theta+\widetilde{\eta}} y(x)\,g(x)\,dx, \end{aligned}$$

which is further rewritten as (1). ■

We observe, for instance, that the individual obtains a flexibility gain from a delay in the investment either if $\widetilde{\eta}$ can take only two values, as in the example presented in Introduction, or if it takes values on the range $[-\eta,\eta]$ according to a symmetric distribution. More generally, there exists a flexibility gain if the rate of change of $y(x)$ is lower than that of the likelihood $g(x)$, namely if $y'(x)/y(x) < g'(x)/g(x)$. In particular, because $y(x)$ is a decreasing function for a risk averse individual, this condition is satisfied if the cumulative distribution function $G(x)$ is convex.

The possible existence of a flexibility gain ($w(\theta,\eta) > 0$) is explained by the irreversible nature of the investment decision. By committing today to a capacity of $y(\theta)$, the individual renounces to the opportunity of purchasing only $y(\theta+\eta) < y(\theta)$ capacity units tomorrow, if she finds out that the cost of procuring capacity is high ($\theta+\eta > \theta$). Therefore, $\mathbb{E}\left[y\left(\theta+\widetilde{\eta}\right)\right] > y(\theta)$ and such that the individual bears more risk if she commits to a capacity choice today, rather than delaying the decision. Instead, if she were to choose between some exogenous capacity of y today and $y+\widetilde{\varepsilon}$ tomorrow, where $\mathbb{E}\left[\widetilde{\varepsilon}\right] = 0$, then there would be no flexibility gain and, as usual, the risk premium would be positive.

In good substance, the downside risk is reduced when the investment (and, implicitly, the consumption) is delayed rather than being made immediately. We shall now show that the flexibility gain is greater when the unit cost of capacity is smaller if and only if the third derivative of the utility function is high. To do so, we introduce the following definition:

$$\zeta(a,b,c) = [f(a) - f(a+c)] - [f(b) - f(b+c)], \tag{2}$$

where $f(\cdot)$ is the inverse function of $u'(\cdot)$.

Being based on the following lemma, we can next use $\zeta(a,b,c)$ as a measure of the concavity/convexity of the marginal utility function.

Lemma 2 *$\zeta(a,b,c) > 0$ if and only if $u''' > 0$, $\forall a,b,c$ such that $a < c$ and $b > 0$.*

Proof. One has $\zeta(a,b,c) > 0$ if and only if

$$[f(a) - f(a+c)] > [f(b) - f(b+c)]$$
$$\Leftrightarrow$$
$$\int_a^{a+c} f'(x) - f'(x+b-a)\,dx < 0$$

Provided u''' has **a** constant sign, this is also the case of $f'(\cdot)$, and the above condition holds if and only if $f'(x) < f'(x+b-a)$, for any given $x \in [a, a+c]$. This is equivalent to $u''(y(x)) > u''(y(x+b-a))$. Because $b > a$ and $u'' < 0$, $y(x) > y(x+b-a)$. Hence, $u''(y(x)) > u''(y(x+b-a))$ is equivalent to $u''' > 0$. ∎

We are now ready to show that the degree of flexibility, which the individual enjoys if she delays the investment decision, is related to her prudence.

Proposition 1 *$\frac{dw(\theta,\eta)}{d\theta} < 0$ if and only if*

$$\zeta(\theta-\eta,\theta,\eta) \geq \psi(\theta,\eta), \tag{3}$$

where

$$\psi(\theta,\eta) \equiv y(\theta-\eta)\frac{g(\theta)-g(\theta-\eta)}{g(\theta)} - y(\theta+\eta)\frac{g(\theta+\eta)-g(\theta)}{g(\theta)}$$

