Multi-object auction design: revenue maximization with no wastage*

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February 9, 2017

Abstract

A seller is selling multiple objects to a set of agents. Each agent can buy at most one object and his utility over consumption bundles (i.e., (object,transfer) pairs) need not be quasilinear. The seller considers the following desiderata for her mechanism, which she terms *desirable*: (a) dominant strategy incentive compatibility, (b) ex-post individual rationality, (c) equal treatment of equals, (d) no wastage (every object is allocated to some agent). The minimum Walrasian equilibrium price (MWEP) mechanism is desirable. We show that the MWEP mechanism generates more revenue for the seller than any other desirable mechanism satisfying no subsidy at every profile of preferences, i.e., irrespective of the prior of the seller, the MWEP mechanism is revenue-optimal. Our result works for quasilinear type space and for various non-quasilinear type spaces which incorporates positive income effect of agents. We can relax no subsidy in our result for certain type spaces with positive income effect.

*We are grateful to Brian Baisa, Albin Erlanson, Ryan Tierney for useful comments and discussions. The authors thank participants at the Advances in Mechanism Design conference at NYU Abu Dhabi, and seminar participants at Hitotsubashi University and Indian Statistical Institute for their comments.

1 Introduction

One of the most challenging problems in microeconomic theory is the design of revenue maximizing multi-object auction. Ever since the seminal work of Myerson (1981) for solving the revenue maximizing single object auction, advances in the mechanism design literature have convinced researchers that it is difficult to precisely describe a revenue maximizing multi-object auction. We offer a robust resolution to this difficulty by imposing some additional axioms that are appealing in many settings.

We study the problem of auctioning (allocating) m indivisible objects to $n > m$ agents, each of whom can be assigned at most one object (unit demand bidders). Agents in our model can have non-quasilinear preferences over consumption bundles - (object, transfer) pairs. We impose four desiderata on mechanisms: (1) strategy-proofness or dominant strategy incentive compatibility, (2) ex-post individual rationality, (3) equal treatment of equals - two agents having identical preferences must be assigned consumption bundles (i.e., (object, payment) pairs) to which they are indifferent, (d) no wastage (every object is allocated to some agent). Any mechanism satisfying these properties is termed *desirable*.

If the type space is *rich*, then our main result says that the minimum Walrasian equilibrium price (MWEP) mechanism (which we describe in the next paragraph) is revenue maximizing in the class of all desirable and no subsidy mechanisms. No subsidy requires that payment of each bidder is non-negative. The richness in type space captures the quasilinear domain and many type spaces with positive income effect. Further, we show that if the type space includes all positive income effect preferences, then the MWEP mechanism is revenue maximizing in the class of all desirable and *no bankruptcy* mechanisms, where no bankruptcy requires that the sum of payments of all agents across all profiles is bounded below. Notice that no bankruptcy is weaker than no subsidy. Without no bankruptcy, the auctioneer runs the risk of being bankrupt at some profile of preferences. Our revenue maximization result is robust in the following sense: at *every* profile of preferences the revenue from the MWEP mechanism beats the revenue from any other desirable mechanism satisfying no subsidy (or, no bankruptcy if the domain contains all positive income effect preferences). Hence, we can recommend the MWEP mechanism without resorting to any prior-based maximization.

The MWEP mechanism is based on a "market-clearing" notion. A price vector on objects is called a Walrasian equilibrium price vector if there is an allocation of objects such that each agent gets an object from his demand set. Demange and Gale (1985) showed that the set of Walrasian equilibrium price vectors is always a non-empty compact lattice in our model. This means that there is a unique minimum Walrasian equilibrium price vector. [1] The

[1] Results of this kind were earlier known for quasilinear preferences (Shapley and Shubik, 1971; Leonard,

MWEP mechanism selects the minimum Walrasian equilibrium price vector at every profile of preferences and uses a corresponding equilibrium allocation. The MWEP mechanism is desirable (Demange and Gale, 1985) and satisfies no subsidy. We show that in many domains of preferences, it is revenue-optimal among all desirable and no subsidy mechanisms.

Our results stand out in the literature (discussed later in Section 6) in another important way - ours is the first paper to study revenue maximizing multi-object auctions when preferences of agents are not quasilinear. Quasilinearity has been the standard assumption in most of mechanism design. While it allows for analysis of mechanism design problems using standard convex analysis tools (illustrated by the analysis of Myerson (1981)), its practical relevance is debatable in many settings. For instance, in spectrum auctions, the payments of bidders are large sums of money. Firms have limited liquidity to pay these sums and usually borrow from banks at non-negligible interest rates. Such borrowing introduces non-quasilinear preferences over transfers. Moreover, income effects are present in many standard settings and should not be overlooked. By analyzing revenue maximizing auctions without any functional form assumption on preferences, we carry out a "detail-free" mechanism design of our problem. Along with the robustness to distributional assumptions, this brings in another dimension of robustness to our results.

We briefly discuss what drives our surprisingly robust results. The literature on revenue maximizing auctions (single or multiple objects) considers only incentive and participation constraints: Bayesian incentive compatibility and interim individual rationality. We have departed from this by considering stronger former incentive and participation constraints: strategy-proofness and ex-post individual rationality. [2] This is consistent with our objective of providing a robust recommendation of mechanism in our setting. Further, it allows us to stay away from prior-based analysis.

The main drivers for our results are equal treatment of equals, no subsidy, and no wastage. Equal treatment of equals is a natural weak axiom to impose on mechanisms since it only requires a minimal amount of fairness. For instance, Deb and Pai (2016) cite many legal implications of violating such symmetric treatment of bidders in auctions. It is also consistent with some fundamental laws of equity. [3] The no subsidy axiom is standard in almost all

1983).

[2] There is also a large literature (discussed in Section 6) on single agent revenue maximizing mechanism, commonly referred to as the screening problem, where the two solution concepts coincide.

[3] Quoting Aristotle,

> *Justice is considered to mean equality. It does not mean equality - but equality for those who are equal, and not for all.*

auction formats. Further, we show some possibility to weaken it (by using no bankruptcy) in the positive income effect domain of preferences.

Perhaps the most controversial axiom in our results is no wastage. An important aspect of Myerson's optimal auction result for single object sale (in quasilinear domain) is that a Vickrey auction with an *optimally* chosen reserve price is expected revenue-maximizing (Myerson, 1981). In the multi-object auction environment, the structure of incentive and participation constraints (even in the quasilinear environment) becomes quite messy. Among many other difficulties in extending Myerson's result to the multi-object auction environment, one major difficulty is finding the *optimal* reserve prices.

Our no wastage axiom escapes this particular difficulty. Note that it is unclear that imposing no wastage gets rid of all the difficulties in finding an optimal multi-object auction. To our knowledge, the literature is silent on this issue. No wastage is a mild efficiency restriction on the set of allocation rules, and still leaves us with a large set of allocation rules to optimize. Undoubtedly, reserve prices are used in many auctions in real-life. However, the objective of such reserve prices are unclear in many settings. For instance, when governments sell natural resources using auctions, unsold objects and low revenues create a lot of controversies in the public. Moreover, often, the unsold objects are resold - for instance, Indian spectrum auctions reported a large number of unsold spectrum blocks and low revenues in 2016, and all of them are supposed to be re-auctioned. [4] This clearly indicates that a primary mandate in resource allocation by Governments is to not waste any of the available resources and maximize revenue (for redistribution to other welfare programs) from selling the resources. Hence, no wastage seems to be an appropriate axiom in such settings. Our result shows the implication of such a minimal form of efficiency on revenue-maximizing multi-object auction design. In Section 4.3, we give two further motivating examples which seem to fit most of our assumptions in the model.

Our result relies on the fact that the mechanism selects a Walrasian equilibrium allocation. Further, the desirable properties and the no subsidy (or, no bankruptcy) axiom impose nice structure on the set of mechanisms. We exploit these to give simple proofs of our two main results. This is an added advantage of our results.

Finally, the MWEP mechanism can be implemented as a simple ascending price auction - for quasilinear type spaces, see Demange et al. (1986), and for non-quasilinear type spaces, see Morimoto and Serizawa (2015). Such ascending auctions have distinct advantages of practical implementation and are often used in practice - the main selling point seems to be

[4] See the following news article: http://www.livemint.com/Industry/xt5r4Zs5RmzjdwuLUdwJMI/Spectrum-auction-ends-after-lukewarm-response-from-telcos.html

their efficiency properties (Ausubel et al., 2002). Our results provide a *revenue maximizing* and robust foundation for such ascending price auctions.

2 The preferences

A seller has a m objects to sell, denoted by $M := \{1, \ldots, m\}$. There are $n > m$ agents (buyers), denoted by $N := \{1, \ldots, n\}$. Each agent can receive at most one object (unit-demand preference). Let $L \equiv M \cup \{0\}$, where 0 is the null object, which is assigned to any agent who does not receive any object in M - thus, the null object can be assigned to more than one agent.

The (consumption) bundles of every agent is the set $L \times \mathbb{R}$, where a typical element $z \equiv (a, t)$ corresponds to object $a \in L$ and transfer $t \in \mathbb{R}$. Throughout the paper, t will be interpreted as the amount *paid* by an agent to the designer, i.e., a negative t will indicate that the agent receives a transfer of $-t$.

A preference ordering R_i (of agent i) over $L \times \mathbb{R}$, with strict part P_i and indifference part I_i, is **classical** if it satisfies the following assumptions:

1. **Money monotonicity.** for every $t > t'$ and for every $a \in L$, we have $(a, t')\ P_i\ (a, t)$.
2. **Desirability of objects.** for every t and for every $a \in M$, $(a, t)\ P_i\ (0, t)$.
3. **Continuity.** for every $z \in L \times \mathbb{R}$, the sets $\{z' : z'\ R_i\ z\}$ and $\{z' : z\ R_i\ z'\}$ are closed.
4. **Possibility of compensation.** for every $z \in L \times \mathbb{R}$ and for every $a \in L$, there exists t and t' such that $z\ R_i\ (a, t)$ and $(a, t')\ R_i\ z$.

A *quasilinear* preference is classical. In particular, a preference R_i is quasilinear if there exists $v \in \mathbb{R}^{|L|}$ such that for every $a, b \in L$ and $t, t' \in \mathbb{R}$, $(a, t)\ R_i\ (b, t')$ if and only if $v_a - t \geq v_b - t'$. Usually, v is referred to as the valuation of the agent, and v_0 is normalized to 0. The idea of valuation may be generalized as follows for non-quasilinear preferences.

Definition 1 *The* **valuation** *at a classical preference R_i for object $a \in L$ with respect to bundle z is defined as $V^{R_i}(a; z)$, which uniquely solves $(a, V^{R_i}(a; z))\ I\ z$.*

A straightforward consequence of our assumptions is that for every $a \in L$, for every $z \in L \times \mathbb{R}$, and for every classical preference R_i, the valuation $V^{R_i}(a, z)$ exists. For any R and for any $z \in L \times \mathbb{R}$, the valuations at bundle z with preference R is a vector in $\mathbb{R}^{|L|}$.

An illustration of the valuation is shown in Figure 1. In the figure, the horizontal lines cor-

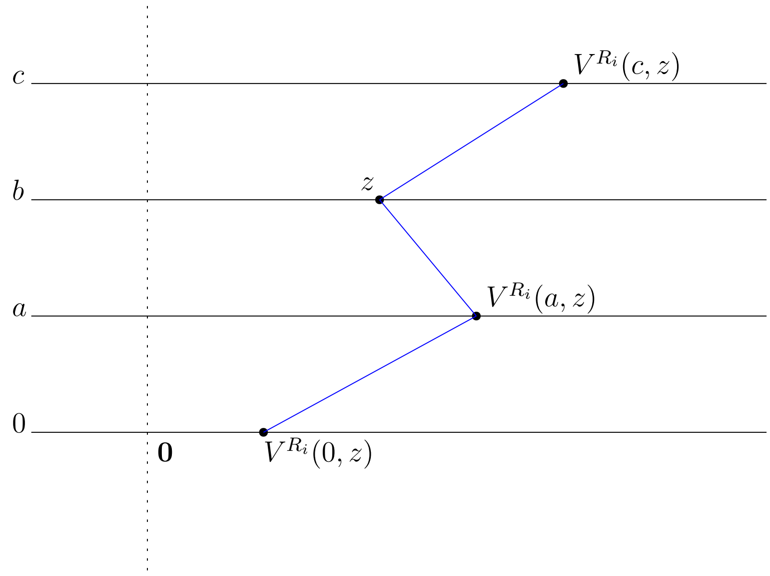

Figure 1: Valuation at a preference

respond to objects: $L = \{0, a, b, c\}$. The horizontal lines indicate transfer amounts. Hence, the four lines are the entire set of consumption bundles of the agent. A preference R_i can be described by drawing (non-intersecting) indifference vectors through these consumption bundles (lines). One such indifference vector passing through z is shown in Figure 1. This indifference vector actually consists of four points: $V^{R_i}(0,z), V^{R_i}(a,z), z, V^{R_i}(c,z)$ as shown. Parts of the curve in Figure 1 which lie between the consumption bundle lines is useless and has no meaning - it is only displayed for convenience.

Our modeling of preferences captures income effects even though we do not model income explicitly. Indeed, as transfer changes, the income levels of agents change and this is automatically reflected in the preferences.

2.1 Desirable mechanisms

Let $\mathcal{R}^C$ denote the set of all classical preferences and $\mathcal{R}^Q$ denote the set of all quasilinear preferences. We will consider an arbitrary class of classical type space $\mathcal{R} \subseteq \mathcal{R}^C$ - we will put specific restrictions on $\mathcal{R}$ later. The type of agent i is a preference $R_i \in \mathcal{R}$. A type profile is just a profile of preferences $R \equiv (R_1, \ldots, R_n)$. The usual notations R_{-i} and $R_{-N'}$ will denote a preference profile without the preference of agent i and without the preferences of agents in $N' \subseteq N$ respectively.

An *object allocation* is an n-tuple $(a_1, \ldots, a_n) \in L^n$, where $a_i \neq a_j$ for all i, j with $a_i, a_j \neq 0$. The set of all object allocations is denoted by A. A (feasible) allocation is an

n-tuple $((a_1,t_1),\ldots,(a_n,t_n)) \in A\times\mathbb{R}$, where (a_i,t_i) is the allocation of agent i. Let Z denote the set of all feasible allocations. For every allocation $(z_1,\ldots,z_n)\in Z$, we will denote by z_i the allocation of any agent i.

A **mechanism** is a map $f:\mathcal{R}^n \to Z$. At a preference profile $R\in\mathcal{R}^n$, we denote the allocation of agent i in mechanism f as $f_i(R)\equiv(a_i(R),t_i(R))$, where $a_i(R)$ and $t_i(R)$ are respectively the object allocated to agent i and the transfer paid by agent i at preference profile R.