Proof. Using (1),

$$\begin{aligned}
\frac{dw(\theta,\eta)}{d\theta} &= \frac{d}{d\theta}\int_{\theta-\eta}^{\theta} [y(x)g(x) - y(x+\eta)g(x+\eta)]\,dx \\
&= [y(\theta)g(\theta) - y(\theta+\eta)g(\theta+\eta)] - [y(\theta-\eta)g(\theta-\eta) - y(\theta)g(\theta)] \\
&= g(\theta)[-\zeta(\theta-\eta,\theta,\eta) + \psi(\theta,\eta)],
\end{aligned}$$

and the result follows. ∎

The lower that the expected unit cost is the higher that the optimal value of $y(\theta+\widetilde{\eta})$ is, for each realization of $\widetilde{\eta}$. In this case, rather than mirroring a lower risk premium required by the individual, a high level of u''' (which determines a high value of $\zeta(\theta-\eta,\theta,\eta)$), mirrors a greater flexibility gain from delaying the capacity decision. Noticeably, the level of u''', which separates the region where $dw(\theta,\eta)/d\theta$ has a negative sign from that where its sign is positive, is not necessarily zero. Therefore, one cannot use Proposition 1 to provide a definition of prudence, which would be equivalent to $u'''>0$. The reason is that not only does the flexibility gain depend on the fact that the values of $y(\theta+\widetilde{\eta})$ are shifted upwards. It also depends on how the frequency of each state $y(\theta+\widetilde{\eta})$ changes as the support of values is shifted upwards. Hence, it depends on the characteristics of the distribution function $g(\widetilde{\eta})$. In Introduction, we provided a simple example in which $\widetilde{\eta}$ can only take two equally likely values so that $\psi(\theta,\eta)=0$. In that case, $dw(\theta,\eta)/d\theta$ is negative if and only if the individual is prudent. That example belongs to the category of uniform distributions captured by the following remark.

Remark 1 *Assume that $g(\cdot)$ is symmetric. Then, $\psi(\theta,\eta)\geq 0$, with $\psi(\theta,\eta)=0$ if $g(\cdot)$ is uniform.*

It should thus be apparent that the notion of prudence has a broader interpretation than usually considered in the literature. When the distribution of cost values is symmetric around the mean the investor enjoys a flexibility gain from delaying the investment decision if and only if she is prudent. The reason why the investor postpones her decision is that, by delaying the investment, she faces a lower downside risk.[2]

Before concluding, we show that the link between the flexibility gain and the preferences of the individual goes beyond the sole definition of prudence. Indeed, as we now turn to highlight, the third derivative of the utility function provides a measure of how much the individual is ready to pay to be faced with a cost distribution associated with a lower expected cost θ, or with a higher spread η, or with some combination of θ and η, taking into account that each of these two parameters contributes to the magnitude of the flexibility gain the individual enjoys in her capacity decision.

Suppose that the unknown unit cost is $\theta_i+\widetilde{\eta}_j$, where $i\in\{1,2\}$ and $j\in\{1,2\}$, such that $\theta_1<\theta_2$ and $\eta_1<\eta_2$. Further define

$$D_{ij/i'j'}=\mathbb{E}\left[U_{ij}-U_{i'j'}\right],$$

where

$$U_{ij}=u\left(y\left(\theta_i+\widetilde{\eta}_j\right)\right)-\left(\theta_i+\widetilde{\eta}_j\right)y\left(\theta_i+\widetilde{\eta}_j\right).$$

For instance, $D_{1j/2j}$ measures the additional gain that a technology associated with an expected unit cost of θ_1 grants, relative to one associated with an expected unit cost of θ_2,

[2]Notice also that, unlike in the case of the expected unit cost, the influence of the spread of the cost on the flexibility gain is unrelated to u'''. For instance, if the distribution is uniform, then the option to delay is more important the higher that the spread between future costs is and regardless of the characteristics of the utility function (a standard result in the literature on investment under uncertainty).

for any given value η_j of the spread. Suppose that some innovation is available at some cost $I > 0$, which allows for the use of a technology $\theta_1 + \widetilde{\eta}_j$ rather than $\theta_2 + \widetilde{\eta}_j$. Then, the higher that $D_{1j/2j}$ is, the more likely that the individual will be ready to pay that cost to acquire the innovation. According to the following result, the willingness to pay I is related to the third derivative of the utility function.