Definition 2 *A mechanism $f:\mathcal{R}^n\to Z$ is* **desirable** *if it satisfies the following properties:*

1. **Strategy-proof or dominant strategy incentive compatibility.** *for every $i\in N$, for every $R_{-i}\in\mathcal{R}^{n-1}$, and for every $R_i,R_i'\in\mathcal{R}$, we have*
$$f_i(R_i,R_{-i})\ R_i\ f_i(R_i',R_{-i}).$$
2. **Ex-post individual rationality (IR).** *for every $i\in N$, for every $R\in\mathcal{R}^n$, we have $f_i(R)\ R_i\ (0,0)$.*
3. **Equal treatment of equals (ETE).** *for every $i,j\in N$, for every $R\in\mathcal{R}^n$ with $R_i=R_j$, we have $f_i(R)\ I_i\ f_j(R)$.*
4. **No wastage (NW).** *for every $R\in\mathcal{R}^n$ and for every $a\in M$, there exists some $i\in N$ such that $a_i(R)=a$.*

Out of the four properties of a desirable mechanism, strategy-proofness and IR are standard constraints imposed on a mechanism. Most of the literature considers Bayesian incentive compatibility and interim individual rationality. As a consequence, one ends up working in the "reduced-form" problems (Border, 1991), and one needs to put additional constraints, commonly referred to as "Border constraints", in the optimization program. The multi-object analogues of the Border constraints are difficult to characterize (Che et al., 2013). Working with strategy-proof and ex-post IR, we get around these problems. [5]

ETE is a very mild form of fairness requirement. It states that two agents with identical preferences must be assigned bundles to which they should be indifferent. As argued in the introduction, such minimal notion of fairness is often required by law. The desirability of NW is debatable, and the readers are referred back to the Introduction section for more discussions on this. Besides desirability, for some of our results, we will require some form of restrictions on payments.

[5] On a related note, in the single object case, there is strong equivalence between the set of strategy-proof and Bayesian incentive compatible mechanisms (Manelli and Vincent, 2010; Gershkov et al., 2013). But this equivalence is lost in the multi-object problem.

Definition 3 *A mechanism $f : \mathcal{R}^n \to Z$ satisfies* **no subsidy** *if for every $R \in \mathcal{R}^n$ and for every $i \in N$, we have $t_i(R) \geq 0$.*

A mechanism $f : \mathcal{R}^n \to Z$ satisfies **no bankruptcy** *if there exists $\ell \leq 0$ such that for every $R \in \mathcal{R}^n$, we have $\sum_{i \in N} t_i(R) \geq \ell$.*

No subsidy can be considered desirable to exclude "fake" agents, who participate in auctions just to take away available subsidy. As was discussed earlier, it is an axiom satisfied by most standard auctions in practice. Obviously, no subsidy is a stronger property than no bankruptcy. Both these properties are motivated by settings where the auctioneer has no or limited means to finance the auction participants. no bankruptcy allows the auctioneer to collectively finance the bidders up to a certain limit $-\ell$.

3 The Walrasian equilibrium

In this section, we define the notion of a Walrasian equilibrium, and use it to define a desirable mechanism. A price vector $p \in \mathbb{R}^{|L|}_{+}$ defines a price for every object with $p_0 = 0$. At any price vector p, let $D(R_i, p) := \{a \in L : (a, p_a) \ R_i(b, p_b) \ \forall \ b \in L\}$ denote the demand set of agent i with preference R_i at price vector p. [6]

Definition 4 *An object allocation $(a_1, \ldots, a_n)$ and a price vector p is a* **Walrasian equilibrium** *at a preference profile $R \in \mathcal{R}^n$ if*

1. *$a_i \in D(R_i, p)$ for all $i \in N$ and*

2. *for all $a \in M$ with $a_i \neq a$ for all $i \in N$, we have $p_a = 0$.*

In this case, we refer to p as a **Walrasian equilibrium price vector** *at R.*

Since we assume $n > m$, the first condition of Walrasian equilibrium implies that for all $a \in M$, we have $a_i = a$ for some $i \in N$.

A price vector p is a **minimum Walrasian equilibrium price vector** at preference profile R if for every Walrasian equilibrium price vector p' at R, we have $p_a \leq p'_a$ for all $a \in L$. Demange and Gale (1985) prove that if R is a profile of classical preferences, then a Walrasian equilibrium exists at R, and the set of Walrasian equilibrium price vectors forms a

[6] A more traditional definition of demand set using the notion of a budget set is also possible. Here, we define the budget set of each agent at price vector p as $B(p) := \{(a, p_a) : a \in L\}$ and the demand set of agent i is just the maximal bundles in the budget set according to preference R_i.

lattice with a unique minimum and a unique maximum. We denote the minimum Walrasian equilibrium price vector at R as $p^{min}(R)$. Notice that if $n > m$, then for every $a \in A$, we have $p^{min}_a(R) > 0$. [7]

We give an example to illustrate the notion of minimum Walrasian equilibrium price vector. Suppose $N = \{1, 2, 3\}$ and $M = \{a, b\}$. Figure 2 shows some indifference vectors of a preference profile $R \equiv (R_1, R_2, R_3)$ and the corresponding minimum Walrasian equilibrium price vector $p^{min}(R) \equiv p^{min} \equiv (p^{min}_0 = 0, p^{min}_a, p^{min}_b)$.

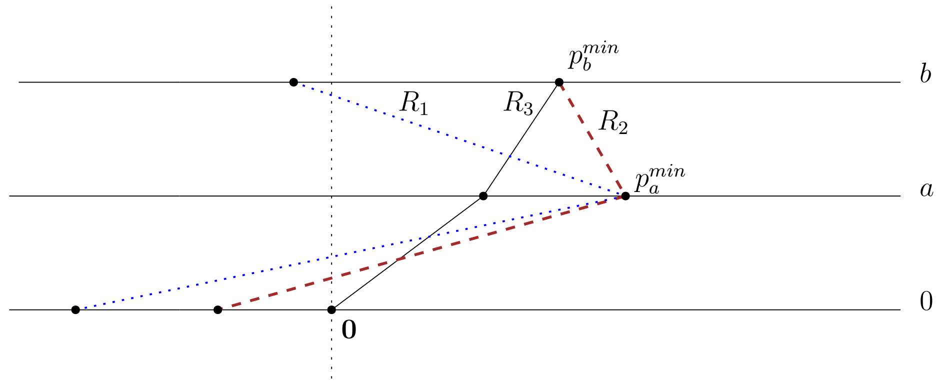

Figure 2: The minimum Walrasian equilibrium price vector

First, note that

$$D_1(R_1, p^{min}) = \{a\}, D_2(R_2, p^{min}) = \{a, b\}, D_3(R_3, p^{min}) = \{0, b\}.$$

Hence, a Walrasian equilibrium is the allocation where agent 1 gets object a, agent 2 gets object b, and agent 3 gets the null object at the price vector p^{min}. Also, p^{min} is the minimum such Walrasian equilibrium price vector. To see this, if price of object a only is decreased, then both agents 1 and 2 will demand only object a, which contradicts Walrasian equilibrium. But if price of object b is decreased, then no agent will demand the null object. Hence, this will contradict Walrasian equilibrium again. This means that p^{min} is the minimum Walrasian equilibrium price vector.

3.1 A desirable mechanism

In this section, we present a desirable mechanism satisfying no subsidy. The mechanism picks a minimum Walrasian equilibrium allocation at every profile of preferences. Although the

[7] To see this, suppose $p^{min}_a(R) = 0$, then any agent $i \in N$ who is not assigned in the Walrasian equilibrium will prefer $(a, 0)$ to $(0, 0)$ contradicting the fact that he is assigned a bundle from his demand set. Indeed, this argument holds for any Walrasian equilibrium price vector.

minimum Walrasian equilibrium price vector is unique at every preference profile, there may be multiple supporting object allocation - all these object allocations must be indifferent to all the agents. To handle this multiplicity problem, we introduce some notation. Let $Z^{min}(R)$ denote the set of all allocations at a minimum Walrasian equilibrium at preference profile R. Note that if $((a_1, \ldots, a_n), p) \in Z^{min}(R)$ then $p = p^{min}(R)$.

DEFINITION **5** *A mechanism* $f^{min} : \mathcal{R}^n \to Z$ *is a* **minimum Walrasian equilibrium price (MWEP)** *mechanism if*

$$f^{min}(R) \in Z^{min}(R) \; \forall \; R \in \mathcal{R}^n.$$

Demange and Gale (1985) showed that every MWEP mechanism is strategy-proof. Clearly, it also satisfies individual rationality, no subsidy, and ETE. We document this fact below.