Proposition 2 *i) $D_{1j/2j'}$ increases with u''', $\forall j \geq j'$.*

ii) $D_{i2/i'1}$ increases with u''', $\forall i \leq i'$.

iii) $D_{11/22}$ increases with u''' if and only if $\theta_2 - \theta_1 > \eta_2 - \eta_1$

Proof. Being based on the definition of U_{ij}, we can compute

$$\begin{aligned}
D_{ij/i'j'} &= \mathbb{E}\left[U_{ij} - U_{i'j'}\right] \\
&= -\mathbb{E}\left[\int_{\theta_i+\widetilde{\eta}_j}^{\theta_{i'}+\widetilde{\eta}_{j'}} u'\left(y\left(x\right)\right) y'\left(x\right) dx\right] - \mathbb{E}\left[\left(\theta_i + \widetilde{\eta}_j\right) y\left(\theta_i + \widetilde{\eta}_j\right)\right] \\
&\quad + \mathbb{E}\left[\left(\theta_{i'} + \widetilde{\eta}_{j'}\right) y\left(\theta_{i'} + \widetilde{\eta}_{j'}\right)\right] \\
&= -\mathbb{E}\int_{\theta_i+\widetilde{\eta}_j}^{\theta_{i'}+\widetilde{\eta}_{j'}} x y'\left(x\right) dx - \mathbb{E}\left[\left(\theta_i + \widetilde{\eta}_j\right) y\left(\theta_i + \widetilde{\eta}_j\right)\right] + \mathbb{E}\left[\left(\theta_{i'} + \widetilde{\eta}_{j'}\right) y\left(\theta_{i'} + \widetilde{\eta}_{j'}\right)\right],
\end{aligned}$$

which further reduces to

$$D_{ij/i'j'} = \mathbb{E}\int_{\theta_i+\widetilde{\eta}_j}^{\theta_{i'}+\widetilde{\eta}_{j'}} y\left(x\right) dx. \tag{4}$$

Proof of (i) and (ii). Using (4) and (2), we can further develop

$$\begin{aligned}
D_{1j/2j} &= \mathbb{E}\int_{\theta_1+\widetilde{\eta}_j}^{\theta_2+\widetilde{\eta}_j} y\left(x\right) dx \\
&= \mathbb{E}\left[\int_{\theta_1+\widetilde{\eta}_j}^{\theta_1} y\left(x\right) dx + \int_{\theta_1}^{\theta_2} y\left(x\right) dx + \int_{\theta_2}^{\theta_2+\widetilde{\eta}_j} y\left(x\right) dx\right] \\
&= \frac{1}{2}\int_{\theta_1-\eta_j}^{\theta_1} \zeta\left(x, x+\Delta\theta, n_j\right) dx + \int_{\theta_1}^{\theta_2} y\left(x\right) dx,
\end{aligned}$$

where $\Delta\theta = \theta_2 - \theta_1$. In the expression of $D_{1j/2j}$ the term that depends on u''' is $\zeta\left(\cdot,\cdot,\cdot\right)$, which is defined in (2)) and increases with u'''. Also,

$$\begin{aligned}
D_{i2/i1} &= -\mathbb{E}\left[\int_{\theta_i+\widetilde{\eta}_1}^{\theta_i+\widetilde{\eta}_2} y\left(x\right) dx\right] \\
&= \frac{1}{2}\left[\int_{\theta_i-\eta_2}^{\theta_i} y\left(x\right) dx - \int_{\theta_i}^{\theta_i+\eta_2} y\left(x\right) dx\right] - \frac{1}{2}\left[\int_{\theta_i-\eta_1}^{\theta_i} y\left(x\right) dx - \int_{\theta_i}^{\theta_i+\eta_1} y\left(x\right) dx\right] \\
&= \frac{1}{2}\int_{\theta_i-\eta_2}^{\theta_i} \zeta\left(x, x+\Delta\eta, \eta_2\right) dx + \frac{1}{2}\left[\int_{\theta_i}^{\theta_i+\Delta\eta} y\left(x\right) dx - \int_{\theta_i+\eta_1}^{\theta_i+\eta_1+\Delta\eta} y\left(x\right) dx\right],
\end{aligned}$$

where $\Delta\eta = \eta_2 - \eta_1$. Again, the term that depends on u''' is $\zeta\left(\cdot,\cdot,\cdot\right)$, which increases with

u'''.