FACT **1 (Demange and Gale (1985); Morimoto and Serizawa (2015))** *Every MWEP mechanism is desirable and satisfies no subsidy.*

Although it is difficult to describe the set of desirable mechanisms satisfying no subsidy, there are *discontinuous* desirable mechanisms satisfying no subsidy even in the restricted domain of quasilinear preferences - see an example following Remark 1 in Tierney (2016). Indeed, the set of all desirable mechanisms satisfying no subsidy seems quite complicated to describe in the quasilinear domain of preferences. Our main result shows that every MWEP mechanism is revenue-optimal in a strong sense in the class of desirable and no subsidy mechanisms.

4 The results

In this section, we formally state our results. The proofs of our results will be presented in Section 5. Before we state our result, we define some extra notations and the richness in type space necessary for our results. For any mechanism $f : \mathcal{R}^n \to Z$, we define the **revenue** at preference profile $R \in \mathcal{R}^n$ as

$$\text{REV}^f(R) := \sum_{i \in N} t_i(R).$$

The domain of preferences (type space) that we consider for our first result is the following. [8]

[8] For every price vector $p \in \mathbb{R}_+^{|L|}$, we assume that $p_0 = 0$. Further, for any pair of price vectors $p, \hat{p} \in \mathbb{R}_+^{|L|}$, we write $p > \hat{p}$ if $p_a > \hat{p}_a$ for all $a \in M$.

Definition **6** *A domain of preferences $\mathcal{R}$ is* **rich** *if for all $a \in M$ and for every price vector $\hat{p}$ with $\hat{p}_a > 0, \hat{p}_b = 0$ for all $b \neq a$ and for every price vector $p > \hat{p}$, there exists $R_i \in \mathcal{R}$ such that*

$$D(R_i, \hat{p}) = \{a\} \text{ and } D(R_i, p) = \{0\}.$$

In words, richness requires that if there are two price vectors $p > \hat{p}$, where the only positive price object at $\hat{p}$ is object a, then there is a preference ordering where the agent only demands a at $\hat{p}$ and demands nothing at p. The richness can be trivially satisfied if a domain contains the quasilinear domain - for instance, consider a quasilinear preference where we pick a value for object a between $\hat{p}_a$ and p_a and value for all other objects arbitrarily close to zero. Later, we show that this richness condition can be satisfied for many non-quasilinear preferences also.

Figure 3 illustrates this notion of richness with two objects a and b - two possible price vectors p and $\hat{p}$ are shown and two indifference vectors of a preference R_i are shown such that $D(R_i, p) = \{0\}$ and $D(R_i, \hat{p}) = \{a\}$.

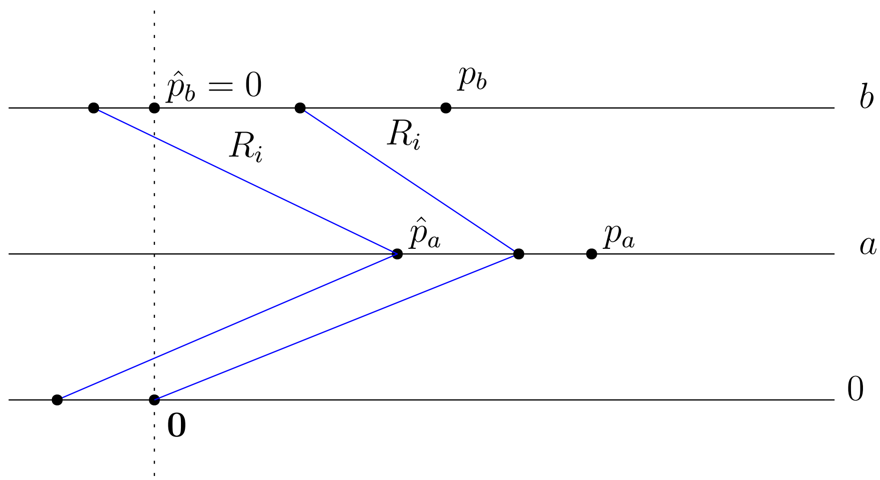

Figure 3: Illustration of richness

We are ready to state one of our main results now.

Theorem **1** *Suppose $\mathcal{R}$ is a rich domain of preferences. For every desirable mechanism $f : \mathcal{R}^n \to Z$ satisfying no subsidy, the following holds:*

$$\text{Rev}^{f^{min}}(R) \geq \text{Rev}^{f}(R) \quad \forall\ R \in \mathcal{R}^n,$$

where $f^{min} : \mathcal{R}^n \to Z$ is an MWEP mechanism.

All the omitted proofs are in Section 5.

Theorem 1 clearly implies that even if we do *expected* revenue maximization with respect to *any* prior on the preferences of agents, we will only get an MWEP mechanism among the class of desirable and no subsidy mechanisms.

4.1 Richness and income effects

We now discuss some specific domains where our richness condition holds. We also show how Theorem 1 can be strengthened in some specific rich domains.

DEFINITION **7** *A preference R_i satisfies* **positive income effect** *if for every $a, b \in L$ and for every t, t' with $t < t'$ and $(a,t)\ I_i\ (b,t')$, we have*

$$(a, t-\delta)\ P_i\ (b, t'-\delta) \quad \forall\ \delta > 0.$$

A preference R_i satisfies **non-negative income effect** *if for every $a, b \in L$ and for every t, t' with $t < t'$ and $(a,t)\ I_i\ (b,t')$, we have*

$$(a, t-\delta)\ R_i\ (b, t'-\delta) \quad \forall\ \delta > 0.$$

Let $\mathcal{R}^{++}$ and $\mathcal{R}^{+}$ denote the set of all positive income effect and non-negative income effect domain of preferences respectively.

Positive (non-negative) income effects are natural restrictions to impose in settings where the objects are normal goods. Our next claim shows that the richness condition is satisfied in a variety of type spaces containing positive income effect preferences. Since the proof is straightforward, we skip it.

CLAIM **1** *A domain of preferences $\mathcal{R}$ satisfies richness if any of the following conditions holds: (1) $\mathcal{R} \supseteq \mathcal{R}^Q$; (2) $\mathcal{R} \supseteq \mathcal{R}^+$; (3) $\mathcal{R} \supseteq \mathcal{R}^{++}$; (4) $\mathcal{R} \supseteq \mathcal{R}^C \setminus \mathcal{R}^Q$.*

Next, we show that if the domain contains all the positive income effect preferences, then our result can be strengthened - we can replace no subsidy in Theorem 1 by no bankruptcy.

THEOREM **2** *Suppose $\mathcal{R} \supseteq \mathcal{R}^+$. For every desirable mechanism $f : \mathcal{R}^n \to Z$ satisfying no bankruptcy, the following holds:*

$$\text{REV}^{f^{min}}(R) \geq \text{REV}^f(R) \quad \forall\ R \in \mathcal{R}^n,$$

where $f^{min} : \mathcal{R}^n \to Z$ is an MWEP mechanism.

4.2 Pareto efficiency

Since no wastage is a minimal form of efficiency axiom, it is natural to explore the implications of stronger forms of efficiency. We now discuss the implications of Pareto efficiency in our problem and relate it to our results. Before we formally define it, we must state the obvious fact that no wastage is a much weaker but more testable axiom in practice than Pareto efficiency. Our results establish that even if an auctioneer maximizes her revenue with this weak form of efficiency, it will be forced to use a Pareto efficient mechanism.

DEFINITION **8** *A mechanism* $f : \mathcal{R}^n \to Z$ *is* **Pareto efficient** *if at every preference profile* $R \in \mathcal{R}^n$*, there exists no allocation* $((\hat{a}_1, \hat{t}_1), \ldots, (\hat{a}_n, \hat{t}_n))$ *such that*

$$(\hat{a}_i, \hat{t}_i) \; R_i \; f_i(R) \qquad \forall \; i \in N$$
$$\sum_{i \in N} \hat{t}_i \geq \textsc{Rev}^f(R),$$

with either the second inequality holding strictly or some agent i *strictly preferring* $(\hat{a}_i, \hat{t}_i)$ *to* $f_i(R_i)$.