Rewriting

$$D_{1j/2j'} = D_{1j/2j} + D_{2j/2j'} = \begin{cases} D_{1j/2j}, & \text{if } j' = j, \\ D_{12/22} + D_{22/21}, & \text{if } j' = 1 < j = 2 \end{cases}$$

and

$$D_{i2/i'1} = D_{i2/i1} + D_{i1/i'1} = \begin{cases} D_{i2/i1}, & \text{if } i = i', \\ D_{12/11} + D_{11/21}, & \text{if } i = 1 < i' = 2 \end{cases},$$

and considering that $D_{1j/2j}$, $\forall j$, and $D_{i2/i1}$, $\forall i$, increase with u''', we deduce that this is the case of $D_{1j/2j'}$ and $D_{i2/i'1}$ as well.

Proof of (iii). Using (4) and (2), we can compute

$$\begin{aligned} D_{11/22} &= \mathbb{E}\left[\int_{\theta_L+\widetilde{\eta}_L}^{\theta_H+\widetilde{\eta}_H} y(x)\,dx\right] \\ &= \frac{1}{2}\int_{\theta_L-\eta_L}^{\theta_H-\eta_H} \zeta(x, x+\eta_L, x+2\eta_L)\,dx + \int_{\theta_L}^{\theta_L+\Delta\theta-\Delta\eta} y(x)\,dx \end{aligned}$$

If $\Delta\theta > \Delta\eta$, then $D_{11/22}$ is positive and increases with $\zeta(\cdot,\cdot,\cdot)$; if $\Delta\theta < \Delta\eta$, then it is negative and decreases with $\zeta(\cdot,\cdot,\cdot)$. Provided that $\zeta(\cdot,\cdot,\cdot)$ increases with u''', the result follows. ■

Therefore, how much an individual prefers being faced with a set $\{\theta_i - \eta_j, \theta_i + \eta_j\}$, rather than with a set $\{\theta_{i'} - \eta_{j'}, \theta_{i'} + \eta_{j'}\}$, is related to u'''. Remarkably, one cannot use the usual notions of stochastic dominance to relate a preference over distributions to u'''. Indeed, it is easy to show that, if two distributions share the same support but are ordered in the sense of first-order stochastic dominance, or in the sense of a mean preserving spread, then the preference of the individual for one distribution is unrelated to u'''.

3 Applications

We hereafter propose a few examples, in which the utility function of the decision maker is given by $u(y) - xy$, where $u(y)$ is a constant relative risk aversion function, defined as follows:

$$u(y) = \frac{1}{1-\gamma}y^{1-\gamma},$$

for some $\gamma \in (0,1)$. Accordingly, we have $u' = y^{-\gamma} > 0$, $u'' = -\gamma y^{-\gamma-1} < 0$ and $u''' = \gamma(\gamma+1)y^{-\gamma-2} > 0$. Considering a utility function with these properties is convenient in that it permits to look at variations in u''' through variations in γ. Indeed, one has

$$\frac{du'''}{d\gamma} = (2\gamma+1)\,y^{-\gamma-2} + \gamma(\gamma+1)\,y^{-\gamma-2}\ln y,$$

which is strictly positive if the quantity y is above one. The second and the third derivatives of $u(\cdot)$ are also the second and the third derivatives of $u(y) - xy$.

3.1 Delegation with unknown cost in the contracting stage

A principal who delegates the production activity to an agent obtains the gross utility $u(y)$ from consumption of the y units produced by the agent. By assigning a profit of $\pi = t(x, y) - xy$ to the agent, where $t(x, y)$ is a transfer, the principal obtains a net utility of $u(y) - xy - \pi$.

Take $x_0 = \theta$ to be known and $x_1 \in \{\theta - \eta, \theta + \eta\}$ with equal probabilities. From the previous analysis, we know that, if $\pi = 0$, then the principal strictly prefers to condition the production quantity on x_1, rather than on x_0, since, by doing so, she obtains a greater flexibility gain. Moreover, according to Proposition 1, higher values of θ are associated with a smaller flexibility gain.