The above definition is the appropriate notion of Pareto efficiency in this setting: (a) the first set of inequalities just say that no agent i prefers the allocation $(\hat{a}_i, \hat{t}_i)$ to that of the mechanism and (b) the second inequality ensures that the auctioneer's revenue is not better in the proposed allocation. Without the second inequality, there is always an allocation where some money is distributed to all the agents to make them better off than the allocation in the mechanism.

The MWEP mechanism is Pareto efficient - first welfare theorem, see also Morimoto and Serizawa (2015). An immediate corollary of our results is the following.

COROLLARY **1** *Let* $f : \mathcal{R}^n \to Z$ *be a desirable mechanism. If* $\mathcal{R}$ *is rich and* f *satisfies no subsidy, then consider the following statements.*

1. $f = f^{min}$.
2. $\textsc{Rev}^f(R) \geq \textsc{Rev}^{f'}(R)$ *for any desirable mechanism* $f' : \mathcal{R}^n \to Z$ *satisfying no subsidy.*
3. f *is Pareto efficient.*

Statements (1) and (2) are equivalent, and each of them imply Statement (3).

If $\mathcal{R} = \mathcal{R}^+$ *and* f *satisfies no bankruptcy, then the same equivalence between (1) and (2) holds with no subsidy weakened to no bankruptcy in (2), and each of them still imply (3).*

In other words, even if the auctioneer maximizes her revenue among the set of all desirable mechanisms satisfying no subsidy (or no bankruptcy in the positive income effect domain), it will be forced to use a Pareto efficient mechanism. Hence, we get Pareto efficiency as a corollary without imposing it explicitly.

If Pareto efficiency is explicitly imposed, then the following two results are known in the literature, and using them, we can strengthen Corollary 1 further.

1. In the quasilinear domain, every strategy-proof and Pareto efficient mechanism is a Groves mechanism (Holmstrom, 1979). Imposing individual rationality and no subsidy immediately implies that the pivotal or the Vickrey-Clarke-Groves (VCG) mechanism is the unique strategy-proof mechanism satisfying Pareto efficiency, individual rationality, and no subsidy - notice that equal treatment of equals is not needed for this result and no wastage is implied by Pareto efficiency. The MWEP mechanism coincides with the VCG mechanism in the quasilinear domain.

2. In the classical domain $\mathcal{R}^C$ (containing *all* classical preferences), the MWEP mechanism is the unique mechanism satisfying strategy-proofness, individual rationality, Pareto efficiency, and no subsidy (Morimoto and Serizawa, 2015) - again, equal treatment of equals is not needed for this result and no wastage is implied by Pareto efficiency.

Both these results imply the following strengthening of Corollary 1 in quasilinear and classical domains.

COROLLARY **2** *Let $f : \mathcal{R}^n \to Z$ be a desirable mechanism. If $\mathcal{R} \in \{\mathcal{R}^Q, \mathcal{R}^C\}$ and f satisfies no subsidy, then the following statements are equivalent.*

1. $f = f^{min}$.

2. $\textsc{Rev}^f(R) \geq \textsc{Rev}^{f'}(R)$ *for any desirable mechanism $f' : \mathcal{R}^n \to Z$ satisfying no subsidy.*

3. *f is Pareto efficient.*

4.3 Discussions on applicability of the results

As discussed in the introduction, our results are driven by a particular set of assumptions we have made in the paper, which are different from the literature. Here, we give two real-life examples of auctions, where most of the assumptions made in the paper appear to make sense.

Indian Premier League Auctions. A professional cricket league, called the *Indian Premier League (IPL)* was started in India in 2007. [9] Eight Indian cities were chosen and it was decided to have a team from each of those cities (i.e., eight heterogeneous objects were sold). An auction was held to sell these teams to interested owners (bidders). The auctions, whose details are not available in public domain, fetched more than 700 million US Dollars in revenue to IPL. Clearly, it does not make sense for two teams to have the the same owner - so, the unit demand assumption in our model is satisfied in this problem. The huge sums of bids implied that most of these teams were financed out of loans from banks, which implies non-quasilinear preferences of bidders. Further, when IPL was starting out, it must be interested in starting with teams in as many cities as possible - else, it would have sent a wrong signal to its future prospects. Indeed, all the teams were sold with high bid prices. So, a natural objective for IPL seems to be revenue maximization with no wastage. Finally, as is common in such settings, IPL did not subsidize any bidders.

Google's Sponsored Search Auction. Google sells billions of dollars worth of keywords using auctions for advertisement slots (Edelman et al., 2007). Usually, each advertisement slot is awarded a unique bidder - so, the unit demand assumption is satisfied. Google does not use reserve prices and sells all the slots to advertisers. So, it is fair to say that Google aims to maximize revenue from its sale of advertisement slots under no wastage. The bidders are usually given a fixed budget to work with, and this results in an extreme form of non-quasilinearity. This has started a big literature on auctions with budget constraints in the computer science community (Ashlagi et al., 2010; Dobzinski et al., 2012; Lavi and May, 2012). Finally, Google does not subsidize any of its bidders.

These examples reinforce the fact that even though a precise description to revenue maximizing multi-object auction is impossible in many settings, for a variety of problems where no wastage makes sense, the MWEP mechanism is a strong candidate.

In both these examples, the seller is not the Government. It makes more sense for such a seller to maximize her revenue. Corollaries 1 and 2 establish that even if such a seller maximizes her revenue, under the assumptions of our model, she would be forced to pick a Pareto efficient mechanism.

[9] Interested readers can read the Wiki entry for IPL: https://en.wikipedia.org/wiki/Indian_Premier_League and a news article here: http://content-usa.cricinfo.com/ipl/content/current/story/333193.html.

5 The proofs

In this section, we present all the proofs. The proofs, though tedious and far from trivial, do not require any sophisticated mathematical tool. This is an added advantage of our approach, and makes the results even more surprising. The proofs use the following fact very crucially: the MWEP mechanism chooses a Walrasian equilibrium outcome.

5.1 Proof of Theorem 1

We start with a series of Lemmas before providing the main proof. Throughout, we assume that $\mathcal{R}$ is a rich domain of preferences and f is a desirable mechanism satisfying no subsidy on $\mathcal{R}^n$. For the lemmas, we need the following definition. A preference R_i is (a,t)-**favoring** for $t > 0$ and $a \in M$ if for price vector p with $p_a = t, p_b = 0$ for all $b \neq a$, we have $D(R_i, p) = \{a\}$. An equivalent way to state this is that R_i is (a,t)-favoring for $t > 0$ and $a \in M$ if $V^{R_i}(b,(a,t)) < 0$ for all $b \neq a$.

Lemma **1** *For every preference profile R, for every $i \in N$ with $f_i(R) \neq 0$, and for every R'_i such that R'_i is an $f_i(R)$-favoring preference, we have $f_i(R'_i, R_{-i}) = f_i(R)$.*

Proof: If $a_i(R'_i, R_{-i}) = a_i(R)$, then strategy-proofness implies $t_i(R'_i, R_{-i}) = t_i(R)$, and we are done. Suppose $a = a_i(R) \neq a_i(R'_i, R_{-i}) = b$. By strategy-proofness,

$$\big[(b, t_i(R'_i, R_{-i}))\ R'_i\ (a, t_i(R))\big] \Rightarrow\ \big[t_i(R'_i, R_{-i}) \leq V^{R'_i}(b,(a,t_i(R)))\big].$$

Since R'_i is $(a, t_i(R))$-favoring, we must have $V^{R'_i}(b,(a,t_i(R))) < 0$. This implies that $t_i(R'_i, R_{-i}) < 0$, which is a contradiction to no subsidy. ■

Lemma **2** *For every preference profile R and for every $i \in N$ with $f_i(R) \neq 0$, there is no $j \neq i$ such that R_j is $f_i(R)$-favoring.*

Proof: Assume for contradiction that there is $j \neq i$ such that R_j is $f_i(R)$-favoring. Consider $R'_i \equiv R_j$. By equal treatment of equals $f_i(R'_i, R_{-i})\ I_j\ f_j(R'_i, R_{-i})$. Also, by Lemma 1, $f_i(R'_i, R_{-i}) = f_i(R)$. Hence, $f_i(R)\ I_j\ f_j(R'_i, R_{-i})$. Note that $a = a_i(R) = a_i(R'_i, R_{-i}) \neq a_j(R'_i, R_{-i}) = b$. Then, $t_j(R) = V^{R_j}(b, f_i(R)) < 0$, where the strict inequality followed from the fact that R_j is $f_i(R)$-favoring and $b \neq a_i(R)$. But this contradicts no subsidy. ■

Lemma **3** *For every preference profile R, for every $i \in N$, for every (a,t) with $a = a_i(R) \neq 0$ and $t > 0$, if there exists $j \neq i$ such that R_j is (a,t)-favoring, then $t_i(R) > t$.*

Proof: Suppose $t_i(R) \leq t$. Since R_j is (a,t)-favoring, $t_i(R) \leq t$ implies that R_j is also $f_i(R) \equiv (a, t_i(R))$-favoring. This is a contradiction to Lemma 2. ■

For the proof, we use a slightly stronger version of (a,t)-favoring preference.

DEFINITION **9** *For every bundle (a,t) with $t > 0$ and for every $\epsilon > 0$, a preference $R_i \in \mathcal{R}$ is a $(a,t)^\epsilon$-favoring preference if it is a (a,t)-favoring preference and*

$$V^{R_i}(a;(0,0)) < t + \epsilon$$
$$V^{R_i}(b;(0,0)) < \epsilon \;\forall\; b \in M \setminus \{a\}.$$

The following lemma shows that if $\mathcal{R}$ is rich, then $(a,t)^\epsilon$-favoring preferences exist for every (a,t) and ϵ.

LEMMA **4** *Suppose $\mathcal{R}$ is rich. Then, for every bundle (a,t) with $t > 0$ and for every $\epsilon > 0$, there exists a preference $R_i \in \mathcal{R}$ such that it is $(a,t)^\epsilon$-favoring.*

Proof: Define $\hat{p}$ as follows:

$$\hat{p}_a = t, \quad \hat{p}_b = 0 \quad \forall\; b \neq a.$$

Define p as follows:

$$p_a = t + \epsilon, \quad p_0 = 0, \quad p_b = \epsilon \;\; \forall\; b \in M \setminus \{a\}.$$

By richness, there exists R_i such that $D(R_i, \hat{p}) = \{a\}$ and $D(R_i, p) = \{0\}$. But this implies that R_i is (a,t)-favoring and

$$V^{R_i}(a;(0,0)) < t + \epsilon$$
$$V^{R_i}(b;(0,0)) < \epsilon \;\forall\; b \in M \setminus \{a\}.$$

Hence, R_i is $(a,t)^\epsilon$-favoring. ■

We will now prove Theorem 1 using these four lemmas.

PROOF OF THEOREM 1

Proof: Fix a desirable mechanism $f : \mathcal{R}^n \to Z$ satisfying no subsidy, where $\mathcal{R}$ is a rich domain of preferences. Fix a preference profile $R \in \mathcal{R}^n$. Let $(z_1, \ldots, z_n) \equiv f^{min}(R)$ be the allocation chosen by the MWEP mechanism at R. For simplicity of notation, we will denote $z_j \equiv (a_j, p_j)$, where $p_j \equiv p^{min}_{a_j}(R)$, for all $j \in N$. We prove that $f_i(R)\; R_i\; z_i$ for all $i \in N$. Note that by the Walrasian equilibrium property $z_i\; R_i\; (a_i(R), p_{a_i(R)})$ for all $i \in N$. Hence, proving the above property will imply that for all $i \in N$, $(a_i(R), t_i(R))\ R_i\ (a_i(R), p_{a_i(R)})$, which in turn implies that $t_i(R) \leq p_{a_i(R)}$. This means that

$$\textsc{Rev}^f(R) = \sum_{i \in N} t_i(R) \leq \sum_{i \in N} p_{a_i(R)} = \textsc{Rev}^{f^{min}}(R).$$

To prove that $f_i(R)\ R_i\ z_i$ for all $i \in N$, assume for contradiction that there is some agent, without loss of generality agent 1, such that $z_1\ P_1\ f_1(R)$. We first construct a finite sequence of agents and preferences: $(1, R'_1), (2, R'_2), \ldots, (n, R'_n)$ such that for every $k \in \{1, \ldots, n\}$, [10]

1. $z_k\ P_k\ f_k(R)$ if $k = 1$ and $z_k\ P_k\ f_k(R'_{N_{k-1}}, R_{-N_{k-1}})$ if $k > 1$, where $N_{k-1} \equiv \{1, \ldots, k-1\}$.

2. $a_k \neq 0$,

3. R'_k is z_k^ϵ-favoring for some $\epsilon > 0$ but arbitrarily close to zero.

Now, we construct this sequence inductively.

Step 1 - Constructing $(1, R'_1)$. Pick $\epsilon > 0$ but arbitrarily close to zero and consider a z_1^ϵ-favoring preference R'_1 - by Lemma 4, such R'_1 can be constructed. By our assumption, $z_1\ P_1\ f_1(R)$. Suppose $a_1 = 0$. Then, $z_1 = (0, 0)\ P_1\ f_1(R)$, which contradicts individual rationality. Hence, $a_1 \neq 0$.

Step 2 - Constructing (k, R'_k) for $k > 1$. We proceed inductively - suppose, we have already constructed $(1, R'_1), \ldots, (k-1, R'_{k-1})$ satisfying Properties (1), (2), and (3). Consider agent j such that $a_j(R'_{N_{k-1}}, R_{-N_{k-1}}) = a_{k-1}$.

If $j = k-1$, then individual rationality implies that

$$t_{k-1}(R'_{N_{k-1}}, R_{-N_{k-1}}) \leq V^{R'_{k-1}}(a_{k-1}, (0, 0)) < p_{k-1} + \epsilon.$$

Further, by our induction hypothesis, $z_{k-1}\ P_{k-1}\ f_{k-1}(R'_{N_{k-2}}, R_{-N_{k-2}})$, and we get

$$p_{k-1} < V^{R_{k-1}}(a_{k-1}, f_{k-1}(R'_{N_{k-2}}, R_{-N_{k-2}})).$$

Since ϵ is arbitrarily close to zero, we get $t_{k-1}(R'_{N_{k-1}}, R_{-N_{k-1}}) < V^{R_{k-1}}(a_{k-1}; f_{k-1}(R'_{N_{k-2}}, R_{-N_{k-2}}))$. But this implies that $f_{k-1}(R'_{N_{k-1}}, R_{-N_{k-1}})\ P_{k-1}\ f_{k-1}(R'_{N_{k-2}}, R_{-N_{k-2}})$, which contradicts strategy-proofness. Hence, $j \neq k-1$.

If $j \in N_{k-2}$, then by individual rationality, we get $t_j(R'_{N_{k-1}}, R_{-N_{k-1}}) \leq V^{R'_j}(a_{k-1}; (0, 0)) < \epsilon$. Since ϵ is arbitrarily close to zero, we get $t_j(R'_{N_{k-1}}, R_{-N_{k-1}}) < \epsilon < p_{k-1}$. But this is a contradiction to Lemma 3 because $j \neq k-1$.

[10]Here, the agents are labeled $1, \ldots, n$ in sequence without loss of generality.