Suppose next that the principal can choose between an agent producing at a cost of $\theta_1 + \widetilde{\eta}_1$ and an agent producing at a cost of $\theta_2 + \widetilde{\eta}_2$, where $\theta_1 < \theta_2$ and $\eta_1 < \eta_2$. We saw that, if the principal can leave zero profit to the agent regardless of the latter's cost, then she prefers a cost of $\theta_1 + \widetilde{\eta}_1$ if and only if $\Delta\theta > \Delta\eta$. Moreover, the gain increases with u'''. Indeed, using $u' = \theta + \widetilde{\eta}$ and $u' = y^{-\gamma}$, we see that $y(\theta + \widetilde{\eta}) = (\theta + \widetilde{\eta})^{-\frac{1}{\gamma}}$ and

$$D_{11/22} = E \int_{\theta_1 + \widetilde{\eta}_1}^{\theta_2 + \widetilde{\eta}_2} x^{-\frac{1}{\gamma}} dx$$

together with

$$\frac{dD_{11/22}}{d\gamma} = \frac{1}{\gamma^2} \mathbb{E} \int_{\theta_1 + \widetilde{\eta}_1}^{\theta_2 + \widetilde{\eta}_2} x^{-\frac{1}{\gamma}} \ln(x)\, dx,$$

which is positive if the unit cost is above 1 in all states. Since $du'''/d\gamma > 0$, we can say that a greater value of u''' is associated with a greater value of $D_{11/22}$, as in Proposition 2.

Why is this relevant? Suppose that the principal runs an auction to select the agent who will accomplish the task. If she does not know which type of agent she is facing, then she will obviously prefer to favour an agent of type $\theta_1 + \widetilde{\eta}_1$, if that type exists. If the principal faces one agent with two possible types, then the greater that u''' is the higher the information rent that she will prefer to concede to type $\theta_1 + \widetilde{\eta}_1$ to solve the usual trade-off between rent extraction and efficiency loss. Indeed, denoting Π_1 and Π_2 the profits designed for the two types and setting $\Pi_2 = 0$, it is easy to verify that

$$\Pi_1 = \Delta\theta \mathbb{E}\left[y(\theta_2 + \widetilde{\eta}_2)\right] - \Delta\sigma \mathbb{E}\left[y(\theta_2 - \eta_2) - y(\theta_2 + \eta_2)\right].$$

Replacing $y(\theta + \widetilde{\eta}) = (\theta + \widetilde{\eta})^{-\frac{1}{\gamma}}$, this becomes

$$\Pi_1 = \Delta\theta \mathbb{E}\left[(\theta_2 + \widetilde{\eta}_2)^{-\frac{1}{\gamma}}\right] - \Delta\sigma \mathbb{E}\left[(\theta_2 - \eta_2)^{-\frac{1}{\gamma}} - y(\theta_2 + \eta_2)^{-\frac{1}{\gamma}}\right].$$

Then

$$\frac{d\Pi_1}{d\gamma} = \frac{1}{2\gamma^2}\left[(\Delta\theta - \Delta\sigma)\left((\theta_2 - \eta_2)^{-\frac{1}{\gamma}}\ln(\theta_2 - \eta_2)\right) + (\Delta\theta + \Delta\sigma)\left(y(\theta_2 + \eta_2)^{-\frac{1}{\gamma}}\ln(\theta_2 + \eta_2)\right)\right] > 0.$$

which confirms that the principal prefers to assign a higher information rent the greater that u''' is.[3][4]

Price discrimination with unknown preferences

The example presented above can be framed within the recent literature on principal-agent problems with privately known distributions. Most of those studies are about price discrimination in the relationship between a monopolist and a consumer, none of whom knows the consumer's valuation for the good in the contracting stage, whereas the consumer has private information on the distribution of his valuation. The pioneering study is that of Courty and Li [2]. They assume that the monopolist receives a fixed payment a at the time when the consumer is uninformed of his valuation. This might be followed by a reimbursement k, which the consumer can require in a later stage, after learning his true valuation. Of course, the consumer will want to be reimbursed, and will thus renounce to consume, if and only if k exceeds his valuation. If the consumer does not renounce, then the monopolist will bear a cost of c to provide the service.