Thus, we have established $j \notin N_{k-1}$. Hence, we denote $j \equiv k$, and note that

$$z_k \ R_k \ z_{k-1} \ P_k \ f_k(R'_{N_{k-1}}, R_{-N_{k-1}}),$$

where the first inequality follows from the Walrasian equilibrium property and the second follows from the fact that $a_k(R'_{N_{k-1}}, R_{-N_{k-1}}) = a_{k-1}$ and $p_{k-1} < t_{k-1}(R'_{N_{k-1}}, R_{-N_{k-1}})$ (Lemma 3). Hence Property (1) is satisfied for agent k. Next, if $a_k = 0$, then $(0,0) = z_k \ P_k \ f_k(R'_{N_{k-1}}, R_{-N_{k-1}})$ contradicts individual rationality. Hence, Property (2) also holds. Now, we satisfy Property (3) by constructing R'_k, which is z^ϵ_k-favoring for some $\epsilon > 0$ but arbitrarily close to zero - by Lemma 4, such R'_k can be constructed.

Thus, we have constructed a sequence $(1, R'_1), \ldots, (n, R'_n)$ such that $a_k \neq 0$ for all $k \in N$. This is impossible since $n > m$, giving us the required contradiction. ■

5.2 Proof of Theorem 2

We now fix a desirable mechanism $f : (\mathcal{R}^+)^n \to Z$ defined on the positive income effect domain $\mathcal{R}^+$. Further, we assume that f satisfies no bankruptcy, where the corresponding bound as $\ell \leq 0$. We start by proving an analogue of Lemma 3.

Lemma **5** *For every preference profile $R \in (\mathcal{R}^+)^n$, for every $i \in N$, and every $(a,t) \in M \times \mathbb{R}_+$ with $a = a_i(R) \neq 0$ and $t > 0$, if there exists $j \neq i$ such that*

$$V^{R_j}(b, (a,t)) < -n\big(\max_{k \in N} \max_{c \in M} V^{R_k}(c, (0,0))\big) + \ell,$$

then $t_i(R) > t$.

Proof: Assume for contradiction $t_i(R) \leq t$. Consider $R'_i = R_j$. By strategy-proofness, $f_i(R'_i, R_{-i}) \ R'_i \ f_i(R) = (a, t_i(R))$. By equal treatment of equals,

$$f_j(R'_i, R_{-i}) \ I_j \ f_i(R'_i, R_{-i}) \ R_j \ (a, t_i(R)).$$

Note that either $a_i(R'_i, R_{-i}) \neq a$ or $a_j(R'_i, R_{-i}) \neq a$. Without loss of generality, assume that $a_j(R'_i, R_{-i}) = b \neq a$. Then, using the fact that $(b, t_j(R'_i, R_{-i})) \ R_j \ (a, t_i(R))$ and $t_i(R) \leq t$, we get

$$\begin{aligned} t_j(R'_i, R_{-i}) &\leq V^{R_j}(b, (a, t_i(R))) \\ &\leq V^{R_j}(b, (a,t)) \\ &< -n\big(\max_{k \in N} \max_{c \in M} V^{R_k}(c, (0,0))\big) + \ell. \end{aligned}$$

By individual rationality

$$t_i(R'_i, R_{-i}) \leq V^{R'_i}(a_i(R'_i, R_{-i}), (0,0)) \leq \max_{c \in M} V^{R'_i}(c, (0,0)).$$

Further, individual rationality also implies that for all $k \notin \{i, j\}$,

$$t_k(R'_i, R_{-i}) \leq V^{R_k}(a_i(R'_i, R_{-i}), (0,0)) \leq \max_{c \in M} V^{R_k}(c, (0,0)).$$

Adding these three sets of inequalities above, we get

$$\begin{aligned}
&\sum_{k \in N} t_k(R'_i, R_{-i}) \\
&< -n\big(\max_{k \in N} \max_{c \in M} V^{R_k}(c, (0,0))\big) + \ell + \max_{c \in M} V^{R'_i}(c, (0,0)) + \sum_{k \in N \setminus \{i,j\}} \max_{c \in M} V^{R_k}(c, (0,0)) \\
&= -n\big(\max_{k \in N} \max_{c \in M} V^{R_k}(c, (0,0))\big) + \max_{c \in M} V^{R_j}(c, (0,0)) + \sum_{k \in N \setminus \{i,j\}} \max_{c \in M} V^{R_k}(c, (0,0)) \\
&= -n\big(\max_{k \in N} \max_{c \in M} V^{R_k}(c, (0,0))\big) + (n-1)\big(\max_{k \in N \setminus \{i\}} \max_{c \in M} V^{R_k}(c, (0,0))\big) \\
&\leq \ell.
\end{aligned}$$

This contradicts no bankruptcy. ■

Using Lemma 5, we can mimic the proof of Theorem 1 to complete the proof of Theorem 2. We start by defining a class of positive income effect preferences by strengthening the notion of $(a,t)^\epsilon$-favoring preference. For every $(a,t) \in M \times \mathbb{R}_+$, for each $\epsilon > 0$, and for each $\delta > 0$, define $\mathcal{R}((a,t), \epsilon, \delta)$ be the set of preferences such that for each $\hat{R}_i \in \mathcal{R}((a,t), \epsilon, \delta)$, the following holds:

1. $\hat{R}_i$ is $(a,t)^\epsilon$-favoring and
2. $V^{\hat{R}_i}(b, (a,t)) < -\delta$ for all $b \neq a$.

A graphical illustration of $\hat{R}_i$ is provided in Figure 4. Since $\delta > 0$, it is clear that a $\hat{R}_i$ can be constructed in $\mathcal{R}((a,t), \epsilon, \delta)$ such that it exhibits positive income effect. Hence, $\mathcal{R}^+ \cap \mathcal{R}((a,t), \epsilon, \delta) \neq \emptyset$.

Proof of Theorem 2

Proof: Now, we can mimic the proof of Theorem 1. We only show parts of the proof that requires some change. As in the proof of Theorem 1, we show that for every profile of

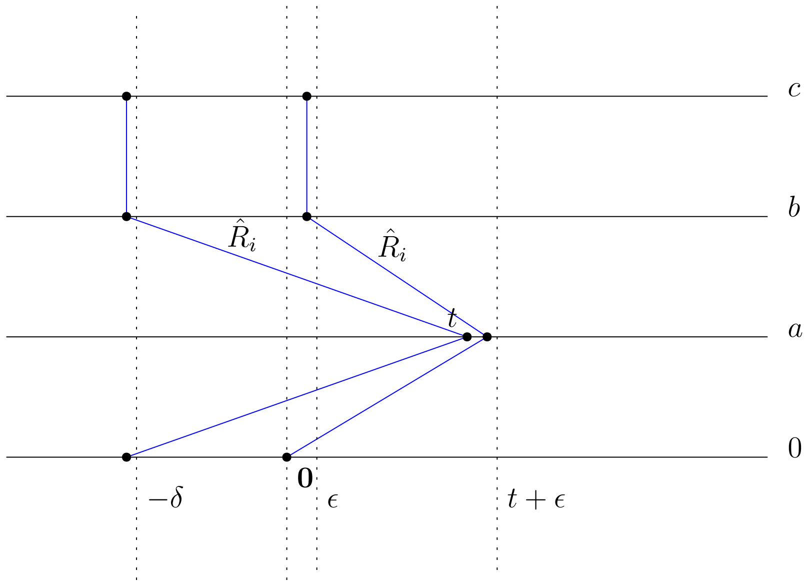

Figure 4: Illustration of $\hat{R}_i$

preferences R and for every $i \in N$, $f_i^{min}(R)\ R_i\ f(R)$. Assume for contradiction that there is some profile of preferences R and some agent, without loss of generality agent 1, such that $z_1\ P_1\ f_1(R)$, where $(z_1, \ldots, z_n) \equiv f^{min}(R)$ be the allocation chosen by the MWEP mechanism at R. For simplicity of notation, we will denote $z_j \equiv (a_j, p_j)$, where $p_j \equiv p_{a_j}^{min}(R)$, for all $j \in N$.

Define $\bar{\delta} > 0$ as follows:

$$\bar{\delta} := n\big(\max_{k \in N} \max_{c \in M} V^{R_k}(c, (0,0))\big) - \ell.$$

We first construct a finite sequence of agents and preferences: $(1, R'_1), (2, R'_2), \ldots, (n, R'_n)$ such that for every $k \in \{1, \ldots, n\}$,

1. $z_k\ P_k\ f_k(R)$ if $k = 1$ and $z_k\ P_k\ f_k(R'_{N_{k-1}}, R_{-N_{k-1}})$ if $k > 1$, where $N_{k-1} \equiv \{1, \ldots, k-1\}$.

2. $a_k \neq 0$,

3. $R'_k \in \mathcal{R}^+ \cap \mathcal{R}(z^k, \epsilon, \bar{\delta})$ for some $\epsilon > 0$ but arbitrarily close to zero.

Now, we can complete the construction of this sequence inductively as in the proof of Theorem 1 (using Lemma 5 instead of Lemma 3), giving us the desired contradiction. ■

6 Relation to the literature

Our paper is related to two strands of literature in mechanism design: (1) multi-object revenue maximization literature and (2) literature on mechanism design without quasilinearity. We discuss them in some detail below.

Revenue maximization literature. Ever since the work of Myerson (1981), various extensions of his work to multi-object case have been attempted. Note that quasilinearity is a central assumption in this literature whereas our results also work for non-quasilinear preferences. The work has mainly focused on the single agent (or, screening problem of a monopolist) with additive valuations (value for a bundle of objects is the sum of values of objects). Armstrong (1996, 2000) are early papers on showing the difficulty in extending Myerson's optimal auctions to multiple objects case - he illustrates the role played by "diagonal" non-local incentive constraints in such models and solves the optimal auction for special classes of distributions. [11] Rochet and Choné (1998) show how to extend the convex analysis techniques in Myerson's work to multidimensional environment and point out various difficulties in the derivation of an optimal auction. These difficulties are more precisely formulated in the following line of work for the single agent additive valuation case: (1) optimal mechanism may require randomization (Thanassoulis, 2004; Manelli and Vincent, 2007); (2) simple auctions like selling each good separately (Daskalakis et al., 2016) and selling all the goods as a grand bundle (Manelli and Vincent, 2006) are optimal for very specific distributions; (3) there is inherent revenue non-monotonicity of the optimal auction - if we take two distributions with one first-order stochastic-dominating the other, the optimal auction revenue may not increase (Hart and Reny, 2015); (4) the optimal auction may require an infinite menu of prices (Hart and Nisan, 2013). These difficulties have started a parallel literature in computer science and economics in showing the approximate optimality of simple auction forms. For the simple single agent and multiple object problem with additive valuations, Hart and Nisan (2012) show how selling separately and selling as a grand bundle can lead to *approximately* optimal auctions for a class of distributions. Carroll (2016) shows that selling separately is an optimal mechanism if the optimality criteria incorporates a certain kind of worst-case robustness.

Our work considers a problem with multiple agents. Indeed, equal treatment of equal axiom is vacuous and no wastage axiom makes no sense in the single agent setting. Further, it is unclear how some of these single agent results can be extended to the case of multiple

[11] Whenever we say optimal auctions, we mean, like in Myerson (1981), an expected revenue maximizing auction under incentive and participation constraints with respect to some prior distribution.

agents. In the multiple agent problems, the set of feasible allocations starts interacting with the incentive constraints of the agents. Further, the standard Bayesian incentive compatibility constraints become challenging to handle. Note that in the single agent problem, these notions of incentive compatibility are equivalent, and for one-dimensional mechanism design problems, they are equivalent in a useful sense (Manelli and Vincent, 2010; Gershkov et al., 2013). Because we work in a model without quasilinearity, we are essentially operating in an "infinite" dimensional type space. Hence, we should expect the problems discussed in quasilinear environment to appear in an even more complex way in our model. Indeed, in a companion paper (Kazumura et al., 2017), we investigate mechanism design without quasilinearity more abstractly and illustrate the difficulty of solving the single object optimal auction problem. Hence, solving for full optimality without imposing the additional axioms that we put seems to be even more challenging in our model. In that sense, our results provide a useful resolution to this complex problem.

It is also worth mentioning that to circumvent these difficulties, a literature in computer science has developed approximately optimal mechanisms for our model - multiple objects and multiple agents with unit demand bidders (but with quasilinearity). Contributions in this direction include Chawla et al. (2010a,b); Briest et al. (2010); Cai et al. (2012). Most of these papers allow for randomization and show that random mechanisms can do better than deterministic mechanisms. Further, these approximately optimal mechanisms involve reserve prices and violate no wastage axiom. It is unlikely that these results extend to environments without quasilinearity.

Non-quasilinearity literature. There is a short but important literature on auction design with non-quasilinear preferences. Baisa (2016a) considers the single object auction model and allows for randomization with non-quasilinear preferences. He introduces a novel mechanism in his setting and studies its optimality properties (in terms of revenue maximization). We do not consider randomization and our solution concept is different from his. Further, ours is a model with multiple objects.

The literature with non-quasilinear preferences and multiple object auctions have traditionally looked at Pareto efficient mechanisms. As discussed earlier, the closest paper is Morimoto and Serizawa (2015) who consider the same model as ours. They characterize the MPWE mechanism using Pareto efficiency, individual rationality, incentive compatibility, and no subsidy if the domain includes *all* classical preferences - see an extension of this characterization in a smaller type space in Zhou and Serizawa (2016). Pareto efficiency and the *complete* class of classical preferences play a critical role in pinning down the MPWE mechanism in these papers. As Tierney (2016) points out even in the quasilinear domain of preferences, there are desirable mechanisms satisfying no subsidy. Only by imposing revenue maximization as an objective, we get the MPWE mechanism in our model, and Pareto efficiency is obtained as an implication (Corollaries 1 and 2). Finally, our results work for a variety of non-quasilinear preferences, and not restricted to the complete class of classical preferences.

In the single object auction model, earlier papers have carried out axiomatic treatment similar to Morimoto and Serizawa (2015) - work along this line includes Saitoh and Serizawa (2008); Sakai (2008, 2013b,a); Adachi (2014); Ashlagi and Serizawa (2011).

When the set of preferences include all or a very rich class of non-quasilinear preferences and we consider multiple object auctions where agents can consume more than one object, strategy-proofness and Pareto efficiency (along with other axioms) have been shown to be incompatible - (Kazumura and Serizawa, 2016) show this for multi-object auction problems where agents can be allocated more than one object; (Baisa, 2016b) shows this for homogeneous object allocation problems; and Dobzinski et al. (2012); Lavi and May (2012) show similar results for hard budget-constrained auction of a single object. Pareto efficiency along with other axioms play a crucial role in such impossibility results.

There is a literature in auction theory and algorithmic game theory on single object auctions with budget-constrained bidders - see Che and Gale (2000); Pai and Vohra (2014); Ashlagi et al. (2010); Lavi and May (2012). The budget-constraint in these papers introduces a particular form of non-quasilinearity in preferences of agents. Further, the budget-constraint in these models is *hard*, i.e., the utility from any payment above the budget is minus infinity. This assumption is not satisfied by the preferences considered in our model since it leads to discontinuities. Further, these papers focus on single object auction.

7 Conclusion

We circumvent the technical difficulties of designing optimal multiple object auction by imposing additional axioms on mechanisms. We believe that these additional axioms are appealing in a variety of auction environment. A consequence of these assumptions is that we provide robust recommendations on revenue maximizing mechanism: the MWEP mechanism is revenue-maximal profile-by-profile, and the preferences of agents need not be quasilinear. Our proofs are elementary and without any convex analysis techniques used in the literature. Whether we can weaken some of these axioms and further strengthen our results is a question for future research.

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