Essentially, in Courty and Li [2], the economic issue is how to choose the future disbursement k and the current revenue a, which is more in line with the classical savings-consumption model than with the issue of our interest. However, because this problem belongs to the kind of principal-agent models considered in the previous example, for the sake of completeness, we show that u''' plays a role in the solution adopted by the principal also in the problem of price discrimination with unknown preferences. To that end, we restrict attention to the case of symmetric information between players.

Whereas Courty and Li [2] and more recent studies assume that the monopolist is risk neutral, we consider a risk averse monopolist, whose utility $u(\cdot)$ is expressed as a function

[3] Although little apparent from our presentation, there is also an additional aspect of the incentive problem which is related to u'''. That is, if both θ and η are privately known to the agent, then a greater u''' will also reflect the fact that adjacent incentive constraints are tighter than other incentive constraints. For instance, reporting $\theta_1 + \widetilde{\eta}_1$ is more attractive to a type $\theta_1 + \widetilde{\eta}_2$ than reporting $\theta_2 + \widetilde{\eta}_2$. A complete analysis is developed by Danau and Vinella [3], who show that the study of the optimal delegation in this context is lengthy and complicated, unless it is related to u'''.

[4] Remarkably, the results in the delegation example here proposed extend naturally to the regulation of a monopolist. In that framework, $u(\cdot)$ would indicate consumer surplus; $u'(\cdot)$ would measure the consumer willingness to pay for the good sold by the monopolist and, hence, the (inverse) demand for the good. Variations in u''' would represent variations in the price elasticity of the market demand rather than variations in the preferences of a risk averse decision maker. One would find that $\gamma = 1/\varepsilon$, where ε is the constant price elasticity of the demand. Therefore, a greater value of u''' would be associated with a less elastic demand and the results we presented in the example follow accordingly.

of money, as defined above. The total benefit of the monopolist is:

$$u(a) - \upsilon u(k) - (1-\nu) u(c),$$

where ν is the probability of a high valuation, namely $\upsilon > 0$, $(1-\nu)$ is the probability of a low valuation, namely 0, and the reimbursement k is supposed to take values in $(0, \upsilon)$ at optimum. Under symmetric information, if the risk neutral consumer has zero outside opportunity, then the monopolist chooses a fixed payment such that:

$$a(k) = (1-\nu) k + \nu\upsilon$$

The first-order condition of the maximization problem of the principal is given by:

$$u'(k_t + \nu(\upsilon - k_t)) \geq \frac{\nu}{1-\nu} u'(k_t)$$

For a positive solution to exist, it is necessary and sufficient that the high valuation is less likely than the low valuation: $\upsilon < 1/2$. Otherwise, the monopolist will choose $a = \nu\upsilon$ without conceding any reimbursement. Accordingly, we take $\nu < 1/2$. Then, $k > 0$ involving that $a(k) > \nu\upsilon$. Replacing $u'(y) = y^{-\gamma}$, we obtain the following solution:

$$[k + \nu(\upsilon - k)]^{-\gamma} = \frac{\upsilon}{1-\upsilon} k^{-\gamma} \Leftrightarrow k^* = \frac{\nu}{\left(\frac{1-\upsilon}{\upsilon}\right)^{\frac{1}{\gamma}} - (1-\nu)} \upsilon,$$

which is lower than υ and confirms our previous hypothesis. We see that $dk^*/d\gamma > 0$. Hence, the greater that u''' is the higher the value k^* that the solution takes. This is interpreted as follows. The monopolist is more prone to grant a reimbursement to the consumer in a later stage to be able to appropriate a higher certain payment $a(k^*)$ today.

3.2 Investment timing and prudence

We now consider an investment timing problem in which, unlike in the basic model previously used, uncertainty lasts forever. Whereas in the two-period setting we found that u''' is related to the magnitude of the flexibility gain the individual enjoys by delaying the investment, in this example we show that, with infinite uncertainty, u''' is related to how much the individual prefers to delay the investment. As very common in real options analysis, we take x to follow a Geometric Brownian Motion, such that

$$dx_t = \alpha x_t dt + \sigma x_t dz_t, \tag{5}$$

where z_t is a simple Brownian Motion and $\alpha < r$, with r the risk-adjusted discount rate. Starting from an initial capacity of y, the individual must decide at which point in time she will invest in a capacity increment, which would cost x_t at time t. The individual invests immediately if $x_0 < x^*$, for some x^* to be determined; otherwise, she delays the investment until the date $T^* > 0$, defined as follows:

$$T^* = \inf\{t \geq 0, \ s.t. \ x_t = x^*\}.$$

This is the stochastic moment when x^* is reached for the first time. For simplicity, we do not allow for disinvestment. The optimal value x^*, which triggers the investment, is defined by the equation

$$u'(y) = \frac{\beta_2}{\beta_2 - 1} x^*,$$

where β_2 is the negative root of the quadratic equation $\beta(\beta - 1)\frac{\sigma^2}{2} + \alpha\beta = r$.[5] Considering that $u'(y) = y^{-\gamma}$, we can write

$$x^* = \frac{\beta_2 - 1}{\beta_2} y^{-\gamma}$$

and

$$\frac{dx^*}{d\gamma} = -\frac{\beta_2 - 1}{\beta_2} y^{-\gamma} \ln y < 0$$

This suggests that the greater that u''' is the lower that x^* will be. Hence, a greater u''' is associated with a later investment.[6]

An observation is again in order. A greater u''' is not necessarily associated with a later investment in any investment problem under uncertainty. This is or not the case depending on the number of decisions to be made by the investor and on the degree of irreversibility of each such decision. To see this, we now take $u(\cdot)$ to be the individual utility in one single period and $\widetilde{x}$ to represent a unit cost of operation rather than a cost of investment. In this framework, the production quantity y_t changes in each period according to the rule $u'(y_t) = x_t$, involving that $y(x_t) = x_t^{-1/\gamma}$ and

$$u(y(x_t)) - x_t y(x_t) = \delta x_t^{-\delta}, \text{ where } \delta \equiv \frac{\gamma}{1-\gamma}.$$

We see that

$$\frac{d\delta}{d\gamma} = \frac{1}{(1-\gamma)^2} > 0.$$

The expected discounted value of the project at time t is given by

$$V_t = \delta \frac{x_t^{-\delta}}{r + \alpha\delta - \frac{1}{2}\sigma^2\delta(\delta + 1)},$$

[5] The steps to the identification of the solution are standard and thus omitted. See, for instance, the basic model of Dixit and Pindyck [5] presented in Chapter 5, Section 2. The example we provide is similar to their model, except that here the stochastic variable is the unit cost of investment rather than the discounted value of the project.

[6] Dixit [4] determines $\mathbb{E}[T^*]$ analytically and shows that it is inversely related to x^*.

under the assumption that the denominator is positive.[7] The investment trigger is given by

$$\frac{x^{-\delta}}{\frac{r}{\delta}+\alpha-\frac{1}{2}\sigma^2(\delta+1)}=\frac{\beta_2}{\beta_2-1}I.$$

Hence, one obtains

$$x_t^{-\delta}=\frac{\beta_2}{\beta_2-1}I\left[\frac{r}{\delta}+\alpha-\frac{1}{2}\sigma^2(\delta+1)\right].$$

The derivatives of both the left-hand side and the right-hand side decrease with δ and it is not clear that the investment trigger is monotonic in δ. This is because, unlike the investment decision, the decision concerning the scale of operation is not irreversible. Therefore, as compared to the example illustrated above, there is now an additional benefit to investing immediately, which might offset the benefit associated with the investment delay.

4 Conclusion

We showed that the notion of prudence extends to situations the literature has not considered so far. Specifically, provided that an individual prefers future outcomes to current ones, the third derivative of the utility function (and, implicitly, prudence) is a measure of that preference. An individual is likely to prefer future outcomes to current ones when the attainment of the outcomes requires making an irreversible investment.

Relying on a simple model, we pinned down the equivalence between prudence and a specific type of behavior, in the same vein as prudence is proven to be a specific preference over simple lotteries in the previous literature. We showed that, in the setting we considered, prudence is equivalent to a more pronounced preference for a flexibility gain the lower that the expected unit cost is, if the individual's utility is concave in consumption and the cost of investment is linear.

We provided two examples of a preference profile with these features, which we drew from two different domains of literature, without the ambition of being exhaustive. This helped us illustrate that the notion of prudence can be employed along a novel research direction.

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